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# On the circumcenters of triangular orbits in elliptic billiard

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## Abstract

On an elliptic billiard, we study the set of the circumcenters of all triangular orbits and we show that this is an ellipse. This article follows [18], which proves the same result with the incenters, and [6], which among others introduces the theory of complex reflection in the complex projective plane. The result we present was found at the same time by Ronaldo Garcia in an article to appear in American Mathematical Monthly (no preprint available). His proof uses completely different methods of real differential calculus.

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## 1 Overview of the problem

The famous great theorem of Poncelet, cf [1] and [15], asserts that *if there exists an  $n$ -sided polygon inscribed in a conic  $\mathcal{K}_i$  and circumscribed about another one  $\mathcal{K}_c$ , then there are infinitely many such polygons, and you can find such one for*

each point of  $\mathcal{K}_i$  chosen to be one of its vertices. A classical proof of it can be found in [1]. [9] and [19] give a way to prove it using complex methods.

It has a lot of consequences (see [1], [3]), especially in billiard theory, since it gives a condition to the existence of particular  $n$ -periodical orbits in conics. In particular, given an ellipse  $\mathcal{E}$ , one can find a confocal ellipse  $\gamma$  to  $\mathcal{E}$ , such that each triangular orbit on  $\mathcal{E}$  is circumscribed about it; and conversely one can complete each tangent line to  $\gamma$  to a triangular orbit of  $\mathcal{E}$ .

We study here the set of circumcenters, which are the centers of the circumscribed circles, of all triangular orbits in such an elliptic billiard  $\mathcal{E}$ . We want to prove the following:

**Theorem 1.1.** *The set  $\mathcal{C}$  of the circumcenters of all triangular orbits of the billiard within an ellipse is also an ellipse.*

*Remark 1.2.* Theorem 1.1 is obvious in the particular case where the ellipse is a circle, because then the set of circumcenters is reduced to a single point. Thus, from now on *we will assume that the ellipse is not a circle.*

There are many other results similar to theorem 1.1. Dan Reznik discovered experimentally the same result for the incenters of triangular orbits, see the video [16] and the github page [17] written with Jair Koiller. Romaskevitch (see [18]) confirmed these observations by proving them and her proof widely inspired ours. Tabachnikov and Schwartz, in [20], proved that the loci of the centers of mass (and of an other particular point) of a 1-parameter family of Poncelet  $n$ -gons in an ellipse is an ellipse homothetic to the previous one. They also mention that a similar result was proved by Zaslowski, Kosov and Muzafarov for the orthocenters ([21], reference from [20]). And Garcia (see [5]) uses explicit calculations to prove that the loci of incenters and orthocenters of triangular orbits are ellipses, and describes them precisely.

Before going into details, we give here a brief summary of the proof, which is inspired by [18], and in which we use the same complex methods. We consider a projective complexified version of  $\mathcal{C}$ , denoted by  $\hat{\mathcal{C}}$ , which turns out to be an algebraic curve as a consequence of Remmert proper mapping theorem, see [10] p. 34. Then we show that the intersection of the complex curve  $\hat{\mathcal{C}}$  with the foci line of the boundary ellipse  $\mathcal{E}$  is reduced to two points, each one of them corresponding to a single triangular orbit. Further algebraic arguments on the intersection type of  $\hat{\mathcal{C}}$  with the foci line of  $\mathcal{E}$  allow to conclude that it is a conic, using Bezout theorem. It's then easy to check that  $\hat{\mathcal{C}}$  is an ellipse since its real part is bounded.

As explained, one considers the projective complex Zariski closure of the ellipse  $\mathcal{E}$  and a complexified version of  $\mathcal{C}$ ,  $\hat{\mathcal{C}}$ . In order to define  $\hat{\mathcal{C}}$  and to prove the first statement concerning the intersection with the foci line, we study an extension of the reflection law and of the triangular orbits to complex domain, as in [18], and we use some of the results contained in the later article such as Proposition 2.14. Complex reflection law and complex planar billiards were introduced and studied by A. Glutsyuk in [6] and [7]. See also [8] where they were applied to solve the two-dimensional Tabachnikov's Commuting Billiard conjecture and a particular case of two-dimensional Plakhov's Invisibility conjecture with four reflections.

In [2] section III, a billiard reflection law was defined in purely projective terms using polar duality with respect to quadrics. Their definition makes sense in any dimension and over any number field ( $\mathbb{R}, \mathbb{C}, \dots$ ). However, the definition of [2] is given only for billiards within quadrics, but it can be adapted to any smooth hypersurface except at certain points, as we will prove in Subsection 2.4. Later a complex reflection law in two dimensions was introduced in another way by Glutsyuk [6, 7] which also takes into account these exceptional points. In dimension 2 and in the complex case, the two definitions coincide as we will prove in Subsection 2.4. The author follows Glutsyuk's definition in this work.

**Section 2** is devoted to the complex reflection law and to complex orbits in a complexified ellipse: in **Subsection 2.1**, we introduce the complex reflection law; **Subsection 2.2** recalls some results about complexified conics; we further define what is a triangular complex orbit in **Subsection 2.3**; then, in **Section 3** we introduce the definition and we study properties of complex circumscribed circles to such orbits: Proposition 3.6 is the main result of this section. Finally, **Section 4** is devoted to the proof of Theorem 1.1, using previous results.

## 2 Complex triangular orbits on an ellipse

### 2.1 Complex reflection law

Considering an affine chart whose coordinates will be denoted by  $(x, y)$ , we have the inclusion  $\mathbb{R}^2 \subset \mathbb{C}^2 \subset \mathbb{CP}^2$ , and  $\mathbb{CP}^2 = \mathbb{C}^2 \sqcup \overline{\mathbb{C}}_\infty$ , where  $\overline{\mathbb{C}}_\infty$  is the line at infinity. As introduced and explained in [6], and studied in [18], the reflection law in  $\mathbb{R}^2$  can be extended to  $\mathbb{CP}^2$  by considering the complexified version of the euclidean squared metric, that is the quadratic form

$$Q(z, w) = z^2 + w^2$$

which is a non degenerate quadratic form on  $\mathbb{C}^2$ . In a similar way to the euclidean case, it leads to construct a notion of symmetry in  $\mathbb{C}^2$ . It has two isotropic subspaces - which by definition are spaces where the quadratic form vanishes identically - of dimension 1 : namely  $F_I := \mathbb{C}(1, i)$  and  $F_J := \mathbb{C}(1, -i)$ . Since both spaces  $F_I$  and  $F_J$  are contained in their orthogonal spaces, the definition of symmetry of a line with respect to  $F_I$  or  $F_J$  cannot be introduced as in the usual euclidean way. A similar notion of symmetry with isotropic spaces is studied in [12] and [4] in pseudo-Euclidean and pseudo-Riemannian cases.

**Definition 2.1** ([6], definition 1.2). Define the *isotropic points at infinity* (or *cyclic points*) to be either  $I = [1 : i : 0]$  or  $J = [1 : -i : 0]$ . A line in  $\mathbb{CP}^2$  is said to be *isotropic* if it contains either  $I$  or  $J$ , and *non-isotropic* if not. Notice that the line at infinity is automatically isotropic.

**Definition 2.2** ([6], definition 2.1). The *symmetry* with respect to a line  $L \subset \mathbb{C}^2$  is defined as follows:

- Case 1:  $L$  is non-isotropic. The *symmetry acting on  $\mathbb{C}^2$*  is the unique non-trivial complex isometric involution (for the above quadratic form) fixing the points of the line  $L$ . It induces the same symmetry acting on lines.

- Case 2:  $L$  is isotropic. We define the *symmetry of lines through a finite point*  $x \in L$ : two lines  $l$  and  $l'$  which contain  $x$  are called symmetric if there are sequences  $(L_n)_n, (l_n)_n, (l'_n)_n$  of lines through points  $x_n$  so that  $L_n$  is non-isotropic,  $l_n$  and  $l'_n$  are symmetric with respect to  $L_n$ ,  $l_n \rightarrow l$ ,  $l'_n \rightarrow l'$ ,  $L_n \rightarrow L$  and  $x_n \rightarrow x$ .

We recall now lemma 2.3 [6] which gives an idea of this notion of symmetry in the case of an isotropic line through a finite point.

**Lemma 2.3** ([6], lemma 2.3). *If  $L$  is an isotropic line through a finite point  $x$  and  $l, l'$  are two lines which contain  $x$ , then  $l$  and  $l'$  are symmetric with respect to  $L$  if and only if either  $l = L$ , or  $l' = L$ .*

This complex reflection law allows to talk about complex billiard orbits on the ellipse, as it will be done in subsection 2.3. Before studying those orbits, it is necessary to introduce some geometric notions about projective conics.

## 2.2 Preliminary results on complexified conics

One needs to present here useful results on confocal conics. They can be found in [1] and [13]. [6] also cites them in subsection 2.4. This section allows to understand some links between an ellipse and its Poncelet ellipse of triangular orbits, when both are complexified. Thus, by conic (resp. ellipse) we mean here the complex projective closure of a real regular conic (resp. ellipse). This choice of definition is due to the fact that the ellipse in which we study billiard orbits is a real ellipse. Later, in order to define circumscribed circle (see section 3), we will understand conics as general complex conics (not just complexified real ones).

**Proposition 2.4** ([1] subsection 17.4.2.1). *A conic is a circle if and only if some of the points  $I$  or  $J$  belong to it. Furthermore, if a conic is a circle, then both  $I$  and  $J$  belong to it.*

In fact, a circle has two isotropic tangent lines intersecting at its center (see the following propositions).

**Proposition 2.5** ([1] subsection 17.4.3.1). *A focus  $f$  of a conic lies in the intersection of two isotropic tangent lines to the conic.*

**Proposition 2.6** ([13], p. 179). *Two complexified confocal ellipses have the same tangent isotropic lines, which are four isotropic lines taken with multiplicities: one pair intersecting at a focus, and the other one - at the other focus.*

This brings us to the following redefinition of the foci:

**Definition 2.7** ([1] subsection 17.4.3.2). The *complex foci* of an ellipse are the intersection points of its isotropic tangent lines.

*Remark 2.8.* The complex projective closure of a real ellipse has four complex foci, including two real ones.

**Corollary 2.9.** *A conic has at most four distinct finite isotropic tangent lines, each two of them intersecting either at a focus, or at an isotropic point at infinity.*

## 2.3 Complex orbits

We have enough material at this stage to study complex triangular orbits. See [6], definition 1.3, for a more general definition of periodic orbits.

**Definition 2.10** ([6], definition 1.3). A *non-degenerate triangular complex orbit* on a complex conic  $\mathcal{E}$  is an ordered triple of points  $A_1, A_2, A_3$  on  $\mathcal{E}$  so that for all  $i$ , one has  $A_i \neq A_{i+1}$ , the tangent line  $T_{A_i}\mathcal{E}$  is non isotropic, and the complex lines  $A_iA_{i+1}$  and  $A_iA_{i-1}$  are symmetric with respect to  $T_{A_i}\mathcal{E}$  (with the obvious convention  $A_0 := A_3$  and  $A_4 := A_1$ ). A *side* of a non-degenerate orbit is a complex line  $A_iA_{i+1}$ .

*Remark 2.11.* The vertices of a non-degenerate orbit are not collinear since a line intersects the ellipse in at most two points.

*Remark 2.12.* As explained in [7], the reflection with respect to a non-isotropic line permutes the isotropic directions  $I$  and  $J$ . This argument implies that a non-degenerate triangular orbit has no isotropic side.

We will also study the limit orbits of the above defined orbits, which will be called degenerate orbits.

**Definition 2.13** ([6]). A *degenerate triangular complex orbit* on a complex conic  $\mathcal{E}$  is an ordered triple of points in  $\mathcal{E}$  which is the limit of non-degenerate orbits and which is not a non-degenerate orbit. We define the *sides* of a degenerate orbit as the limits of the sides of the non-degenerate orbits which converge to it. If  $A_j = A_{j+1}$ , then it is natural to define the side  $A_jA_{j+1}$  as the tangent line  $T_{A_j}\mathcal{E}$ .

**Proposition 2.14** ([18], lemma 3.4). *A degenerate triangular orbit of an ellipse  $\mathcal{E}$  has an isotropic side  $A$  which is tangent to  $\mathcal{E}$ , and two coinciding non-isotropic sides  $B$ .*

During the proof, it will be convenient to distinguish two types of orbits : the ones with no points at infinity, and the others, with at least one point at infinity:

**Definition 2.15.** An *infinite triangular complex orbit* on a complex conic  $\mathcal{E}$  is an orbit which has at least one vertex on the line at infinity. The orbits with only finite vertices are called *finite orbits*.

**Proposition 2.16.** *An infinite orbit is not degenerate, and has exactly one vertex at infinity.*

*Proof.* Suppose two vertices,  $\alpha, \beta$ , of the orbit are at infinity. Then,  $\alpha\beta$  is the line at infinity. But the tangent  $T_\beta$  to the ellipse  $\mathcal{E}$  in  $\beta$  is not isotropic, and the line at infinity reflects to itself through the reflection by  $T_\beta$ . Hence, the orbit is  $\{\alpha, \beta\} = \overline{\mathbb{C}}_\infty \cap \mathcal{E}$ , which should be a degenerate orbit. But it cannot be a degenerate orbit by Proposition 2.14 since the tangent lines to its vertices  $\alpha, \beta$  are not isotropic. Thus, only one vertex lies at infinity.

Therefore, if it is a degenerate orbit, it has two vertices,  $\alpha, \beta$ , corresponding by Proposition 2.14 to two sides,  $A$  which is isotropic and tangent to the ellipse in  $\alpha$ , and  $B$  which is a line containing  $\alpha$  and  $\beta$ . Since the tangency points of

isotropic tangent lines are finite,  $\alpha$  is finite. Thus  $\beta$  is infinite (because the orbit is supposed infinite). Then  $B$  and the tangent line  $T_\beta\mathcal{E}$  to the ellipse in  $\beta$  are collinear (since they have the same intersection point at infinity). But both are stable by the complex reflection by  $T_\beta$ , hence  $T_\beta\mathcal{E} = B$  which is impossible since  $B$  is not tangent to the ellipse.  $\square$

## 2.4 Digression: reflection law on a quadric $Q_1$ with respect to another quadric $Q_2$

In [2] section III, the authors define the reflection law *inside* a quadric  $Q_1$  with respect to another quadric  $Q_2$  in  $\mathbb{R}^n$  using polar considerations. They use this definition to prove what they call the *Full Poncelet's Theorem*:

**Theorem 2.17** (Chang, Crespi, Shi, see [2]). *Let  $Q_1, \dots, Q_m$  be confocal quadrics in  $\mathbb{R}^n$  and  $x_1x_2 \dots x_m$  be a polygon such that  $x_i$  lie on  $Q_i$  and the successive sides  $\overrightarrow{x_{i-1}x_i}, \overrightarrow{x_ix_{i+1}}$  satisfy the usual reflection law. Then all the sides of the polygon are tangent to a confocal quadric  $Q$ . Furthermore, for these quadrics  $Q_1, \dots, Q_m, Q$ , there is an  $(n-1)$ -parameters family of polygons satisfying the same property.*

Their definition of reflection has a natural extension to other fields  $k$  than  $\mathbb{R}$ , and we would like to compare it when  $k = \mathbb{C}$  to the complex reflection law which is described in Subsection 2.1. We are really grateful to the referee who suggested us to read the above-mentioned article.

In the real case one can define locally an *inside* and an *outside* of a quadric  $Q$  (since a real hypersurface of  $\mathbb{R}^n$  locally separates a small ball centered at its points into two domains). Therefore one can define an orientation on incident and reflected rays, so that the reflected ray is directed to the side where the incident ray arrived from. But in complex spaces this is not the case anymore, and in the following descriptions we will omit any considerations on orientations of rays.

Consider the field  $k = \mathbb{R}$  or  $\mathbb{C}$  and let  $Q$  be a non-degenerate quadric in  $\mathbb{P}^n(k)$ . For any hyperplane  $u \subset \mathbb{P}^n(k)$  one can define its pole  $z$  with respect to  $Q$  (see [2] for more details). Now suppose  $Q_1$  and  $Q_2$  are two quadrics. If  $x \in Q_1$ , one can consider the tangent space  $u = T_xQ_1$  to  $Q_1$  at  $x$ , and denote by  $z$  its pole with respect to  $Q_2$ .

**Definition 2.18** (section III of [2] without orientations). Note  $u = T_xQ_1$  and suppose that  $z \notin u$ . Let  $\ell_1$  and  $\ell_2$  be two lines intersecting  $Q_1$  at  $x$  transversally. We say that  $\ell_1$  and  $\ell_2$  satisfy the law of reflection at  $x$  on  $Q_1$  about  $Q_2$  if the lines  $\ell_1, \ell_2, xz$  are contained in a plane  $P$ , and if the lines  $\ell_1, \ell_2, xz, P \cap u$  form a harmonic set.

As explained in [2], when  $k = \mathbb{R}$  and  $Q_1$  and  $Q_2$  are two confocal quadrics (the definition of confocal quadrics is recalled below, see formula (1)), this definition is the same as the usual billiard reflection. We would like to show that this result holds in the case when  $Q_1$  and  $Q_2$  are complex confocal quadrics, namely that *generically two lines  $\ell_1$  and  $\ell_2$  intersecting at a point  $x \in Q_1$  satisfy the law of reflection at  $x$  about  $Q_2$  (as in Definition 2.18) if and only if they are symmetric by the complex reflection law at  $x$  (as defined in Subsection 2.1).*

On  $\mathbb{CP}^n_{(z_0:\dots:z_n)}$ , we choose a standard affine chart  $U = \{z_n = 1\} \subset \mathbb{CP}^n$ . Consider the complex quadratic form  $Q = \sum_{k=0}^{n-1} dz_k^2$  on  $\mathbb{C}^n$  which can be seen as a quadratic form on  $U \simeq \mathbb{C}^n$ .

**Proposition 2.19.** *Let  $Q_1, Q_2$  be two distinct confocal quadrics in  $\mathbb{CP}^n$ ,  $x \in Q_1 \cap U$ ,  $u = T_x Q_1$  and  $z$  be the pole of  $u$  with respect to  $Q_2$ . Then in the chart  $U$ ,  $u$  and  $xz$  are orthogonal for  $Q$ .*

*Remark 2.20.* We believe this result is well-known by specialists and can be shown using elementary results of polar geometry, but we were unable to find any proof of it. This result is also true in the real case, but we haven't found any proof of it, even in [2] where it is also mentioned without any proof.

*Proof.* Any point in  $\mathbb{CP}^n$  can be written as the projection  $\bar{x} = (x_0 : \dots : x_n)$  of a point  $x \in \mathbb{C}^{n+1}$ . We suppose that  $Q_i = \{q_i = 0\}$  for  $i = 1, 2$  where  $q_1, q_2$  are the quadratic forms defined for all  $x = (x_0, \dots, x_n)$  by

$$q_1(x) = \sum_{k=0}^{n-1} \frac{x_k^2}{a_k} - x_n^2 \quad \text{and} \quad q_2(x) = \sum_{k=0}^{n-1} \frac{x_k^2}{a_k + \lambda} - x_n^2 \quad (1)$$

with  $a_1, \dots, a_n, \lambda$  being complex numbers such that  $a_k \neq 0$  and  $a_k + \lambda \neq 0$ . Write  $M_1$  (respectively  $M_2$ ) the diagonal  $(n+1) \times (n+1)$  matrix whose diagonal coefficients are the  $a_k^{-1}$  (respectively the  $(a_k + \lambda)^{-1}$ ),  $k = 0 \dots n$ , and  $-1$ . Both matrices  $M_1$  and  $M_2$  are such that  $q_i(x) = \langle x | M_i x \rangle$  and the bilinear form associated to  $q_i$  is  $b_i(x, x') = \langle x | M_i x' \rangle$ , where we set  $\langle x | y \rangle = \sum_{k=0}^n x_k y_k$ . Note that for any matrix  $M$  one has  $\langle x | M y \rangle = \langle M^T x | y \rangle$  where  $M^T$  is the transpose of  $M$ . Therefore, for a  $\bar{x} \in Q_1 \cap U$ , the equation of  $T_{\bar{x}} Q_1$  is given by the set of all  $\bar{y}$  verifying

$$0 = b_1(x, y) = \langle M_1 x | y \rangle = \langle M_1 x | M_2^{-1} M_2 y \rangle = \langle M_2^{-1} M_1 x | M_2 y \rangle = b_2(M_2^{-1} M_1 x, y).$$

Hence the pole of  $u = T_{\bar{x}} Q_1$  with respect to  $Q_2$  is the point  $\bar{z}$  where  $z = M_2^{-1} M_1 x$ . Now the diagonal forms of the matrices allow to compute easily that  $z = M_2^{-1} M_1 x = x + \lambda u$  where

$$u = \left( \frac{x_0}{a_0}, \dots, \frac{x_{n-1}}{a_{n-1}}, 0 \right).$$

Therefore  $xz$  is the line containing  $x$  and directed by  $u$ . Now the vector subspace of  $\mathbb{C}^n \simeq U$  parallel to  $T_{\bar{x}} Q_1 \cap U$  is defined by the equation  $\sum_{k=0}^{n-1} \frac{x_k}{a_k} dx_k = 0$  which is the equation of the  $Q$ -orthogonal space to  $u$ . Hence  $xz$  and  $T_{\bar{x}} Q_1$  are  $Q$ -orthogonal.  $\square$

Choose  $n = 2$ , consider a standard affine chart  $\mathbb{C}^2 \subset \mathbb{CP}^2$ . On this chart define the same quadratic form  $Q(z, w) = z^2 + w^2$  as in subsection 2.1. Let  $C$  be a conic and  $x \in C$ . Proposition 2.19 implies the following

**Corollary 2.21.** *The  $Q$ -normal line to the tangent line  $u = T_x C$  to  $C$  is the line  $xz$ , where  $z$  is the pole of  $u$  with respect to any confocal conic  $C'$  to  $C$ .*



Yet if two lines  $\ell_1$  and  $\ell_2$  intersect at  $x$ , they are symmetric with respect to  $u$  (for the complex reflection law as defined in Subsection 2.1) if and only if the lines  $\ell_1$ ,  $\ell_2$ ,  $u$  and the  $Q$ -normal line to  $C$  at  $x$  form a harmonic set. Note that if  $u$  is an isotropic tangent line to  $C$ , then  $u$  and its  $Q$ -normal line through  $x$  are the same, and the above mentioned result on harmonic sets makes no sense. But in the other case we have the

**Corollary 2.22.** *If  $u = T_x C$  is not an isotropic tangent line to  $C$ , then two lines  $\ell_1$  and  $\ell_2$  through  $x$  are symmetric with respect to  $u$  for the complex reflection law introduced in Subsection 2.1 if and only if  $\ell_1$  and  $\ell_2$  satisfy the law of reflection at  $C$  about any confocal conic  $C'$  (Definition 2.18).*

Therefore Definition 2.18 coincide with the complex reflection law introduced in Subsection 2.1 in the case when the reflection takes place on a conic, at a point  $x$  satisfying a generic condition of non-isotropy of its tangent line. From Proposition 2.19 we deduce a way to extend Definition 2.18 to any smooth hypersurface  $\Gamma$  which needs not to be a quadric.

**Definition 2.23.** Let  $\Gamma \subset \mathbb{C}\mathbb{P}^n$  be a germ of  $\mathcal{C}^1$ -smooth hypersurface at a point  $x \in \mathbb{C}\mathbb{P}^n$ . Consider a quadric  $Q_1$  containing  $x$  and such that  $T_x Q_1 = u$ . We say that two lines  $\ell_1$  and  $\ell_2$  through  $x$  are *CCS-symmetric with respect to  $u$*  if and only if  $\ell_1$  and  $\ell_2$  satisfy the law of reflection at  $Q_1$  about any confocal quadric  $Q_2$  given by Definition 2.18.

By Proposition 2.19, this definition doesn't depend on the choice of the quadric  $Q_2$ . We still have the

**Proposition 2.24.** *Let  $\Gamma \subset \mathbb{C}\mathbb{P}^2$  be a germ of  $\mathcal{C}^1$ -smooth curve and  $x \in \Gamma$  such that  $u = T_x \Gamma$  is not isotropic. Then two lines  $\ell_1$  and  $\ell_2$  through  $x$  are symmetric with respect to  $u$  for the complex reflection law introduced in Subsection 2.1 if and only if  $\ell_1$  and  $\ell_2$  are CCS-symmetric with respect to  $u$ .*

We conclude that, in the complex setting, Definition 2.23 extends the law introduced in [2] to any smooth hypersurface  $\Gamma$  of  $\mathbb{C}\mathbb{P}^n$ , and a similar work can be done for hypersurfaces of  $\mathbb{R}\mathbb{P}^n$ . In the two-dimensional complex case, this extension coincide with the complex reflection law introduced in [6, 7] except that it is not well-defined at points of isotropic tangency.

### 3 Circumcircles and circumcenters of complex orbits

Here we present the last part of the required definitions, which concerns the complex circles circumscribed to triangular orbits. This part is different from the previous one, because here the considered conics are complex and not necessarily complexified versions of real conics.

**Definition 3.1.** A *complex circle* is a regular complex conic passing through both isotropic points at infinity. Its *center* is the intersection point of its tangent lines at the isotropic points.

**Proposition 3.2.** *For a non-degenerate finite orbit, there is a unique complex circle passing through the vertices of the orbit and both isotropic points at infinity. It is called the circumscribed circle or circumcircle to the non-degenerate orbit.*

*Proof.* Denote by  $\alpha, \beta, \gamma$  the vertices of the orbit. We have to prove that no three points of  $\alpha, \beta, \gamma, I, J$  are collinear. Indeed, as no vertices are on the line at infinity, we only need to study two different cases:

1.  $\alpha, \beta, \gamma$  are not collinear because they are distinct and they lie on the ellipse which has at most two intersection points with any line.
2.  $\alpha, \beta, I$  are not collinear or else the line  $\alpha\beta$  would be isotropic. But this is impossible for a non-degenerate triangular orbit by Remark 2.12.

We then exclude all other possible combinations of two vertices of the orbit with  $I$  or  $J$ , using the same arguments.

□

Let us extend this definition to degenerate orbits.

**Definition 3.3.** Let  $T$  be a degenerate or infinite orbit. A *circumscribed circle* of  $T$  is the limit (in the space of conics) of a converging sequence of circumscribed circles of non-degenerate finite orbits converging to  $T$ . If a sequence of complex circles converges to a conic so that their centers converge to a point  $c \in \mathbb{CP}^2$ , then  $c$  is called a center of the limit conic. A *circumcenter* of  $T$  is a center of its circumscribed circle.

*Remark 3.4.* *A priori*, a limit conic  $\mathcal{K}$  may have several centers in the sense of this definition. Indeed,  $c$  depends on the choice of the sequence of circles converging to  $\mathcal{K}$ . See Case 4 of Proposition 3.5 and its proof for more details.

Even if they are called *circles*, the circumscribed circles to a degenerate or infinite orbit can degenerate into pairs of lines, as described below.

**Proposition 3.5.** *The limit of a converging sequence of complex circles is one of the following:*

1. a regular circle ;
2. a pair of isotropic non-parallel finite lines ; the corresponding center lies on their intersection ;
3. the infinite line and a finite line  $d$  ; the center  $c$  lies on the line at infinity and represents a direction which is orthogonal to  $d$  ;
4. the line at infinity taken twice : its center can be an arbitrary point in  $\mathbb{CP}^2$ .

*Proof.* The equation of a regular circle  $\mathcal{D}$  is of the form

$$a(x^2 + y^2) + pxz + qyz + rz^2 = 0$$

where  $a, p, q, r \in \mathbb{C}$ ,  $a \neq 0$  and  $4ar \neq p^2 + q^2$ . Both isotropic tangent lines to  $\mathcal{D}$  have equations  $2a(x \pm iy) + (p \pm iq)z = 0$ , whose intersection is  $c = (p : q : -2a)$ , which is the center of  $\mathcal{D}$  by definition.

If we take a limit of regular circles, the equation of the limit circle is of the same type, that is

$$a(x^2 + y^2) + pxz + qyz + rz^2 = 0$$

but maybe with  $a = 0$  or  $4ar = p^2 + q^2$ . And the center  $c$  is still of coordinates  $(q : p : -2a)$ .

If  $a = 0$ , the limit circle is the union of the line at infinity ( $z = 0$ ) and the line  $d$  of equation  $px + qy + rz = 0$ . The line  $d$  is finite if and only if  $(p, q) \neq 0$ , and in this case it has direction  $(q, -p)$ . Since  $c = (p : q : 0)$ , the direction represented by  $c$  is orthogonal to  $d$ . If  $d$  is infinite, the limit circle is the (double) line at infinity. Note that in this case the center can be an arbitrary point.

If  $a \neq 0$ , but  $4ar = p^2 + q^2$ , the equation of the limit circle becomes

$$\left(x + \frac{p}{2a}z\right)^2 + \left(y + \frac{q}{2a}z\right)^2 = 0$$

which is the equation of two isotropic non collinear lines intersecting at the point  $(-\frac{p}{2a} : -\frac{q}{2a} : 1) = (p : q : -2a) = c$ .

If  $a \neq 0$  and  $4ar \neq p^2 + q^2$ , the limit circle is regular.  $\square$

Now let us find which triangular orbits have their center on the line of real foci of  $\mathcal{E}$ .

**Proposition 3.6.** *Suppose that  $T$  is a complex triangular orbit whose circumcenter lies on the real foci line. Then  $T$  is finite, non-degenerate, symmetric with respect to the real foci line of  $\mathcal{E}$ , and has a vertex on it.*

*Proof.* Let  $T$  be a triangular orbit with a circumscribed circle  $C$  having a center  $c$  on the real foci line of  $\mathcal{E}$ .

**First case : Suppose  $T$  is finite and non-degenerate.** We follow the arguments of Romaskevich [18] who treated the similar case for incenters. Indeed, at least two vertices should lie outside the foci line. If the line through them is not orthogonal to the foci line, then this pair of vertices together with their symmetric points and the remaining third vertex in  $T$  are five distinct points contained in the intersection  $\mathcal{E} \cap C$ . This is impossible, since  $\mathcal{E}$  is not a circle. Finally, the remaining vertex has to be on the foci line, or else we could find two distinct orbits sharing a common side, which is impossible by definition of the reflection law with respect to non-isotropic lines.

**Second case : Suppose  $T$  is infinite.** Then the line at infinity cuts  $C$  in three distinct points, hence  $C$  is degenerate. By Proposition 3.5,  $C$  contains the line at infinity. Since  $T$  has only one infinite vertex  $\alpha$  by Proposition 2.16, and two other finite vertices  $\beta, \gamma$ , the other line  $d \subset C$  is not the line at infinity. Again by Proposition 3.5, the center is infinite and represents the orthogonal direction to  $d$ . Since it is on the real foci line, the latter is orthogonal to  $d$ . Thus  $d$  intersects the infinity line at the same point as the line orthogonal to the foci line. This point does not lie in  $\mathcal{E}$ , and in particular,  $d$  does not contain  $\alpha$ . Hence, we have  $d = \beta\gamma$

is a side of  $T$ ,  $\alpha \notin d$  and by the same symmetry argument as in the first case  $\alpha$  should belong to the real foci line. But this is impossible since the latter intersects  $\mathcal{E}$  in only two finite points.

**Last case : Suppose  $T$  is degenerate.** Then  $C$  cannot be a regular circle, otherwise the latter would be tangent to  $\mathcal{E}$  in a point of isotropic tangency (by Proposition 2.14): this would imply that this point of isotropic tangency is  $I$  or  $J$ , which is impossible since they do not belong to  $\mathcal{E}$ , assumed not to be a circle.

The circumcircle  $C$  cannot be the union of the line at infinity and another line  $d$ . Otherwise, by the same arguments as in the second case, this line would be othogonal to the real foci line. Since  $T$  is finite (Proposition 2.16),  $d$  goes through its both vertices, implying that they are symmetric with respect to the foci line. Therefore, both vertices are points of isotropic tangency but this cannot happen for a degenerate triangular orbit.

Finally suppose  $C$  is the union of two isotropic lines having different directions.

**Lemma 3.7.** *Let  $C_n$  be a sequence of circles containing two distinct points  $M_n$  and  $N_n$  of  $\mathcal{E}$  converging to the same finite point  $\alpha$ . Suppose  $C_n$  has a center  $c_n$  converging to a finite point  $c \neq \alpha$ . Then the line  $c\alpha$  is orthogonal to the line  $T_\alpha\mathcal{E}$ .*

*Proof.* The tangent line to  $C_n$  at  $M_n$  is orthogonal to the line  $M_n c_n$  hence the same is true for their limits. The limit of  $T_{M_n} C_n$  is obviously the limit of the line  $M_n N_n$ . Since  $M_n$  and  $N_n$  are on  $\mathcal{E}$ , the line  $M_n N_n$  also converges to the tangent line  $T_\alpha\mathcal{E}$ . Hence  $T_\alpha\mathcal{E}$  is orthogonal to  $c\alpha$ .  $\square$

Thus if  $\alpha$  is a vertex of isotropic tangency of the orbit, Lemma 3.7 implies that  $c\alpha$  is orthogonal to  $T_\alpha\mathcal{E}$ , hence  $ac = T_\alpha\mathcal{E}$  since the latter is isotropic. Recall that  $\alpha$  does not lie in the real foci line. Since both isotropic lines constituting the circle go through  $c$ , one of them is  $T_\alpha\mathcal{E}$ . Hence, they are both tangent to  $\mathcal{E}$  by symmetry with respect to the real foci line. Thus the other vertex of  $T$  is a point of isotropic tangency of  $\mathcal{E}$ , which is not possible by the previous arguments (such an orbit is not closed).  $\square$

## 4 Proof of Theorem 1.1

We recall that  $\mathcal{E}$  is a complexified ellipse, which we will identify with  $\overline{\mathbb{C}}$ . Denote by  $\gamma$  the real ellipse inscribed in all triangular real periodic orbits. We use the same notation  $\gamma$  for its complexified version.

Consider the Zariski closure  $\mathcal{T}$  of the set of real triangular orbits (which are circumscribed about  $\gamma$ ). Let  $\mathcal{T}_3$  denote the set of triangles with vertices in  $\mathcal{E}$  that are circumscribed about  $\gamma$ . It is a Zariski closed subset of  $\mathcal{E}^3 \simeq (\mathbb{CP}^1)^3$  that contains the real orbits and can be identified with the set of pairs  $(A, L)$ , where  $A$  is a point of the complexified ellipse  $\mathcal{E}$  and  $L$  is a line through  $A$  that is tangent to  $\gamma$ . The set of the above pairs  $(A, L)$  is identified with an elliptic curve, and each pair extends to a circumscribed triangle as above, see the complex Poncelet Theorem and its proof in [9] for more details. Hence  $\mathcal{T}_3$  is an irreducible algebraic curve. Each triangle in  $\mathcal{T}$  is circumscribed about  $\gamma$ , by definition and since this is

true for the real triangular orbits and the tangency condition of the edges with  $\gamma$  is algebraic. Thus  $\mathcal{T} \subset \mathcal{T}_3$ . Hence  $\mathcal{T} = \mathcal{T}_3$ , by definition and since the curve of real triangular orbits (which is contained in  $\mathcal{T}$ ) is Zariski dense in  $\mathcal{T}_3$  (irreducibility). Now the set  $\hat{\mathcal{T}} \subset \mathcal{T}$  of complex non-degenerate triangular orbits circumscribed about the Poncelet ellipse  $\gamma$  is a subset of  $\mathcal{T}_3 = \mathcal{T}$ , Zariski open in  $\mathcal{T}$  (because  $\mathcal{T} \setminus \hat{\mathcal{T}}$  is defined by polynomial equations). Note that  $\mathcal{T} \setminus \hat{\mathcal{T}}$  is finite (since it is a proper Zariski closed subset of an algebraic curve  $\mathcal{T}$ ), and  $\hat{\mathcal{T}}$  is dense in  $\mathcal{T}$  for the usual topology. Thus the analytic map  $\phi : \hat{\mathcal{T}} \rightarrow \mathbb{C}\mathbb{P}^2$  which assigns to a non-degenerate orbit its circumcenter can be extended to a holomorphic map  $\mathcal{T} \rightarrow \mathbb{C}\mathbb{P}^2$ , being a rational map. And by Remmert proper mapping theorem (see [10]), its image denoted by  $\hat{\mathcal{C}}$  is an irreducible analytic curve of  $\mathbb{C}\mathbb{P}^2$ , hence it is an irreducible algebraic curve by Chow theorem (see [10]).

Let us show that  $\hat{\mathcal{C}}$  is a conic, using Bezout theorem and studying its intersection with the real foci line of  $\mathcal{E}$ . In fact, we already know two distinct points lying on this intersection: the circumcenters  $c_1$  and  $c_2$  of both triangular real orbits  $T_1$  and  $T_2$  circumscribed about Poncelet's ellipse  $\gamma$  and having a vertex on the foci line.

**Lemma 4.1.** *The foci line of the ellipse intersects  $\hat{\mathcal{C}}$  in only  $c_1$  and  $c_2$  which are distinct, and for each  $i$  the only triangular orbit of  $\mathcal{T}$  having  $c_i$  as a circumcenter is  $T_i$ .*

*Proof.* Take a point  $c$  of  $\hat{\mathcal{C}}$  lying on the foci line. Then by Proposition 3.6, an orbit of center  $c$  is finite, non-degenerate, and has a vertex on the foci line. If this orbit is in  $\mathcal{T}$ , it is circumscribed about  $\gamma$ . One of its vertices lies on the foci line, hence coincides with a vertex of some  $T_i$ . Hence it is  $T_1$  or  $T_2$ , otherwise we could find a number strictly greater than two of tangent lines to  $\gamma$  containing a vertex of  $\mathcal{E}$ . Furthermore, if  $c_1 = c_2$ , the circumcircle of  $T_1$  would be the same as the one of  $T_2$  by symmetry, and  $\mathcal{E}$  would share six distinct points with the former, which is impossible. The result follows.  $\square$

**Theorem 4.2.** *The set  $\hat{\mathcal{C}} \subset \mathbb{C}\mathbb{P}^2$  is an ellipse.*

*Proof.* Let us show that  $c_1$  is a regular point of  $\hat{\mathcal{C}}$ , and that the latter intersects the foci line transversally. Fix an order on the vertices of  $T_1$  and consider the germ  $(\mathcal{T}, T_1)$ . The latter is irreducible (because parametrized by  $\gamma$ ), hence the germ  $(V, c_1) \subset (\hat{\mathcal{C}}, c_1)$  defined as  $\phi(\mathcal{T}, T_1)$  is also irreducible. By Lemma 4.1, any other irreducible component  $V'$  of  $(\hat{\mathcal{C}}, c_1)$  is parametrized locally by  $\phi$  and a germ  $(\mathcal{T}, T'_1)$ , where  $T'_1$  is obtained from  $T_1$  by a permutation of its vertices. Thus  $V' = V$  since  $\phi$  doesn't change by permutation of the vertices of the orbits:  $(\hat{\mathcal{C}}, c_1)$  is irreducible.

We fix a local biholomorphic parametrization  $P(t)$  of the complexified ellipse  $\mathcal{E}$ , so that  $P_0 = P(0)$  is a vertex of the real ellipse  $\mathcal{E}$  that is also a vertex of the real triangular orbit  $T_1$ . This gives local parametrizations of the orbits  $T(P)$  whose first vertex is  $P$  and of their circumcenters  $c(t) = \phi(T(P(t)))$ . We restrict  $P$  to the curve  $P(t)$  parametrizing the real points of  $\mathcal{E}$ . We can suppose that  $P(t)$  and  $P(-t)$  are symmetric with respect to  $\mathcal{F}$ . Write  $r(t) = |P(t)c(t)|$  for the radius of the circumscribed circle to  $T(t)$ . Thus we have  $c(0) = \phi(T_1) = c_1$ , and we need to

show that  $c'(0) \neq 0$  and that  $c'(0)$  has not the same direction as the line of real foci of  $\mathcal{E}$ .

First, we have  $r(t) = r(-t)$  by symmetry, and  $r$  is smooth around 0 since  $P(0) \neq c(0)$ . Thus,  $r'(0) = 0$ . This implies that the vector  $c'(0) - P'(0)$  is orthogonal to the line  $c(0)P(0)$ , which is the real foci line by definition. But  $P'(0)$  is already orthogonal to the foci line (being a vector tangent to  $\mathcal{E}$  at its vertex  $P_0$ ), hence the same hold for  $c'(0)$ . It's then enough to show that  $c'(0) \neq 0$ .

Suppose the contrary, i.e.  $c'(0) = 0$ . We use again  $r'(0) = 0$ . If we denote by  $Q(t)$  one of the other vertices of  $T(t)$  and  $Q_0 = Q(0)$ , then since also  $r(t) = |Q(t)c(t)|$ , the equality  $r'(0) = 0$  gives that the line  $Q_0c_1$  is orthogonal to  $T_{Q_0}\mathcal{E}$ . It means that the circumscribed circle  $\mathcal{D}$  to  $T_1$  has the same tangent line in  $Q_0$  as  $\mathcal{E}$ . Since this is also true in  $P_0$  and in the third point of  $T_1$  (same proof), we get that  $\mathcal{E}$  and  $\mathcal{D}$  have three common points with the same tangent lines, which means that  $\mathcal{E}$  is a circle. But this case was excluded at the beginning (remark 1.2).

Hence  $c'(0) \neq 0$  and  $c'(0)$  is orthogonal to the line of real foci. The proof is the same for  $c_2$ . Hence by Bezout theorem,  $\hat{\mathcal{C}}$  is an ellipse.  $\square$

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