



**HAL**  
open science

# Spiraling of sub-Riemannian geodesics around the Reeb flow in the 3D contact case

Yves Colin de Verdière, Luc Hillairet, Emmanuel Trélat

► **To cite this version:**

Yves Colin de Verdière, Luc Hillairet, Emmanuel Trélat. Spiraling of sub-Riemannian geodesics around the Reeb flow in the 3D contact case. 2021. hal-03151365

**HAL Id: hal-03151365**

**<https://hal.science/hal-03151365v1>**

Preprint submitted on 24 Feb 2021

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Spiraling of sub-Riemannian geodesics around the Reeb flow in the 3D contact case

Yves Colin de Verdière\*    Luc Hillairet †    Emmanuel Trélat‡

## Abstract

We consider a closed three-dimensional contact sub-Riemannian manifold. The objective of this note is to provide a precise description of the sub-Riemannian geodesics with large initial momenta: we prove that they “spiral around the Reeb orbits”, not only in the phase space but also in the configuration space. Our analysis is based on a normal form along any Reeb orbit due to Melrose.

## 1 Introduction and main result

Let  $(M, D, g)$  be a closed sub-Riemannian (sR) manifold of dimension 3, where  $D$  is a contact distribution endowed with a Riemannian metric  $g$ . We assume for simplicity that  $D$  is oriented, and we denote by  $\alpha_g$  the unique one-form defining  $D$  so that  $d\alpha_g|_D$  is the volume form induced by the metric  $g$  on  $D$ . The Reeb vector field  $Z$  is then characterized by the relations  $\alpha_g(Z) = 1$  and  $d\alpha_g(Z, \cdot) = 0$ . Equivalently, given any positive  $g$ -orthonormal local frame  $(X, Y)$  of  $D$ ,  $Z$  is the unique vector field such that  $[X, Z] \in D$ ,  $[Y, Z] \in D$  and  $[X, Y] = -Z \pmod{D}$ .

We recall that the cometric  $g^* : T^*M \rightarrow \mathbb{R}^+$  associated with the sR metric  $g$  is defined by

$$g^*(q, p) = \|p|_{D(q)}\|_{g(q)}^2.$$

The Hamiltonian  $G = \sqrt{g^*}$ , which is homogeneous of degree 1, generates the sR geodesic flow: the projections onto  $M$  of the integral curves of the associated Hamiltonian vector field  $\vec{G}$  are the sR geodesics with speed 1. Note that the function  $G$  is not smooth along the line bundle  $\Sigma = G^{-1}(0) = D^\perp$  (the annihilator of  $D$ ). The geodesic flow  $G_t = \exp(t\vec{G})$  is homogeneous of degree 0, and thus is defined and smooth on  $S^*M \setminus S\Sigma$ . Here,  $S^*M$  is the unit cotangent bundle, and  $S\Sigma$  is the sphere bundle of  $\Sigma$  (quotient of  $\Sigma$  by positive homotheties).

The Reeb vector field  $Z$  has the following dynamical interpretation. If  $v_0 \in D(q_0)$ , then there exists a one-parameter family of geodesics associated with the Cauchy data  $(q_0, v_0)$ , all of them being the projections of the integral curves of  $\vec{G}$  with Cauchy data  $(q_0, p_0)$  and  $(p_0)|_{D_{q_0}} = g(v_0, \cdot)$ . For every  $s \in \mathbb{R}$ , the projections on  $M$  of the integral curves of  $\vec{G}$  with Cauchy data  $(q_0, p_0 + s\alpha_g)$  in the cotangent space have the same Cauchy data  $(q_0, v_0)$  in the tangent space. As  $s \rightarrow \pm\infty$ , they spiral around the integral curves of  $\mp Z$ .

---

\*Université de Grenoble-Alpes, Institut Fourier, Unité mixte de recherche CNRS-UJF 5582, BP 74, 38402-Saint Martin d'Hères Cedex, France ([yves.colin-de-verdiere@univ-grenoble-alpes.fr](mailto:yves.colin-de-verdiere@univ-grenoble-alpes.fr)).

†Université d'Orléans, Institut Denis Poisson, route de Chartres, 45067 Orléans Cedex 2, France ([luc.hillairet@univ-orleans.fr](mailto:luc.hillairet@univ-orleans.fr)).

‡Sorbonne Université, CNRS, Université de Paris, Inria, Laboratoire Jacques-Louis Lions (LJLL), F-75005 Paris, France ([emmanuel.trelat@sorbonne-universite.fr](mailto:emmanuel.trelat@sorbonne-universite.fr)).

In this paper, our objective is to give a precise description of these sR geodesics with large initial momenta: we will explain in what precise sense they spiral around the Reeb orbits.

To establish this feature, we will use an exact local normal form along an arc of a Reeb orbit. This result is originally due to Melrose (see [8, Proposition 2.3]); his proof is however rather sketchy and we will give here a full proof of it (which is far from being obvious) in Section 2.

In Section 3, we will deduce from this normal form the spiraling property of sR geodesics around Reeb trajectories, not only in the phase space  $T^*M$  (the cotangent bundle of  $M$ ), but also in the configuration space  $M$ . We will then be in a position to explain more precisely what *spiraling* means.

## 2 Melrose normal form along a Reeb orbit

We denote by  $\omega$  the canonical symplectic form of  $T^*M$ . In local symplectic coordinates  $(q, p)$ , we have  $\omega = dq \wedge dp$  (see Appendix A.1 for the sign conventions that are used here and throughout). The codimension-2 manifold  $\Sigma$  is a symplectic subcone of  $T^*M$ , endowed with the restriction  $\omega|_\Sigma$ . Note that

$$\Sigma = \{(q, s\alpha_g(q)) \in T^*M \mid q \in M, s \in \mathbb{R}\}.$$

We define the Hamiltonian  $\rho : \Sigma \rightarrow \mathbb{R}$  by  $\rho(s\alpha_g) = |s|$ . The projection onto  $M$  of the Hamiltonian vector field  $\vec{\rho}$  is  $\pm Z$ , depending on the sign of  $s$ . We restrict our study to  $\Sigma^+ = \rho^{-1}((0, +\infty))$ , the cone of positive multiples of  $\alpha_g$ . The cone  $\Sigma^- = -\Sigma^+$  is obtained by changing the orientation of  $D$ . Given an open subset  $U \subset M$ , we denote by  $\Sigma_U^+$  the cone  $\Sigma_U^+ = \{(q, s\alpha_g(q)) \mid q \in U, s > 0\}$ .

We consider the symplectic conic manifold  $\Sigma^+ \times \mathbb{R}_{u,v}^2$  endowed with the symplectic form  $\tilde{\omega} = \omega|_\Sigma + du \wedge dv$  and with the conic structure defined by  $\lambda \cdot (q, s\alpha_g(q), u, v) = (q, \lambda s\alpha_g(q), \sqrt{\lambda}u, \sqrt{\lambda}v)$  for any  $\lambda > 0$ . We define the function  $I$  on  $\Sigma^+ \times \mathbb{R}_{u,v}^2$  by  $I(\sigma, u, v) = u^2 + v^2$ , for any  $\sigma \in \Sigma^+$  and  $(u, v) \in \mathbb{R}^2$ . is the Hessian of the function  $(u, v) \mapsto g^*(\sigma, u, v)$  (which vanishes as well as its differential at  $(0, 0)$ ) for any fixed  $\sigma \in \Sigma^+$ .

We are going to establish the following Melrose normal form.

**Theorem 2.1.** *Let  $\Gamma_0$  be a closed arc of a Reeb orbit in  $M$ , diffeomorphic to  $[0, 1]$ . There exist a neighborhood  $U$  of  $\Gamma_0$  and a homogeneous symplectic diffeomorphism  $\chi$  of a conic neighborhood  $C$  of  $\Sigma_U^+$  in  $T^*U$  to a conical neighborhood  $C'$  of  $\Sigma_U^+ \times \{0\}$  in  $\Sigma_U^+ \times \mathbb{R}^2$ , satisfying  $\chi(\sigma) = (\sigma, 0)$  for every  $\sigma \in \Sigma_U^+$ , such that  $g^* \circ \chi^{-1} = \rho I$ .*

The proof is done in Sections 2.1 and 2.2 hereafter. But Section 2.1 also contains results on automorphisms preserving the normal form, and on a parallel transport property along Reeb trajectories.

### 2.1 Birkhoff normal form and parallel transport

Let us first recall the Birkhoff normal form derived in [3]. In what follows, given  $k \in \mathbb{N} \cup \{+\infty\}$ , given smooth maps  $f_1$  and  $f_2$  on  $T^*M$  (or on  $\Sigma^+ \times \mathbb{R}_{u,v}^2$ ) with values in some manifold, the notation  $f_1 = f_2 + O_\Sigma(k)$  means that  $f_1$  coincides with  $f_2$  along  $\Sigma$  (or along  $\Sigma \times \{0\}$ ) at order  $k$ , at least.

**Theorem 2.2** ([3]). *Let  $q_0 \in M$  be arbitrary. There exist a conic neighborhood  $C$  of  $(q_0, \alpha_g(q_0)) \in \Sigma^+$  in  $T^*M \setminus \{0\}$  and a smooth homogeneous symplectomorphism  $\chi : C \rightarrow \chi(C) \subset \Sigma^+ \times \mathbb{R}_{u,v}^2$ , satisfying  $\chi(\sigma) = (\sigma, 0)$  for every  $\sigma \in \Sigma^+ \cap C$ , such that  $g^* \circ \chi^{-1} = \rho I + O_\Sigma(\infty)$ .*

In other words, in local coordinates  $(q, s, u, v)$  as above, we have  $g^* \circ \chi^{-1}(q, s, u, v) = s(u^2 + v^2) + O_\Sigma(\infty)$ . Note that the proof of Theorem 2.2, done in [3], is quite long and technical, and consists in deriving a normal form with remainder terms, in using the Darboux-Weinstein lemma

and then in improving the remainder terms to a flat remainder term  $O_\Sigma(\infty)$  by solving an infinite series of cohomological equations. The final step of the proof of Theorem 2.1 will in particular consist of removing the flat term  $O_\Sigma(\infty)$  (see Section 2.2).

Now, we give a result on symplectomorphisms (canonical transforms) preserving the above Birkhoff normal form. We identify  $\mathbb{R}_{u,v}^2 \sim \mathbb{C}$  with  $(u, v) \sim u + iv$  for convenience.

**Theorem 2.3.** *With the notations of Theorem 2.2, let  $\Theta : \chi(C) \rightarrow \chi(C)$  be a smooth homogeneous symplectomorphism (for the symplectic form  $\tilde{\omega}$ ), satisfying  $\Theta(\sigma, 0) = (\sigma, 0)$  for every  $\sigma \in \Sigma^+$ , such that  $g^* \circ \chi^{-1} \circ \Theta^{-1} = \rho I + O_\Sigma(\infty)$ , that is, such that  $\rho I \circ \Theta = \rho I + O_\Sigma(\infty)$ . Then there exist smooth mappings  $F : C \rightarrow \Sigma^+$  satisfying  $F(\sigma, 0) = \sigma$  for every  $\sigma \in \Sigma^+$ , and  $\theta_j : \Sigma^+ \cap C \rightarrow \mathbb{R}$  such that  $\{\rho, \theta_j\}_\omega = 0$  for every  $j \in \mathbb{N}$ , i.e., all functions  $\theta_j$  are constant along the Reeb orbits, and such that*

$$\Theta(\sigma, u + iv) = \left( F(\sigma, u + iv), e^{i \sum_{j=0}^{+\infty} \theta_j(\sigma) I(u,v)^j} (u + iv) \right) + O_\Sigma(\infty),$$

for all  $(\sigma, u, v) \in C$ . In particular, not only  $\rho I$  but also  $I$  and  $\rho$  are preserved by  $\Theta$  modulo  $O_\Sigma(\infty)$ .

Theorem 2.3 actually follows from the Lewis-Sternberg theorem (see [7, 12], see also [5]), or more precisely from a variant of it where we assume that  $\chi$  is the identity along  $\Sigma^+$ . But we provide hereafter a more direct proof.

*Proof of Theorem 2.3.* We start with the following lemma. Recall that  $(\Sigma^+, \omega|_\Sigma)$  is a symplectic conic manifold of dimension 4.

**Lemma 2.1.** *Let  $N \geq 2$  be an arbitrary integer. Let  $\Theta_N : \chi(C) \rightarrow \chi(C)$  be a smooth symplectomorphism (for the symplectic form  $\tilde{\omega}$ ), satisfying  $\Theta_N = \text{id} + R_N + O_\Sigma(N+1)$ , with  $R_N$  homogeneous of degree  $N \geq 2$  in  $(u, v)$ . Then there exists a smooth function  $S : \chi(C) \rightarrow \mathbb{R}$ , homogeneous of degree  $N + 1$  in  $(u, v)$ , such that  $\Theta_N = \exp(\vec{S}) + O_\Sigma(N + 1)$ .*

*Proof of Lemma 2.1.* Let us choose local symplectic coordinates  $(q, p)$  with  $q = (q_1, q_2)$ ,  $p = (p_1, p_2)$  so that  $\Sigma = \{q_2 = p_2 = 0\}$ . We write  $\Theta_N(q, p) = (q_1, q_2 + A_N(q, p), p_1 + B_N(q, p)) + O_\Sigma(N + 1)$ . Using the fact that  $\Theta_N$  is symplectic, we get  $dA_N \wedge dp - dB_N \wedge dq = O_\Sigma(N)$ . Hence, there exists  $S$  homogeneous of degree  $N + 1$  in the variable  $(q_2, p_2)$  so that  $dS = A_N dp - B_N dq + O_\Sigma(N + 1)$ . This gives the result, because  $\exp(\vec{S})(x) = x + \vec{S}(x) + O_\Sigma(2N)$ . □

Hereafter, we use local coordinates  $(\sigma, u, v)$ , with  $\sigma = (q, s)$ . In order to avoid heavy notations, we denote (without any index) by  $\{ , \}$  the Poisson bracket with respect to the symplectic form  $\tilde{\omega} = \omega|_\Sigma + du \wedge dv$ . Note that the variables  $\sigma$  and  $(u, v)$  are symplectically orthogonal, and that the coordinates  $u$  and  $v$  are symplectically conjugate and  $\{u, v\} = 1$ .

In these local coordinates  $(q, s, u, v)$ , we have  $g^* \circ \chi^{-1}(q, s, u, v) = s(u^2 + v^2) + O_\Sigma(\infty)$ . Setting  $\Theta(q, s, u, v) = (q', s', u', v')$ , we have also  $g^* \circ \chi^{-1} \circ \Theta^{-1}(q', s', u', v') = s'(u'^2 + v'^2) + O_\Sigma(\infty)$ . It follows that  $s'(u'^2 + v'^2) = s(u^2 + v^2) + O_\Sigma(\infty)$ , which can be written in short as  $\rho I \circ \Theta = \rho I + O_\Sigma(\infty)$ .

Besides, since  $\Theta(\sigma, 0) = (\sigma, 0)$  and the Hamiltonian vector field of  $\rho$  is preserved, we infer that  $\sigma' = \sigma + O_\Sigma(2)$ . Therefore we get that  $u'^2 + v'^2 = u^2 + v^2 + O_\Sigma(4)$ . As a consequence, there exists  $\theta_0(\sigma) \in \mathbb{R}$  such that  $u' + iv' = e^{i\theta_0(\sigma)}(u + iv) + O_\Sigma(3)$ , where  $\sigma = (q, s)$ . Defining the smooth 2-homogeneous function  $S_2 = \theta_0 I / 2$ , i.e.,  $S_2(\sigma, u, v) = \theta_0(\sigma)(u^2 + v^2) / 2$ , the corresponding Hamiltonian vector field  $\vec{S}_2$  generates a rotation of angle  $\theta_0(\sigma)$  in the coordinates  $(u, v)$ , and we infer that

$$\Theta = \exp(\vec{S}_2) + O_\Sigma(3).$$

Now, writing  $\exp(-\vec{S}_2) \circ \Theta = \text{id} + R_2 + O_\Sigma(4)$  with  $R_2$  homogeneous of degree 3 in  $(u, v)$ , applying Lemma 2.1 with  $N = 3$  yields the existence of a smooth 4-homogeneous function  $S_3$  such that

$$\Theta = \exp(\vec{S}_2) \circ \exp(\vec{S}_3) + O_\Sigma(4). \quad (1)$$

Since  $S_2 = \theta_0 I/2$ , we have  $\{S_2, \rho I\} = \frac{1}{2}\{\theta_0, \rho\}I^2$  and  $\{S_2, \{S_2, \rho I\}\} = \frac{1}{4}\{\{\theta_0, \rho\}\}I^3 = O_\Sigma(6)$ , and thus

$$\rho I \circ \exp(\vec{S}_2) = \exp(\{S_2, \cdot\}) \cdot \rho I = \rho I + \frac{1}{2}\{\theta_0, \rho\}I^2 + O_\Sigma(6).$$

Now,  $S_3(\sigma, u, v)$  is a sum of terms of the kind  $a_3(\sigma)Q_3(u, v)$  with  $a_3$  smooth and  $Q_3$  homogeneous in  $(u, v)$  of degree 3 (running over  $u^3, u^2v, uv^2, v^3$ ). Taking one of them,  $S_3(\sigma, u, v) = a_3(\sigma)Q_3(u, v)$ , we compute

$$\begin{aligned} \rho I \circ \exp(\vec{S}_2) \circ \exp(\vec{S}_3) &= \exp(\{S_3, \cdot\}) \cdot \left( \rho I + \frac{1}{2}\{\theta_0, \rho\}I^2 + O_\Sigma(6) \right) \\ &= \rho I + \frac{1}{2}\{\theta_0, \rho\}I^2 + a_3\rho\{Q_3, I\} + O_\Sigma(5). \end{aligned} \quad (2)$$

Computing  $\{u^3, I\} = 6u^2v$ ,  $\{u^2v, I\} = -2u^3 + 4uv^2$ ,  $\{uv^2, I\} = -4u^3v + 2v^3$ ,  $\{v^3, I\} = -6uv^2$ , we note that  $\{Q_3, I\}$  is homogeneous of degree 3. Now, since we must have  $\rho I \circ \Theta = \rho I + O_\Sigma(\infty)$ , it follows from (1) that  $\rho I = \rho I \circ \exp(\vec{S}_2) \circ \exp(\vec{S}_3) + O_\Sigma(5)$ , and using (2) we infer that  $\frac{1}{2}\{\theta_0, \rho\}I^2 + a_3\rho\{Q_3, I\} = O_\Sigma(5)$ . Since  $I^2$  is 4-homogeneous and  $\{Q_3, I\}$  is 3-homogeneous, it follows that  $\{\theta_0, \rho\} = 0$  and  $a_3\{Q_3, I\} = 0$ . Then we have either  $a_3 = 0$  and thus  $S_3 = 0$ , or  $\{Q_3, I\} = 0$ , and in this case, it follows from Remark A.1 (in Appendix A.2) that  $Q_3 = 0$ . Then, in all cases, we have  $S_3 = 0$ .

Since  $S_3 = 0$ , writing now  $\exp(-\vec{S}_2) \circ \Theta = \text{id} + O_\Sigma(3) = \text{id} + R_3 + O_\Sigma(4)$  with  $R_3$  homogeneous of degree 3, applying again Lemma 2.1 with  $N = 3$  yields the existence of a smooth 4-homogeneous function  $S_4$  such that

$$\Theta = \exp(\vec{S}_2) \circ \exp(\vec{S}_4) + O_\Sigma(4).$$

As above, setting  $Q_4(\sigma, u, v) = a_4(\sigma)Q_4(u, v)$ , we compute  $\rho I \circ \exp(\vec{S}_2) \circ \exp(\vec{S}_4) = \rho I + a_4\rho\{Q_4, I\} + O_\Sigma(6)$ , and similarly, since we must have  $\rho I \circ \Theta = \rho I + O_\Sigma(\infty)$ , it follows that  $a_4\rho\{Q_4, I\} = 0$ . Then, either  $a_4 = 0$  or  $\{Q_4, I\} = 0$ . In the latter case, by Remark A.1 we have  $AQ_4 = 0$ , and by Lemma A.1 we get that  $Q_4 \in P_4^{\text{inv}}$ , that is,  $Q_4 = cI^2$  for some  $c \in \mathbb{R}$ .

The reasoning continues by iteration, and is even simpler now that we know that  $\{\theta_0, \rho\} = 0$  and thus that  $\rho I \circ \exp(\vec{S}_2) = \rho I + O(\infty)$ . In passing, we note that this implies immediately that  $I \circ \exp(\vec{S}_2) = I + O(\infty)$  and  $\rho \circ \exp(\vec{S}_2) = \rho + O(\infty)$ . We only describe the next step, then the recurrence is immediate.

Writing  $\exp(-\vec{S}_4) \circ \exp(-\vec{S}_2) \circ \Theta = \text{id} + R_4 + O_\Sigma(5)$  with  $R_4$  that is 4-homogeneous, applying Lemma 2.1 with  $N = 4$  yields the existence of a smooth 5-homogeneous function  $S_5$  such that

$$\Theta = \exp(\vec{S}_2) \circ \exp(\vec{S}_4) \circ \exp(\vec{S}_5) + O_\Sigma(5).$$

Taking  $S_4(\sigma, u, v) = a_4(\sigma)Q_4(u, v)$ , we compute  $\rho I \circ \exp(\vec{S}_2) \circ \exp(\vec{S}_4) \circ \exp(\vec{S}_5) = \rho I + \{a_4, \rho\}I^3 + a_5\rho\{Q_5, I\} + O_\Sigma(7)$ . Therefore, reasoning as above, we infer that  $\{a_4, \rho\} = 0$  and  $a_5\rho\{Q_5, I\} = 0$ . Then either  $a_5 = 0$  or  $\{Q_5, I\} = 0$ . In the latter case, we get  $Q_5 = 0$  by Remark A.1. Hence  $S_5 = 0$ .

Since  $S_5 = 0$ , writing now  $\exp(-\vec{S}_4) \circ \exp(-\vec{S}_2) \circ \Theta = \text{id} + O_\Sigma(5) = \text{id} + R_5 + O_\Sigma(6)$  with  $R_5$  homogeneous of degree 5, applying again Lemma 2.1 with  $N = 5$  yields the existence of a smooth 6-homogeneous function  $S_6$  such that  $\Theta = \exp(\vec{S}_2) \circ \exp(\vec{S}_4) \circ \exp(\vec{S}_6) + O_\Sigma(6)$ . Reasoning as for  $S_4$ , we establish that  $S_6(\sigma, u, v) = a_6(\sigma)I(u, v)^3$ .

A recurrence argument concludes the proof.  $\square$

**Parallel transport along Reeb trajectories.** An interesting consequence of Theorem 2.3 is that this result reveals a property of parallel transport of  $D$  along Reeb trajectories: this transport is the image by the differential of  $\Pi \circ \Theta^{-1}$  of the trivial transport in the  $(u, v)$  planes. Indeed, Theorem 2.3 implies that  $d\Theta(\sigma, 0)$  is a rotation in the  $(u, v)$  plane with constant angle  $2s_1(\sigma)$  along the orbits of  $\vec{\rho}$ . Projecting this property onto  $M$  gives the invariance of the parallel transport under the change of charts preserving the Birkhoff normal form.

## 2.2 End of the proof of Theorem 2.1 by Nelson's trick

To end the proof and in particular to remove the flat remainder term  $O_\Sigma(\infty)$  in the Birkhoff normal form given in Theorem 2.2, we use the nice *scattering* method due to Edward Nelson (see [10]), thanks to which we will establish an exact normal form in a Reeb flow box.

The Birkhoff normal form given in Theorem 2.2 is valid in some conic neighborhood of  $\Sigma_U^+$ , where  $U$  is a flow box for the Reeb flow, i.e.,  $U$  is diffeomorphic to  $S_x \times (-2c, T_0 + 2c)_y$  for some  $c > 0$ , with  $Z = \partial_y$ . Theorem 2.2 provides us with a homogenous symplectic diffeomorphism that we use as a coordinate system, so that we now work in  $\Sigma_U^+ \times \mathbb{R}_{u,v}^2$ .

We define the Hamiltonian  $H = \sqrt{\rho I}$  on  $\Sigma^+ \times (\mathbb{R}^2 \setminus \{0\})$ . We set  $J = \sqrt{I/\rho}$  and we define  $\theta$  so that  $(J, \theta)$  are polar coordinates in  $\mathbb{R}_{u,v}^2$ . The function  $J$  is homogeneous of degree 0 and is a measure of the angular distance to  $\Sigma$ . The sphere bundle  $S(\Sigma_U^+ \times (\mathbb{R}^2 \setminus \{0\}))$  is thus parametrized by  $(m, J, \theta) = [m, \alpha_g(m); J, \theta]$  with  $m \in U$ .

We denote by  $H_t$  the flow of the Hamiltonian vector field

$$\vec{H} = \frac{J}{2} \vec{\rho} + \frac{1}{J} \frac{\partial}{\partial \theta},$$

which is homogeneous of degree 0 and hence is defined on the sphere bundle. It is explicitly given by

$$H_t(m, J, \theta) = (\mathcal{R}_{Jt/2}(m), J, \theta + t/J),$$

where  $(\mathcal{R}_s)_{s \in \mathbb{R}}$  is the Reeb flow on  $M$ .

Let  $V_0 \subset V$  be defined by  $V_0 = S_0 \times (-c, T_0 + c)$  with  $\bar{S}_0 \subset S$ . Let us choose  $0 < a_0 < a_1$  and denote by  $C_j$  the cones  $C_j = \{(m, J, \theta) \mid m \in U_j, J < a_j\}$ . Using the Birkhoff normal form, recalling that  $G = \sqrt{g^x}$ , we have  $G^2 = H^2 + O_\Sigma(\infty)$ , where the remainder term  $O_\Sigma(\infty)$  is smooth. This can be rewritten as

$$G^2 = H^2(1 + O_\Sigma(\infty)),$$

so that the geodesic Hamiltonian satisfies  $G = H + O_\Sigma(\infty)$  in  $C_1$  and the remainder term  $O_\Sigma(\infty)$  in the latter equation is smooth even though  $G$  and  $H$  are not.

We now extend  $G$  to  $\Sigma^+ \times (\mathbb{R}^2 \setminus \{0\})$  as follows: let  $\psi$  be a smooth function that is homogeneous of degree 0 on  $\Sigma^+ \times \mathbb{R}^2$  identically equal to 1 in  $C_0$  and identically equal to 0 outside of  $C_1$ . We define  $\tilde{G} = \psi G + (1 - \psi)H$ , and we check that  $R = \tilde{G} - H = O_\Sigma(\infty)$  and that  $\{J, \tilde{G}\} = \dot{O}_\Sigma(\infty)$ , where the upper dot indicates that we are dealing here with functions on  $\Sigma^+ \times (\mathbb{R}^2 \setminus \{0\})$ . It follows from the definition that the flows  $\tilde{G}_t$  and  $H_t$  are complete and coincide outside of  $C_1$ .

**Lemma 2.2.** *Given any  $z_0$ , we set  $\mathcal{J}(t) = J(\tilde{G}_t(z_0))$ . For every  $N \geq 2$ , there exist  $C_N > 0$  and  $D_N > 0$  such that, if  $0 < \mathcal{J}(0) \leq \frac{1}{2}$  and  $0 \leq t \leq C_N/\mathcal{J}(0)^{2N}$ , then  $\mathcal{J}(t) \leq 2\mathcal{J}(0)$  and  $|\mathcal{J}(t) - \mathcal{J}(0)| \leq D_N \mathcal{J}(0)^{2N+1} t$ .*

*Proof of Lemma 2.2.* Since  $\{J, \tilde{G}\} = \dot{O}_\Sigma(\infty)$  is homogenous of degree 0, for any  $N$  there exists  $C > 0$  such that

$$0 < J(z) \leq 1 \Rightarrow |\{J, \tilde{G}\}(z)| \leq C J(z)^{2N+1}.$$

It follows that  $\dot{\mathcal{J}}(t) \leq C\mathcal{J}^{2N+1}(t)$  as long as  $\mathcal{J}(t) \leq 1$ . Integrating, we get

$$\mathcal{J}(t) \leq \frac{\mathcal{J}(0)}{(1 - 2NCt\mathcal{J}^{2N}(0))^{1/2N}},$$

as long as  $(2NC\mathcal{J}^{2N}(0))t \leq 1/2$  and  $\mathcal{J}(t) \leq 1$ . Hence, there exists  $C_N > 0$  (e.g.,  $C_N = \frac{2^{2N}-1}{2^{2N+1}NC}$ ) such that

$$0 \leq t \leq C_N \frac{1}{\mathcal{J}(0)^{2N}} \quad \text{and} \quad \mathcal{J}(0) \leq \frac{1}{2} \Rightarrow \mathcal{J}(t) \leq 2\mathcal{J}(0).$$

Therefore  $J(t) \leq 2J(0)$ . Using that  $|J'(t)| \leq CJ^{2N+1}(t) \leq C(2J(0))^{2N+1}$ , we get the second estimate (with  $D_N = Z^{2N+1}CC_N$ ).  $\square$

Using the fact that  $\vec{G} - \vec{H} = O_\Sigma(\infty)$ , taking  $a_1$  small enough, we assume that, in the decomposition  $\vec{G}(z) = a\partial_y + V + b\partial_u + c\partial_v$ , where  $V$  is tangent to  $S_1 \times \{y\}$ , we have  $a \geq J/4$  as long as  $z$  is in  $C_1$ .

Lemma 2.2 then implies that, for  $\varepsilon > 0$  small enough, if  $t \leq \varepsilon(\mathcal{J}(0))^{2N}$  then  $\mathcal{J}(t) \geq \frac{\mathcal{J}(0)}{2}$  so that the flow  $\tilde{G}_t$  is going out of  $C_1$  within a time of order at most  $O(1/\mathcal{J}(0))$  in both time directions.

Following the method of Nelson, let us now define  $\chi : \Sigma^+ \times (\mathbb{R}^2 \setminus \{0\}) \rightarrow \Sigma^+ \times (\mathbb{R}^2 \setminus \{0\})$  by

$$\chi(z) = \lim_{t \rightarrow +\infty} (H_t \circ \tilde{G}_{-t})(z).$$

It is well defined and the limit is obtained within a time  $O(1/J(0))$ , because as soon as  $t_0$  is such that  $G_{-t_0}(z)$  has left  $C_1$ , we have  $H_t \circ \tilde{G}_{-t}(z) = H_{t_0} \circ H_{t-t_0} \circ G_{-t+t_0} \circ G_{-t_0}$  for  $t > t_0$ , and the flows of  $H$  and  $\tilde{G}$  coincide outside of  $C_1$ .

By definition we have  $\chi \circ \tilde{G}_t = H_t$ , so that  $\chi \circ G_t(z) = H_t(z)$  for  $t$  small and  $z \in C_0$ , since the flows of  $G$  and  $\tilde{G}$  coincide there.

Moreover,  $\chi$  is a symplectomorphism of  $\Sigma^+ \times \mathbb{R}^2 \setminus \{0\}$  whose inverse is given by  $\chi^{-1} = \lim_{t \rightarrow +\infty} (\tilde{G}_t \circ H_{-t})(z)$ .

Let us finally prove that we can extend  $\chi$  to  $\Sigma^+ \times \mathbb{R}^2$  by the identity on  $\Sigma$  and obtain a smooth symplectomorphism that transforms  $G$  to the desired Melrose normal form in  $C_0$ . The latter fact follows from the definition so that the only issue is the smooth extension.

Let  $\phi$  be a smooth function that is homogeneous of degree 0 on  $\Sigma^+ \times \mathbb{R}^2$ . Following [11, Section 3], we compute the time derivative of  $\mathcal{D}(t)(z) = \phi(H_t \circ \tilde{G}_{-t}(z))$ , given by

$$\mathcal{D}'(t) = -\{\tilde{G}, \mathcal{D}\} + \{H \circ \tilde{G}_{-t}, \mathcal{D}\} = -\{R \circ \tilde{G}_{-t}, \mathcal{D}\} = -\{R, \phi \circ H_t\} \circ \tilde{G}_{-t},$$

where we have used, successively, the invariance of  $\tilde{G}$  under  $\tilde{G}_t$ , and the invariance of the Poisson bracket under  $\tilde{G}_t$ . Using the explicit expression for  $H_t$  and the fact that  $J$  is preserved by the flow of  $H_t$ , we see that, as long as  $J(z) \times t$  is bounded above, the differential of  $H_t$  at  $z$  is  $O(J(z)^{-2})$ . It follows that the Poisson bracket  $\{R, \phi \circ H_t\}$  is  $O(J(z)^\infty)$  as long as  $J(z)t$  is bounded. Using Lemma 2.2, it follows that  $\mathcal{D}'(t) = O(\mathcal{J}(0)^\infty)$ , and thus  $|\mathcal{D}(t) - \mathcal{D}(0)| = O(\mathcal{J}(0)^\infty)$  as long as  $\mathcal{J}(0)t$  is bounded.

Locally in  $z$ , in the expression giving  $\chi(z)$  instead of the limit  $t \rightarrow +\infty$  we can fix  $t = T_0$  chosen of the order of  $1/J(z)$ . Then, we have  $|\mathcal{D}(t) - \mathcal{D}(0)| = O(\mathcal{J}(0)^\infty)$  for  $t = O(1/J(0))$ . Choosing  $\phi$  among a finite set of functions that give local coordinates in  $C_1$ , we obtain that  $d(\chi(z), z) = O(J^\infty)$  for some distance  $d$ . We have then  $d(z_0, \chi(z)) \leq d(z_0, z) + d(z, \chi(z))$  if  $z_0 \in \Sigma \times \{0\}$ . This proves that  $\chi$ , extended by the identity on  $\Sigma^+ \times 0$ , is continuous.

To finish, Nelson's trick consists of constructing extensions of these flows to the  $k$ -jets spaces. For example, if  $k = 1$ , we consider the lifts of both flows to the tangent space  $T(\Sigma^+ \times \mathbb{R}^2)$ . The same properties as above are satisfied, and we get that  $\chi$  can be extended smoothly and its differential is the identity on  $\Sigma^+ \times \{0\}$ . Hence  $\chi$  can be extended to a diffeomorphism such that  $\tilde{G} \circ \chi = H$  and, in particular,  $G \circ \chi(z) = H(z)$  if  $z \in C_0$ . Theorem 2.1 is proved.

### 3 Spiraling along periodic Reeb orbits

In this section, we show how the Melrose normal form along a Reeb orbit given in Theorem 2.1 can be used to describe the spiraling property of geodesics around Reeb trajectories.

Considering the symplectomorphism  $\chi$  given by Theorem 2.1, the geodesics of large initial momenta are images under  $\chi^{-1}$  of the curves

$$t \rightarrow \left( \mathcal{R}_{tJ_0/2}, J_0 e^{it/J_0 + \theta_0} \right),$$

with  $t = O(1/J_0)$ . In order to describe these images in a precise way, it is useful to compute the differential  $\Xi$  of  $\chi^{-1}$  along  $\Sigma^+ \times \{0\}$ . Fixing a positive  $g$ -orthonormal basis  $(X, Y)$  of  $D$  which is *parallel* along the Reeb flow for the parallel transport defined in 2.1, we have

$$\Xi_\sigma(\delta\sigma, \delta u, \delta v) = \delta\sigma + \frac{1}{\sqrt{\rho(\sigma)}}(\delta u \vec{h}_X + \delta v \vec{h}_Y).$$

To state the result below, it is more convenient to define a complex structure on  $D$ : the product by  $i$  is defined as the rotation of angle  $\pi/2$  with respect to the orientation of  $D$  and the metric  $g$ .

**Theorem 3.1.** *Let  $q_0 \in M$  be arbitrary, and let  $(q_0, p_0) \in T_{q_0}^*M$  be the Cauchy data of a geodesic  $t \mapsto \gamma(t)$  starting at  $q_0$  with unit speed  $\dot{\gamma}(0) = X_0 \in D(q_0)$ . We assume that  $h_0 = h_Z(p_0) \gg 1$  (large initial momentum).*

*Then, there exists a point  $Q_0 = Q_0(q_0, p_0) \in M$  close to  $q_0$ , and a vector  $Y_0 \in D(Q_0)$  close to  $X_0$ , such that, denoting by  $\Gamma(\tau) = \mathcal{R}_\tau(Q_0)$  the Reeb orbit of  $Q_0$ , and by  $Y(t)$  the parallel transport of  $Y_0$  along  $\Gamma$ ,*

*we have, using the complex structure on  $D$ , for  $t = O(h_0)$ ,*

$$\gamma(t) = \Gamma(J_0 t/2) - i J_0 e^{it/J_0} Y(J_0 t/2) + O(J_0^2), \quad \dot{\gamma}(t) = e^{it/J_0} Y(J_0 t/2) + O(J_0),$$

*with  $J_0 = h_0^{-1} + O(h_0^{-3})$  and  $Q_0 = q_0 - i h_0^{-1} X_0 + O(h_0^{-2})$ .*

In other words, this result says that, on time intervals of length of the order of  $1/h_0$ , the geodesics spiral along orbits of the Reeb flow taken for time intervals  $O(1)$ .

*Proof.* We have  $g^*(q_0, p_0) = 1$  and thus  $J_0 = 1/\rho_0$ . Using that  $h_Z = \rho + O_\Sigma(2)$ , we get that  $J_0 = h_0^{-1} + O(h_0^{-3})$ . Setting  $z(t) = G_t(q_0, p_0)$ , we have

$$z(t) = \chi^{-1} \left( \mathcal{R}_{J_0 t/2}(Q_0), \frac{1}{J_0} \alpha_g(Q_0), J_0 e^{it/J_0} \right).$$

This defines  $Q_0$  and we have  $\gamma(t) = \pi(z(t))$ , where  $\pi : T^*M \rightarrow M$  is the canonical projection. The result now follows by taking (if necessary) a covering of the Reeb trajectory by a finite number of charts in which the normal form is valid.  $\square$



## 4 A conjecture on periodic geodesics

We consider a periodic orbit  $\Gamma$  of the Reeb flow on  $M$  of primitive period  $T_0 > 0$ . In this section, using the normal form, we derive an approximation of the return Poincaré first return map of the geodesic flow, for geodesics spiraling around  $\Gamma$  and that are almost closed within time  $T_0/J_0$ . Using this, we give a conjecture on the lengths of long closed geodesics close to  $\Gamma$ .

Along  $\Gamma$ , the fiber bundle  $D$  is trivial. We then consider a  $g$ -orthonormal frame of  $D(\Gamma(0))$ , that we transport in a parallel way along  $\Gamma$ . In this way, we obtain a monodromy of angle  $\alpha_0$ . Let us consider geodesics spiraling around  $\Gamma$  that are close to  $\gamma_0(t) = \mathcal{R}_{J_0 t/2}(\Gamma(0)) - iJ_0 e^{it/J_0} Y(J_0 t/2)$ . The conditions for  $\gamma_0$  to be a periodic integral curve of  $\vec{\rho I}$  of period  $T_{j,k}$ , covering  $j$  times  $\Gamma$  and winding  $k$  times around  $\Gamma$ , are given by  $J_0 T_{j,k} = 2jT_0$  and  $\frac{T_{j,k}}{J_0} + j\alpha_0 = 2k\pi$ . It follows that

$$T_{j,k} = 2\sqrt{jk\pi T_0 \left(1 - \frac{j\alpha_0}{2k\pi}\right)}.$$

We formulate the following conjecture:

*If the Reeb periodic orbit is non degenerated, then for every  $j \in \mathbb{N} \setminus \{0\}$  there exists a sequence of closed geodesics covering  $j$  times  $\Gamma$  with lengths close to  $T_{j,k}$  for  $k$  large enough.*

This conjecture is consistent with our computation made in the Heisenberg flat case in [3, Section 3.1] and with the case of the sphere  $S^3$  computed in [6]. We guess that our conjecture holds true at least if  $I$  is globally defined, which is the case in the example described in [4].

## A Appendix

### A.1 Sign conventions in symplectic geometry

Since there are several possible sign conventions in the Hamiltonian formalism, we fix them as follows.

Given a smooth finite-dimensional manifold  $M$ , the canonical symplectic form on the cotangent bundle  $T^*M$  is  $\omega = dq \wedge dp = -d\theta$  with  $\theta = p dq$  in local symplectic coordinates  $(q, p)$ .

Given a (smooth) Hamiltonian function  $h$ , the associated Hamiltonian vector field  $\vec{h}$  is defined by  $\iota_{\vec{h}}\omega = \omega(\vec{h}, \cdot) = dh$ . In local coordinates, we have  $\vec{h} = (\partial_p h, -\partial_x h)$ .

The Poisson bracket of two Hamiltonian functions  $f$  and  $g$  is defined by  $\{f, g\} = \omega(\vec{f}, \vec{g}) = df \cdot \vec{g} = -dg \cdot \vec{f}$ . In local coordinates, we have  $\{f, g\} = \partial_q f \partial_p g - \partial_p f \partial_q g$ . We have  $\overline{\{f, g\}} = -[\vec{f}, \vec{g}]$ .

Given a vector field  $X$  on  $M$ , the Hamiltonian lift is the function defined by  $h_X(q, p) = \langle p, X(q) \rangle$ . Given two vector fields  $X$  and  $Y$  on  $M$ , we have  $\{h_X, h_Y\} = -h_{[X, Y]}$ .

### A.2 A useful lemma for homogeneous polynomials

Given any integer  $k$ , we define  $P_k$  as the set of  $k$ -homogeneous polynomials in two variables  $(u, v)$ . We define

$$P_k^0 = \left\{ Q \in P_k \mid \int_0^{2\pi} Q(\cos \theta, \sin \theta) d\theta = 0 \right\},$$

$$P_k^{\text{inv}} = \{c(u^2 + v^2)^{k/2} \mid c \in \mathbb{R}\}.$$

The set  $P_k^0$  is the set of  $k$ -homogeneous polynomials having zero average along the circle  $u^2 + v^2 = 1$ .

We endow  $P_k$  with the scalar product  $\langle Q_1, Q_2 \rangle = \frac{1}{2\pi} \int_0^{2\pi} Q_1(\cos \theta, \sin \theta) Q_2(\cos \theta, \sin \theta) d\theta$ . Note that  $P_k^0 \perp P_k^{inv}$ . Considering polar coordinates, defining the endomorphism  $A$  of  $P_k$  defined by  $A = u\partial_v - v\partial_u$  (or equivalently, considering the operator  $\partial_\theta$ ), we have the following immediate result.

**Lemma A.1.** • If  $k$  is odd then  $P_k = P_k^0$  and  $A$  is invertible.

• If  $k$  is even then  $P_k = P_k^0 \oplus P_k^{inv}$ , and we have  $\text{range}(A) = P_k^0$  and  $\ker(A) = P_k^{inv}$ .

**Remark A.1.** Setting  $I(u, v) = u^2 + v^2$ , endowing  $\mathbb{R}_{u,v}^2$  with the symplectic form  $du \wedge dv$ , it is useful to note that, given any  $Q \in P_k$ , we have

$$\{I, Q\} = u\partial_v Q - v\partial_u Q = AQ.$$

In particular, when  $k$  is odd, we have  $\{I, Q\} = 0$  if and only if  $Q = 0$ .

## References

- [1] A. Abbondandolo, L. Macarini, M. Mazzucchelli, G. P. Paternain, *Infinitely many periodic orbits of exact magnetic flows on surfaces for almost every subcritical energy level*, to appear in J. Eur. Math. Soc.
- [2] Y. Colin de Verdière, *Magnetic fields and sub-Riemannian geometry*, In preparation (2017).
- [3] Y. Colin de Verdière, L. Hillairet, E. Trélat, *Spectral asymptotics for sub-Riemannian Laplacians I: Quantum ergodicity and quantum limits in the 3D contact case*, Duke Math. J., **167(1)**:109–174 (2018).
- [4] Y. Colin de Verdière, J. Hilgert, T. Weich, *Irreducible representations of  $SL_2(\mathbb{R})$  and the Peyresq's operators*, Work in progress.
- [5] I. Iantchenko, J. Sjöstrand, *Birkhoff normal forms for Fourier Integral Operators II*, American J. Math. **124 (4)** (2002), 817–850.
- [6] D. Klapheck, M. VanValkenburgh, *The length spectrum of the sub-Riemannian three-sphere*, Preprint ArXiv:1507.03041 (2015).
- [7] D. Lewis, *Formal power series transformations* Duke Math. J. **5** (1939), 794–805.
- [8] R. B. Melrose, *The wave equation for a hypoelliptic operator with symplectic characteristics of codimension two*, J. Analyse Math. **44** (1984-1985), 134–182.
- [9] R. Montgomery, *A tour of subriemannian geometries, their geodesics and applications*, Mathematical Surveys and Monographs **91**, American Mathematical Society, Providence, RI, 2002.
- [10] E. Nelson, *Topics in dynamics I: flows*, Mathematical Notes, Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1969, iii+118 pp.
- [11] N. Raymond, S. Vũ Ngọc, *Geometry and spectrum in 2D magnetic wells*, Ann. Inst. Fourier (Grenoble) **65** (2015), no. 1, 137–169.
- [12] S. Sternberg, *Infinite Lie groups and formal aspects of dynamical systems*, J. Math. Mech. **10** (1961), 451–474.