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# Nonlinear Fokker-Planck equation with reflecting boundary conditions

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## Abstract

We study the existence and uniqueness of a mild solution to a nonlinear Fokker-Planck equation with reflecting boundary conditions, by using a monotonicity approach. Then we prove that the mild solution is also a distributional one.

**Keywords:** Fokker-Planck equation, m-accretive operator, mild solution, McKean-Vlasov stochastic differential equation

AMS [2020] Primary 60H30, 60H10, 60G46; secondary 35Q84, 35D99

## 1 Introduction

In the present work we are interested in the study of a nonlinear Fokker-Planck equation with Neumann boundary conditions of the following type

$$\begin{cases} \frac{\partial y}{\partial t} - \frac{1}{2} \Delta \beta(y) - \operatorname{div}(b(\xi)y) = 0, & [0, T] \times \mathcal{O}, \\ \frac{1}{2} \frac{\partial}{\partial n} \beta(y) + (b(\xi) \cdot n(\xi))y = 0, & [0, T] \times \partial \mathcal{O}, \\ y(0, \xi) = y_0(\xi), & \xi \in \mathcal{O}. \end{cases} \quad (1)$$

Here  $\mathcal{O} \subset \mathbb{R}^d$ ,  $d \geq 1$ , is assumed to be an open bounded and convex domain with smooth boundary  $\partial \mathcal{O}$ . We denoted by  $n$  the outward normal to  $\partial \mathcal{O}$  and by  $\frac{\partial}{\partial n}$  the outward normal derivative. The closer of  $\mathcal{O}$  is denoted by  $\overline{\mathcal{O}} = K$ .

### Assumptions

We shall assume the following hypotheses on the functions  $\beta : \mathbb{R} \rightarrow \mathbb{R}$  and  $b : \overline{\mathcal{O}} \rightarrow \mathbb{R}^d$ :

- i)  $\beta \in C(\mathbb{R})$ ,  $\beta(0) = 0$ , and  $\beta$  is increasing maximal monotone such that, for  $m \geq 1$  we have

$$|\beta(r)| \leq \alpha_1 |r|^m + \alpha_2, \text{ for all } r \in \mathbb{R},$$

for some constants  $\alpha_1 > 0$  and  $\alpha_2 \geq 0$ .

- ii)  $j$  is the potential of  $\beta$ , i.e.  $\partial j = \beta$ , and we assume that for the same  $m \geq 1$  we have

$$j(r) \geq \alpha_3 r^{m+1}, \text{ for all } r \in \mathbb{R},$$

for some positive constant  $\alpha_3$ .

- iii)  $b \in C^1(\overline{\mathcal{O}})$  is a given function such that  $\operatorname{div} b \leq 0$  in  $\overline{\mathcal{O}}$ ,  $b \cdot n \geq 0$  on  $\partial\mathcal{O}$  and  $|b|_\infty < 1$ .

Equation (1), where  $y$  is a probability density, is known in the literature as the nonlinear Fokker-Planck equation with reflection. For reader's convenience we shall give some details concerning the connection with a stochastic differential equation.

If we choose the solution  $y : [0, T] \times \mathcal{O} \rightarrow \mathbb{R}$  in the class of probability density functions on  $\mathcal{O}$ , i.e. such that  $y \in L^1(\mathcal{O})$ ,  $y \geq 0$  a.e. on  $\mathcal{O}$  and  $\int_{\mathcal{O}} y(\xi) d\xi = 1$ , then  $y$  can be seen as the probability density of the law  $\mathcal{L}_X$ , where  $X$  is the weak solution (in the stochastic sense) to the following stochastic differential equation with reflection

$$\begin{cases} dX(t) + b(X(t)) dt + N_K(X(t)) dt \ni \sqrt{\frac{\beta(y(t, X(t)))}{y(t, X(t))}} dW_t \\ X(0) = X_0 \end{cases} \quad (2)$$

in a probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$  where  $(\mathcal{F}_t)_t$  is the normal filtration and  $W_t$ ,  $t \geq 0$ , is a Wiener process.

Keeping in mind that  $K = \overline{\mathcal{O}}$ , we recall that  $N_K : K \rightarrow 2^{\mathbb{R}^d}$  is the normal cone to  $K$ , i.e.

$$N_K(x) = \{z \in \mathbb{R}^d; (z, x - y)_{\mathbb{R}^d} \geq 0, \quad \forall y \in K\}.$$

Actually, the reflecting boundary condition from equation (1) gives the second drift term  $N_K(X)$  from the stochastic equation (2).

Concerning the stochastic differential equation, we recall the definition of the solution.

**Definition 1** *We call a strong solution to the stochastic differential equation (2) a pair of continuous  $\mathbb{R}^d$ -valued and  $\mathcal{F}_t$ -adapted processes  $(X(t), \eta(t))$ ,  $t \in [0, T]$  such that*

- 1)  $X(t) \in K$ ,  $\mathbb{P}$ -a.s., for all  $t \in [0, T]$ .

2)  $\exists \eta \in BV([0, T]; \mathbb{R}^d)$ ,  $\mathbb{P}$ -a.s., such that

$$X(t) + \int_0^t b(X(s)) ds + \eta(t) = X_0 + \int_0^t \sqrt{\frac{\beta(y(s, X(s)))}{y(s, X(s))}} dW_s$$

$$\int_0^T (X(s) - Z(s)) \cdot d\eta(s) \geq 0, \quad \forall Z \in C([0, T]; K).$$

As usual,  $BV([0, T]; \mathbb{R}^d)$  is the space of functions with bounded variation and  $\int_0^T (X(s) - Z(s)) \cdot d\eta(s)$  is the Riemann-Stieltjes integral.

Equations as (2) are usually referred as McKean-Vlasov SDEs. For classical results concerning this equation see [12], [16], [17], [19], [21]. Recently there was a large interest concerning this subject in different situations (see e.g. [5], [13], [14], [18]).

One can prove that, if  $y$  is a solution to (1), then there is a martingale solution  $X$  to (2) which has  $y$  as probability density, by using the superposition principle by Trevisan [22] (which is a generalization of the work of Figalli [11]) and by a classical result of Stroock-Varadhan (see [20]). For the general McKean-Vlasov equation see e.g. [6], [7]. For an immediate extension to a stochastic reflection problem as in the present case see [3].

In the present work we are interested to treat the well-posedness of the nonlinear Fokker-Planck equation (1) in  $L^1(\mathcal{O})$ . This work follows the direction opened by Barbu and Röckner in [5], [6], [7], [8] for the Fokker-Planck equation with Dirichlet boundary conditions under different sets of assumptions. This approach is based on the Crandall and Liggett existence theorem for a nonlinear Cauchy problem in a Banach space.

Concerning the Fokker-Planck equation with Neumann boundary condition we refer to [3] for a controllability problem treating an equation with a particular form of the function  $\beta$  which is Lipschitz.

In the present work we study a general Fokker-Planck equation with Neumann boundary conditions, under general growth conditions on  $\beta$ . The main objective of this paper is to prove existence and uniqueness of the mild solution to equation (1) and then to show that it is also a variational solution.

For reader's convenience we shall first recall the following definitions.

**Definition 2** Let  $y_0 \in L^2(\mathcal{O})$ . The function  $y$  is called a mild solution to equation (1) if  $y \in C([0, T]; L^1(\mathcal{O}))$  and

$$y(t) = \lim_{\delta \rightarrow \infty} y_\delta(t) \text{ in } L^1(\mathcal{O}), \text{ uniformly on } [0, T],$$

where  $y_\delta : [0, T] \rightarrow \mathbb{R}$  is a step function such that

$$y_\delta(t) = y_\delta^i, \quad \forall t \in [i\delta, (i+1)\delta), \quad i = 0, 1, \dots, N_\delta - 1,$$

for  $N_\delta = \lceil \frac{T}{\delta} \rceil$  which satisfies

$$\frac{1}{\delta} \int_{\mathcal{O}} (y_\delta^{i+1}(\xi) - y_\delta^i(\xi)) \psi(\xi) d\xi + \int_{\mathcal{O}} \left( \frac{1}{2} \nabla \beta(y_\delta^{i+1}) + b(\xi) y_\delta^{i+1} \right) \nabla \psi(\xi) d\xi = 0,$$

for  $\forall \psi \in H^1(\mathcal{O})$  such that  $y_\delta^0 = y_0$  and  $\beta(y_\delta^{i+1}) \in H^1(\mathcal{O})$ .

**Definition 3** Let  $y_0 \in L^2(\mathcal{O})$ . We say that  $y$  is a distributional solution to (1) if

$$y \in C([0, 1]; L^1(\mathcal{O})), \quad \beta(y) \in L^2(0, T; H_0^1(\mathcal{O}))$$

such that

$$\begin{aligned} \int_0^T \int_{\mathcal{O}} \left( y \frac{\partial \varphi}{\partial t} - \frac{1}{2} \nabla \beta(y) \cdot \nabla \varphi - (b \cdot \nabla \varphi) y \right) d\xi dt \\ + \int_{\mathcal{O}} y_0(\xi) \varphi(0, \xi) d\xi = 0 \end{aligned} \quad (3)$$

for all  $\varphi \in C^1([0, T] \times \overline{\mathcal{O}})$ ,  $\varphi(T, \cdot) = 0$ .

### Notations

For an open subset  $\mathcal{O} \subset \mathbb{R}^d$  we denote by  $L^p(\mathcal{O})$ ,  $1 \leq p \leq \infty$ , the space of standard Lebesgue  $p$ -integrable functions on  $\mathcal{O}$ , and by  $W^{1,p}(\mathcal{O})$  the Sobolev space  $\{u \in L^p(\mathcal{O}), D_i u \in L^p(\mathcal{O}) \text{ for } i = 1, 2, \dots, d\}$  where  $D_i = \frac{\partial}{\partial x_i}$  is taken in the sense of Schwartz distributions. In particular, we denote by  $H^1(\mathcal{O}) = W^{1,2}(\mathcal{O})$ .

## 2 The main results

We can formulate now the main result of this work.

**Theorem 4** Assuming the hypotheses above, for each  $y \in L^1(\mathcal{O})$ , there is a unique mild solution  $y$  to equation (1).

Moreover, if  $y_0 \geq 0$  a.e. on  $\mathcal{O}$  and  $\int_{\mathbb{R}} y_0(\xi) d\xi = 1$ , then  $y(t)$  is a probability density function on  $\mathcal{O}$  for  $\forall t \in [0, T]$ .

Furthermore,  $y$  is also a distributional solution to (1).

The main idea is to rewrite equation (1) as a Cauchy problem of the form

$$\begin{cases} dy + A(y) = 0, & (0, T), \\ y(0) = y_0, \end{cases}$$

in the space  $L^1(\mathcal{O})$ , where  $A : D(A) \subset L^1(\mathcal{O}) \rightarrow L^1(\mathcal{O})$  is an appropriate  $m$ -accretive operator and to apply the Crandall-Liggett theorem. To this purpose we shall first study this operator in the  $L^2(\mathcal{O})$  space framework.

**The  $L^2(\mathcal{O})$  approach**

We define the operator

$$\begin{aligned} A_0 & : D(A_0) \subset L^2(\mathcal{O}) \rightarrow L^2(\mathcal{O}) \\ D(A_0) & = \left\{ u \in L^2(\mathcal{O}); \Delta\beta(u) + \operatorname{div}(bu) \in L^2(\mathcal{O}), \right. \\ & \quad \left. \frac{\partial}{\partial n}\beta(u) + (b \cdot n)u = 0 \text{ a.e. on } \partial\mathcal{O} \right\} \\ A_0(u) & = -\Delta\beta(u) - \operatorname{div}(bu). \end{aligned}$$

For some  $f \in L^2(\mathcal{O})$  arbitrarily fixed and  $u \in H^1(\mathcal{O})$ , if we consider the equation

$$A_0 u = f$$

i.e.

$$\begin{cases} -\frac{1}{2}\Delta\beta(u) - \operatorname{div}(bu) = f, & \text{in } \mathcal{O} \\ \frac{1}{2}\frac{\partial}{\partial n}\beta(u) + (b \cdot n)u = 0, & \text{on } \partial\mathcal{O} \end{cases}$$

we can write its variational formulation as follows

$$\int_{\mathcal{O}} \left( \frac{1}{2}\nabla\beta(u(\xi)) + b(\xi)u(\xi) \right) \nabla\psi(\xi) d\xi = \int_{\mathcal{O}} f(\xi)\psi(\xi) d\xi, \quad \forall \psi \in H^1(\mathcal{O}).$$

**Proposition 5** *Assuming the hypotheses above, there is  $\lambda_0 > 0$  such that, for all  $f \in L^2(\mathcal{O})$  and  $\lambda \in (0, \lambda_0)$  arbitrarily fixed, the equation*

$$u + \lambda A_0 u = f \tag{4}$$

has a unique variational solution  $u$ .

Furthermore, if we denote by  $(I + \lambda A_0)^{-1}(f)$  the solution to equation (4) for each  $\lambda$  and  $f$ , we have the following propertie

$$\left| (I + \lambda A_0)^{-1}(f_1) - (I + \lambda A_0)^{-1}(f_2) \right|_{L^1(\mathcal{O})} \leq |f_1 - f_2|_{L^1(\mathcal{O})}, \quad \forall f_1, f_2 \in L^2(\mathcal{O}). \tag{5}$$

In addition, we have that  $(I + \lambda A_0)^{-1}(f)$  is a probability density function on  $\mathcal{O}$  for all  $f \in L^2(\mathcal{O})$  which is also a probability density function.

**Proof.** We shall first rewrite (4) as

$$\begin{cases} u - \frac{1}{2}\lambda\Delta\beta(u) - \lambda\operatorname{div}(bu) = f, & \text{in } \mathcal{O}, \\ \frac{1}{2}\frac{\partial}{\partial n}\beta(u) + (b \cdot n)u = 0, & \text{on } \partial\mathcal{O}, \end{cases} \tag{6}$$

with the variational formulation

$$\begin{aligned} & \int_{\mathcal{O}} u\psi d\xi + \frac{\lambda}{2} \int_{\mathcal{O}} \nabla\beta(u(\xi)) \nabla\psi d\xi + \lambda \int_{\mathcal{O}} (b(\xi) \cdot \nabla\psi(\xi)) u(\xi) d\xi \quad (7) \\ & = \int_{\mathcal{O}} f(\xi) \psi(\xi) d\xi, \quad \psi \in H^1(\mathcal{O}). \end{aligned}$$

We approximate the operator  $\beta$  by  $\tilde{\beta}_\varepsilon(r) = \beta_\varepsilon(r) + \varepsilon r$  where  $\beta_\varepsilon$  is the Yosida approximation of  $\beta$ , i.e.,

$$\beta_\varepsilon(r) = \frac{1}{\varepsilon} (r - J_\varepsilon(r)) = \beta(J_\varepsilon(r))$$

for

$$J_\varepsilon(r) = (Id + \varepsilon\beta)^{-1}(r), \quad \forall r \in \mathbb{R}.$$

Since  $\beta_\varepsilon$  is Lipschitz and increasing, we can easily see that  $\tilde{\beta}_\varepsilon$  is also Lipschitz and strongly monotone, that is

$$\left(\tilde{\beta}_\varepsilon(r) - \tilde{\beta}_\varepsilon(\bar{r})\right) (r - \bar{r}) \geq \varepsilon |r - \bar{r}|^2, \quad \forall r, \bar{r} \in \mathbb{R}.$$

Note that this property implies that  $\left(\tilde{\beta}_\varepsilon\right)^{-1}$  is also Lipschitz, where  $\left(\tilde{\beta}_\varepsilon\right)^{-1}$  is the inverse of  $\tilde{\beta}_\varepsilon$ .

We approximate equation (6) by

$$\begin{cases} u^\varepsilon - \frac{1}{2} \lambda \Delta \tilde{\beta}_\varepsilon(u^\varepsilon) - \lambda \operatorname{div}(bu^\varepsilon) = f, & \text{in } \mathcal{O}, \\ \frac{1}{2} \frac{\partial}{\partial n} \tilde{\beta}_\varepsilon(u^\varepsilon) + (b \cdot n) u^\varepsilon = 0, & \text{on } \partial\mathcal{O}, \end{cases} \quad (8)$$

with the variational formulation

$$\begin{aligned} & \int_{\mathcal{O}} u^\varepsilon \psi d\xi + \frac{\lambda}{2} \int_{\mathcal{O}} \nabla \tilde{\beta}_\varepsilon(u^\varepsilon(\xi)) \nabla \psi d\xi + \lambda \int_{\mathcal{O}} (b(\xi) \cdot \nabla \psi(\xi)) u^\varepsilon(\xi) d\xi \quad (9) \\ & = \int_{\mathcal{O}} f(\xi) \psi(\xi) d\xi, \quad \psi \in H^1(\mathcal{O}). \end{aligned}$$

If we denote by  $\left(\tilde{\beta}_\varepsilon\right)^{-1}$  the inverse of  $\tilde{\beta}_\varepsilon$ , we can rewrite (8) as follows

$$\begin{cases} \left(\tilde{\beta}_\varepsilon\right)^{-1}(v^\varepsilon) - \frac{1}{2} \lambda \Delta v^\varepsilon - \lambda \operatorname{div}\left(b \left(\tilde{\beta}_\varepsilon\right)^{-1}(v^\varepsilon)\right) = f, & \text{in } \mathcal{O}, \\ \frac{1}{2} \frac{\partial}{\partial n} v^\varepsilon + (b \cdot n) \left(\tilde{\beta}_\varepsilon\right)^{-1}(v^\varepsilon) = 0, & \text{on } \partial\mathcal{O}. \end{cases} \quad (10)$$

Note that since  $\tilde{\beta}_\varepsilon$  is strictly monotone, we can easily see that  $\left(\tilde{\beta}_\varepsilon\right)^{-1}$  is Lipschitz on  $\mathbb{R}$ .

Let us define the operator

$$B^\varepsilon : H^1(\mathcal{O}) \rightarrow (H^1)^*(\mathcal{O})$$

such that, for each  $v \in H^1(\mathcal{O})$  we have

$$\begin{aligned} (H^1)^* \langle B^\varepsilon(v), \varphi \rangle_{H^1} &= \int_{\mathcal{O}} \left( \tilde{\beta}_\varepsilon \right)^{-1}(v) \varphi d\xi \\ &+ \frac{\lambda}{2} \int_{\mathcal{O}} \nabla v \nabla \varphi d\xi \\ &+ \lambda \int_{\mathcal{O}} b(\xi) \cdot \nabla \varphi \left( \tilde{\beta}_\varepsilon \right)^{-1}(v) d\xi, \text{ for } \forall \varphi \in H^1(\mathcal{O}). \end{aligned}$$

We can now rewrite equation (10) as

$$B^\varepsilon(v^\varepsilon) = f. \quad (11)$$

We shall first show that the operator  $B^\varepsilon$  is strictly monotone. We have

$$\begin{aligned} (H^1)^* \langle B^\varepsilon(v) - B^\varepsilon(\bar{v}), v - \bar{v} \rangle_{H^1} &= \int_{\mathcal{O}} \left( \left( \tilde{\beta}_\varepsilon \right)^{-1}(v) - \left( \tilde{\beta}_\varepsilon \right)^{-1}(\bar{v}) \right) (v - \bar{v}) d\xi \\ &+ \frac{\lambda}{2} \int_{\mathcal{O}} \nabla(v - \bar{v}) d\xi \\ &+ \lambda \int_{\mathcal{O}} b(\xi) \cdot \nabla(v - \bar{v}) \left( \left( \tilde{\beta}_\varepsilon \right)^{-1}(v) - \left( \tilde{\beta}_\varepsilon \right)^{-1}(\bar{v}) \right) d\xi. \end{aligned}$$

Since by assumption  $i$ ) we have that

$$\left( \left( \tilde{\beta}_\varepsilon \right)^{-1}(v) - \left( \tilde{\beta}_\varepsilon \right)^{-1}(\bar{v}) \right) (v - \bar{v}) \geq \varepsilon \left| \left( \tilde{\beta}_\varepsilon \right)^{-1}(v) - \left( \tilde{\beta}_\varepsilon \right)^{-1}(\bar{v}) \right|^2$$

we obtain

$$\begin{aligned} (H^1)^* \langle B^\varepsilon(v) - B^\varepsilon(\bar{v}), v - \bar{v} \rangle_{H^1} &\geq \frac{\lambda}{4} \int_{\mathcal{O}} |\nabla(v - \bar{v})|^2 d\xi \\ &+ (\varepsilon - |b|_\infty \lambda) \int_{\mathcal{O}} \left| \left( \tilde{\beta}_\varepsilon \right)^{-1}(v) - \left( \tilde{\beta}_\varepsilon \right)^{-1}(\bar{v}) \right|^2 \\ &\geq 0, \text{ for } \lambda < \frac{\varepsilon}{2|b|_\infty} = \lambda_0. \end{aligned}$$

Keeping in mind that  $\beta(0) = 0$  we have also that

$$(H^1)^* \langle B^\varepsilon(v), v \rangle_{H^1} \geq \frac{\lambda}{4} \int_{\mathcal{O}} |\nabla(v)|^2 d\xi + (\varepsilon - |b|_\infty \lambda) \int_{\mathcal{O}} \left| \left( \tilde{\beta}_\varepsilon \right)^{-1}(v) \right|^2,$$



for  $\lambda < \frac{\varepsilon}{2|b|_\infty} = \lambda_0$ . Since  $\tilde{\beta}_\varepsilon$  is Lipschitz we have that  $\left| \left( \tilde{\beta}_\varepsilon \right)^{-1}(v) \right| \geq C|v|$ , for some positive constant  $C(\varepsilon)$  and for each  $\varepsilon$  fixed, and therefore  $B^\varepsilon$  is coercive from  $H^1$  to  $(H^1)^*$ .

Keeping in mind that  $B^\varepsilon$  is also continuous and monotone, we can now use the Minty-Browder theorem and get that  $B^\varepsilon$  is surjective. Therefore equation (11) has a unique solution  $v^\varepsilon \in H^1(\mathcal{O})$  for each  $\varepsilon$  fixed.

We obtain that  $u^\varepsilon = \tilde{\beta}_\varepsilon(v^\varepsilon) \in L^2(\mathcal{O})$  is a solution to (6).

We shall now prove that

$$|u^\varepsilon(\lambda, f_1) - u^\varepsilon(\lambda, f_2)|_{L^1(\mathcal{O})} \leq |f_1 - f_2|_{L^1(\mathcal{O})}, \quad \forall f_1, f_2 \in L^2(\mathcal{O}),$$

by using a similar approach with the one suggested in [4] [5] and [6]. To this purpose we define the function

$$\chi_\alpha(r) = \begin{cases} 1, & r > \alpha, \\ \frac{r}{\alpha}, & |r| \leq \alpha, \\ -1, & r < -\alpha, \end{cases}$$

for  $\alpha > 0$ .

On the other hand, from the variational form of the solution, we have for  $u_i^\varepsilon = u^\varepsilon(\lambda, f_i)$  where  $f_i \in L^2(\mathcal{O})$ ,  $i = \overline{1, 2}$ , that

$$\begin{aligned} & \int_{\mathcal{O}} (u_1^\varepsilon - u_2^\varepsilon) \psi d\xi + \frac{\lambda}{2} \int_{\mathcal{O}} \left( \nabla \tilde{\beta}_\varepsilon(u_1^\varepsilon) - \nabla \tilde{\beta}_\varepsilon(u_2^\varepsilon) \right) \nabla \psi d\xi \\ & + \lambda \int_{\mathcal{O}} b(\xi) \cdot \nabla \psi (u_1^\varepsilon - u_2^\varepsilon) d\xi \\ & = \int_{\mathcal{O}} (f_1 - f_2) \psi d\xi, \quad \psi \in H^1(\mathcal{O}). \end{aligned}$$

If we take  $\psi = \chi_\alpha \left( \tilde{\beta}_\varepsilon(u_1^\varepsilon) - \tilde{\beta}_\varepsilon(u_2^\varepsilon) \right)$ , in the previous relation, we get

$$\begin{aligned} & \int_{\mathcal{O}} (u_1^\varepsilon - u_2^\varepsilon) \chi_\alpha \left( \tilde{\beta}_\varepsilon(u_1^\varepsilon) - \tilde{\beta}_\varepsilon(u_2^\varepsilon) \right) d\xi \\ & + \frac{\lambda}{2} \int_{\mathcal{O}} \left| \nabla \tilde{\beta}_\varepsilon(u_1^\varepsilon) - \nabla \tilde{\beta}_\varepsilon(u_2^\varepsilon) \right|^2 \chi'_\alpha d\xi \\ & + \lambda \int_{\mathcal{O}} b(\xi) \cdot \nabla \left( \tilde{\beta}_\varepsilon(u_1^\varepsilon) - \tilde{\beta}_\varepsilon(u_2^\varepsilon) \right) \chi'_\alpha (u_1^\varepsilon - u_2^\varepsilon) d\xi \\ & \leq \int_{\mathcal{O}} |f_1 - f_2| d\xi, \end{aligned}$$

where we denoted by  $\chi'_\alpha = \chi'_\alpha \left( \tilde{\beta}_\varepsilon(u_1^\varepsilon) - \tilde{\beta}_\varepsilon(u_2^\varepsilon) \right)$ .

Since  $\chi_\alpha < 1$ ,  $\chi'_\alpha > 0$  and keeping in mind the explicit form of  $\chi_\alpha$  we have

$$\begin{aligned} & \int_{\mathcal{O}} (u_1^\varepsilon - u_2^\varepsilon) \chi_\alpha \left( \tilde{\beta}_\varepsilon(u_1^\varepsilon) - \tilde{\beta}_\varepsilon(u_2^\varepsilon) \right) d\xi \\ & \leq \frac{\lambda}{\alpha} |b|_\infty \int_{[\tilde{\beta}_\varepsilon(u_1) - \tilde{\beta}_\varepsilon(u_2) < \alpha]} \left| \nabla \left( \tilde{\beta}_\varepsilon(u_1^\varepsilon) - \tilde{\beta}_\varepsilon(u_2^\varepsilon) \right) \right| |u_1^\varepsilon - u_2^\varepsilon| d\xi \\ & \quad + \int_{\mathcal{O}} |f_1 - f_2| d\xi. \end{aligned} \tag{12}$$

We calculate the first term from the right hand side of the previous relation.

Since  $\tilde{\beta}_\varepsilon$  is strongly increasing and therefore  $\left( \tilde{\beta}_\varepsilon \right)^{-1}$  is Lipschitz, we have that

$$|u_1^\varepsilon - u_2^\varepsilon| \leq \frac{1}{\varepsilon} \left| \tilde{\beta}_\varepsilon(u_1^\varepsilon) - \tilde{\beta}_\varepsilon(u_2^\varepsilon) \right|,$$

and therefore

$$\begin{aligned} & \lim_{\alpha \rightarrow 0} \frac{\lambda}{\alpha} |b|_\infty \int_{[\tilde{\beta}_\varepsilon(u_1) - \tilde{\beta}_\varepsilon(u_2) < \alpha]} \left| \nabla \left( \tilde{\beta}_\varepsilon(u_1^\varepsilon) - \tilde{\beta}_\varepsilon(u_2^\varepsilon) \right) \right| |u_1^\varepsilon - u_2^\varepsilon| d\xi \\ & \leq \lim_{\alpha \rightarrow 0} \frac{\lambda}{\varepsilon} |b|_\infty \int_{[\tilde{\beta}_\varepsilon(u_1) - \tilde{\beta}_\varepsilon(u_2) < \alpha]} \left| \nabla \left( \tilde{\beta}_\varepsilon(u_1^\varepsilon) - \tilde{\beta}_\varepsilon(u_2^\varepsilon) \right) \right| d\xi = 0. \end{aligned}$$

Going back to (12) and considering that  $\lim_{\alpha \rightarrow 0} \chi_\alpha = \text{sgn}$  and

$$\text{sgn} \left( \tilde{\beta}_\varepsilon(u_1^\varepsilon) - \tilde{\beta}_\varepsilon(u_2^\varepsilon) \right) = \text{sgn}(u_1^\varepsilon - u_2^\varepsilon)$$

we obtain that

$$\lim_{\alpha \rightarrow 0} \int_{\mathcal{O}} (u_1^\varepsilon - u_2^\varepsilon) \chi_\alpha \left( \tilde{\beta}_\varepsilon(u_1^\varepsilon) - \tilde{\beta}_\varepsilon(u_2^\varepsilon) \right) d\xi = |u_1^\varepsilon - u_2^\varepsilon|_{L^1(\mathcal{O})}.$$

This leads to

$$|u_1^\varepsilon - u_2^\varepsilon|_{L^1(\mathcal{O})} \leq |f_1 - f_2|_{L^1(\mathcal{O})}, \quad \forall f_1, f_2 \in L^2(\mathcal{O}). \tag{13}$$

We shall now pass to the limit for  $\varepsilon \rightarrow 0$  in the approximating equation (8) with the variational formulation (9). To this purpose we need the following estimates.

First we take  $\psi = u^\varepsilon$  and we get

$$\int_{\mathcal{O}} |u^\varepsilon|^2 d\xi + \frac{1}{2} \lambda \int_{\mathcal{O}} \nabla \tilde{\beta}_\varepsilon(u^\varepsilon) \nabla u^\varepsilon d\xi + \lambda \int_{\mathcal{O}} (b \cdot \nabla u^\varepsilon) u^\varepsilon d\xi = \int_{\mathcal{O}} f u^\varepsilon d\xi.$$

Since, by assumption *iii*) we have that

$$\frac{1}{2} \int_{\mathcal{O}} b \cdot \nabla (u^\varepsilon)^2 d\xi = -\frac{1}{2} \int_{\mathcal{O}} \text{div} b (u^\varepsilon)^2 d\xi + \frac{1}{2} \int_{\partial \mathcal{O}} (b \cdot n) (u^\varepsilon)^2 d\xi \geq 0$$

we get  $\{u^\varepsilon\}$  is bounded in  $L^2(\mathcal{O})$ .

Then we take  $\psi = \tilde{\beta}_\varepsilon(u^\varepsilon)$  and we get

$$\begin{aligned} & \int_{\mathcal{O}} u^\varepsilon \tilde{\beta}_\varepsilon(u^\varepsilon) d\xi + \frac{1}{2} \lambda \int_{\mathcal{O}} \left| \nabla \tilde{\beta}_\varepsilon(u^\varepsilon) \right|^2 d\xi + \lambda \int_{\mathcal{O}} \left( b \cdot \nabla \tilde{\beta}_\varepsilon(u^\varepsilon) \right) u^\varepsilon d\xi \quad (14) \\ &= \int_{\mathcal{O}} f \tilde{\beta}_\varepsilon(u^\varepsilon) d\xi. \end{aligned}$$

We calculate

$$\begin{aligned} \int_{\mathcal{O}} u^\varepsilon \tilde{\beta}_\varepsilon(u^\varepsilon) d\xi &= \int_{\mathcal{O}} u^\varepsilon \beta_\varepsilon(u^\varepsilon) d\xi + \int_{\mathcal{O}} |u^\varepsilon|^2 d\xi \\ &\geq \int_{\mathcal{O}} (u^\varepsilon - J_\varepsilon(u^\varepsilon)) \beta(J_\varepsilon(u^\varepsilon)) d\xi + \int_{\mathcal{O}} J_\varepsilon(u^\varepsilon) \beta(J_\varepsilon(u^\varepsilon)) d\xi \\ &\geq \alpha_3 \int_{\mathcal{O}} |J_\varepsilon(u^\varepsilon)|^{m+1} d\xi \end{aligned}$$

and

$$\lambda \int_{\mathcal{O}} \left( b \cdot \nabla \tilde{\beta}_\varepsilon(u^\varepsilon) \right) u^\varepsilon d\xi \leq \frac{1}{4} \lambda \int_{\mathcal{O}} \left| \nabla \tilde{\beta}_\varepsilon(u^\varepsilon) \right|^2 d\xi + C \int_{\mathcal{O}} |u^\varepsilon|^2 d\xi.$$

By going back to (14) we get

$$\begin{aligned} & \alpha_3 \int_{\mathcal{O}} |J_\varepsilon(u^\varepsilon)|^{m+1} d\xi + \frac{1}{4} \lambda \int_{\mathcal{O}} \left| \nabla \tilde{\beta}_\varepsilon(u^\varepsilon) \right|^2 d\xi \quad (15) \\ & \leq C + \frac{\rho}{2} \int_{\mathcal{O}} |f|^{m+1} d\xi + \frac{1}{2\rho} \int_{\mathcal{O}} \left| \tilde{\beta}_\varepsilon(u^\varepsilon) \right|^{\frac{m+1}{m}} d\xi. \end{aligned}$$

Since

$$\tilde{\beta}_\varepsilon(u^\varepsilon) = \beta_\varepsilon(u^\varepsilon) + \varepsilon u^\varepsilon$$

we have

$$\int_{\mathcal{O}} \left| \tilde{\beta}_\varepsilon(u^\varepsilon) \right|^{\frac{m+1}{m}} d\xi \leq 2^{\frac{1}{m}} |\beta(J_\varepsilon(u^\varepsilon))|^{\frac{m+1}{m}} + 2^{\frac{1}{m}} \varepsilon^{\frac{m+1}{m}} |u^\varepsilon|^{\frac{m+1}{m}}.$$

On the other hand, since  $|\beta(r)| \leq \alpha_1 |r|^m + \alpha_2, \forall r \in \mathbb{R}$ , we have that

$$|\beta(J_\varepsilon(u^\varepsilon))|^{\frac{m+1}{m}} = \int_{\mathcal{O}} |\beta(J_\varepsilon(u^\varepsilon))|^{\frac{m+1}{m}} d\xi \leq C \left( \int_{\mathcal{O}} |J_\varepsilon(u^\varepsilon)|^{m+1} d\xi + 1 \right).$$

By replacing in (15) we get that, for a constant  $\rho$  sufficiently large, that

$$\frac{\alpha_3}{2} \int_{\mathcal{O}} |J_\varepsilon(u^\varepsilon)|^{m+1} d\xi + \frac{1}{4} \lambda \int_{\mathcal{O}} \left| \nabla \tilde{\beta}_\varepsilon(u^\varepsilon) \right|^2 d\xi \leq C.$$

From the previous estimates, it follows that

$$\{u_\varepsilon\} \text{ bounded in } L^2(\mathcal{O})$$

$$\begin{aligned} \{J_\varepsilon(u^\varepsilon)\} &\text{ bounded in } L^{m+1}(\mathcal{O}) \\ \{\beta_\varepsilon(u^\varepsilon)\} &\text{ bounded in } L^{\frac{m+1}{m}}(\mathcal{O}) \\ \{\nabla\beta_\varepsilon(u^\varepsilon)\} &\text{ bounded in } L^2(\mathcal{O}) \subset L^{\frac{m+1}{m}}(\mathcal{O}). \end{aligned}$$

Since the Sobolev space  $W^{1, \frac{m+1}{m}}(\mathcal{O})$  is compactly embedded in  $L^{\frac{m+1}{m}}(\mathcal{O})$  we have that

$$\beta_\varepsilon(u^\varepsilon) \rightarrow \eta \text{ strongly in } L^{\frac{m+1}{m}}(\mathcal{O})$$

and therefore  $\beta(u) = \eta$ .

We can get (5) by passing to the limit in (13).

In order to conclude the proof of the proposition, it only remain to show that  $u$  is a probability density function on  $\mathcal{O}$  for each  $f \in L^2(\mathcal{O})$  which is a probability density function.

We assume that  $f$  is a probability density, i.e.  $f \geq 0$  a.e. in  $\mathcal{O}$  and  $\int_{\mathcal{O}} f(\xi) d\xi = 1$ .

Let us first take  $\psi = u^-$  in (7). We obtain

$$\begin{aligned} & - \int_{[u<0]} |u|^2 d\xi - \frac{\lambda}{2} \int_{[u<0]} \nabla\beta(u) \nabla u d\xi + \lambda \int_{\mathcal{O}} b \cdot \nabla u^- u d\xi \quad (16) \\ & = - \int_{[u<0]} f u d\xi. \end{aligned}$$

We calculate

$$\begin{aligned} & \lambda \int_{\mathcal{O}} b \cdot \nabla u^- u d\xi \\ & = -\lambda \int_{\mathcal{O}} b \cdot \nabla u^- u^- d\xi \\ & = -\frac{\lambda}{2} \int_{\mathcal{O}} b \cdot \nabla (u^-)^2 d\xi \\ & = \frac{\lambda}{2} \int_{\mathcal{O}} \operatorname{div} b (u^-)^2 d\xi - \frac{\lambda}{2} \int_{\partial\mathcal{O}} b \cdot n (u^-)^2 d\xi \\ & = \frac{\lambda}{2} \int_{[u<0]} \operatorname{div} b (u^-)^2 d\xi - \frac{\lambda}{2} \int_{\partial\mathcal{O}} (b \cdot n) (u^-)^2 d\xi < 0. \end{aligned}$$

Keeping in mind the assumptions on  $b$ , we obtain that  $u^- = 0$  a.e. on  $\mathcal{O}$  since the left hand side of (16) is negative and  $-\int_{[u<0]} f u d\xi > 0$ . It is then immediate that

$$\int_{\mathcal{O}} u(\xi) d\xi = \int_{\mathcal{O}} f(\xi) d\xi = 1$$

and we can conclude that  $u$  is a probability density function. The proof of the proposition is now complete. ■

**The  $L^1(\mathcal{O})$  approach**

Let us first consider the operator  $\bar{A}: D(\bar{A}) \subset L^1(\mathcal{O}) \rightarrow L^1(\mathcal{O})$  defined by

$$\bar{A}(u) = -\Delta\beta(u) + \operatorname{div}(bu)$$

where

$$D(\bar{A}) = \left\{ u \in L^1(\mathcal{O}); \Delta\beta(u) + \operatorname{div}(bu) \in L^1(\mathcal{O}), \right. \\ \left. \frac{\partial}{\partial n}\beta(u) + (b \cdot n)u = 0 \text{ a.e. on } \partial\mathcal{O} \right\},$$

where  $\Delta$  is taken in the sense of distributions on  $\mathcal{O}$ .

We can prove now the following result.

**Proposition 6** *For all  $f \in L^1(\mathcal{O})$  and  $\forall \lambda \in (0, \lambda_0)$ , the equation*

$$u + \lambda\bar{A}u = f$$

*has a solution  $u = (I + \lambda\bar{A})^{-1}f$ .*

*Moreover*

$$\left| (I + \lambda\bar{A})^{-1}f_1 - (I + \lambda\bar{A})^{-1}f_2 \right|_{L^1(\mathcal{O})} \leq |f_1 - f_2|_{L^1(\mathcal{O})}, \quad (17)$$

$\forall f_1, f_2 \in L^1(\mathcal{O})$ .

**Proof.** For  $f \in L^1(\mathcal{O})$  arbitrarily fixed we have a sequence  $\{f_n\} \subset L^2(\mathcal{O})$  such that  $f_n \rightarrow f$  strongly in  $L^1(\mathcal{O})$  for  $n \rightarrow \infty$ . Since we have existence of a solution for the following equation

$$u_n + \lambda\bar{A}u_n = f_n,$$

we can rewrite it as

$$u_n = (I + \lambda\bar{A})^{-1}f_n.$$

Using Proposition 5 we have that

$$\left| (I + \lambda A_0)^{-1}f_1 - (I + \lambda A_0)^{-1}f_2 \right|_{L^1(\mathcal{O})} \leq |f_1 - f_2|_{L^1(\mathcal{O})},$$

for all  $f_1, f_2 \in L^2(\mathcal{O})$  which implies directly that

$$|u_n - u_m|_{L^1(\mathcal{O})} \leq |f_n - f_m|_{L^1(\mathcal{O})}, \quad \forall n, m \in \mathbb{N},$$

and then  $(u_n)_n$  converges strongly in  $L^1(\mathcal{O})$  to some  $u \in L^1(\mathcal{O})$ . Since  $(I + \lambda A_0)^{-1}(f) = (I + \lambda\bar{A})^{-1}(f)$ , for  $f \in L^2(\mathcal{O})$  we get that  $\bar{A}u_n \rightarrow \bar{A}u$  in  $L^1(\mathcal{O})$  and

$$u + \lambda\bar{A}u = f.$$

Hence the operator  $\bar{A}$  is closed in  $L^1 \times L^1$ . The estimate follows by the previous result. ■

We can finally define the operator

$$A : D(A) \subset L^1(\mathcal{O}) \rightarrow L^1(\mathcal{O})$$

$$Ay = \bar{A}y, \quad \forall y \in D(A) = (I + \lambda \bar{A})^{-1}(L^1(\mathcal{O}))$$

for some  $\lambda > 0$  (equivalently, for each  $\lambda > 0$ ).

Concerning the operator above we have the following result.

**Proposition 7** *The operator  $A$  is  $m$ -accretive in  $L^1(\mathcal{O})$ , that is,  $R(I + \lambda A) = L^1(\mathcal{O})$ , for  $\forall \lambda > 0$  and*

$$\left| (I + \lambda A)^{-1} f_1 - (I + \lambda A)^{-1} f_2 \right|_{L^1(\mathcal{O})} \leq |f_1 - f_2|_{L^1(\mathcal{O})},$$

for  $\forall f_1, f_2 \in L^1(\mathcal{O})$ ,  $\lambda > 0$ .

The proof of this result follows directly from the previous one by the definition of the operator  $A$ .

**Proof of the Theorem 4.** In order to use the Crandall-Liggett theorem for the Cauchy problem

$$\begin{cases} \frac{dy}{dt} + A(y) = 0, & t \in (0, T), \\ y(0) = y_0, \end{cases}$$

where  $y_0 \in L^1(\mathcal{O})$ , we need to have  $\overline{D(A)} = L^1(\mathcal{O})$ , but this follows by a similar argument as the one used in Theorem 2 from [4].

For reader's convenience we shall sketch it here. We first consider for each  $\varepsilon > 0$  the approximating operator

$$A_\varepsilon : D(A_\varepsilon) \subset L^2(\mathcal{O}) \rightarrow L^2(\mathcal{O})$$

defined by

$$A_\varepsilon(u) = -\Delta \tilde{\beta}_\varepsilon(u) - \operatorname{div}(b \cdot u), \quad \text{for } \forall u \in D(A_\varepsilon)$$

where

$$D(A_\varepsilon) = \left\{ u \in H^1(\mathcal{O}); \tilde{\beta}_\varepsilon(u) \in H^2(\mathcal{O}), \right. \\ \left. \frac{\partial}{\partial n} \tilde{\beta}_\varepsilon(u) + (b \cdot n)u = 0, \text{ a.e. on } \partial\mathcal{O} \right\}.$$

By Proposition 5 we have that

$$|u_\varepsilon^1 - u_\varepsilon^2|_{L^1(\mathcal{O})} \leq |f_1 - f_2|_{L^1(\mathcal{O})}, \quad \text{for } \forall f_1, f_2 \in L^2(\mathcal{O}),$$

where  $u_\varepsilon^i$  is the solution of  $u_\varepsilon^i + \lambda A_\varepsilon (u_\varepsilon^i) = f_i$ ,  $i = \overline{1, 2}$  and also that

$$\lim_{\varepsilon \rightarrow 0} u_\varepsilon = u = J_\lambda (f) \text{ in } L^1(\mathcal{O}), \quad \forall f \in L^2(\mathcal{O}), \quad \lambda \in (0, \lambda_0).$$

Since  $C_0^\infty(\mathbb{R}^d) \subset D(A_\varepsilon)$  it follows that  $D(A_\varepsilon)$  is dense in  $L^1(\mathcal{O})$  and therefore so is  $D(A)$ .

We get that, for each  $y_0 \in L^1(\mathcal{O}) = \overline{D(A)}$ , the problem above has a unique mild solution  $y \in C([0, T]; L^1(\mathcal{O}))$ . Moreover,

$$y(t) = \lim_{n \rightarrow \infty} \left( I + \frac{t}{n} A \right)^{-n} y_0, \quad \text{in } L^1(\mathcal{O}),$$

uniformly in  $t$  on compact intervals (see e.g. [1]).

Keeping in mind the definition of a mild solution, this means that

$$y(t) = \lim_{\delta \rightarrow 0} y_\delta(t), \quad \text{in } L^1(\mathcal{O})$$

uniformly on  $[0, T]$ , where

$$y_\delta(t) = y_\delta^{i+1}, \quad \forall t \in [i\delta, (i+1)\delta),$$

where  $i = 0, 1, \dots, N_\delta - 1$  for  $N_\delta = \lceil \frac{T}{\delta} \rceil$ , and

$$\begin{cases} y_\delta^{i+1} + \delta A(y_\delta^{i+1}) = y_\delta^i, & i = 0, 1, \dots, N_\delta - 1 \\ y_\delta^0 = y_0. \end{cases} \quad (18)$$

Moreover  $y = y(t, y_0)$  generates a semigroup of contraction in  $L^1(\mathcal{O})$ , i.e.

$$|y(t, y_0^1) - y(t, y_0^2)|_{L^1(\mathcal{O})} \leq |y_0^1 - y_0^2|_{L^1(\mathcal{O})}, \quad \forall y_0^1, y_0^2 \in L^1(\mathcal{O}).$$

If we take  $y_0 \in L^2(\mathcal{O})$  and keep in mind that by Proposition 6 we have

$$(I + \lambda \bar{A})^{-1} f = (I + \lambda A)^{-1} f, \quad \forall f \in L^2(\mathcal{O}),$$

it follows that  $y_\delta^{i+1} \in D(A)$  and

$$\frac{1}{\delta} (y_\delta^{i+1} - y_\delta^i) = -A y_\delta^{i+1}, \quad \forall i = 0, 1, \dots, N_\delta - 1.$$

This means that  $y$  is the mild solution to equation (1) in the sense of Definition 2. By the accretivity of the operator  $A$  in  $L^1(\mathcal{O})$  it is easily seen that we have also uniqueness of the solution.

In order to conclude the proof of the theorem we only need to show that  $y$  is also a distributional solution to equation (1).

From the definition of the mild solution we have that

$$\frac{1}{\delta} \int_{\mathcal{O}} (y_\delta^{i+1} - y_\delta^i) \beta(y_\delta^{i+1}) d\xi + \frac{1}{2} \int_{\mathcal{O}} |\nabla \beta(y_\delta^{i+1})|^2 d\xi$$

$$+ \int_{\mathcal{O}} b(\xi) \cdot \nabla \beta(y_\delta^{i+1}) y_\delta^{i+1} d\xi = 0,$$

and keeping in mind that  $\partial j = \beta$  we can compute

$$\begin{aligned} & \int_{\mathcal{O}} j(y_\delta^{i+1}) d\xi - \int_{\mathcal{O}} j(y_\delta^i) d\xi + \frac{\delta}{2} \int_{\mathcal{O}} |\nabla \beta(y_\delta^{i+1})|_2^2 d\xi \\ & + \delta \int_{\mathcal{O}} b(\xi) \cdot \nabla \beta(y_\delta^{i+1}) y_\delta^{i+1} d\xi = 0. \end{aligned}$$

If we take the sum for  $i = \overline{1, k}$  we get

$$\begin{aligned} & \int_{\mathcal{O}} j(y_\delta^{k+1}) d\xi - \int_{\mathcal{O}} j(y_\delta^0) d\xi + \frac{\delta}{2} \sum_{i=0}^k |\nabla \beta(y_\delta^{i+1})|_2^2 \\ & + \delta \sum_{i=0}^k \int_{\mathcal{O}} b(\xi) \cdot \nabla \beta(y_\delta^{i+1}) y_\delta^{i+1} d\xi = 0, \end{aligned}$$

which leads to

$$\begin{aligned} & \int_{\mathcal{O}} j(y_\delta^{k+1}) d\xi + \frac{\delta}{2} \sum_{i=0}^k |\nabla \beta(y_\delta^{i+1})|_2^2 \\ & \leq \int_{\mathcal{O}} j(y_\delta^0) d\xi + \delta \sum_{i=0}^k |b|_\infty |y_\delta^{i+1}|_2 |\nabla \beta(y_\delta^{i+1})|_2, \end{aligned}$$

for  $\forall k = 0, 1, \dots, N_\delta - 1$ .

We obtain that

$$\int_{\mathcal{O}} j(y_\delta^{k+1}) d\xi + \frac{\delta}{4} \sum_{i=0}^k |\nabla \beta(y_\delta^{i+1})|_2^2 \leq \int_{\mathcal{O}} j(y_\delta^0) d\xi + \delta |b|_\infty \sum_{i=0}^k |y_\delta^{i+1}|_2^2.$$

Since  $y_\delta(t) = y_\delta^i$  for  $t \in [i\delta, (i+1)\delta)$  we get by the previous estimate that

$$\begin{aligned} & \int_{\mathcal{O}} j(y_\delta(t)) d\xi + \frac{1}{4} \int_0^t |\nabla \beta(y_\delta(s))|_2^2 ds \\ & \leq \int_{\mathcal{O}} j(y_0) d\xi + |b|_\infty \int_0^t |y_\delta(s)|_2^2 ds, \quad \forall t \in [0, T]. \end{aligned}$$

From the assumptions we have on  $j$  we get that

$$\begin{aligned} & \sup_{t \in [0, T]} |y_\delta(t)|_2^2 + \frac{1}{4} \int_0^T |\nabla \beta(y_\delta(s))|_2^2 ds \\ & \leq \int_{\mathcal{O}} j(y_0) d\xi + |b|_\infty^2 T \sup_{t \in [0, T]} |y_\delta(s)|_2^2 ds, \end{aligned}$$



and therefore

$$\left(1 - |b|_\infty^2\right) \sup_{t \in [0, T]} \|y_\delta(t)\|_2^2 + \frac{1}{4} \int_0^T \|\nabla \beta(y_\delta(s))\|_2^2 ds \leq \int_{\mathcal{O}} j(y_0) d\xi$$

We know that

$$\begin{aligned} y_\delta &\rightarrow y \text{ in } C([0, T]; L^1(\mathcal{O})) \\ y_\delta &\rightharpoonup y \text{ weakly in } L^\infty(0, T; L^2(\mathcal{O})) \\ \beta(y_\delta) &\rightharpoonup \eta \text{ weakly in } L^2(0, T; H^1(\mathcal{O})) \\ \beta &\text{ maximal monotone in } L^2(\mathcal{O}) \times L^2(\mathcal{O}). \end{aligned}$$

Since  $\beta$  is continuous and

$$y_\delta(t, \xi) \rightarrow y(t, \xi) \text{ a.e. on } (0, T) \times \mathcal{O}$$

it follows that

$$\eta(t, \xi) = \lim_{\delta \rightarrow 0} \beta(y_\delta(t, \xi)) \text{ a.e. on } (0, T) \times \mathcal{O}.$$

We can now conclude the proof that  $y$  is also a distributional solution. We shall use the previous estimate to pass to the limit in the mild solution in order to obtain that it is also a distributional solution. More precisely, if we take

$$\frac{1}{\delta} \int_{\mathcal{O}} (y_\delta^{i+1} - y_\delta^i) \psi d\xi + \int_{\mathcal{O}} \left( \frac{1}{2} \nabla \beta(y_\delta^{i+1}) + b(\xi) y_\delta^{i+1} \right) \cdot \nabla \psi d\xi = 0$$

and keep in mind that

$$y_\delta(t) = y_\delta^i, \quad t \in [i\delta, (i+1)\delta)$$

we obtain, for all  $\psi \in C^1([0, T] \times \overline{\mathcal{O}})$  such that  $\psi(T, \cdot) \equiv 0$ , that

$$\begin{aligned} &\frac{1}{\delta} \int_{\mathcal{O}} \int_0^{T-\delta} (y_\delta(t+\delta) - y_\delta(t)) \psi(t, \xi) d\xi \\ &+ \int_{\mathcal{O}} \int_0^{T-\delta} \left( \frac{1}{2} \nabla \beta(y_\delta(t+\delta)) + b(\xi) y_\delta(t+\delta) \right) \cdot \nabla \psi(t, \xi) d\xi = 0. \end{aligned}$$

After some elementary calculus we get for the first term that

$$\begin{aligned} &\frac{1}{\delta} \int_{\mathcal{O}} \int_0^{T-\delta} y_\delta(t+\delta) \psi(t, \xi) d\xi - \frac{1}{\delta} \int_{\mathcal{O}} \int_0^{T-\delta} y_\delta(t) \psi(t, \xi) d\xi \\ &= -\frac{1}{\delta} \int_{\mathcal{O}} \int_0^{T-\delta} y_\delta(t) (\psi(t, \xi) - \psi(t-\delta, \xi)) d\xi \\ &\quad - \frac{1}{\delta} \int_{\mathcal{O}} \int_0^\delta y_\delta(t) \psi(t, \xi) d\xi \\ &\quad + \frac{1}{\delta} \int_{\mathcal{O}} \int_{T-\delta}^T y_\delta(t) \psi(t, \xi) d\xi. \end{aligned}$$

By going back to the previous relation and replacing this expression we get that

$$\begin{aligned} & -\frac{1}{\delta} \int_{\mathcal{O}} \int_0^{T-\delta} y_\delta(t) (\psi(t, \xi) - \psi(t - \delta, \xi)) d\xi \\ & -\frac{1}{\delta} \int_{\mathcal{O}} \int_0^\delta y_\delta(t) \psi(t, \xi) d\xi + \frac{1}{\delta} \int_{\mathcal{O}} \int_{T-\delta}^T y_\delta(t) \psi(t, \xi) d\xi \\ & + \int_{\mathcal{O}} \int_0^{T-\delta} \left( \frac{1}{2} \nabla \beta(y_\delta(t + \delta)) + b(\xi) y_\delta(t + \delta) \right) \cdot \nabla \psi(t, \xi) d\xi = 0 \end{aligned}$$

and finally, by letting  $\delta \rightarrow 0$ , it follows that  $y$  is a variational solution in the sense of Definition 3. ■

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