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Telveenus Antony

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HALIDON RINGS AND THEIR APPLICATIONS

A. TELVEENUS

Kingston University London International Study Centre, Kingston Hill Campus, KT2 7LB,UK
e-mail address: t.fernandezantony@kingston.ac.uk

ABSTRACT. Halidon rings are rings with a unit element, containing a primitive m^{th} root of unity and m is invertible in the ring. The field of complex numbers is a halidon ring with any index $m \geq 1$. This article examines different applications of halidon rings. The main application is the extension of Maschke's theorem. In representation theory, Maschke's theorem has an important role in studying the irreducible subrepresentations of a given group representation and this study is related to a finite field of characteristic which does not divide the order of the given finite group or the field of real or complex numbers. The second application is the computational aspects of halidon rings and group rings which enable us to verify Maschke's theorem. Some computer codes have been developed to establish the existence of halidon rings which are not fields and the computation of units, involutions and idempotents in both halidon rings and halidon group rings. The third application of halidon rings is in solving Rososhek's problem related to some cryptosystems. The applications in Bilinear forms and Discrete Fourier Transform (DFT) with two computer codes developed to calculate DFT and inverse DFT have also been discussed.

1. INTRODUCTION

In 1940, the famous celebrated mathematician Graham Higman published a theorem [3] in group algebra which is valid only for a field or an integral domain with some specific conditions. In 1999, the author noticed that this theorem can be extended to a rich class of rings called halidon rings[9]. In complex analysis, $|z| = 1$ is the set of all points on a circle with centre at the origin and radius 1. The solutions of $z^n = 1$, which are usually called the n^{th} roots of unity, which can be computed by DeMoivre's theorem, will form a regular polygon with vertices as the solutions. As $n \rightarrow \infty$ this regular polygon will become a unit circle. This is the ordinary concept of n^{th} of unity in the field of complex numbers. For a ring, we need to think differently.

A primitive m^{th} root of unity in a ring with unit element is completely different from that of in a field, because of the presence of nonzero zero divisors. So we need a separate

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Former Prof. & Head, Dept. of Mathematics, Fatima Mata National College, University of Kerala, Kollam, S. India . Supported by no grants from any sources.

definition for a primitive m^{th} root of unity. An element ω in a ring R is called a *primitive m^{th} root* if m is the least positive integer such that $\omega^m = 1$ and

$$\begin{aligned} \sum_{r=0}^{m-1} \omega^{r(i-j)} &= m, \quad i = j \pmod{m} \\ &= 0, \quad i \neq j \pmod{m}. \end{aligned}$$

A ring R with unity is called a *halidon ring* with index m if there is a primitive m^{th} root of unity and m is invertible in R . The ring of integers is a halidon ring with index $m = 1$ and $\omega = 1$. The halidon ring with index 1 is usually called a *trivial halidon ring*. The field of real numbers is a halidon ring with index $m = 2$ and $\omega = -1$. The field $\mathbb{Q}(i) = \{a + ib \mid a, b \in \mathbb{Q}\}$ is a halidon ring with $\omega = i$ and $m = 4$. \mathbb{Z}_{65} is an example of a halidon ring with index 4 and $\omega = 8$ which is not a field. In general \mathbb{Z}_{4r^2+1} is a halidon ring with index 4 and $\omega = 2r$ for each integer $r > 0$. \mathbb{Z}_p is a halidon ring with index $p - 1$ for every prime p . Interestingly, \mathbb{Z}_{p^k} is also a halidon ring with same index for any integer $k > 0$ and it is not a field if $k > 1$. Note that if ω is a primitive m^{th} root of unity, then ω^{-1} is also a primitive m^{th} root of unity.

2. PRELIMINARY RESULTS

We state the following propositions without proof. The proofs are available in [9] and [10]. Let $U(R)$ and $ZD(R)$ denote the unit group of R and the set of zero divisors in R respectively. Clearly, they are disjoint sets for a finite commutative halidon ring R .

Proposition 1. *A finite commutative ring R with unity is a halidon ring with index m if and only if there is a primitive m^{th} root of unity ω such that $m, \omega^r - 1 \in U(R)$; the unit group of R for all $r = 0, 1, \dots, m - 1$. If m is a prime number it is enough to have $m, \omega - 1 \in U(R)$.*

Corollary 1. *Let R be a finite commutative halidon ring with index m . Then m divides the number of nonzero zero divisors in R .*

Corollary 2. *Let R be a commutative halidon ring with index m and let $k > 1$ be a divisor of m . Then R is also a halidon ring with index k .*

Proposition 2. *Let R be a finite commutative halidon ring with index m . Then $R = U(R) \cup ZD(R)$ with $U(R) \cap ZD(R) = \{\}$; the empty set.*

Proof. Since R is finite and m is invertible in R , the proof is evident. □

Proposition 3. *Let R be a finite commutative halidon ring with index m . Then $|R| \equiv 1 \pmod{m}$.*

Proposition 4. *Let R_1 and R_2 be two halidon rings with same index m , then $R_1 \oplus R_2$ is also a halidon ring with same index m .*

Proposition 5. *Let $h: R \rightarrow R'$ be a homomorphism from a halidon ring with index m to another ring R' such that the unit elements correspond. Then $h(R) \subset R'$ is also a halidon ring with same index.*

Corollary 3. *Let S be a multiplicatively closed subset of a halidon ring R with index m , then $R_S = \{a/s \mid a \in R, s \in S\}$ is also a halidon ring with same index.*

Corollary 4. *Let I be an ideal of a halidon ring with index m . Then R/I is also a halidon with same index m .*

Corollary 5. *Let R be a ring and G be a group. The group ring RG is a halidon group ring with index m if and only if R is a halidon ring with the index m .*

Proposition 6. *Let M be an R -module over a halidon ring with index m and let $\text{End } M$ be the ring of endomorphisms of M , then $\text{End } M$ is also a halidon ring with same index that of R .*

Proposition 7. *Let R be a finite commutative halidon ring with index m such that $U(R) = \langle \omega \rangle$, where $\langle \omega \rangle$ is the cyclic group generated by ω ; a primitive m^{th} root of unity. Then*

- (1) $R = T \oplus L$, where T is a subfield of R and L is a subspace of R as a vector space over T ,
- (2) R is semisimple.

Theorem 1. *A ring R is a finite field if and only if R is a finite commutative halidon ring with index m such that $U(R) = \langle \omega \rangle$.*

Let

$$u = \sum_{i=1}^m \alpha_i g_i$$

be an element in the group algebra RG and let

$$\lambda_r = \sum_{i=1}^m \alpha_{m-i+2} (\omega^{(i-1)})^{(r-1)}$$

where $\omega \in R$ is a primitive m^{th} root of unity. Then u is said to be depending on $\lambda_1, \lambda_2, \dots, \lambda_m$.

Proposition 8. *Let S be a subring with unity of a halidon ring R with index m and let $G = \langle g \rangle = \{g_1 = 1, g_2 = g, \dots, g_m = g^{m-1}\}$ be a cyclic group of order m generated by g . Let*

$$u = \sum_{i=1}^m \alpha_i g_i \in U(RG) \cap SG$$

be depending on $\lambda_1, \lambda_2, \dots, \lambda_m$. Then $u^{-1} \in SG$ if and only if $\lambda = \lambda_1 \lambda_2 \dots \lambda_m$ is invertible in S .

The next theorem is about the existence of units in integral group rings which is an important area of study in group rings or group algebras.

Theorem 2. *Let $\omega \in \mathbb{C}$ be a complex primitive m^{th} root of unity and let $G = \langle g \rangle = \{g_1 = 1, g_2 = g, \dots, g_m = g^{m-1}\}$ be a cyclic group of order m generated by g . Let*

$$u = \sum_{i=1}^m \alpha_i g_i \in \mathbb{C}G$$

be depending on $\lambda_1, \lambda_2, \dots, \lambda_m$. Let

$$\lambda_i^* = \prod_{r=1, r \neq i}^m \lambda_r$$

and

$$\lambda = \prod_{r=1}^m \lambda_r$$

Then

$$u = \sum_{i=1}^m \alpha_i g_i \in U(\mathbb{Z}G)$$

if and only if $\lambda = \pm 1$. If $\lambda_m^{-1} = f(\omega)$; a polynomial function of ω , then $u^{-1} = f(g)$.

Remark 1. When we use the formula $u^{-1} = f(g)$, ensure that the use of

$$\sum_{r=0}^{m-1} \omega^r = 0$$

should be avoided in the calculation of λ_i or λ in order to make the behaviour of ω exactly same as that of g . For an example see [9].

Theorem 3. Let R be a halidon ring with index m and $G = \langle g \rangle = \{g_1 = 1, g_2 = g, \dots, g_m = g^{m-1}\}$ be a cyclic group of order m generated by g .

Let

$$u = \sum_{i=1}^m \alpha_i g_i$$

be an element in the group algebra RG depending on $\lambda_1, \lambda_2, \dots, \lambda_m$ such that $u^m = 1$. Then there is a homomorphism $\phi : RG \rightarrow RG$ and an element

$$y = \sum_{i=1}^m y_i g_i \in \text{Ker} \phi$$

such that

$$\begin{aligned} y_i &= (-1)^{m-i+1} S_{m-i+1} & \text{if } i \neq 1 \\ &= 1 + (-1)^m S_m & \text{if } i = 1 \end{aligned}$$

where S_i 's are the elementary symmetric polynomials in $\lambda_1, \lambda_2, \dots, \lambda_m$.

Corollary 6. If $R = \mathbb{Z}_p$, the ring of integer modulo p where $p > 2$ is a prime integer in the above theorem and $m = p - 1$, then the trace of $u^r = 0$ (that is the coefficient of identity in G of u^r), for $r = 1, 2, \dots, p - 2$.

Corollary 7. If $R = \mathbb{Z}_p$, the ring of integer modulo p where $p > 2$ is a prime integer in the above theorem and $m = p - 1$, then $S_r = 0 \pmod{p}$ for $r = 1, 2, \dots, p - 2$ [**Lagrange's Theorem**] and $1 + (p - 1)! = 0 \pmod{p}$ [**Wilson's Theorem**].

Theorem 4. Let R be a commutative halidon ring with index m and let G be a cyclic group of order m . Then $RG \cong R^m$ as R -algebras.

Theorem 5 (Higman's Theorem [3], [5]). Let R be a commutative halidon ring with index m and let G be an finite abelian group of order n with exponent m and n is a unit in R . Then $RG \cong R^n$ as R -algebras.

Let R be a commutative halidon ring with index m and let G be an abelian group of order n with exponent m and n is a unit in R . By the fundamental theorem on finite abelian groups, G is a direct product of cyclic groups say $\langle \zeta_1 \rangle, \langle \zeta_2 \rangle, \dots, \langle \zeta_k \rangle$, where ζ_s is of order n_s and $n = n_1 n_2 n_3 \dots n_k$. Assume that $n_1 \leq n_2 \leq n_3 \leq \dots \leq n_k$. Then $m = n_k$. Since $n_1, n_2, n_3, \dots, n_k$ divide n and since n is a unit in R , $n_1, n_2, n_3, \dots, n_k$ are also units in R . An arbitrary element of $z \in G$ can be uniquely written as $z = \zeta_1^{\alpha_1} \zeta_2^{\alpha_2} \dots \zeta_k^{\alpha_k}$ where $0 \leq \alpha_s < n_s$. Let ω be a primitive m^{th} root of unity over R . Corresponding to each k tuples $r = [r_1, r_2, \dots, r_k]$ where $0 \leq r_s < n_s$, $\Gamma^{[r]}$ be a map of G into $U(R)$ defined by

$$\Gamma^{[r]}(z) = \prod_{s=1}^k [\omega^{\alpha_s (\frac{d_s}{c})}]^{r_s} \quad (*)$$

where

$$d_i = \prod_{s=1, s \neq i}^k n_s$$

and c is given by $n = cm$. We define

$$e_{[r]} = \frac{1}{n} \sum_{x \in G} \Gamma^{[r]}(x)x$$

Then $\{e_{[r]}\}$ forms an orthogonal R-basis for RG .

Theorem 6. *Let R be a halidon ring with index m and let G be a finite abelian group of order n with exponent m such that n is invertible in R . Let*

$$u = \sum_{x \in G} u_x x$$

be a unit in RG with inverse

$$v = \sum_{x \in G} v_x x,$$

then

$$v_x = \frac{1}{n} \sum_s \beta_{[s]}^{-1} \Gamma^{[s]}(x)$$

where

$$\beta_{[s]} = \sum_{x \in G} u_x \Gamma^{[s]}(x^{-1}).$$

Theorem 7. *Let R be a halidon ring with index m and let G be a finite abelian group of order n and exponent m such that n is invertible in R . Let*

$$u = \sum_{x \in G} u_x x$$

be an idempotent in RG , then

$$u_x = \frac{1}{n} \sum_s \beta_{[s]} \Gamma^{[s]}(x)$$

where $\beta_{[s]}$ are idempotents in R .

Let $D_{2m} = \langle a, b \mid a^m = b^2 = 1, abab = 1 \rangle$ be the *dihedral group* of order $2m$ and let R be a halidon ring with index m . Any element in RD_{2m} can be taken as

$$u = \sum_{i=1}^m \alpha_i a^{i-1} + \sum_{i=1}^m \alpha_{m+i} a^{i-1} b.$$

Let $\Lambda_u = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{bmatrix}$, where

$$\Lambda_{11} = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_m)$$

and

$$\lambda_i = \sum_{r=1}^m \alpha_{m-r+2} (\omega^{(i-1)})^{(r-1)}$$

where $\omega \in R$ is a primitive m^{th} root of unity, $m - r + 2$ is being taken reduction modulo m , zero is treated as m and $i = 1, 2, \dots, m$.

$$\Lambda_{12} = \text{diag}(\gamma_1, \gamma_2, \dots, \gamma_m)$$

and

$$\gamma_j = \sum_{r=1}^m \alpha_{m+r} (\omega^{(j-1)})^{(r-1)} ; j = 1, 2, \dots, m$$

$$\Lambda_{21} = \text{diag}(\delta_1, \delta_2, \dots, \delta_m)$$

and

$$\delta_k = \sum_{r=1}^m \alpha_{m+(m-r+2)} (\omega^{(i-1)})^{(r-1)} ; k = 1, 2, \dots, m$$

where $m - r + 2$ is being taken reduction modulo m and zero is treated as m .

$$\Lambda_{22} = \text{diag}(\eta_1, \eta_2, \dots, \eta_m)$$

and

$$\eta_s = \sum_{r=1}^m \alpha_r (\omega^{(s-1)})^{(r-1)} ; s = 1, 2, \dots, m.$$

Theorem 8. $RD_{2m} \cong V$, as R -algebras, where $V = \{\Lambda_u \mid u \in RD_{2m}\}$ is a subalgebra of the algebra of matrices $\text{Mat}_{2m}(R)$, of order $2m$ over R .

Theorem 9. $U(\mathbb{Z}D_{2m}) \cong \{\Lambda_u \mid |\Lambda_u| = \pm 1, u \in \mathbb{Z}D_{2m}\}$.

Let $T_{2m} = \langle a, b \mid a^m = 1, b^2 = a^{\frac{m}{2}}, b^{-1}ab = a^{-1} \rangle$ be the *dicyclic group* of order $2m$ where $m = 2n$ and let R be a halidon ring with index m .

Let $Q_{2m} = \langle a, b \mid a^m = 1, b^2 = a^p, b^{-1}ab = a^{-1}, m = 2^{n+1}, p = 2^n \rangle$ be the *generalised quaternion group* of order $2m$ and $n > 1$. Clearly Q_{2m} arises as a particular case of *dicyclic group*.

Theorem 10. $RT_{2m} \cong V_1$, as R -algebras, where V_1 is a subalgebra of the algebra of matrices $\text{Mat}_{2m}(R)$, of order $2m$ over R .

Let $S_{2m} = \langle a, b \mid a^m = 1, b^2 = a^p, b^{-1}ab = a^{-1}, m = 2^{n+1}, p = 2^n - 1 \rangle$ be the *semidihedral group* of order $2m$ and $n > 1$ and let R be a halidon ring with index m .

Theorem 11. $RS_{2m} \cong V_2$, as R -algebras, where V_2 is a subalgebra of the algebra of matrices $\text{Mat}_{2m}(R)$, of order $2m$ over R .

3. MAIN RESULT- MASCHKE’S THEOREM

In this section, we discuss the characters of a group over halidon rings and Maschke’s Theorem. Usually, the character of a group is defined over a real field \mathbb{R} or a complex field \mathbb{C} . But, here we define character of a group over a halidon ring which need not be a field. Since the field of complex numbers is a halidon ring with any index greater than 1, all the properties of characters over halidon rings will automatically satisfied over the field of complex numbers. Now we are looking into those properties of characters of a group over a complex field which are also true over halidon rings.

Let R be a commutative halidon ring with index m and let G be a finite group of order n with exponent m such that n is invertible in R . A homomorphism $\rho : G \rightarrow GL(k, R)$ is defined as a *representation* of G over R of *degree* k , where $GL(k, R)$ is the general linear group of invertible matrices of order k over R . The *character* of G is defined as a homomorphism $\chi : G \rightarrow R$ such that $\chi(g) = tr(\rho(g))$ for $\forall g \in G$ where $tr(\rho(g))$ is the trace of the matrix $\rho(g)$. A representation is *faithful* if it is injective. When $k = 1$, we call the character χ as *linear* character and it is actually a homomorphism $\chi : G \rightarrow u(R)$. In the standard definition, by a *linear* character χ of G , mean a homomorphism $\chi : G \rightarrow \mathbb{C} - \{0\}$. If $R = \mathbb{C}$, then $U(R) = \mathbb{C} - \{0\}$. So the linear characters are precisely $\Gamma^{[r]}$. Thus we have the following theorem which is already known to be true for linear characters of a group over \mathbb{C} .

Theorem 12. *Let R be a commutative halidon ring with index m and let G be a group of order n and exponent m such that n is invertible in R . Let G^* be the set of homomorphisms $\Gamma^{[s]}$ of G into $U(R)$, which are given by (*). Then G^* is a group under the product $\Gamma^{[s]}\Gamma^{[t]} = \Gamma^{[s+t]}$ and $G^* \cong G$.*

Proof. Refer to [10] for the proof. □

Suppose that $\rho : G \rightarrow GL(k, R)$ is a *representation* of G over R . Write $V = R^k$, the module of all row vectors $(\lambda_1, \lambda_2, \dots, \lambda_k)$ with each $\lambda_i \in R$. For all $v \in R$ and $g \in G$, the matrix product

$$v\rho(g),$$

of the row vector v with the matrix $\rho(g)$ is a row vector in V . With the multiplication $vg = v\rho(g)$, V becomes an RG -module. Refer to [2].

The next theorem is the Maschke’s Theorem for a cyclic group over a halidon ring.

Theorem 13. *Let G be a finite cyclic group of order m and let R be a commutative halidon ring with index m . Let $V = R^m$ be an RG -module. Then there are m irreducible RG -submodules U_1, U_2, \dots, U_m such that $V = U_1 \oplus U_2 \oplus \dots \oplus U_m$.*

Proof. Let ρ be the regular representation of $G = \langle g | g^m = 1 \rangle$. Then

$$\rho(g) = \text{cirulant}(0, 0, 0, \dots, 1) = \begin{pmatrix} 0 & 0 & 0 \dots & 0 & 1 \\ 1 & 0 & 0 \dots & 0 & 0 \\ \cdot & \cdot & \dots & \cdot & \cdot \\ 0 & 0 & 0 \dots & 1 & 0 \end{pmatrix}$$

is a square matrix of order m and each row is obtained by moving one position right and wrapped around of the row above. The eigen values are

$$1, \omega, \omega^2, \dots, \omega^{m-1},$$

where ω is a primitive m^{th} root of unity and with corresponding eigen vectors $(1, 1, \dots, 1)$,
 $(1, \omega, \omega^2, \dots, \omega^{m-1})$, $(1, \omega^2, (\omega^2)^2, \dots, (\omega^{m-1})^2), \dots$,
 $(1, \omega^{m-1}, (\omega^{m-1})^2, \dots, (\omega^{m-1})^{m-1})$.

Let $U_r = \text{span}\{(1, \omega^{r-1}, (\omega^{r-1})^2, \dots, (\omega^{r-1})^{m-1})\}$. Then any element u_r of U_r can be taken as $\lambda(1, \omega^{r-1}, (\omega^{r-1})^2, \dots, (\omega^{r-1})^{m-1})$ for some $\lambda \in R$. Therefore

$$\begin{aligned} ug &= u\rho(g) \\ &= \lambda(1, \omega^{r-1}, (\omega^{r-1})^2, \dots, (\omega^{r-1})^{m-1})\text{cirulant}(0, 0, 0, \dots, 1) \\ &= \lambda\omega^{r-1}(1, \omega^{r-1}, (\omega^{r-1})^2, \dots, (\omega^{r-1})^{m-1}) \in U_r. \end{aligned}$$

This means that U_r is an RG -submodule of V for each $r = 1, 2, 3, \dots, m$.

We define $\pi_r : V \rightarrow V$ by

$$\pi_r(v) = u_r \in U_r.$$

Then $\pi_r^2 = \pi_r$ and therefore $V = U_1 \oplus U_2 \oplus \dots \oplus U_m$. \square

Corollary 8. *Let G be a finite cyclic group of order k and let R be a commutative halidon ring with index m . Let $V = R^k$ be an RG -module where $k \mid m$. Then there are k irreducible RG -submodules U_1, U_2, \dots, U_k such that $V = U_1 \oplus U_2 \oplus \dots \oplus U_k$.*

Proof. Since $k \mid m$, we can take $m=ck$. Then $\omega_1 = \omega^c$ is a primitive k^{th} root of unity. Applying the above theorem, the result follows. \square

Theorem 14 (Mascheke's Theorem). *Let G be a finite group of order n with exponent m , let R be a commutative halidon ring with index m and n is invertible in R . Let $V = R^m$ be an RG -module. If U is an RG -submodule of V , then there is an RG -submodule W such that $V = U \oplus W$.*

Proof. First choose any submodule W_0 of V such that $V = U \oplus W_0$. Since V is a free module with rank m , we can find a basis $\{v_1, \dots, v_p\}$ of U . Next we have to show that this basis can be extended to a basis $\{v_1, \dots, v_p, v_{p+1}, \dots, v_m\}$ of V . Let $v_{p+1} \notin \text{span}\{v_1, \dots, v_p\}$. Suppose that

$$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_p v_p + \alpha_{p+1} v_{p+1} = 0.$$

Then

$$\alpha_{p+1} v_{p+1} = -\alpha_1 v_1 - \alpha_2 v_2 - \dots - \alpha_p v_p$$

If $\alpha_{p+1} = 0$, then v_1, \dots, v_p, v_{p+1} are linearly independent.

If $\alpha_{p+1} \in U(R)$; the unit group of R , then $v_{p+1} \in \text{span}\{v_1, \dots, v_p\}$, which is a contradiction.

So $\alpha_{p+1} \notin U(R)$.

If $\alpha_{p+1} \neq 0 \in ZD(R)$; the set of zero divisors of R , then there exists

a $\beta_{p+1} \neq 0 \in ZD(R)$ such that $\alpha_{p+1}\beta_{p+1} = 0$.

$$\Rightarrow -\beta_{p+1}\alpha_1 v_1 - \beta_{p+1}\alpha_2 v_2 - \dots - \beta_{p+1}\alpha_p v_p = 0$$

$\Rightarrow \beta_{p+1}\alpha_i = 0$ for all $i = 1, 2, 3, \dots, p$. Since the above is true for all $i = 1, 2, 3, \dots, p$, β_{p+1} must be zero. This is a contradiction as $\beta_{p+1} \neq 0$. Therefore α_{p+1} must be zero. So v_1, \dots, v_p, v_{p+1} are linearly independent. Continuing like this we can extend a basis $\{v_1, \dots, v_p\}$ of U to a basis $\{v_1, \dots, v_p, v_{p+1}, \dots, v_m\}$ of V . Then $W_0 = \text{span}\{v_{p+1}, \dots, v_m\}$.

Rest of the proof is in line with [2]. Now for all $v \in V$ there exist unique vectors $u \in U$

and $w \in W_0$ such that $v = u + w$. We define $\phi : V \rightarrow V$ by setting $\phi(v) = u$. Recall from algebra that if $V = U \oplus W$, and if we define $\pi : V \rightarrow V$ by

$$\pi(u + w) = u \quad \text{for all } u \in U, w \in W$$

then π is an endomorphism of V . Moreover, $Im(\pi) = U$, $Ker(\pi) = W$ and $\pi^2 = \pi$. With this in mind we see that ϕ is a projection of V with kernel W_0 and image U . Our aim is to modify the projection ϕ to create an RG -homomorphism from $V \rightarrow V$ with image U .

Define $\tau : V \rightarrow V$ by

$$\tau(v) = \frac{1}{n} \sum_{g \in G} g^{-1} \phi(gv), \quad v \in V$$

Then τ is an endomorphism of V and $Im(\tau) \subseteq U$. Now we will show that τ is an RG -homomorphism. For $v \in V$ and $x \in G$ we have

$$\tau(xv) = \frac{1}{n} \sum_{g \in G} g^{-1} \phi(gxv)$$

As g runs through the elements of G , so does $h = gx$. Thus we have

$$\begin{aligned} \tau(xv) &= \frac{1}{n} \sum_{h \in G} xh^{-1} \phi(hv) \\ &= \frac{1}{n} x \left(\sum_{h \in G} h^{-1} \phi(hv) \right) \\ &= x\tau(v) \end{aligned}$$

Thus τ is an RG -homomorphism.

It remains to show that τ is a projection with image U . To show that τ is a projection it suffices to demonstrate that $\tau^2 = \tau$. Note that given $u \in U$ and $g \in G$, we have $gu \in U$, so $\phi(gu) = gu$. Using this we see that:

$$\begin{aligned} \tau(u) &= \frac{1}{n} \sum_{g \in G} g^{-1} \phi(gu) \\ &= \frac{1}{n} \sum_{g \in G} g^{-1} gu \\ &= \frac{1}{n} \sum_{g \in G} u \\ &= u \end{aligned}$$

Now let $v \in V$. Then $\tau(v) \in U$, so we have $\tau(\tau(v)) = \tau(v)$. We have shown that $\tau^2 = \tau$. □

In [2], the authors have stated the Maschke's theorem for vector spaces over the field of real numbers \mathbb{R} or complex numbers \mathbb{C} and provided an example where Maschke's theorem can fail (see chapter 7) if the field is not a \mathbb{R} or \mathbb{C} . In the light of the theorem 14, we can still have a vector space over the field \mathbb{Z}_p ; which is a halidon ring with index $m = p - 1$ for prime p . By theorem 1, there is a primitive m^{th} root of unity ω such that $U(\mathbb{Z}_p) = \langle \omega \rangle$. Let $G = C_m = \langle a : a^m = 1 \rangle$ and let $R = \mathbb{Z}_p$. Note that we cannot take G as $C_p = \langle a : a^p = 1 \rangle$ as $|G| = p$ has no multiplicative inverse in \mathbb{Z}_p . Let $V = R^2$ and let

$\{v_1, v_2\}$ be the standard basis for V . We define $\sigma : G \rightarrow GL(2, R)$ by $\sigma(a^j) = \begin{pmatrix} \omega^j & 0 \\ 0 & \omega^j \end{pmatrix}$ for $j = 1, 2, 3, \dots, m$. Clearly σ is a representation of G . Also, $U = \text{span}\{v_1\}$ is an RG -submodule of V . Define $W = \text{span}\{v_1 + v_2\}$, which is also an RG -submodule of V . Any element v in V can be written as $v = (\alpha_1 - \alpha_2)v_1 + \alpha_2(v_1 + v_2)$ for some $\alpha_1, \alpha_2 \in R$. If $x \in U \cap W$, then $x \in U$ and $x \in W$. So $x = \lambda_1 v_1 = (\lambda_1, 0)$ and $x = \lambda_2(v_1 + v_2) = (\lambda_2, \lambda_2)$ for some $\lambda_1, \lambda_2 \in R$. This implies $x = (0, 0)$ and hence $V = U \oplus W$ as desired.

Definition 1. Let R be a commutative halidon ring with index $m > 2$ and let S_n be the symmetric group on n symbols such that $|S_n| = m$. Let $S_n = \{g_1, g_2, g_3, \dots, g_m\}$ be in some order and let $\{e_{g_1}, e_{g_2}, e_{g_3}, \dots, e_{g_m}\}$ be the standard basis for $V = R^m$. Define $\rho(g)(h) = e_{gh}$ for all $g, h \in S_n$. This is called the **permutation representation** of S_n .

Example 1. Let us order the elements of S_3 as follows:

$$S_3 = \{g_1 = id, g_2 = (1, 2), g_3 = (1, 3), g_4 = (2, 3), g_5 = (1, 2, 3), g_6 = (1, 3, 2)\}.$$

Then the composition table is given as below.

	g_1	g_2	g_3	g_4	g_5	g_6
g_1	g_1	g_2	g_3	g_4	g_5	g_6
g_2	g_2	g_1	g_5	g_6	g_3	g_4
g_3	g_3	g_6	g_1	g_5	g_4	g_2
g_4	g_4	g_5	g_6	g_1	g_2	g_3
g_5	g_5	g_4	g_2	g_3	g_6	g_1
g_6	g_6	g_3	g_4	g_2	g_1	g_5

Let R be a halidon ring with index $m=6$ and $V = R^6$ (for example, $R = Z_{49}$). Let $\{e_{g_1}, e_{g_2}, e_{g_3}, \dots, e_{g_6}\}$ be the standard basis for V . The representation ρ is given by $\rho(g_i)g_j = e_{g_i g_j}$. Using the composition table, for example, we can see that

$$\rho(g_3) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Let $W_0 = \text{span}\{e_{g_1}, e_{g_2}, e_{g_3}, e_{g_4}, e_{g_5}\}$ and $U = \text{span}\{e_{g_1} + e_{g_2} + e_{g_3} + e_{g_4} + e_{g_5} + e_{g_6}\}$. Then clearly $V = W_0 \oplus U$. We define a projection $\phi(e_{g_i}) = 0$, for $i = 1, 2, 3, 4, 5$ and $\phi(e_{g_6}) = e_{g_1} + e_{g_2} + e_{g_3} + e_{g_4} + e_{g_5} + e_{g_6}$. From the proof of Maschke's Theorem, we have $\tau(e_{g_i}) = \frac{1}{6}(e_{g_1} + e_{g_2} + e_{g_3} + e_{g_4} + e_{g_5} + e_{g_6})$ for all $i = 1, 2, 3, 4, 5, 6$. Let $W = \text{Ker } \tau = \text{span}\{e_{g_1} - e_{g_2}, e_{g_2} - e_{g_3}, e_{g_3} - e_{g_4}, e_{g_4} - e_{g_5}, e_{g_5} - e_{g_6}\}$. Then $wg \in W$ for all $w \in W$ and $g \in S_3$. Therefore W is an RS_3 submodule and $V = U \oplus W$ as expected from the Maschke's Theorem.

Proposition 9. Let R and D_{2m} be as in theorem 8. The isomorphism ρ between RD_{2m} and $V = \{\Lambda_u | u \in RD_{2m}\}$ is defined by $\rho(u) = \Lambda_u$. Then $\text{tr}(\rho(u))$ is $2m\alpha_1$.

Proof. The trace function

$$\begin{aligned}
 tr(\rho(u)) = tr\Lambda_u &= \sum_{i=1}^m \lambda_i + \sum_{i=1}^m \eta_i \\
 &= \sum_{i=1}^m \sum_{r=1}^m \alpha_{m-r+2} (\omega^{(i-1)})^{(r-1)} + \sum_{s=1}^m \sum_{r=1}^m \alpha_r (\omega^{(s-1)})^{(r-1)} \\
 &= \sum_{r=1}^m (\alpha_{m-r+2} + \alpha_r) \sum_{i=1}^m (\omega^{(i-1)})^{(r-1)} \\
 &= 2m\alpha_1
 \end{aligned}$$

as

$$\begin{aligned}
 \sum_{r=1}^m \omega^{(i-1)(r-1)} &= m, \quad r = 1 \\
 &= 0, \quad r \neq 1
 \end{aligned}$$

□

Corollary 9. *If $\chi = \rho|_{D_{2m}}$, the restriction of ρ to D_{2m} , then χ is a character of D_{2m} such that*

(1) $\chi(1) = 2m$

(2) $\chi(g) = 0 \quad \forall \quad g \neq 1 \in D_{2m}$.

Proof.

$$\begin{aligned}
 \alpha_1 &= 1 \quad \text{if } g = 1 \\
 &= 0 \quad \text{if } g \neq 1.
 \end{aligned}$$

□

Remark 2. *The results in the proposition 9 and corollary 9 are also valid for RT_{2m} and RS_{2m} .*

4. ROSOSHEK'S PROBLEM

In speaking at the Ergolol-2007 conference, Prof. S. K. Rososhek came up with a new cryptosystem, which is based on the use of group rings [8]. Let R be a commutative ring and G be a group. We say that an automorphism $\psi : RG \rightarrow RG$ is *standard* if it is defined by some ring automorphism $\alpha : R \rightarrow R$ and by a group automorphism $\sigma : G \rightarrow G$, so that

$$\psi(x) = \sum_{g \in G} \alpha(a_g) \sigma(g)$$

for every

$$x = \sum_{g \in G} a_g g \in RG$$

Rososhek's problem: For which finite commutative rings R and finite groups G does the group ring RG have standard automorphisms only?

We refer to group rings RG possessing just standard automorphisms as *automorphically*

rigid, and the problem then consists in defining finite *automorphically rigid* group rings. In [8], the author has proved the following theorem for a finite field:

Theorem 15. *Let K be a finite field of characteristic p and G a finite group. A group algebra KG is automorphically rigid if and only if one of the following conditions holds:*

- (1) G is a trivial group, i.e., $G = e$;
- (2) G is a cyclic group of odd prime order q , i.e., $G = C_q$, $q = 2$, and $K = F_p$ is a prime field of characteristic p , with p a primitive root modulo q ;
- (3) G is a direct product of an order 2 cyclic group and a cyclic group C_q of odd prime order $q = 2$, i.e., $G = C_2 \times C_q$, with 2 a primitive root modulo q , and $K = F_2$ is a two-element field;
- (4) G is a permutation group on three symbols, i.e., $G = S_3$, and $K = F_2$ is a two-element field.

Still, the question remains unanswered for a finite commutative ring which is not a field.

Proposition 10. *Let $R = \frac{\mathbb{Z}_n[X]}{(X^2 - 1)}$. Then the number of automorphisms of R , $|AutR|$ is given by*

$$\begin{aligned} |AutR| &= 2 \text{ if } n = p^s \text{ where } s \geq 1 \text{ is an integer and } p \text{ is an odd prime number} \\ &= 2^k \text{ if } n = p_1^{e_1} p_2^{e_2} p_3^{e_3} \dots p_k^{e_k} \text{ with prime numbers } 2 < p_1 < p_2 < \dots < p_k. \end{aligned}$$

Proof. This proof is somewhat similar to example 2 in [7]. Let x be the image of X in R . Any element in R can be uniquely written as $ax + b$ with $a, b \in \mathbb{Z}_n$. Let $\sigma \in AutR$. Then $\sigma(a) = a$ for every $a \in \mathbb{Z}_n$. Therefore $\sigma(x) = ax + b$ for some $a, b \in \mathbb{Z}_n$. Since σ is an automorphism, there exists an element $px + q$ with $p, q \in \mathbb{Z}_n$ such that $x = \sigma(px + q) = p\sigma(x) + q = pax + pb + q$. then we get $pa = 1$ and so a must be a unit in \mathbb{Z}_n . Further, if $\sigma(x) = ax + b$ with $a \in U(\mathbb{Z}_n)$, we must also have $0 = \sigma(x^2 - 1) = \sigma(x^2) - 1 = (ax + b)^2 - 1 = a^2x^2 + 2abx + b^2 - 1 = a^2(x^2 - 1) + 2abx + a^2 + b^2 - 1 = 2abx + a^2 + b^2 - 1$.

Since n is odd, $2ab = 0 \implies b = 0$ as $a \in U(\mathbb{Z}_n)$. This implies $a^2 = 1$. Therefore $|AutR| = \text{no. of involutions in } \mathbb{Z}_n$. But, the number of involutions in \mathbb{Z}_n clearly follows the statement in the proposition. Programme-3 mentioned below is useful to compute involutions in \mathbb{Z}_n . \square

If \mathbb{Z}_n is halidon ring with index m , then so is $\mathbb{Z}_n[X]$. By corollary 4, $\frac{\mathbb{Z}_n[X]}{(X^2 - 1)}$ is also a halidon ring with index m .

Proposition 11. *Let $R = \frac{\mathbb{Z}_n[X]}{(X^2 - 1)}$ be a halidon ring with index m and let $G = C_m$ be a cyclic group of order m . If $m! = 2^k \times \phi(m)$ for some positive integer k , then RG is automorphically rigid.*

Proof. Clearly R is not a field. By theorem 5, $RG \cong R^m$. So $|AutRG| = |AutR^m|$. But, $AutR^m$ is just the collection of automorphisms which permute the positions of the elements of R^m . Thus we have $|AutR^m| = m!$. By proposition 10, we have $|AutR| = 2^k$ for some positive integer k . So $m! = 2^k \times \phi(m)$ implies $|AutRG| = |AugR| \times |AugG|$ and therefore RG is automorphically rigid. \square

The only solution to the above proposition is $m = 2$. So one of the solutions (there may have other solutions) to the Rososhek's problem is given by $R = \frac{\mathbb{Z}_{p^s}[X]}{(X^2 - 1)}$ where p is an odd prime, $s \geq 1$ is an integer and $\omega = p^s - 1 = -1 \pmod{p^s} \in \mathbb{Z}_{p^s}$ is a primitive m^{th} root of unity and $G = C_2$.

5. THE COMPUTATIONAL ASPECTS OF HALIDON RINGS AND HALIDON GROUP RINGS

The main purpose this section is to verify Maschke's Theorem using some computer codes. The computer programme-5, is very useful to verify the Maschke's theorem for cyclic group. Throughout this section, let $R = \mathbb{Z}_n$ be a halidon ring with index m and primitive m^{th} root of unity ω . Let $G = \langle g \rangle = \{g_1 = 1, g_2 = g, \dots, g_m = g^{m-1}\}$ be a cyclic group of order m generated by g . We study the computational aspects of finding halidon rings for any integer $n > 2$ and computing the units and idempotents in the halidon group ring RG based on the related theorems.

Theorem 16. *A finite commutative ring R with unity is a halidon ring with index m if and only if there is a primitive m^{th} root of unity ω such that $m, \omega^d - 1 \in U(R)$; the unit group of R for all divisors d of m and $d < m$.*

Proof. If R is a finite commutative halidon ring with index m , then it is evident that there is a primitive m^{th} root of unity ω such that $m, \omega^d - 1 \in U(R)$; the unit group of R for all divisor d of m .

Conversely, assume that there is a primitive m^{th} root of unity ω such that $m, \omega^d - 1 \in U(R)$; the unit group of R for all divisor d of m . We would like to show that $\omega^r - 1 \in U(R)$, for $r = 1, 2, 3, \dots, m - 1$. Let g be the greatest common divisor of r and m . Then there are integers u and v such that $ur + vm = g$. Using the well known geometric series, we have

$$(p - 1)(1 + p + p^2 + \dots + p^{k-1}) = p^k - 1.$$

Put $p = \omega^r$ and $k = u$. Then we get $(\omega^r - 1).a = \omega^{ru} - 1$ for some $a \in R$. This means that $(\omega^r - 1)$ divides $\omega^{ru} - 1 = \omega^{ru}.\omega^{vm} - 1 = \omega^{ru+vm} - 1 = \omega^g - 1$. Now we can choose a divisor d of m such that g divides d . Let $p = \omega^g$ and $k = \frac{d}{g}$. Thus we have

$$(\omega^g - 1).b = \omega^d - 1$$

for some $b \in R$. If $(\omega^g - 1).c = 0$, multiplying by b , we get $(\omega^d - 1).c = 0$. Since $(\omega^d - 1) \in U(R)$, $c = 0$. Therefore $(\omega^g - 1)$ is not a zero divisor. Since R is finite and using proposition 2, we get $(\omega^g - 1) \in U(R)$.

Also, $(\omega^r - 1)$ divides $\omega^g - 1$. Therefore $(\omega^r - 1)$ is a unit in R . Thus we have $(\omega^r - 1)$ is a unit in R for $r = 1, 2, 3, \dots, m - 1$. By theorem 1, R is a halidon ring with index m . \square

Definition 2. *Let $p_1, p_2, p_3, \dots, p_k$ be odd primes and let $\phi(x)$ be the Euler's totient function. We define the halidon function*

$$\psi(n) = \begin{cases} \gcd\{\phi(p_1^{e_1}), \phi(p_2^{e_2}), \phi(p_3^{e_3}), \dots, \phi(p_k^{e_k})\}, & n = p_1^{e_1} p_2^{e_2} p_3^{e_3} \dots p_k^{e_k} \\ 1, & n \text{ is even} \end{cases}$$

Proposition 12. *Let n be as in 2. Then the halidon function*

$$\psi(n) = \gcd\{p_1 - 1, p_2 - 1, p_3 - 1, \dots, p_k - 1\},$$

which is independent of the exponents $e_1, e_2, e_3, \dots, e_k$.

Proof. The proof follows immediately from the fact that $p_1, p_2, p_3, \dots, p_k$ are distinct primes. \square

It is well-known that the *carmichael function* $\lambda(n)$ is the exponent of $U(\mathbb{Z}_n)$. The proof of the following proposition is evident.

Proposition 13. *If $\lambda(n)$ is the carmichael function, then*

- (1) $\psi(n)$ divides $\lambda(n)$,
- (2) $\psi(n^k) = \psi(n)$ if n is odd,
- (3) $\psi(p_1 p_2 p_3 \dots p_s) = \psi(p_1^{d_1} p_2^{d_2} p_3^{d_3} \dots p_s^{d_s})$ if each integer $d_i > 0$ for $i = 1, 2, 3, \dots, s$.

I have developed 5 computer programme codes in c++ (Microsoft Visual Studio 2019) based on the theorems 16, 18 and 2 programmes for Discrete Fourier Transforms. The programme codes are included in the appendix.

The computer programme-1 can be used to find a halidon ring Z_n of given order $n \geq 1$. The author would like to provide its algorithm as follows:

ALGORITHM for Programme-1

Input: n .

Output: Z_n is a trivial halidon ring or not.

1. If n is even, then Z_n is a trivial halidon ring.

2. Else

for $i=1, 2, \dots, n-1$

for $j=1, 2, 3, \dots, n-1$ do compute $i * j \text{ mod } n$

if $i*j \text{ mod } n=1$, $w \leftarrow i$

3. for $i=1, 2, 3, \dots, n-1$ do compute w^i

4. if $w^i \text{ mod } n = 1$, $m \leftarrow i$

5. compute divisors d of m and $d < m$

6. compute $w^d - 1$ and if $w^d - 1 = 1 \text{ mod } n$ for all d , then Z_n is a nontrivial halidon ring with index m .

As there are two for loops, the complexity of the above algorithm is in polynomial time $O(n^2)$.

There are infinitely many halidon rings which are not fields. For example, using the programme-1, we can see that:

- (1) \mathbb{Z}_{49} is halidon ring with index $m = 6$ and $\omega = 19$,
- (2) \mathbb{Z}_{2001} is a trivial halidon ring with index $m = 2$ and $\omega = 2000$,
- (3) \mathbb{Z}_{2501} is halidon ring with index $m = 20$ and $\omega = 8$ or 2493 ,
- (4) \mathbb{Z}_{3601} is halidon ring with index $m = 12$ and $\omega = 1350$ or 2528 ,
- (5) \mathbb{Z}_{10001} is halidon ring with index $m = 8$ and $\omega = 10$ or 9220 ,
- (6) \mathbb{Z}_{100001} is halidon ring with index $m = 10$ and $\omega = 26364$ or 73728 (running time 35 minutes).

By running the same programme several times for different values of n , the author has come to the conclusion of the following conjecture.

Conjecture 1. *If $R = \mathbb{Z}_n$ and $n = p_1^{e_1} p_2^{e_2} p_3^{e_3} \dots p_k^{e_k}$ with primes $p_1 < p_2 < p_3 < \dots < p_k$ including 2, then R is a halidon ring with maximal index $m_{max} = \psi(n)$.*

Theorem 17. *Let*

$$u = \sum_{i=1}^m \alpha_i g_i \in U(RG)$$

be depending on $\lambda_1, \lambda_2, \dots, \lambda_m$. Let

$$v = \sum_{i=1}^m \beta_i g_i$$

be the multiplicative inverse of u in RG . Then

$$\beta_i = \frac{1}{m} \sum_{r=1}^m \lambda_r^{-1} (\omega^{i-1})^{r-1}.$$

The computer programme-2 can be used to test whether a given element u in RG is a unit or not. If it is a unit, then the programme will give the multiplicative inverse v in RG .

Input: $n = 121, m = 10, m^{-1} = 109, \omega = 94, a[1] = 62, a[2] = 21, a[3] = 22, a[4] = 85, a[5] = 81, a[6] = 95, a[7] = 24, a[8] = 30, a[9] = 1, a[10] = 65$

Output: The multiplicative inverse of $a = 62 + 21g + 22g^2 + 85g^3 + 81g^4 + 95g^5 + 24g^6 + 30g^7 + g^8 + 65g^9$ is $b = 102 + 68g + 34g^2 + 61g^3 + 73g^4 + 54g^5 + 102g^6 + 109g^7 + 18g^8 + 455g^9$.

Input: $n = 121, m = 10, m^{-1} = 109, \omega = 94, a[1] = 72, a[2] = 71, a[3] = 89, a[4] = 48, a[5] = 54, a[6] = 0, a[7] = 2, a[8] = 105, a[9] = 25, a[10] = 19$

Output: The multiplicative inverse of $a = 72 + 71g + 89g^2 + 48g^3 + 54g^4 + 0g^5 + 2g^6 + 105g^7 + 25g^8 + 19g^9$ is $b = 72 + 71g + 89g^2 + 48g^3 + 54g^4 + 0g^5 + 2g^6 + 105g^7 + 25g^8 + 19g^9$, which is an involution.

Input: $n = 121, m = 10, m^{-1} = 109, \omega = 94, a[1] = 5, a[2] = 7, a[3] = 2, a[4] = 40, a[5] = 22, a[6] = 90, a[7] = 20, a[8] = 25, a[9] = 10, a[10] = 55$

Output: The element $a = 5 + 7g + 2g^2 + 40g^3 + 22g^4 + 90g^5 + 20g^6 + 25g^7 + 10g^8 + 56g^9$ has no multiplicative inverse.

A direct calculation shows that all outputs are correct.

Theorem 18. *Let*

$$u = \sum_{i=1}^m \alpha_i g_i \in RG$$

be depending on $\lambda_1, \lambda_2, \dots, \lambda_m$. Then

- (1) $u \in U(RG)$ if and only if each $\lambda_i \in U(R)$,
- (2) $u \in E(RG)$ if and only if each $\lambda_i \in E(R)$, where $E(RG)$ is the set of idempotents in RG .

More over, $|U(RG)| = |U(R)|^{|G|}$ and $|E(RG)| = |E(R)|^{|G|}$.

Proof. The proof follows from the isomorphism $\rho(u) = (\lambda_1, \lambda_2, \dots, \lambda_m)$ from RG onto R^m . This is the isomorphism used to prove theorem 4. \square

In order to find a unit element or an involution or an idempotent in R , we can use the programme-3.

Input: $n=25$

Output: Involutions are 1 and 24.

Idempotents are 0 and 1.

The units and their inverses are (1, 1), (2, 13), (3, 17), (4, 19), (6, 21), (7, 18), (8, 22), (9, 14), (11, 16), (12, 23), (13, 2), (14, 10), (16, 11), (17, 3), (18, 7),

(9, 4), (21, 5), (22, 20), (23, 12) and (24, 24).

After finding the units and involutions in R , using the programme-4, we can compute the units and involutions in RG .

Input: $n = 25, m = 4, m^{-1} = 19, \omega = 7, l[1] = 7, l[2] = 3, l[3] = 13, l[4] = 21, l1[1] = 18, l1[2] = 17, l1[3] = 2, l1[4] = 5$

Output: The multiplicative inverse of $a = 11 + 17g + 24g^2 + 5g^3$ is $b = 23 + 12g^2 + 8g^3$.

Input: $n = 25, m = 4, m^{-1} = 19, \omega = 7, l[1] = 1, l[2] = 24, l[3] = 24, l[4] = 1, l1[1] = 1, l1[2] = 24, l1[3] = 24, l1[4] = 1$

Output: The multiplicative inverse of $a = 22g + 4g^3$ is $b = 22g + 4g^3$, which is an involution. All outputs can be verified through direct calculations.

After finding the idempotents in R , using the programme-5, we can compute the idempotents in RG .

Input: $n = 49, m = 6, m^{-1} = 41, \omega = 19, l[1] = 1, l[2] = 0, l[3] = 0, l[4] = 0, l[5] = 0, l[6] = 0$

Output: $e_1 = 41 + 41g + 41g^2 + 41g^3 + 41g^4 + 41g^5$

Input: $n = 49, m = 6, m^{-1} = 41, \omega = 19, l[1] = 0, l[2] = 1, l[3] = 0, l[4] = 0, l[5] = 0, l[6] = 0$

Output: $e_2 = 41 + 44g + 3g^2 + 8g^3 + 5g^4 + 46g^5$

Input: $n = 49, m = 6, m^{-1} = 41, \omega = 19, l[1] = 0, l[2] = 0, l[3] = 1, l[4] = 0, l[5] = 0, l[6] = 0$

Output: $e_3 = 41 + 3g + 5g^2 + 41g^3 + 3g^4 + 5g^5$

Input: $n = 49, m = 6, m^{-1} = 41, \omega = 19, l[1] = 0, l[2] = 0, l[3] = 0, l[4] = 1, l[5] = 0, l[6] = 0$

Output: $e_4 = 41 + 8g + 41g^2 + 8g^3 + 41g^4 + 8g^5$

Input: $n = 49, m = 6, m^{-1} = 41, \omega = 19, l[1] = 0, l[2] = 0, l[3] = 0, l[4] = 0, l[5] = 1, l[6] = 0$

Output: $e_5 = 41 + 5g + 3g^2 + 41g^3 + 5g^4 + 3g^5$

Input: $n = 49, m = 6, m^{-1} = 41, \omega = 19, l[1] = 0, l[2] = 0, l[3] = 0, l[4] = 0, l[5] = 0, l[6] = 1$

Output: $e_6 = 41 + 46g + 5g^2 + 8g^3 + 3g^4 + 44g^5$

Clearly e_1, e_2, e_3, e_4, e_5 and e_6 are orthogonal idempotents such that $e_1 + e_2 + e_3 + e_4 + e_5 + e_6 = 1$. Therefore $\mathbb{Z}_{49}G = U_1 \oplus U_2 \oplus U_3 \oplus U_4 \oplus U_5 \oplus U_6$ where $U_1 = \text{span}\{41 + 41g + 41g^2 + 41g^3 + 41g^4 + 41g^5\} = \text{span}\{1 + g + g^2 + g^3 + g^4 + g^5\}$,

$U_2 = \text{span}\{41 + 44g + 3g^2 + 8g^3 + 5g^4 + 46g^5\} = \text{span}\{1 + \omega g + \omega^2 g^2 + \omega^3 g^3 + \omega^4 g^4 + \omega^5 g^5\}$,

$U_3 = \text{span}\{41 + 3g + 5g^2 + 41g^3 + 3g^4 + 5g^5\} = \text{span}\{1 + (\omega)^2 g + (\omega^2)^2 g^2 + (\omega^2)^3 g^3 + (\omega^2)^4 g^4 + (\omega^2)^5 g^5\}$,

$U_4 = \text{span}\{41 + 8g + 41g^2 + 8g^3 + 41g^4 + 8g^5\} = \text{span}\{1 + (\omega)^3 g + (\omega^3)^2 g^2 + (\omega^3)^3 g^3 + (\omega^3)^4 g^4 + (\omega^3)^5 g^5\}$,

$U_5 = \text{span}\{41 + 5g + 3g^2 + 41g^3 + 5g^4 + 3g^5\} = \text{span}\{1 + (\omega)^4 g + (\omega^4)^2 g^2 + (\omega^4)^3 g^3 + (\omega^4)^4 g^4 + (\omega^4)^5 g^5\}$,

$U_6 = \text{span}\{41 + 46g + 5g^2 + 8g^3 + 3g^4 + 44g^5\} = \text{span}\{1 + (\omega)^5 g + (\omega^5)^2 g^2 + (\omega^5)^3 g^3 + (\omega^5)^4 g^4 + (\omega^5)^5 g^5\}$, after multiplying by 6, the inverse of 41. This confirms the theorem 13.

Input: $n = 65, m = 4, m^{-1} = 49, \omega = 8, l[1] = 1, l[2] = 26, l[3] = 40, l[4] = 26$

Output: The idempotent in $\mathbb{Z}_{65}G$ is $e = 7 + 39g + 46g^2 + 39g^3$.

A direct calculation verifies that all outputs are correct.

6. BILINEAR FORMS AND CIRCULANT MATRICES

Let ring R be a commutative halidon ring with index m and primitive m^{th} root of unity ω . Let G be a cyclic group of order m , generated by g . We define $g_i = g^{i-1}$. $\therefore g_i g_j = g_{i+j-1}$; $i = 1, 2, 3, \dots, m$. By the extension theorem of Higman, we have $RG \cong R^m$ as R -algebras

and the isomorphism ρ is given by

$$\rho \left(\sum_{i=1}^m \alpha_i g_i \right) = (\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_m)$$

where

$$\lambda_i = \sum_{r=1}^m \alpha_{m-r+1} (\omega^{i-1})^{r-1}.$$

Since $\{g_i\}$ is an R -basis for RG , $\{\rho(g_i)\}$ is a basis for R^m and

$$\begin{aligned} s_1 &= \rho(g_1) = (1, 1, 1, \dots, 1), \\ s_2 &= \rho(g_2) = (1, \omega^{m-1}, (\omega^{m-1})^2, \dots, (\omega^{m-1})^{m-1}), \\ s_3 &= \rho(g_3) = (1, \omega^{m-2}, (\omega^{m-2})^2, \dots, (\omega^{m-2})^{m-1}), \\ &\dots\dots\dots, \\ s_m &= \rho(g_m) = (1, \omega, \omega^2, \dots, \omega^{m-1}). \end{aligned}$$

Let $\{e_i\}$ be the standard basis in R^m . Then

$$\begin{aligned} s_1 &= e_1 + e_2 + e_3 + \dots + e_m, \\ s_2 &= e_1 + \omega^{m-1}e_2 + (\omega^{m-1})^2e_3 + \dots + (\omega^{m-1})^{m-1}e_m, \\ s_3 &= e_1 + \omega^{m-2}e_2 + (\omega^{m-2})^2e_3 + \dots + (\omega^{m-2})^{m-1}e_m, \\ &\dots\dots\dots, \\ s_m &= e_1 + \omega e_2 + \omega^2 e_3 + \dots + \omega^{m-1}e_m. \end{aligned}$$

This gives

$$\begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ \dots \\ s_m \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega^{m-1} & (\omega^{m-1})^2 & \dots & (\omega^{m-1})^{m-1} \\ 1 & \omega^{m-2} & (\omega^{m-2})^2 & \dots & (\omega^{m-2})^{m-1} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & \omega & (\omega)^2 & \dots & (\omega^{m-1}) \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ \dots \\ e_m \end{pmatrix}$$

$$\begin{aligned} s^T &= \Phi^* e^T \\ \therefore e^T &= \frac{1}{m} \Phi s^T \end{aligned}$$

where

$$\Phi = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega & \omega^2 & \dots & \omega^{m-1} \\ 1 & \omega^2 & (\omega^2)^2 & \dots & (\omega^2)^{m-1} \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot \\ 1 & \omega^{m-1} & (\omega^{m-1})^2 & \dots & (\omega^{m-1})^{m-1} \end{pmatrix}$$

and Φ^* is the Φ conjugate transposed[1]. Thus we have the following theorem.

Theorem 19. *Let ring R be commutative halidon ring with index m and primitive m^{th} root of unity ω . Let G be a cyclic group of order m , generated by g . We define $g_i = g^{i-1}$; $i = 1, 2, 3, \dots, m$ and let $\{s_i\}$ be the image of $\{g_i\}$ under the isomorphism $RG \cong R^m$ and let $\{e_i\}$ be the standard basis for R^n . Then $s^T = \Phi^* e^T$ or $e^T = \frac{1}{m} \Phi s^T$.*

For each $u = (u_1, u_2, u_3, \dots, u_m) \in R^m$, we define $C_u = \text{circu}(u_1, u_2, u_3, \dots, u_m)$. Also, we define $f_u : R^m \times R^m \rightarrow R$ by

$$f_u(x, y) = \langle x, y \rangle_u = x C_u y^T$$

for every $x = (x_1, x_2, x_3, \dots, x_m), y = (y_1, y_2, y_3, \dots, y_m) \in R^m$. We adopt some standard definitions of bilinear form for $\langle x, y \rangle_u$. $\langle x, y \rangle_u$ is said to *symmetric* if $\langle x, y \rangle_u = \langle y, x \rangle_u$ for every $x, y \in R^m$. It is said to be *skewsymmetric* if $\langle x, y \rangle_u = -\langle y, x \rangle_u$ for every $x, y \in R^m$. $\langle x, y \rangle_u$ is said to *alternating* if $\langle x, x \rangle_u = 0$ for every $x \in R^m$ [6].

Theorem 20. f_u is a bilinear form and

$$\begin{aligned} \langle s_i, s_j \rangle_u &= m(u_1 + u_2\omega^{i-1} + u_3(\omega^{i-1})^2 + \dots + u_m(\omega^{i-1})^{m-1}), \\ &\text{if } i + j = 2 \pmod{m} \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

Proof. It is clear that f_u is a bilinear form.

$$\begin{aligned} \langle e_i, e_j \rangle_u = e_i C_u e_j^T &= u_{j-i+1} \quad \text{if } i \leq j \\ &= u_{m+j-i+1} \quad \text{if } i > j. \end{aligned}$$

$$\begin{aligned} \langle s_i, s_j \rangle_u &= \langle e_1 + \omega^{m-i+1}e_2 + (\omega^{m-i+1})^2e_3 + \dots + (\omega^{m-i+1})^{m-1}e_m, \\ &e_1 + \omega^{m-j+1}e_2 + (\omega^{m-j+1})^2e_3 + \dots + (\omega^{m-j+1})^{m-1}e_m \rangle_u \\ &= (u_1 + \omega^{m-j+1}u_2 + (\omega^{m-j+1})^2u_3 + \dots + (\omega^{m-j+1})^{m-1}u_m) \\ &+ (\omega^{m-i+1})(u_m + \omega^{m-j+1}u_1 + (\omega^{m-j+1})^2u_2 + \dots + (\omega^{m-j+1})^{m-1}u_{m-1}) \\ &+ (\omega^{m-i+1})^2(u_{m-1} + \omega^{m-j+1}u_m + (\omega^{m-j+1})^2u_1 + \dots + (\omega^{m-j+1})^{m-2}u_{m-1}) \\ &+ \dots + (\omega^{m-i+1})^{m-1}(u_2 + \omega^{m-j+1}u_3 + (\omega^{m-j+1})^2u_4 + \dots + (\omega^{m-j+1})^{m-1}u_1) \\ &= u_1(1 + \omega^{2m-i-j+2} + (\omega^{2m-i-j+2})^2 + \dots + (\omega^{2m-i-j+2})^{m-1}) \\ &+ u_2\omega^{m-i+1}(1 + \omega^{2m-i-j+2} + (\omega^{2m-i-j+2})^2 + \dots + (\omega^{2m-i-j+2})^{m-1}) \\ &+ u_3(\omega^{m-i+1})^2(1 + \omega^{2m-i-j+2} + (\omega^{2m-i-j+2})^2 + \dots + (\omega^{2m-i-j+2})^{m-1}) \\ &+ \dots + u_m(\omega^{m-i+1})^{m-1}(1 + \omega^{2m-i-j+2} + (\omega^{2m-i-j+2})^2 + \dots + (\omega^{2m-i-j+2})^{m-1}) \\ &= (1 + \omega^{2m-i-j+2} + (\omega^{2m-i-j+2})^2 + \dots + (\omega^{2m-i-j+2})^{m-1}) \\ &(u_1 + u_2\omega^{m-j+1} + u_3(\omega^{m-j+1})^2 + \dots + u_m(\omega^{m-j+1})^{m-1}) \end{aligned}$$

$$\begin{aligned} \omega^{2m-i-j+2} &= 1 \quad \text{if } i + j = 2 \pmod{m} \\ &\neq 1 \quad \text{otherwise} \end{aligned}$$

$$\begin{aligned} \therefore \langle s_i, s_j \rangle_u &= m(u_1 + u_2\omega^{i-1} + u_3(\omega^{i-1})^2 + \dots + u_m) \quad \text{if } i + j = 2 \pmod{m} \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

Hence the proof. □

Corollary 10. $\langle s_i, s_j \rangle_u = 0$ for all $i, j \in \{1, 2, 3, \dots, m\}$ if and only if $u = 0$.

Proof. If $u = 0$, there is nothing to prove.

By theorem 20, $\langle s_i, s_j \rangle_u = 0$ for all i, j other than $i + j = 2 \pmod{m}$. So it is enough to consider $\langle s_i, s_j \rangle_u = 0$ for $i + j = 2 \pmod{m}$. Since m is invertible in R , $\langle s_i, s_j \rangle_u = 0$ for $i + j = 2 \pmod{m}$ implies

$$\begin{aligned}
 u_1 + u_2 + u_3 + \dots + u_m &= 0 \\
 u_1 + u_2\omega + u_3(\omega)^2 + \dots + u_m(\omega)^{m-1} &= 0 \\
 u_1 + u_2\omega^2 + u_3(\omega^2)^2 + \dots + u_m(\omega^2)^{m-1} &= 0 \\
 &\dots\dots\dots \\
 u_1 + u_2\omega^{m-1} + u_3(\omega^{m-1})^2 + \dots + u_m(\omega^{m-1})^{m-1} &= 0
 \end{aligned}$$

This can be put into the matrix form $\Phi u^T = 0$. Since Φ^{-1} exists, $u^T = 0$ and therefore $u = 0$. □

We write $x \perp y$ if $\langle x, y \rangle_u = 0$. We define $(R^m)^\perp = \{x \in R^m \mid \langle x, y \rangle_u = 0 \text{ for all } y \in R^m\}$. We say that $\langle x, y \rangle_u$ is *nondegenerate* if $(R^m)^\perp = \{0\}$.

Corollary 11. $\langle x, y \rangle_u$ is a nondegenerate bilinear form if $\langle s_i, s_j \rangle_u \in U(R)$ for all i and j such that $i + j = 2 \pmod m$.

Proof. $\langle x, y \rangle_u = \sum_{i,j} x_i y_j \langle s_i, s_j \rangle_u = \sum_{i+j=2 \pmod m} x_i y_j \langle s_i, s_j \rangle_u$ by 20.

$$\begin{aligned}
 \text{Therefore } \langle x, y \rangle_u &= x_1 y_1 \langle s_1, s_1 \rangle_u + x_2 y_m \langle s_2, s_m \rangle_u + \\
 &x_3 y_{m-1} \langle s_3, s_{m-1} \rangle_u + \dots + x_m y_2 \langle s_m, s_2 \rangle_u \\
 \langle x, y \rangle_u = 0 &\implies x_1 y_1 \langle s_1, s_1 \rangle_u + x_2 y_m \langle s_2, s_m \rangle_u + \\
 &x_3 y_{m-1} \langle s_3, s_{m-1} \rangle_u + \dots + x_m y_2 \langle s_m, s_2 \rangle_u = 0 \\
 \implies &\left(\begin{matrix} x_1 \langle s_1, s_1 \rangle_u & x_2 \langle s_2, s_m \rangle_u & x_3 \langle s_3, s_{m-1} \rangle_u & \dots & x_m \langle s_m, s_2 \rangle_u \end{matrix} \right)
 \end{aligned}$$

$$\begin{pmatrix} y_1 \\ y_m \\ y_{m-1} \\ \cdot \\ y_2 \end{pmatrix} = 0. \text{ Since this is true for all } y = (y_1, y_2, y_3, \dots, y_m),$$

$$\begin{pmatrix} x_1 \langle s_1, s_1 \rangle_u & x_2 \langle s_2, s_m \rangle_u & x_3 \langle s_3, s_{m-1} \rangle_u & \dots & x_m \langle s_m, s_2 \rangle_u \end{pmatrix} = (0, 0, \dots, 0)$$

$$\implies x_1 \langle s_1, s_1 \rangle_u = 0, x_2 \langle s_2, s_m \rangle_u = 0, x_3 \langle s_3, s_{m-1} \rangle_u = 0..$$

$$x_m \langle s_m, s_2 \rangle_u = 0$$

$x_1 = x_2 = x_3 = \dots = x_m = 0$ only when $\langle s_i, s_j \rangle_u \in U(R)$ for $i + j = 2 \pmod m$.

Thus $(R^m)^\perp = \{0\}$ and therefore $\langle x, y \rangle_u$ is a nondegenerate bilinear form. □

Corollary 12. Let $\langle x, y \rangle_u$ be a nondegenerate bilinear form. Then $M = (\langle s_i, s_j \rangle)$ is an invertible matrix of order m .

Proof. By theorem 20, the matrix M can be written as

$$M = \begin{pmatrix} \langle s_1, s_1 \rangle_u & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & \langle s_2, s_m \rangle_u \\ 0 & 0 & 0 & \dots & \langle s_3, s_{m-1} \rangle_u & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \langle s_m, s_2 \rangle_u & 0 & \dots & 0 & 0 \end{pmatrix}$$

Clearly $D = \det M = \pm \langle s_1, s_1 \rangle_u \langle s_2, s_m \rangle_u \langle s_3, s_{m-1} \rangle_u \dots \langle s_m, s_2 \rangle_u$ and the sign

is depending on m . By corollary 11, $D \in U(R)$. So M^{-1} exists and M^{-1} is given by

$$M^{-1} = \begin{pmatrix} \langle s_1, s_1 \rangle_u^{-1} & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & \langle s_m, s_2 \rangle_u^{-1} \\ 0 & 0 & 0 & \dots & \langle s_{m-1}, s_3 \rangle_u^{-1} & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \langle s_2, s_m \rangle_u^{-1} & 0 & \dots & 0 & 0 \end{pmatrix}. \quad \square$$

Corollary 13. *Let $R = Z_n$ be the ring integers modulo n . It is a halidon ring with maximum index $m_{max} = \psi(n)$; where $\psi(n)$ is the halidon function. Then the number of nondegerate bilinear forms $\langle x, y \rangle_u$ is $\phi(n)^{\psi(n)}$.*

Proof. By 11, $\langle x, y \rangle_u$ is nondegenrate if and only if $\langle s_i, s_j \rangle_u \in U(R)$ for all i and j such that $i + j = 2 \pmod{m}$. Here $|U(R)| = \phi(n)$ and $m = \psi(n)$. Therefore $|\langle x, y \rangle_u| = \phi(n)^{\psi(n)}$. \square

Theorem 21. *Let $C = \{C_u | u = (u_1, u_2, u_3, \dots, u_m) \in R^m\}$ and let G be as in theorem 19. Then C is an R -algebra.*

Proof. Let $u = (u_1, u_2, u_3, \dots, u_m), v = (v_1, v_2, v_3, \dots, v_m) \in R^m$ be any two elements in R . Since $RG \cong R^m$, we can identify the elements u and v as $\sum_{i=1}^m \alpha_i g_i$ and $\sum_{i=1}^m \beta_i g_i$ respectively, where

$$u_i = \sum_{r=1}^m \alpha_{m-r+2} (\omega^{i-1})^{r-1}.$$

and

$$v_i = \sum_{r=1}^m \beta_{m-r+2} (\omega^{i-1})^{r-1}.$$

Since R is a halidon ring with index m and ω is a primitive m^{th} root of unity, the circulant matrix C_u can be written as

$$C_u = \frac{1}{m} \Phi \Lambda_u \Phi^*,$$

where

$$\Lambda_u = \text{diag}(\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_m)$$

such that

$$\lambda_i = \sum_{r=1}^m u_i (\omega^{(i-1)})^{(r-1)}$$

and Φ^* is the conjugate transposed of Φ [1] and $\frac{1}{m} \Phi \Phi^* = I = \frac{1}{m} \Phi^* \Phi$. Clearly $\Lambda_u \Lambda_v = \Lambda_{uv}$.

$$\begin{aligned} \therefore C_{uv} &= \frac{1}{m} \Phi \Lambda_{uv} \Phi^* \\ &= \frac{1}{m} \Phi \Lambda_u \Lambda_v \Phi^* \\ &= \frac{1}{m} \Phi \Lambda_u \frac{1}{m} \Phi^* \Phi \Lambda_v \Phi^* \\ &= \left(\frac{1}{m} \Phi \Lambda_u \Phi^* \right) \left(\frac{1}{m} \Phi \Lambda_v \Phi^* \right) \\ &= C_u C_v \end{aligned}$$

We define $h : R^m \rightarrow C$ by $h(u) = C_u$; which is clearly an algebra isomorphism. $\therefore R^m \cong C$. And hence the theorem. \square

Theorem 22. Let $B = \{ \langle x, y \rangle_u \mid \langle x, y \rangle_u = xC_u y^T, \text{ for each } u \in R^m, x, y \in R^m \}$. Then B is an R -module.

Proof. Let $\alpha \in R$ be any element in R . Then $\langle x, y \rangle_{u+v} = \langle x, y \rangle_u + \langle x, y \rangle_v$ and $\langle x, y \rangle_{\alpha u} = \alpha \langle x, y \rangle_u$. Therefore B is an R -module. \square

7. DISCRETE FOURIER TRANSFORMS

In this section, we deal with the ring of polynomials over a halidon ring which has an application in Discrete Fourier Transforms [4]. Throughout this section, let R be a finite commutative halidon ring with index m and $R[x]$ denotes the ring of polynomials degree less than m over R .

Definition 3. [4] Let $\omega \in R$ be a primitive m^{th} root of unity in R and let $f(x) = \sum_{j=0}^{m-1} f_j x^j \in R[x]$ with its coefficients vector $(f_0, f_1, f_2, \dots, f_{m-1}) \in R^m$. The **Discrete Fourier Transform (DFT)** is a map

$$DFT_\omega : R[x] \rightarrow R^m$$

defined by

$$DFT_\omega(f(x)) = (f_0(1), f_1(\omega), f_2(\omega^2), \dots, f_{m-1}(\omega^{m-1})).$$

Remark 3. Clearly DFT_ω is a R -linear map as $DFT_\omega(af(x) + bg(x)) = aDFT_\omega(f(x)) + bDFT_\omega(g(x))$ for all $a, b \in R$. Also, if $R = \mathbb{C}$, the field of complex numbers, then $\omega = \cos(\frac{2\pi}{m}) + i\sin(\frac{2\pi}{m}) = e^{i\frac{2\pi}{m}}$ and the Fourier series will become the ordinary series of \sin and \cos functions.

Definition 4. [4] The **convolution** of $f(x) = \sum_{j=0}^{m-1} f_j x^j$ and $g(x) = \sum_{k=0}^{m-1} g_k x^k$ in $R[x]$ is

defined by $h(x) = f(x) * g(x) = \sum_{l=0}^{m-1} h_l x^l \in R[x]$ where $h_l = \sum_{j+k=l \pmod m} f_j g_k = \sum_{j=0}^{m-1} f_j g_{l-j}$

for $0 \leq l < m$.

The notion of convolution is equivalent to polynomial multiples in the ring $R[x]/\langle x^m - 1 \rangle$. The l^{th} coefficient of the product $f(x)g(x)$ is $\sum_{j+k=l \pmod m} f_j g_k$ and hence

$$f(x) * g(x) = f(x)g(x) \pmod{(x^m - 1)}.$$

Proposition 14. [4] For polynomials $f(x), g(x) \in R[x]$, $DFT_\omega(f(x)*g(x)) = DFT_\omega(f(x)).DFT_\omega(g(x))$, where \cdot denotes the pointwise multiplication of vectors.

Proof. $f(x) * g(x) = f(x)g(x) + q(x)(x^m - 1)$ for some $q(x) \in R[x]$. Replace x by ω^j , we get

$$\begin{aligned} f(\omega^j) * g(\omega^j) &= f(\omega^j)g(\omega^j) + 0. \\ \therefore DFT_\omega(f(x) * g(x)) &= DFT_\omega(f(x)).DFT_\omega(g(x)). \end{aligned}$$

\square

Theorem 23. For a polynomial $f(x) \in R[x]$, $DFT_\omega^{-1}(f(x)) = \frac{1}{m}DFT_{\omega^{-1}}(f(x))$.

Proof. The matrix of the transformation $DFT_\omega(f(x))$ is

$$[DFT_\omega(f(x))] = \phi = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega & \omega^2 & \dots & \omega^{m-1} \\ 1 & \omega^2 & (\omega^2)^2 & \dots & (\omega^2)^{m-1} \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot \\ 1 & \omega^{m-1} & (\omega^{m-1})^2 & \dots & (\omega^{m-1})^{m-1} \end{pmatrix}$$

The matrix ϕ is the well known Vandermonde matrix and its inverse is $\frac{1}{m}\phi^*$, where ϕ^* is the matrix transpose conjugated [1]. Since ϕ is a square matrix and the conjugate of ω is ω^{-1} , we have $DFT_\omega^{-1}(f(x)) = \frac{1}{m}DFT_{\omega^{-1}}(f(x))$. \square

Example 2. We know that $R = Z_{49}$ is a halidon ring with index $m = 6$ and $\omega = 19$. Also, $\omega^{-1} = \omega^5 = 31$. Let $f(x) = 2 + x + 2x^2 + 3x^3 + 5x^4 + 10x^5 \in R[x]$. Then $DFT_\omega(f(x))$ can be expressed as

$$\begin{aligned} \begin{pmatrix} F_0 \\ F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \end{pmatrix} &= \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \omega^4 & \omega^5 \\ 1 & \omega^2 & \omega^4 & 1 & \omega^2 & \omega^4 \\ 1 & \omega^3 & 1 & \omega^3 & 1 & \omega^3 \\ 1 & \omega^4 & \omega^2 & 1 & \omega^4 & \omega^2 \\ 1 & \omega^5 & \omega^4 & \omega^3 & \omega^2 & \omega \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{pmatrix} \Rightarrow \begin{pmatrix} F_0 \\ F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \end{pmatrix} = \begin{pmatrix} 23 \\ 24 \\ 32 \\ 44 \\ 9 \\ 27 \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{pmatrix} = \\ 6^{-1} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^5 & \omega^4 & \omega^3 & \omega^2 & \omega \\ 1 & \omega^4 & \omega^2 & 1 & \omega^4 & \omega^2 \\ 1 & \omega^3 & 1 & \omega^3 & 1 & \omega^3 \\ 1 & \omega^2 & \omega^4 & 1 & \omega^2 & \omega^4 \\ 1 & \omega & \omega^2 & \omega^3 & \omega^4 & \omega^5 \end{pmatrix} \begin{pmatrix} F_0 \\ F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \end{pmatrix} \\ \Rightarrow \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{pmatrix} &= 41 \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 31 & 30 & 48 & 18 & 19 \\ 1 & 30 & 18 & 1 & 30 & 18 \\ 1 & 48 & 1 & 48 & 1 & 48 \\ 1 & 18 & 30 & 1 & 18 & 30 \\ 1 & 19 & 18 & 48 & 30 & 31 \end{pmatrix} \begin{pmatrix} F_0 \\ F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \end{pmatrix} \\ \text{If } \begin{pmatrix} F_0 \\ F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \end{pmatrix} &= \begin{pmatrix} 23 \\ 24 \\ 32 \\ 44 \\ 9 \\ 27 \end{pmatrix}, \text{ then a direct calculation gives } \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 2 \\ 3 \\ 5 \\ 10 \end{pmatrix} \\ &\text{as expected.} \end{aligned}$$

The programme-6 and programme-7 will enable us to calculate Discrete Fourier Transform and its inverse. We can cross-check the programmes against example 2.

If $R = Z_{100001}$, $m = 10$, $\omega = 26364$ and $f(x) = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + 7x^6 + 8x^7 + 9x^8 + x^9 \in R[x]$, then

$$\begin{pmatrix} F_0 \\ F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \\ F_8 \\ F_9 \end{pmatrix} = \begin{pmatrix} 46 \\ 19019 \\ 3314 \\ 10082 \\ 48017 \\ 4 \\ 80347 \\ 18172 \\ 68413 \\ 52627 \end{pmatrix}.$$

Also, we can verify the inverse DFT using the above data.

8. CONCLUSIONS

The halidon rings are useful to extend the two famous theorems of Graham Higman(1940) and Maschke(1898). Since Higman’s theorem and Maschke’s theorem have a wide range of applications in group rings, group algebras and representation theory, there is a big scope of wider applications of halidon rings. The field of complex numbers is an infinite halidon ring with any index $m > 0$. The field of real numbers is an infinite halidon ring with index 2 and the ring integers is a trivial halidon ring with index $m=1$. Using the field of complex numbers, we can create infinite halidon rings of square matrices with any index $m > 0$. This will open new vistas of applications in algebra and number theory. Another area of application is the coding theory on which the author is currently working with.

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APPENDIX A.

Programme-1 : To check whether $Z(n)$ is a trivial or nontrivial halidon ring

```

#include <iostream>
#include <cmath>
using namespace std;
int main() {
cout << "To check whether Z(n) is a trivial or nontrivial halidon ring." << endl;
unsigned long long int t = 0, n = 1, w = 1, hcf, hcf1,
d = 1, k = 1, q = 1, p = 1, b=0, c=0, temp = 1;
cout << "Enter an integer n >0: ";
cin >> n;
if (n % 2 == 0) {
cout << "Z(" << n << ") is a trivial halidon ring." << endl;
}
for (w = 1; w < n; ++w) {
for (int i = 1; i <= n; ++i) {
if (w % i == 0 && n % i == 0) {
hcf = i;

}
} if (hcf == 1) {
++t; // cout << " " << w << " ";
}
}
for (w = 1; w < n; ++w) {
for (int i = 1; i <= n; ++i) {
if (w % i == 0 && n % i == 0) {
hcf = i;

}
}
} if (hcf == 1) {
for (int k = 1; k <= t; ++k) {
q = q * w; q = q % n;
if (q == 1) { if (temp <= k) { temp = k; } break; }
}
}
}

for (w = 2; w < n; ++w) {
for (int i = 1; i <= n; ++i) {
if (w % i == 0 && n % i == 0) {
hcf = i;
}
}
}
}

```

```

if (hcf == 1) {
for (int k = 1; k <= temp; ++k) {
q = q * w; q = q % n; if (q == 1) {
for (int i = 1; i <= n; ++i) {
if (k % i == 0 && n % i == 0) {
hcf1 = i;
}
}
if (hcf1 == 1) {
for (int j = 1; j < k; ++j)
{
if (k%j == 0) {
d = j;
for (int l = 1; l <= d; ++l)
{
p = (p*w); p = p % n;
}
for (int i = 1; i <= n; ++i) {
if ((p - 1) % i == 0 && n % i == 0) {
hcf = i;
}
}
}
                if (hcf == 1) {

p = 1; b = b + 1;
}
else p = 1;
c = c + 1;
        }
}
if (c == b) { cout << "    Z(" << n << ")" <<

" is a halidon ring with index m= " << k <<

" and w= " << w << "."<< endl; } {p = 1; c = 0; b = 0; }

break;
}
}
}
}
} return 0;
}

```

Programme-2: To check whether an element in Z_n ; G is a cyclic group of order m has a multiplicative inverse or not
#include<iostream>

```

#include<cmath>
using namespace std;
int main() {
cout << "To check whether an element in Z(n)G;" <<
"G is a cyclic group of order m" <<
      \has a multiplicative inverse or not" << endl;
int a[100], b[100], c[100], d[100], e[100], w1[100], m = 1,
t = 0, x = 1, s = 0, s1 = 0, l = 0, m1 = 1, hcf = 1,
n = 1, i = 1, k = 0, q = 1, p = 1, r = 1, w = 1;
cout << "Enter n =" << endl;
cin >> n;
cout << "Enter index m =" << endl;
cin >> m;
cout << "Enter m(-1) =" << endl;
cin >> m1;
cout << "Enter primitive m th root w =" << endl;
cin >> w;
for (i = 0; i < m; ++i)
    { w1[i] = ( (long long int)pow(w, i)) % n;
cout << "w1[" << i << "]" << w1[i] << endl; }
for (int i = 1; i < m + 1; ++i) {
cout << "Enter a["<<i<<"]=" << endl;
cin >> a[i];
}
a[0] = a[m];
for (int r = 1; r < m + 1; ++r) {
for (int j = 1; j < m + 1; ++j)
{
l = (m - j + 2) % m;
x = ((j - 1) * (r - 1)) % m;
k = k + (a[l] * w1[x]) % n; k = k % n;
// cout << "k=" << k << endl;
} c[r] = k; cout << "c[" << r << "]" << c[r] << endl;
k = 0;
}
for (r = 1; r < m+1; ++r) {
for (int i = 1; i <= n; ++i) {
if (c[r] % i == 0 && n % i == 0) {
hcf = i;
}
}
}
if (hcf == 1) {
cout << "c[" << r << "]" is a unit" << endl;
}
else { cout << "c[" << r <<
"] is a not unit. So there is no multiplicative inverse." <<
endl; t = 1; }
}

```

```

}
for (r = 1; r < m + 1; ++r) {
for (int i = 1; i <= n; ++i) {
e[r] = (c[r] * i) % n;
if (e[r] == 1) {
b[r] = i;
cout << " The inverse of c[" << r << "] is " << b[r] << endl;
}
}
}
b[0] = b[m];
for (int r = 1; r < m + 1; ++r) {
for (int j = 1; j < m + 1; ++j)
{
l = (m - j + 2) % m;
x = (m*m-(j - 1) * (r - 1)) % m; cout << "x= " << x << endl;
k = k + (m1*b[l] * w1[x]) % n; k = k % n;
cout << "k=" << k << endl;
}
d[r] = k; cout << "d[" << r << "]=" << d[r] << endl;
k = 0;
}
if (t == 1) {
s = m;
mylabel2:
cout << a[m - s + 1] << "g^(" << m - s << ") + ";
s--;
if (s > 0) goto mylabel2; cout <<
"has no multiplicative inverse." << endl;
}
else {cout << "The inverse of ";
s = m;
mylabel:
cout << a[m - s + 1] << "g^(" << m - s << ") + ";
s--;
if (s > 0) goto mylabel; cout << "is" << endl;
s1 = m;
mylabel1:
cout << d[m - s1 + 1] << "g^(" << m - s1 << ") + ";
s1--;
if (s1 > 0) goto mylabel1; cout << "." << endl; }
return 0;
}

```

Programme-3: To find idempotents, involutions and units in Z_n

```

#include<iostream>
#include<cmath>

```

```

using namespace std;
int main() {
cout << "To find idempotents, involutions and units in Z(n)." << endl;
long long int i, j=0,n, k=0, x, y;
cout << "Enter n=" << endl;
cin >> n;
cout << "The involutions are ";
for (i = 1; i < n + 1; ++i) {
x = (i * i) % n;
if (x == 1) { cout << i << " "; }
} cout << "." << endl;
cout << "The idempotents are ";
for (i = 0; i < n + 1; ++i) {
x = (i * i) % n;
if (x == i) { cout << i << " "; j++; }
} cout << "." << endl; cout <<
    "Number of idempotents = " << j << endl;
cout << "The units and its inverse are " << endl;
for (y = 1; y < n + 1; ++y) {
for (i = 1; i < n + 1; ++i) {
x = (y * i) % n;
    if (x == 1) { cout << y << " , " << i << endl; k++; }
}
} cout << "Number of units = " << k;
return 0;
}

```

Programme-4: To find the inverse of an element in $Z(n)G$; G is a cyclic group of order m through lambda units

```

#include<iostream>
#include<cmath>
using namespace std;
int main() {
cout << "To find the inverse of an element in Z(n)G;" <<
"G is a cyclic group of order m through lambda units." << endl;
int a[100], b[100], l[100], ll[100], w1[100],
m = 1, t = 0, x = 1, y=1,s = 0, s1 = 0, m1 = 1, hcf = 1, n = 1,
i = 1, k = 0, q = 1, p = 1, r = 1, w = 1;
cout << "Enter n =" << endl;
cin >> n;
cout << "Enter index m =" << endl;
cin >> m;
cout << "Enter  $m^{-1}$  =" << endl;
cin >> m1;
cout << "Enter primitive m th root w =" << endl;
cin >> w;

```

```

for (i = 0; i < m; ++i) { w1[i] = ((long long int)pow(w, i)) % n;
cout << "w1[" << i << "]= " << w1[i] << endl; }
cout << "Enter lambda values which have inverse" << endl;
for (int i = 1; i < m + 1; ++i) {
cout<< "l[" << i << "]= " << endl;
cin >> l[i];
}
cout << "Enter lambda inverse values " << endl;
for (int i = 1; i < m + 1; ++i) {
cout << "l1[" << i << "]= " << endl;
cin >> l1[i];
}
for (int r = 1; r < m + 1; ++r) {
for (int j = 1; j < m + 1; ++j)
{
x = ((j - 1) * (r - 1)) % m;
k = k + (m1*l[j] * w1[x]) % n; k = k % n;
// cout << "k=" << k << endl;
} a[r] = k; cout << "a[" << r << "]= " << a[r] << endl;
k = 0;
}
for (int r = 1; r < m + 1; ++r) {
for (int j = 1; j < m + 1; ++j)
{
x = ((j - 1) * (r - 1)) % m; cout << "x= " << x << endl;
k = k + (m1 * l1[j] * w1[x]) % n; k = k % n;
cout << "k=" << k << endl;
} b[r] = k; cout << "b[" << r << "]= " << b[r] << endl;

k = 0;
}
cout << "The inverse of a= ";
s = 1;
mylabel:
cout << a[s ] << "g^(" << s-1 << ") + ";
s++;
if (s < m+1) goto mylabel; cout << endl; cout << "is b=";
s1 = m;
mylabel1:
cout << b[m - s1 + 1] << "g^(" << m - s1 << ") + ";
s1--;
if (s1 > 0) goto mylabel1; cout << "." << endl;
cout << "Note: Please neglect the last + as" <<
"it is unavoidable for a for loop.";
return 0;
}

```

Programme-5: To find the idempotent elements in Z_nG ; G is a cyclic group of order m through
 λ takes idempotent values in Z_n

```
\begin{verbatim}
#include<iostream>
#include<cmath>
using namespace std;
int main() {
cout << "To find the idempotent elements in Z(n)G;"<<
"G is a cyclic group of order m through" <<
"lamda takes idempotent values in Z(n)." << endl;
int a[100], l[100], w1[100], m = 1, t = 0, x = 1, y = 1, s = 0,
s1 = 0, m1 = 1, n = 1, i = 1, k = 0, r = 1, w = 1;

cout << "Enter n =" << endl;
cin >> n;
cout << "Enter index m =" << endl;
cin >> m;
cout << "Enter m(-1) =" << endl;
cin >> m1;
cout << "Enter primitive m th root w =" << endl;
cin >> w;
for (i = 0; i < m; ++i) { w1[i] = ((long long int)pow(w, i)) % n;
cout << "w1[" << i << "]"<< w1[i] << endl; }
cout << "Enter lambda values which are idempotents" << endl;
for (int i = 1; i < m + 1; ++i) {
cout << "l[" << i << "]"<< endl;
cin >> l[i];
}
for (int r = 1; r < m + 1; ++r) {
for (int j = 1; j < m + 1; ++j)
{
x = ((j - 1) * (r - 1)) % m;
k = k + (m1 * l[j] * w1[x]) % n; k = k % n;
}
a[r] = k; cout << "a[" << r << "]"<< a[r] << endl;
k = 0;
}
cout << "The idempotent element in RG is e= ";
s = 1;
mylabel:
cout << a[s] << "g^(" << s - 1 << ") + ";
s++;
if (s < m + 1) goto mylabel; cout << endl;
cout << endl;
cout << "Note: Please neglect the last + as it is" <<
"unavoidable for a for loop.";
return 0;
}
```

```
}

```

Programme-6: Discrete Fourier Transform

```
#include <iostream>
#include<cmath>
using namespace std;
int main()
{
    cout << "Discrete Fourier Transform" << endl;
    unsigned long long int a[200][200], b[200][200],
    mult[200][200],q=1,m=1, n=1, w2=1,w=1, r1, c1, r2,
    c2, i, j, k, t=1;
    cout << "Enter n,m,w: ";
    cin >> n >> m >> w;
    r1 = m; c1=m;
    r2 = m; c2=1;
    for (i = 0; i < r1; ++i)
    for (j = 0; j < c1; ++j)
    {
        t = (i*j)%m;
        if (t == 0) a[i][j] = 1;
        else
        for (q = 1; q < t + 1; ++q) { w2 = (w2 * w) % n; }
        a[i][j] = w2; w2 = 1;
    }
    for (i = 0; i < r1; ++i)
    for (j = 0; j < c1; ++j)
    {
        cout<<" a"<<i+1<<" "<<j+1<<"="<<a[i][j] ;
        if (j == c1 - 1)
        cout << endl;
    }
    cout << endl << "Enter coefficient vector of
    the polynomial:" << endl;
    for (i = 0; i < r2; ++i)
    for (j = 0; j < c2; ++j)
    {
        cout << "Enter element f" << i << " = ";
        cin >> b[i][j];
    }
    for (i = 0; i < r1; ++i)
    for (j = 0; j < c2; ++j)
    {
        mult[i][j] = 0;
    }
    for (i = 0; i < r1; ++i)

```

```

for (j = 0; j < c2; ++j)
for (k = 0; k < c1; ++k)
{
    mult[i][j] += (a[i][k]) * (b[k][j]);
}
cout << endl << "DFT Output: " << endl;
for (i = 0; i < r1; ++i)
for (j = 0; j < c2; ++j)
{
    cout << "F"<< i << "="<< mult[i][j]%n;
    if (j == c2 - 1)
    cout << endl;
}
return 0;
}

```

Programme-7: Inverse Discrete Fourier Transform

```

#include <iostream>
#include<cmath>
using namespace std;
int main()
{
    cout << "Inverse Discrete Fourier Transform" << endl;
    unsigned long long int a[100][100], b[100][100],
        mult[100][100], p=1, q=1, l=1, m = 1, m1 = 1, w1 = 1,
        w2=1, n = 1, w = 1, r1, c1, r2, c2, i, j, k,
        c=1,t = 1;
    cout << "Enter n,m, w: ";
    cin >> n >> m >> w;
    for (l = 1; l < n; ++l)
    {
        c = (l * m) % n;
        if (c == 1)
        {
            m1 = 1;
        }
    }
    for (p = 1; p < m; ++p)
    {
        w1 = (w1 * w) % n;
    }
    r1 = m; c1 = m;
    r2 = m; c2 = 1;
    for (i = 0; i < r1; ++i)
    for (j = 0; j < c1; ++j)
    {

```

```

t = (i * j) % m;
if (t == 0) a[i][j] = 1;
else
for (q = 1; q < t + 1; ++q) { w2 = (w2 * w1) % n; }
a[i][j] = w2; w2 = 1;
}
for (i = 0; i < r1; ++i)
for (j = 0; j < c1; ++j)
{
cout << " a" << i + 1 << j + 1 << "=" << a[i][j];
if (j == c1 - 1)
cout << endl;
}
cout << endl << "Enter DFT vector :" << endl;
for (i = 0; i < r2; ++i)
for (j = 0; j < c2; ++j)
{
cout << "Enter element F" << i << " = ";
cin >> b[i][j];
}
for (i = 0; i < r1; ++i)
for (j = 0; j < c2; ++j)
{
mult[i][j] = 0;
}
for (i = 0; i < r1; ++i)
for (j = 0; j < c2; ++j)
for (k = 0; k < c1; ++k)
{
mult[i][j] += (a[i][k]) * (m1 * b[k][j]);
}
cout << endl << "Polynomial vector: " << endl;
for (i = 0; i < r1; ++i)
for (j = 0; j < c2; ++j)
{
cout << "f" << i << "=" << mult[i][j] % n;
if (j == c2 - 1)
cout << endl;
}
return 0;
}

```