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REGULARITY AND SYMMETRY FOR SEMILINEAR ELLIPTIC EQUATIONS IN BOUNDED DOMAINS.

LOUIS DUPAIGNE AND ALBERTO FARINA

RÉSUMÉ. In the present paper, we investigate the regularity and symmetry properties of weak solutions to semilinear elliptic equations which are locally stable.

1. INTRODUCTION AND MAIN RESULTS

In the present paper, we investigate the regularity and symmetry properties of weak solutions to semilinear elliptic equations. We shall focus on the following class :

Definition 1. Let $N \geq 1$, $\Omega \subset \mathbb{R}^N$ denote an open set and $f \in C^1(\mathbb{R})$. Assume that $u \in H_{loc}^1(\Omega)$, $f(u) \in L_{loc}^1(\Omega)$ and that u solves

$$(1) \quad -\Delta u = f(u) \quad \text{in } \mathcal{D}'(\Omega).$$

We say that u is **locally stable** in Ω if $f'(u) \in L_{loc}^1(\Omega)$ and if for every $x \in \Omega$, there exists an open neighborhood $\omega \subset \Omega$ of x such that for every $\varphi \in C_c^1(\omega)$, there holds

$$(2) \quad \int_{\omega} f'(u) \varphi^2 \leq \int_{\omega} |\nabla \varphi|^2.$$

A solution is stable in Ω if the above inequality holds for $\omega = \Omega$ and for every $\varphi \in C_c^1(\Omega)$.

As shown by the following examples, the class of locally stable solutions is natural and wide enough to encompass various interesting families of solutions (naturally) arising in the study of PDEs.

- (1) Smooth solutions are locally stable, thanks to the (sharp) Poincaré inequality.
- (2) More generally, for $N \geq 3$, weak solutions such that $f'(u) \in L_{loc}^1(\Omega)$ and $f'(u)^+ \in L_{loc}^{N/2}(\Omega)$ are locally stable. Indeed, choosing ω so small that $\|f'(u)^+\|_{L^{\frac{N}{2}}(\omega)} \leq \frac{N(N-2)|B_1|}{4}$ and applying Hölder's and Sobolev's inequalities, we have

$$\int_{\omega} f'(u) \varphi^2 \leq \int_{\omega} f'(u)^+ \varphi^2 \leq \|f'(u)^+\|_{L^{\frac{N}{2}}(\omega)} \|\varphi\|_{L^{\frac{2N}{N-2}}(\omega)}^2 \leq \int_{\omega} |\nabla \varphi|^2$$

When $N = 2$, the local stability follows from Moser-Trudinger inequality if $f'(u) \in L_{loc}^1(\Omega)$ and $f'(u)^+ \in L_{loc}^p(\Omega)$ for some $p > 1$.

- (3) If $N \geq 3$, $f(u) = 2(N-2)e^u$ and $u = -2 \ln |x|$, then $f'(u) = \frac{2(N-2)}{|x|^2} \in L_{loc}^1$ but just fails to belong to $L^{N/2}$ near the origin. By the optimality of Hardy's inequality, u is never locally stable in any open set containing the origin whenever $3 \leq N \leq 9$.
- (4) Local minimizers are stable : $u \in H_{loc}^1(\Omega)$ is a local minimizer if for any $\Omega' \subset\subset \Omega$ and for all $\varphi \in C_c^1(\Omega')$, $t = 0$ is a point of minimum of the function $t \mapsto e(t) := \mathcal{E}_{\Omega'}(u + t\varphi)$, where $\mathcal{E}_{\Omega'}(v) = \int_{\Omega'} (\frac{1}{2}|\nabla v|^2 - F(v))$ and $F' = f$. Therefore (2) holds (since $e''(0) \geq 0$).
- (5) If $u \in H_{loc}^1(\Omega)$ has finite Morse index¹ in Ω , then u is locally stable in Ω , see Proposition 1.5.1 in [10] (or Proposition 2.1 in [9]). In addition, u is stable outside a compact set, see Remark 1 in [12]. But there are also locally stable solutions of infinite Morse index. This is the case e.g. when Ω is the punctured unit ball, $f(u) = 2(N-2)e^u$, $u(x) = -2 \ln |x|$ and $3 \leq N \leq 9$.

1. We recall that a solution u to (1) has Morse index equal to $K \geq 0$, if $f'(u) \in L_{loc}^1(\Omega)$ and K is the maximal dimension of a subspace X_K of $C_c^1(\Omega)$ such that $\int_{\Omega} |\nabla \psi|^2 < \int_{\Omega} f'(u) \psi^2$ for any $\psi \in X_K \setminus \{0\}$. In particular u is stable if and only if its Morse index is zero.

Our first result concerns the complete classification of nonnegative stable solutions $u \in H_0^1(\Omega)$ to (1), when f is a convex function satisfying $f(0) = 0$.

Theorem 1. *Let Ω be a bounded domain of \mathbb{R}^N , $N \geq 1$, let $f \in C^1(\mathbb{R})$ be a convex function such that $f(0) = 0$ and let λ_1 be the principal eigenvalue of $-\Delta$ with homogeneous Dirichlet boundary conditions. Assume that $u \in H_0^1(\Omega)$, $f(u) \in L_{loc}^1(\Omega)$ and that u is a stable solution to*

$$(3) \quad \begin{cases} -\Delta u = f(u) & \text{in } \mathcal{D}'(\Omega) \\ u \geq 0 & \text{a.e. on } \Omega. \end{cases}$$

Then, either $u \equiv 0$ or $f(t) = \lambda_1 t$ on $(0, \sup_{\Omega} u)$ and $u \in C^\infty(\Omega) \cap H_0^1(\Omega)$ is a positive first eigenfunction of $-\Delta$ with homogeneous Dirichlet boundary conditions.

Remark 2. If $u \equiv 0$, then necessarily $f'(0) \leq \lambda_1$, by Lemma 14 in section 3 below. Also observe that for any $\alpha \leq \lambda_1$ there is a convex function f satisfying $f(0) = 0$, $f'(0) = \alpha$ and such that $u \equiv 0$ is a stable solution to (3). An example is provided by $f(u) = u^2 + \alpha u$.

The latter result is a consequence of the following general theorem which holds true for any convex function f of class C^1 and for distributional solutions merely in $H^1(\Omega)$.

Theorem 3. *Let Ω be a bounded domain of \mathbb{R}^N , $N \geq 1$ and let $f \in C^1([0; +\infty))$ be a convex function. Assume that $u, v \in H^1(\Omega)$ satisfy $u - v \in H_0^1(\Omega)$, $0 \leq v \leq u$ a.e. on Ω , $f(u), f(v) \in L_{loc}^1(\Omega)$ and both u and v are solution to*

$$(4) \quad -\Delta w = f(w) \quad \text{in } \mathcal{D}'(\Omega).$$

If $f'(u) \in L_{loc}^1(\Omega)$ and u is stable, then either $u \equiv v$ or $f(t) = a + \lambda_1 t$ for all $t \in (\inf_{\Omega} v, \sup_{\Omega} u)$ and some² $a \in \mathbb{R}$, $u, v \in C^\infty(\Omega)$ and $u - v$ is a positive first eigenfunction of $-\Delta$ with homogeneous Dirichlet boundary conditions.

Remark 4. According to Theorem 1.3 and Corollary 3.7 in [7], there exists a C^1 , positive, increasing but non-convex nonlinearity f with two distinct and ordered (classical) stable³ solutions $0 \leq u \leq v$. In other words, the convexity assumption cannot be completely removed from the above theorem.

Another important consequence of Theorem 3 is the following approximation result which, in turn, motivated our definition of local stability (see Definition 1). This result will be central in the proof of our main regularity results for locally stable solutions to (1).

Theorem 5. *Assume $\alpha \in (0, 1)$ and $N \geq 1$.*

Let Ω be a bounded domain of \mathbb{R}^N and let $f \in C^1([0, +\infty))$ be a convex function such that $f(0) \geq 0$. Assume that $u \in H^1(\Omega)$, $f(u) \in L_{loc}^1(\Omega)$ and that u is a stable solution to

$$(5) \quad \begin{cases} -\Delta u = f(u) & \text{in } \mathcal{D}'(\Omega) \\ u \geq 0 & \text{a.e. on } \Omega. \end{cases}$$

Then, there is a nondecreasing sequence (f_k) of convex functions in $C^1([0, +\infty)) \cap C^{0,1}([0, +\infty))$ such that $f_k \nearrow f$ pointwise in $[0; +\infty)$ and a nondecreasing sequence (u_k) of functions in $H^1(\Omega) \cap C^2(\Omega)$ such

2. Actually the real number a is unique and its value is given by $-\lambda_1 \frac{\int_{\Omega} \phi_1 h}{\int_{\Omega} \phi_1} \leq 0$, where ϕ_1 is a positive first eigenfunction of $-\Delta$ with homogeneous Dirichlet boundary conditions and $h \in H^1(\Omega)$ is the unique weak solution of $-\Delta h = 0$ in Ω with $u - h \in H_0^1(\Omega)$ (to see this, use ϕ_1 as test function in the weak formulation of $-\Delta u = a + \lambda_1 u$ and the fact that h is harmonic) and also note that h is nonnegative by the maximum principle. In particular, $u, v \in H_0^1(\Omega) \iff a = 0$. Also note that, for every $a \leq 0$ there exist solutions u, v for which the second alternative of the theorem occurs. Indeed, the functions $u_t := -\frac{a}{\lambda_1} + t\phi_1$, $t \geq 0$ are suitable.

3. Indeed, applying Corollary 3.7 in [7], we see that in the notations of that corollary, for $f(u) = \lambda \bar{g}(u)$, \bar{u}_λ is minimal hence stable. In addition, by minimality, $\lambda \mapsto \bar{u}_\lambda$ is nondecreasing and so \bar{u}_λ converges to a stable solution v as $\lambda \searrow \underline{\lambda}$ such that $v \geq \bar{u}_{\underline{\lambda}}$. Then, take $u = \bar{u}_{\underline{\lambda}}$.

that u_k is a stable weak solution⁴ to

$$(6) \quad \begin{cases} -\Delta u_k = f_k(u_k) & \text{in } \Omega, \\ u_k - u \in H_0^1(\Omega), \\ 0 \leq u_k \leq u & \text{a.e. on } \Omega, \end{cases}$$

and

$$(7) \quad u_k \longrightarrow u \quad \text{in } H^1(\Omega), \quad u_k \nearrow u \quad \text{a.e. on } \Omega.$$

Moreover, if f is nonnegative, then any function f_k is nonnegative too.

Remark 6. (1) It follows from (7) that under the assumptions of the proposition, locally stable solutions are automatically lower semi-continuous.

(2) The proposition recovers and extends Corollary 3.2.1. in [10].

(3) The result is *not true* if we drop the assumption $u \in H^1(\Omega)$. To see this, consider Example 3.2.1 in [10] in the light of Theorem 7 below.

(4) We do not know if the assumption f convex can be dropped.

(5) Theorem 5 generalizes Proposition 21 below, in which the approximating nonlinearity is taken of the form $f_k = (1 - \epsilon_k)f$, with $\epsilon_k \rightarrow 0$ at the expense of additionally assuming that f is nondecreasing.

Theorem 5 can be combined with the following *a priori* estimate due to [5] in order to establish smoothness of locally stable solutions when f is nonnegative, convex and $N \leq 9$.

Theorem A ([5]). *Let B_1 be the unit ball of \mathbb{R}^N , $N \geq 1$. Assume that $u \in C^2(B_1)$ is a stable solution of (1) in $\Omega = B_1$, where $f : \mathbb{R} \rightarrow \mathbb{R}$ is locally Lipschitz and nonnegative. If $1 \leq N \leq 9$, then*

$$(8) \quad \|u\|_{C^\alpha(\overline{B_{1/2}})} \leq C \|u\|_{L^1(B_1)},$$

where $\alpha \in (0, 1)$, $C > 0$ are dimensional constants.

More precisely we have the following interior regularity result :

Theorem 7. *Let $f \in C^1([0, +\infty))$ be a nonnegative convex function. Let Ω be an open set of \mathbb{R}^N , $N \geq 1$. Assume that $u \in H_{loc}^1(\Omega)$, $f(u) \in L_{loc}^1(\Omega)$ and that u is a locally stable solution of (1) such that $u \geq 0$ a.e. in Ω .*

If $1 \leq N \leq 9$, then $u \in C_{loc}^{2,\beta}(\Omega)$ for all $\beta \in (0, 1)$. In particular, any finite Morse index solution is smooth in Ω .

Remark 8. The result is optimal since for $N \geq 10$, $f(u) = 2(N - 2)e^u$ and $\Omega = B_1$, $u(x) = -2 \ln |x|$ is a singular stable solution in $H_{loc}^1(B_1)$. Also observe that the above theorem fails if we do not assume that u belongs to $H_{loc}^1(\Omega)$, see e.g. Example 3.2.1 in [10].

A priori estimates near the boundary are more subtle, as the following result shows.

Theorem 9. *Let $f \in C^1([0, +\infty))$ be a nonnegative convex function. Let Ω be an open set of \mathbb{R}^N , $N \geq 1$. Assume that $u \in H_0^1(\Omega)$, $f(u) \in L_{loc}^1(\Omega)$ and that u is a finite Morse index solution of (1) such that $u \geq 0$ a.e. in Ω .*

(1) *Let Ω be a bounded uniformly convex domain of class $C^{2,\alpha}$, for some $\alpha \in (0, 1)$. Then there exists constants $\rho, \gamma > 0$, depending only on Ω , such that*

$$(9) \quad \|u\|_{L^\infty(\Omega_\rho)} \leq \frac{1}{\gamma} \|u\|_{L^1(\Omega)}$$

where $\Omega_\rho := \{x \in \Omega : \text{dist}(x, \partial\Omega) < \rho\}$. In particular, $u \in C^{2,\alpha}(\Omega_\rho \cup \partial\Omega)$.

(2) *If $1 \leq N \leq 9$ and either Ω is $C^{2,\alpha}$ and convex or f is nondecreasing and Ω is C^3 , then there exists constants $\rho, \gamma > 0$, depending only on Ω , such that (9) holds.*

4. That is a function u_k satisfying $\int_\Omega \nabla u_k \nabla \varphi = \int_\Omega f_k(u_k) \varphi$, for all $\varphi \in H_0^1(\Omega)$.

- (3) Fix $N \geq 11$. For every sequence $(\rho_n) \subset \mathbb{R}_+^*$ converging to zero, there exists a sequence of bounded C^1 convex domains $\Omega^n \subset \mathbb{R}^N$, $n \in \mathbb{N}^*$, such that the corresponding stable solution u_n to (1) with $f(u) = 2(N-3)e^u$ and $\Omega = \Omega^n$ satisfies

$$\|u_n\|_{L^\infty((\Omega^n)_{\rho_n})} \rightarrow +\infty \quad \text{yet} \quad \frac{1}{|\Omega^n|} \|u_n\|_{L^1(\Omega^n)} \quad \text{remains bounded.}$$

The last point of the above theorem shows that in dimension $N \geq 11$, no universal *a priori* estimate of the type (9) can hold near the boundary if the domain Ω is merely convex and the constant γ depends on the dimension N and the volume $|\Omega|$ only. The case $N = 10$ is open. We do not know either if locally stable solutions are smooth near the boundary of convex domains, although the universal *a priori* estimate fails.

When Ω is bounded and rotationally invariant we can prove the following classification result.

Theorem 10. (1) Let $R > 0$, $N \geq 1$, B be the open ball $B(0, R) \subset \mathbb{R}^N$ and let $f \in C^1([0, +\infty))$ be a convex function. Assume that $u \in H_0^1(B)$, $f(u) \in L_{loc}^1(B)$ and that u is a stable solution to

$$(10) \quad \begin{cases} -\Delta u = f(u) & \text{in } \mathcal{D}'(B) \\ u \geq 0 & \text{a.e. on } B. \end{cases}$$

Then, either $u \equiv 0$ or $u \in C^3(\overline{B} \setminus \{0\})$, $u > 0$ and u is radially symmetric and radially strictly decreasing. Furthermore, if $N \leq 9$ and f is nonnegative, then $u \in C^2(\overline{B})$.

- (2) Let $R > 0$, $1 \leq N \leq 9$, B be the open ball $B(0, R) \subset \mathbb{R}^N$ and let $f \in C^1([0, +\infty))$ be a nonnegative convex function. Assume that $u \in H_0^1(B)$, $f(u) \in L_{loc}^1(B)$ and that u solves

$$(11) \quad \begin{cases} -\Delta u = f(u) & \text{in } \mathcal{D}'(B) \\ u \geq 0 & \text{a.e. on } B. \end{cases}$$

If u has finite Morse index, then either $u \equiv 0$ or $u \in C^2(\overline{B})$, $u > 0$ and u is radially symmetric and radially strictly decreasing.

- (3) Let $N \geq 2$, $\Omega \subset \mathbb{R}^N$ be an open annulus centered at the origin and let $f \in C^1([0, +\infty))$ be a convex function. Assume that $u \in H_0^1(\Omega)$, $f(u) \in L_{loc}^1(\Omega)$ and that u is a stable solution to

$$(12) \quad \begin{cases} -\Delta u = f(u) & \text{in } \mathcal{D}'(\Omega) \\ u \geq 0 & \text{a.e. on } \Omega. \end{cases}$$

Then, either $u \equiv 0$ or $u \in C^2(\overline{\Omega})$, $u > 0$ and u is radially symmetric.⁵

Remark 11. (1) The conclusion that u is radially symmetric in item (1) of Theorem 10 was already known to hold true in the special case where $u \in C^2(\overline{B})$ (and with no additional sign assumption on u), see e.g. [1].

- (2) Item (2) of Theorem 10 is sharp. Indeed, for $N \geq 10$, $f(u) = 2(N-2)e^u$ and $\Omega = B_1 = B(0, 1)$, $u(x) = -2 \ln |x|$ is a singular stable solution in $H_0^1(B_1)$. The finite Morse index assumption is also essential : for $N = 3$ and $f(u) = 2e^u$, there exists a family of nonradial singular solutions in $\Omega = B_1$ of the form $u(x) = -2 \ln |x - x_0| + v(x)$, where $x_0 \neq 0$ and $v \in L^\infty(B) \cap H^1(B)$, see (in details) the proofs in [18]. In particular, $u \in H_0^1(B)$. It follows from our result that u cannot have finite Morse index. In contrast, note that, for $3 \leq N \leq 9$, $\Omega = B_1$ and $f(u) = 2(N-2)e^u$, there exist infinitely many smooth and positive solutions to (1) such that $u = 0$ on $\partial\Omega$ and with finite and non-zero Morse index. See chapter 2 of [10] (and the references therein) for a detailed discussion of this topic.

- (3) It will be clear from the proof that :

- (a) the radial symmetry in item (1) is still true if we replace $u \in H_0^1(\Omega)$ by any member u of $H^1(\Omega)$ having constant trace $c \geq 0$ on ∂B .

5. Furthermore, if Ω is the annulus $\{x \in \mathbb{R}^N : 0 < a < |x| < b\}$ and $u(x) = v(|x|)$, then there is a unique $r_0 \in (a, b)$ such that $v' > 0$ in (a, r_0) , $v'(r_0) = 0$ and $v' < 0$ in (r_0, b) . The result follows as in the proof of item 1). For this reason we omit it.

- (b) the radial symmetry in item (3) is still true if we replace $u \in H_0^1(\Omega)$ by any member u of $H^1(\Omega)$ having a nonnegative constant trace on each of the two connected components of the boundary of the annulus (possibly with different values on the two connected component).
- (c) Note that, if $c > 0$, we do not claim any monotonicity or special property about the radial profile (as it happens when $c = 0$.)

A crucial step in the proof of the above Theorem is the following general symmetry result. As we shall see below, this result also enables us to prove further symmetry results for stable solutions in "symmetric" bounded domains.

Proposition 12. *Assume $N \geq 1$ and let $f \in C^1([0, +\infty))$ be a convex function. Let $\rho \in O(N)$ and let $\Omega \subset \mathbb{R}^N$ be a ρ -invariant bounded domain, i.e., a bounded domain such that $\rho(\Omega) = \Omega$. Assume that $u \in H_c^1(\Omega)$ ⁶, $f(u) \in L_{loc}^1(\Omega)$ and that u is a stable solution to*

$$(13) \quad \begin{cases} -\Delta u = f(u) & \text{in } \mathcal{D}'(\Omega) \\ u \geq 0 & \text{a.e. on } \Omega. \end{cases}$$

If ρ has a fixed point in Ω , then u is ρ -invariant, namely, $u(x) = u(\rho x)$ for almost every $x \in \Omega$.

Remark 13. Below we provide a (non-exhaustive) list of bounded domains to which the above result applies :

- (1) if Ω is any bounded domain symmetric with respect to a hyperplane, then u inherits the same symmetry.
- (2) if Ω is an open ball minus its center x_0 , then u is radially symmetric with respect to x_0 .
- (3) if Ω is a n -sided regular polygon, with $n \geq 3$, then u is invariant with respect to the dihedral group D_n (of order $2n$).
- (4) if Ω is the product of rotationally invariant bounded domains, i.e., $\Omega = \omega_1 \times \dots \times \omega_m$, where ω_j is a bounded rotationnaly invariant domain⁷ of \mathbb{R}^{n_j} , with $n_j \geq 1$ and $N = n_1 + \dots + n_m$, then u inherits the same symmetry, i.e., $u(x) = v(|x^1|, \dots, |x^m|)$ a.e. in Ω .⁸ This case was already addressed under the additional assumption that u is smooth in Remark 2.1 in [6].
- (5) if Ω is a cylinder with ρ -invariant cross section, i.e., $\Omega = \omega \times U$, where ω is a ρ -invariant bounded domain of \mathbb{R}^k , $1 \leq k \leq N-1$ and U is a domain of \mathbb{R}^{N-k} , then $u(x) = u(\rho(x^1), x_{k+1}, \dots, x_N)$ a.e. on Ω (here $x^1 := (x_1, \dots, x_k) \in \mathbb{R}^k$.)
- (6) any bounded domain of "revolution".

2. PROOFS

Proof of Theorem 5.

We distinguish two case : either $f'(t) \leq 0$ for any $t > 0$ or there exists $\bar{t} > 0$ such that $f'(\bar{t}) > 0$. In the first case, by convexity of f , we have that $f'(0) \leq f'(t) \leq 0$ for any $t \geq 0$, then f is also globally Lipschitz-continuous on $[0, +\infty)$. So $f(u) \in L^2(\Omega)$ and $u \in C^2(\Omega)$ by standard elliptic estimates (plus bootstrap and Sobolev imbedding). The claim follows by taking $f_k = f$ and $u_k = u$ for any integer $k \geq 1$.

In the second case, the convexity of f implies the existence of $t_0 > \bar{t}$ that $f(t), f'(t) > 0$ for any $t \geq t_0$. Set $k_0 := [t_0] + 1$ (here by $[t_0]$ we denote the integer part of t_0) and, for any integer $k \geq k_0$ and $t \geq 0$, we set

$$(14) \quad f_k(t) := \begin{cases} f(t) & \text{if } t \leq k, \\ f(k) + f'(k)(t - k) & \text{if } t > k. \end{cases}$$

6. Here $H_c^1(\Omega)$ denotes the subset of $H^1(\Omega)$ whose members take the constant value $c \geq 0$ on $\partial\Omega$. That is, $H_c^1(\Omega) = \{u \in H^1(\Omega) : u - c \in H_0^1(\Omega)\}$. In particular, for $c = 0$, that set boils down to $H_0^1(\Omega)$.

7. An open ball, an open ball minus its center or an annulus.

8. Here, for any $j \in \{1, \dots, m\}$, x^j denotes a generic point of $\omega_j \subset \mathbb{R}^{n_j}$ in such a way that $x := (x^1, \dots, x^m) \in \mathbb{R}^N$.

Clearly, f_k is a convex function of class $C^1([0, +\infty)) \cap C^{0,1}([0, +\infty))$ and $f_k \nearrow f$ pointwise in $[0; +\infty)$ (recall that $f(t), f'(t) > 0$ for any $t \geq t_0$). Moreover we have

$$(15) \quad f'_k(t) \leq f'(t) \quad \forall k \geq k_0, \quad \forall t \geq 0$$

and

$$(16) \quad f_k(t) \geq \min_{t \in [0, k_0]} f(t) := c_0(f), \quad \forall k \geq k_0, \quad \forall t \geq 0.$$

In particular, if f is nonnegative, then any function f_k is nonnegative too.

Since any f_k is globally Lipschitz-continuous on $[0, +\infty)$, we can use the (standard) method of sub and supersolution in H^1 to obtain a stable weak solution to (6) satisfying (7). To this end we observe that $u \in H^1(\Omega)$ is a nonnegative weak supersolution to

$$(17) \quad \begin{cases} -\Delta v_k = f_k(v_k) & \text{in } \Omega, \\ v_k - u \in H_0^1(\Omega), \end{cases}$$

since $f_k \leq f$ on $[0, +\infty)$ implies that $f_k(u)\varphi \leq f(u)\varphi$ a.e. in Ω , for any nonnegative $\varphi \in C_c^\infty(\Omega)$. So

$$(18) \quad \int_{\Omega} f_k(u)\varphi \leq \int_{\Omega} f(u)\varphi = \int_{\Omega} \nabla u \nabla \varphi \quad \forall \varphi \in C_c^\infty(\Omega), \quad \varphi \geq 0 \quad \text{in } \Omega$$

by (1) and then the above inequality holds true for any nonnegative $\varphi \in H_0^1(\Omega)$ by a standard density argument. Also, 0 is a weak subsolution to (17), since $f(0) \geq 0$ by assumption and $(0 - u)^+ \equiv 0 \in H_0^1(\Omega)$. Since $0 \leq u$ a.e. in Ω , the method of sub and supersolution in H^1 provides a weak solution v_k to (17) such that $0 \leq v_k \leq u$ a.e. in Ω and which is *minimal*⁹ in the following sense : given any weak supersolution $\bar{u} \in H^1$ of (17) such that $0 \leq \bar{u} \leq u$ a.e. in Ω , we have that $v_k \leq \bar{u}$ a.e. in Ω . From the latter property we immediately infer that $v_k \leq v_{k+1} \leq u$ a.e. in Ω . Also, by standard elliptic estimates we have $v_k \in C_{loc}^{2,\alpha}(\Omega)$ for any $\alpha \in (0, 1)$. By the convexity of f_k and (15) we see that $f'_k(v_k) \leq f'_k(u) \leq f'(u)$ a.e. in Ω , hence

$$(19) \quad \int_{\Omega} f'_k(v_k)\varphi^2 \leq \int_{\Omega} f'(u)\varphi^2 \leq \int_{\Omega} |\nabla \varphi|^2 \quad \forall \varphi \in C_c^1(\Omega)$$

and so v_k is a stable weak solution to (6).

To prove (7) we test (17) with $u - v_k \in H_0^1(\Omega)$ to obtain

$$(20) \quad \int_{\Omega} \nabla v_k \nabla (u - v_k) = \int_{\Omega} f_k(v_k)(u - v_k)$$

and so

$$(21) \quad \int_{\Omega} |\nabla v_k|^2 = \int_{\Omega} \nabla v_k \nabla u - \int_{\Omega} f_k(v_k)(u - v_k) \leq \int_{\Omega} \nabla v_k \nabla u - c_0(f) \int_{\Omega} (u - v_k)$$

thanks to (16) and $u - v_k \geq 0$ a.e. in Ω . Then

$$\int_{\Omega} |\nabla v_k|^2 \leq \frac{1}{2} \int_{\Omega} |\nabla v_k|^2 + \frac{1}{2} \int_{\Omega} |\nabla u|^2 + |c_0(f)| \int_{\Omega} u$$

hence

$$(22) \quad \|\nabla v_k\|_{L^2(\Omega)}^2 \leq \|\nabla u\|_{L^2(\Omega)}^2 + 2|c_0(f)|\|u\|_{L^1(\Omega)}.$$

We also have $\|v_k\|_{L^2(\Omega)} \leq \|u\|_{L^2(\Omega)}$ since $0 \leq v_k \leq u$ a.e. in Ω . Then (v_k) is bounded in $H^1(\Omega)$ and therefore we may and do suppose that (up to subsequences) $v_k \rightharpoonup v$ in $H^1(\Omega)$, $v_k \rightarrow v$ in $L^2(\Omega)$, $v_k \nearrow v$ a.e. on Ω for some $v \in H^1(\Omega)$, as $k \rightarrow \infty$ (actually the whole original sequence converges a.e. to v , since we already know that it is monotone nondecreasing). In particular we have $0 \leq v \leq u$ a.e. in Ω . Also note that $v - u \in H_0^1(\Omega)$, since $v_k - u \rightharpoonup v - u$ in $H_0^1(\Omega)$.

To prove that $v_k \rightarrow v$ in $H^1(\Omega)$ we test (17) with $v - v_k \in H_0^1(\Omega)$ to get

$$(23) \quad \int_{\Omega} \nabla v_k \nabla (v - v_k) = \int_{\Omega} f_k(v_k)(v - v_k)$$

⁹. Just consider the solution obtained by the standard monotone iterations procedure starting from the subsolution $\underline{u} \equiv 0$.

and so

$$(24) \quad \int_{\Omega} |\nabla v_k|^2 = \int_{\Omega} \nabla v_k \nabla v - \int_{\Omega} f_k(v_k)(v - v_k) \leq \int_{\Omega} \nabla v_k \nabla v + \int_{\Omega} |c_0(f)|(v - v_k)$$

thanks to (16) and $v - v_k \geq 0$ a.e. in Ω . Recalling that $0 \leq v_k \leq v$ a.e. on Ω we then have

$$(25) \quad \int_{\Omega} |v_k|^2 + |\nabla v_k|^2 \leq \int_{\Omega} |v|^2 + \int_{\Omega} \nabla v_k \nabla v + \int_{\Omega} |c_0(f)|(v - v_k)$$

and so

$$(26) \quad \limsup \|v_k\|_{H^1} \leq \|v\|_{H^1}$$

since $v_k \rightharpoonup v$ in $H^1(\Omega)$ and $v_k \rightarrow v$ in $L^2(\Omega)$. Inequality (26) and $v_k \rightharpoonup v$ in $H^1(\Omega)$ imply the strong convergence in $H^1(\Omega)$.

To proceed further we note that $f(v) \in L^1_{loc}(\Omega)$ since

$$(27) \quad c_0(f) \leq f_k(v) \leq f(v) = f(v)\mathbf{1}_{\{v \leq k_0\}} + f(v)\mathbf{1}_{\{v > k_0\}} \leq \sup_{t \in [0, k_0]} f(t) + f(u),$$

where in the latter we have used that $f'(t) > 0$ for $t \geq k_0$ and $v \leq u$ a.e. in Ω .

Next we prove that $-\Delta v = f(v)$ in $\mathcal{D}'(\Omega)$. This follows by passing to the limit in (17) by the Lebesgue's dominated convergence theorem, after having observed that $f_k(v_k) \rightarrow f(v)$ a.e. in Ω , and $c_0(f) \leq f_k(v_k) \leq f(v_k) = f(v_k)\mathbf{1}_{\{v_k \leq k_0\}} + f(v_k)\mathbf{1}_{\{v_k > k_0\}} \leq \sup_{t \in [0, k_0]} f(t) + f(u)$ holds true.

So far we have proved that

$$(28) \quad \begin{cases} -\Delta v = f(v) & \text{in } \mathcal{D}'(\Omega) \\ v - u \in H^1_0(\Omega), \\ 0 \leq v \leq u & \text{a.e. on } \Omega, \end{cases}$$

and therefore from Theorem 3 we deduce that either $u = v$ in Ω or $f(t) = a + \lambda_1 t$ for all $t \in (\inf_{\Omega} v, \sup_{\Omega} u)$ and some $a \in \mathbb{R}$, $u, v \in C^\infty(\Omega)$ and $u - v$ is a positive first eigenfunction of $-\Delta$ with homogeneous Dirichlet boundary conditions. In the first case we are done, while in the second one we distinguish two subcases : either $\sup u = +\infty$ or not. In the first subcase f is necessarily globally Lipschitz-continuous on $[0, +\infty)$ and the conclusion follows by taking $f_k = f$ and $u_k = u$ for every k . If $\sup u < +\infty$, the conclusion follows by taking the sequences $(f_k)_{k \geq \bar{k}}$ and $(u_k)_{k \geq \bar{k}}$, where \bar{k} is any integer satisfying $\bar{k} > \sup u$.

Proof of Theorem 7.

When $N = 1$, any $u \in H^1_{loc}(\Omega)$ is continuous by Sobolev imbedding and so is $f(u)$. This implies $u \in C^2$ and $f(u) \in C^1$, so u is of class C^3 by using equation (1). So assume that $N \geq 2$ and that u is locally stable. For any point $x_0 \in \Omega$ pick a ball $B(x_0, r_0) \subset \Omega$ in which u is stable and set $B = B(x_0, r_0)$.

By Theorem 5 there is a nondecreasing sequence (f_k) of functions in $C^1([0, +\infty)) \cap C^{0,1}([0, +\infty))$ such that $f_k \nearrow f$ pointwise in $[0, +\infty)$ and a nondecreasing sequence (u_k) of functions in $H^1(\Omega) \cap C^2(\Omega)$ such that u_k is a stable weak solution¹⁰ to (6) such that (7) holds.

Since $N \leq 9$ an application of Theorem A to u_k yields

$$\|u_k\|_{C^\beta(\overline{B(x_0, \frac{r_0}{2})})} \leq C \|u_k\|_{L^1(B)} \leq C |B|^{\frac{1}{2}} \|u_k\|_{L^2(B)} \leq C' \|u\|_{L^2(B)}$$

where $\beta \in (0, 1)$ depends only on N , while $C > 0$ depends on N and r_0 . In particular, the sequence (u_k) is bounded in $C^\beta(\overline{B(x_0, \frac{r_0}{2})})$ and the Ascoli-Arzelà's theorem then implies that a subsequence must converge uniformly to some continuous function v on $\overline{B(x_0, \frac{r_0}{2})}$. This function must coincide with u on $\overline{B(x_0, \frac{r_0}{2})}$, since we already know that $u_k \rightarrow u$ a.e. on Ω . The boundedness of u on $B(x_0, \frac{r_0}{2})$ and standard elliptic theory imply $u \in C^{2,\alpha}_{loc}(B(x_0, \frac{r_0}{2}))$ for every $\alpha \in (0, 1)$. Since x_0 is an arbitrary point of Ω , this concludes the proof. \square

Proof of Theorem 10.

1) It is enough to treat the case $u \not\equiv 0$. We first prove that u is radially symmetric. For $\rho \in O(N)$ we set $u_\rho(x) := u(\rho x)$, $x \in B$. Then $u_\rho \in H^1_0(B)$ is a stable solution to (10) since u is so. We can therefore apply

10. That is a function u_k satisfying $\int_{\Omega} \nabla u_k \nabla \varphi = \int_{\Omega} f_k(u_k) \varphi$, for all $\varphi \in H^1_0(\Omega)$

Theorem 17 below to get that u and u_ρ are ordered solutions. If $u \not\equiv u_\rho$, an application of Theorem 3 would give $u, u_\rho \in C^\infty$ and, either $u < u_\rho$ or $u > u_\rho$ in B . The latter are clearly impossible, since $u(0) = u_\rho(0)$. Thus, $u \equiv u_\rho$ for any $\rho \in O(N)$, and so u is radially symmetric in B . Since u is a radially symmetric member of $H_0^1(B)$, we have that $u \in C^0(\overline{B} \setminus \{0\})$ and by standard elliptic regularity, $u \in C_{loc}^{2,\alpha}(\overline{B} \setminus \{0\})$, for any $\alpha \in (0, 1)$. Hence we can write $u(x) = v(r)$, $r = |x| \in (0, R]$, and so $v \in C^2((0, R])$ is a classical solution to the ode $-(r^{N-1}v')' = r^{N-1}f(v)$ in $(0, R]$. The latter clearly implies $v \in C^3((0, R])$ but also that¹¹

$$(29) \quad v' < 0 \quad \text{in} \quad (R - \epsilon, R),$$

for some $\epsilon \in (0, R)$. Indeed, if $f(0) \geq 0$ the Hopf's lemma yields $v'(R) < 0$ (recall that we are supposing that $u \not\equiv 0$) and so (29) follows. When $f(0) < 0$ we have $(r^{N-1}v')' = -r^{N-1}f(v) > 0$ in an interval of the form $(R - \epsilon, R)$, thus $r \rightarrow r^{N-1}v'$ is strictly increasing in $(R - \epsilon, R)$. The latter and the fact that $v'(R) \leq 0$ (recall that $v \geq 0$ in $(0, R)$ and $v(R) = 0$) imply (29).

To conclude it is enough to prove that $v' < 0$ on $(0, R)$ (this also implies that $u > 0$ in $B \setminus \{0\}$). Suppose not, then

$$(30) \quad r_0 := \inf\{r \in (0, R) : v' < 0 \text{ on } (r, R)\}.$$

is well-defined, r_0 belongs to $(0, R)$ and $v'(r_0) = 0$. We have two cases : either there exists $z \in (0, r_0)$ such that $v'(z) = 0$, or v' has a sign on $(0, r_0)$ (i.e., either $v' < 0$ or $v' > 0$ on $(0, r_0)$). Let us show that both of them are impossible. In the first case we observe that $w := u_r$, the radial derivative of u , is of class $C_0^2(\overline{A_{z,r_0}})$ and satisfy

$$(31) \quad -\Delta w + (N-1)\frac{w}{r^2} = f'(u)w \quad \text{on} \quad A_{z,r_0},$$

where $A_{z,r_0} := \{x \in \mathbb{R}^N : z < |x| < r_0\}$. We can then multiply (31) by w , integrate by parts and find

$$(32) \quad \int_{A_{z,r_0}} |\nabla w|^2 - \int_{A_{z,r_0}} f'(u)w^2 = - \int_{A_{z,r_0}} (N-1)\frac{w^2}{r^2}.$$

On the other hand $w \in C_0^2(\overline{A_{z,r_0}})$ and u is stable on A_{z,r_0} , so w can be used as test function in the stability condition satisfied by u to obtain

$$\int_{A_{z,r_0}} |\nabla w|^2 - \int_{A_{z,r_0}} f'(u)w^2 \geq 0.$$

The latter and (32) give $w = 0$ on A_{z,r_0} and so $w \leq 0$ in the open annulus $A_{z,R}$. Since w solves the linear equation $-\Delta w + (\frac{(N-1)}{r^2} - f'(u))w = 0$ on $A_{z,R}$, the strong maximum principle implies $w \equiv 0$ on $A_{z,R}$, and so $v' \equiv 0$ on (z, R) , which contradicts (29). In the second case, if $v' < 0$ on $(0, r_0)$ then $w \leq 0$ on $A_{0,R}$ and, as before, $w \equiv 0$ on $A_{0,R}$ by the strong maximum principle. The latter is again in contradiction with (29). Hence we are left with $v' > 0$ on $(0, r_0)$. In this case, the latter and the definition of r_0 imply $0 \leq u \leq v(r_0)$ on $B \setminus \{0\}$, so $u \in L^\infty(B)$ and then $u \in C^2(\overline{B})$ by standard elliptic estimates. To achieve a contradiction, we first observe that $-\Delta w + (N-1)\frac{w}{r^2} = f'(u)w$ on the annulus $A_{r,r_0} := \{x \in \mathbb{R}^N : r < |x| < r_0\}$, for every $r \in (0, r_0)$, we then multiply the equation by w , integrate by parts to get

$$(33) \quad \int_{A_{r,r_0}} |\nabla w|^2 - \int_{A_{r,r_0}} f'(u)w^2 + \int_{A_{r,r_0}} (N-1)\frac{w^2}{r^2} = \int_{\partial A_{r,r_0}} \frac{\partial w}{\partial \nu} w.$$

Note that $\int_{\partial A_{r,r_0}} \frac{\partial w}{\partial \nu} w \rightarrow 0$, as $r \rightarrow 0^+$. Indeed, the function

$$(34) \quad \tilde{w}(x) := \begin{cases} w(x) & \text{if } x \in B \setminus \{0\}, \\ 0 & \text{if } x = 0, \end{cases}$$

11. In view of (29) we could have used the moving planes procedure to get the strict monotonicity of u in the radial direction. However, we have chosen to give an elementary proof of this fact, which highlights the role played by the stability assumption on u .

is of class $C^{0,1}(B)^{12}$ and therefore $|\int_{\partial A_{r,r_0}} \frac{\partial w}{\partial \nu} w| \leq \int_{S_r} |\nabla \tilde{w}| |w| \leq \|\nabla \tilde{w}\|_{L^\infty(B)} \sigma(S_r) v'(r) \rightarrow 0$ (here S_r is the sphere $\{x \in \mathbb{R}^N : |x| = r\}$ and $\sigma(S_r)$ denotes its measure). Also observe that

$$\int_{A_{r,r_0}} |\nabla w|^2 - \int_{A_{r,r_0}} f'(u) w^2 + \int_{A_{r,r_0}} (N-1) \frac{w^2}{r^2} \rightarrow \int_{B(0,r_0)} |\nabla \tilde{w}|^2 - \int_{B(0,r_0)} f'(u) \tilde{w}^2 + \int_{B(0,r_0)} (N-1) \frac{\tilde{w}^2}{r^2}$$

by monotone and dominated convergence theorems. By gathering together all those information we are led to

$$(35) \quad \int_{B(0,r_0)} |\nabla \tilde{w}|^2 - \int_{B(0,r_0)} f'(u) \tilde{w}^2 = - \int_{B(0,r_0)} (N-1) \frac{\tilde{w}^2}{r^2}.$$

Finally, since u is stable on B , then u is stable on $B(0, r_0)$ too. Thanks to item i) of Proposition 15) we can use $w \in C_0^{0,1}(\overline{B(0, r_0)}) \subset H_0^1(B(0, r_0))$ as test function in the stability condition satisfied by u to get $\int_{B(0,r_0)} |\nabla \tilde{w}|^2 - \int_{B(0,r_0)} f'(u) \tilde{w}^2 \geq 0$. As before, the latter and (35) imply that $w \equiv 0$ on $B(0, r_0) \setminus \{0\}$, which is impossible since we are assuming that $v' > 0$ in $(0, r_0)$.

If $N \leq 9$ and $f \geq 0$, then u is C^2 in a neighborhood of the origin by item (1) of Theorem 7. This completes the proof of item (1).

2) By combining item (1) and item (2) of Theorem 7 we have that $u \in C^2(\overline{B})$. Then, either $u \equiv 0$ or $u > 0$ in B by the strong maximum principle. The desired conclusion then follows by a celebrated result of Gidas, Ni and Nirenberg.

3) Let H be any hyperplane through the origin and let $\rho = \rho(H) \in O(N)$ be the corresponding reflection with respect to H . Set $u_\rho(x) := u(\rho x)$, $x \in \Omega$. Then $u_\rho \in H_0^1(\Omega)$ is a stable solution to (12) since u is so. By Theorem 17, u and u_ρ are ordered solutions. If $u \not\equiv u_\rho$, an application of Theorem 3 would give $u, u_\rho \in C^\infty$ and, either $u < u_\rho$ or $u > u_\rho$ in Ω . The latter are clearly impossible, since $u = u_\rho$ on the hyperplane H . Thus, $u \equiv u_\rho$ and so u is radially symmetric in Ω , since H is arbitrary. Since u is a radially symmetric member of $H_0^1(\Omega)$, we have that $u \in C^0(\overline{\Omega})$ and standard elliptic theory imply $u \in C_{loc}^{2,\alpha}(\overline{\Omega})$, for any $\alpha \in (0, 1)$. The required regularity then follows, as in item (1), by analysing the ode satisfied by the radial profile of u . \square

3. SOME AUXILIARY RESULTS

In this section we prove some results for stable solutions in H^1 . These results have been used in the proofs of our main results and some of them are of independent interest. In what follows, when Ω is bounded, we shall denote by $\lambda_1 = \lambda_1(\Omega) > 0$ the principal eigenvalue of $-\Delta$ with homogeneous Dirichlet boundary conditions.

Lemma 14. *Let Ω be a bounded domain of \mathbb{R}^N , $N \geq 1$ and let $f \in C^1(\mathbb{R})$ such that $f(t) > \lambda_1 t$ for $t > 0$. Assume that $u \in H^1(\Omega)$ and $f(u) \in L_{loc}^1(\Omega)$.*

If u solves

$$(36) \quad \begin{cases} -\Delta u \geq f(u) & \text{in } \mathcal{D}'(\Omega) \\ u \geq 0 & \text{a.e. on } \Omega, \end{cases}$$

then $u \equiv 0$, $f(0) = 0$ and u is a solution to

$$(37) \quad -\Delta u = f(u) \quad \text{in } \mathcal{D}'(\Omega)$$

which is not stable.

12. Since $u \in C^2(\overline{B})$ is a radial function we have $\nabla u(0) = 0$, $w := u_r \in C^1(\overline{B} \setminus \{0\})$ and $|w(x)| \leq C|x|$ in \overline{B} , for some constant $C > 0$. Then $\tilde{w} \in C^0(\overline{B})$, $|v'(r)| \leq Cr$ in $[0, R]$ and so $|v''(r)| \leq C_1$ in $(0, R]$, for some constant $C_1 > 0$, by using the ode satisfied by v . Thus, for any $x, y \in B$ such that $|y| > |x| > 0$ we have : $|w(y) - w(x)| = |v'(|y|) - v'(|x|)| \leq \int_{|x|}^{|y|} |v''(t)| dt \leq \sup_{\xi \in [|x|, |y|]} |v''(\xi)| (|y| - |x|) \leq \sup_{r \in B \setminus \{0\}} |v''(r)| |y - x| \leq C_1 |y - x|$. Then, $\tilde{w} \in C^{0,1}(B)$.

Proof. By the strong maximum principle (for superharmonic functions in H^1) either $u \equiv 0$ or $u > 0$ a.e. on Ω . Let us prove that the latter is impossible. To this end, let us suppose that $u > 0$ and let $h \in H^1(\Omega)$ be the unique weak solution to

$$(38) \quad \begin{cases} -\Delta h = 0 & \text{in } \Omega, \\ u - h \in H_0^1(\Omega). \end{cases}$$

Then $0 \leq h \leq u$ a.e. on Ω by the maximum principle, and $h < u$ a.e. on Ω by the strong maximum principle. Indeed, $h = u$ would imply $0 = -\Delta(u - h) = f(u) > \lambda_1 u > 0$ a.e. on Ω , a contradiction.

Let $\phi_1 \in H_0^1(\Omega)$ be a positive function associated to the eigenvalue λ_1 . Since $f(u) \geq 0$ a.e. on Ω , a standard density argument and Fatou's Lemma imply

$$\int_{\Omega} \nabla u \nabla \varphi \geq \int_{\Omega} f(u) \varphi \quad \forall \varphi \in H_0^1(\Omega), \quad \varphi \geq 0 \text{ a.e. in } \Omega,$$

and therefore we can use ϕ_1 in the latter to get

$$\int_{\Omega} \nabla u \nabla \phi_1 \geq \int_{\Omega} f(u) \phi_1.$$

Then,

$$\int_{\Omega} \lambda_1 u \phi_1 < \int_{\Omega} f(u) \phi_1 \leq \int_{\Omega} \nabla u \nabla \phi_1 = \int_{\Omega} \nabla(u - h) \nabla \phi_1 + \int_{\Omega} \nabla h \nabla \phi_1 = \int_{\Omega} \nabla(u - h) \nabla \phi_1 + 0$$

where we have used the property of f , as well as the fact that h solves (38). In addition, by definition of ϕ_1 and $u - h \in H_0^1(\Omega)$, we also have $\int_{\Omega} \nabla(u - h) \nabla \phi_1 = \int_{\Omega} \lambda_1(u - h) \phi_1$, which leads to

$$\int_{\Omega} \lambda_1 u \phi_1 < \int_{\Omega} f(u) \phi_1 \leq \int_{\Omega} \nabla u \nabla \phi_1 = \int_{\Omega} \nabla(u - h) \nabla \phi_1 = \int_{\Omega} \lambda_1(u - h) \phi_1 \leq \int_{\Omega} \lambda_1 u \phi_1$$

a contradiction. Therefore, $u \equiv 0$ and $f(0) \leq -\Delta u = 0$. The latter and the assumption on f give $f(0) = 0$ and so u solves (37). Since $f'(0) > \lambda_1 = \inf \left\{ \frac{\int_{\Omega} |\nabla u|^2}{\int_{\Omega} u^2} : u \in H_0^1(\Omega), u \not\equiv 0 \right\}$ we immediately see that $u = 0$ cannot be a stable solution to (37). □

Proposition 15. Let Ω be an open set of \mathbb{R}^N , $N \geq 1$, and let $f \in C^1([0; +\infty))$ be a convex function.

i) Let $u \in L_{loc}^1(\Omega)$ such that $u \geq 0$ a.e. in Ω and

$$(39) \quad f'(u) \in L_{loc}^1(\Omega), \quad \int_{\Omega} f'(u) \phi^2 \leq \int_{\Omega} |\nabla \phi|^2 \quad \forall \phi \in C_c^\infty(\Omega).$$

Then

$$(40) \quad f'(u) \varphi^2 \in L^1(\Omega), \quad \int_{\Omega} f'(u) \varphi^2 \leq \int_{\Omega} |\nabla \varphi|^2 \quad \forall \varphi \in H_0^1(\Omega).$$

ii) Let $u, v \in H^1(\Omega)$ such that $f(u), f(v) \in L_{loc}^1(\Omega)$, $(u - v)^+ \in H_0^1(\Omega)$ and $0 \leq v$, $0 \leq u$ a.e. in Ω . Also assume that u satisfies (39). Then, $(f(u) - f(v))(u - v)^+ \in L^1(\Omega)$.

iii) Let $u, v \in L_{loc}^1(\Omega)$ such that $f(u), f(v) \in L_{loc}^1(\Omega)$ and $0 \leq v \leq u$ a.e. in Ω .

Then, $f'(v)(u - v) \in L_{loc}^1(\Omega)$.

Proof.

i) By convexity of f and (39) we have

$$(41) \quad f'(0) \leq f'(u) \quad \text{a.e. in } \Omega$$

and

$$(42) \quad \int_{\Omega} [f'(u)]^+ \phi^2 \leq \int_{\Omega} |\nabla \phi|^2 + \int_{\Omega} [f'(u)]^- \phi^2 \quad \forall \phi \in C_c^\infty(\Omega).$$

From (41) we deduce that $[f'(u)]^- \leq [f'(0)]^-$ and so $[f'(u)]^- \in L^\infty(\Omega)$ and $[f'(u)]^- \varphi^2 \in L^1(\Omega)$ for any $\varphi \in H_0^1(\Omega)$. Now, for any $\varphi \in H_0^1(\Omega)$, let (ϕ_n) be a sequence of functions in $C_c^\infty(\Omega)$ such that $\phi_n \rightarrow \varphi$ in $H_0^1(\Omega)$ and a.e. in Ω . Using $\phi = \phi_n$ in (42) and Fatou's Lemma we immediately get that (42) holds true for any $\varphi \in H_0^1(\Omega)$. In particular, $[f'(u)]^+ \varphi^2 \in L^1(\Omega)$ and the two claims of (40) follow.

ii) The convexity of f and $(u - v)^+ \geq 0$ a.e. on Ω imply

$$(43) \quad f'(0)[(u - v)^+]^2 \leq f'(v)[(u - v)^+]^2 \leq (f(u) - f(v))(u - v)^+ \leq f'(u)[(u - v)^+]^2 \quad \text{a.e. on } \Omega$$

and so

$$(44) \quad (f(u) - f(v))(u - v)^+ \in L^1(\Omega)$$

thanks to $(u - v)^+ \in H_0^1(\Omega)$ and item i).

iii) By convexity of f we have

$$f(u) - f(v) \geq f'(v)(u - v) \geq f'(0)(u - v)$$

which implies the desired conclusion. \square

Theorem 16. *Let Ω be a bounded domain of \mathbb{R}^N , $N \geq 1$ and let $f \in C^1([0; +\infty))$ be a convex function. Assume that $u, v \in H^1(\Omega)$ satisfy $u - v \in H_0^1(\Omega)$, $0 \leq v \leq u$ a.e. on Ω , $f(u), f(v) \in L_{loc}^1(\Omega)$ and both u and v are solution to*

$$(45) \quad -\Delta w = f(w) \quad \text{in } \mathcal{D}'(\Omega).$$

If $f'(u) \in L_{loc}^1(\Omega)$ and u is stable, then either $u \equiv v$ or $f(t) = a + \lambda_1 t$ for all $t \in (\inf_{\Omega} v, \sup_{\Omega} u)$ and some $a \in \mathbb{R}$, $u, v \in C^\infty(\Omega)$ and $u - v$ is a positive first eigenfunction of $-\Delta$ with homogeneous Dirichlet boundary conditions.

Proof.

By item ii) of Proposition 15 we have

$$(46) \quad (f(u) - f(v))(u - v) \in L^1(\Omega),$$

note that $u - v = (u - v)^+$ since $v \leq u$ a.e. in Ω .

We claim that

$$(47) \quad \int_{\Omega} |\nabla(u - v)|^2 = \int_{\Omega} (f(u) - f(v))(u - v).$$

To this end, recall that

$$(48) \quad \int_{\Omega} \nabla(u - v) \nabla \varphi = \int_{\Omega} (f(u) - f(v)) \varphi \quad \forall \varphi \in C_c^\infty(\Omega)$$

by assumption. Then, since $f(u) - f(v) \in L_{loc}^1(\Omega)$, a standard approximation argument and Lebesgue's dominated convergence theorem yield that (48) holds true for any $\varphi \in H^1(\Omega) \cap L^\infty(\Omega)$ with compact support. Now we prove that (48) holds true for any $\varphi \in H^1(\Omega)$ with compact support and such that $0 \leq \varphi \leq u - v$ a.e. on Ω . Indeed, we can apply (48) with $\varphi_n = T_n(\varphi)$, where $T_n = T_n(t)$ denotes the truncation function at level $n \geq 1$, and write

$$(49) \quad \int_{\Omega} \nabla(u - v) \nabla \varphi_n = \int_{\Omega} (f(u) - f(v)) \varphi_n, \quad \forall n \geq 1.$$

Since $\varphi_n \rightarrow \varphi$ in $H_0^1(\Omega)$ and a.e. on Ω , and (46) is in force, an application of Lebesgue's dominated convergence theorem we get

$$(50) \quad \int_{\Omega} \nabla(u - v) \nabla \varphi = \int_{\Omega} (f(u) - f(v)) \varphi \quad \forall \varphi \in H^1(\Omega), \quad \text{supp}(\varphi) \subset \subset \Omega, \quad 0 \leq \varphi \leq u - v \quad \text{a.e. in } \Omega.$$

To conclude the proof of (47) we proceed as follows. Let (ρ_n) be a sequence of nonnegative functions in $C_c^\infty(\Omega)$ such that $\rho_n \rightarrow u - v$ in $H_0^1(\Omega)$ and a.e. on Ω and set $w_n := \min\{u - v, \rho_n\}$. Then, $w_n \in H^1(\Omega)$, $\text{supp}(w_n) \subset \text{supp}(\rho_n) \subset \subset \Omega$ and $0 \leq w_n \leq u - v$ a.e. on Ω . We can therefore apply (50) with $\varphi = w_n$ to get the desired conclusion (47), since $w_n \rightarrow u - v$ in $H_0^1(\Omega)$ and a.e. on Ω , and (46) is in force.

13. Actually the real number a is unique and its value is given by $-\lambda_1 \frac{\int_{\Omega} \phi_1 h}{\int_{\Omega} \phi_1} \leq 0$, where ϕ_1 is a positive first eigenfunction of $-\Delta$ with homogeneous Dirichlet boundary conditions and $h \in H^1(\Omega)$ is the unique weak solution of $-\Delta h = 0$ in Ω with $u - h \in H_0^1(\Omega)$ (to see this, use ϕ_1 as test function in the weak formulation of $-\Delta u = a + \lambda_1 u$ and the fact that h is harmonic) and also note that h is nonnegative by the maximum principle. In particular, $u, v \in H_0^1(\Omega) \iff a = 0$. Also note that, for every $a \leq 0$ there exist solutions u, v for which the second alternative of the theorem occurs. Indeed, the functions $u_t := -\frac{a}{\lambda_1} + t\phi_1$, $t \geq 0$ are suitable.

By convexity of f we have $f'(0) \leq f'(v) \leq f'(u)$ a.e. on Ω , hence $f'(v) \in L^1_{loc}(\Omega)$ and $[f'(v)]^- \leq [f'(0)]^-$ a.e. in Ω . Also observe that $-\Delta(u-v) = f(u) - f(v) \geq f'(v)(u-v)$ in $\mathcal{D}'(\Omega)$ and so also $-\Delta(u-v) + [f'(v)]^-(u-v) \geq [f'(v)]^+(u-v) \geq 0$ in $\mathcal{D}'(\Omega)$, since $u-v \in H^1_0(\Omega)$ is nonnegative a.e. in Ω . The strong maximum principle (see e.g. Theorem 8.19 in [14]) yields either $u \equiv v$, and we are done, or $u > v$ a.e. on Ω . In the remaining part of the proof we assume that the latter possibility is in force and we set $I := \inf_{\Omega} v$, $S := \sup_{\Omega} u$.

Now we combine (47) with (40) with $\varphi = u - v$ to get

$$\int_{\Omega} (f(u) - f(v) - f'(u)(u-v))(u-v) \geq 0.$$

By convexity of f , the integrand in the above inequality is nonpositive and so

$$(f(u) - f(v) - f'(u)(u-v))(u-v) = 0 \quad \text{a.e. in } \Omega$$

and then

$$f(u) - f(v) - f'(u)(u-v) = 0 \quad \text{a.e. in } \Omega$$

since $u > v$ a.e. on Ω . The latter implies that, for almost every $x \in \Omega$, the function f must be affine on the open interval $(v(x), u(x))$. Now we prove that f is an affine function on the interval (I, S) . To this end we first consider the case in which one of the two solutions is a constant function. If v is constant, say $v \equiv c$, then $I = c$ and for almost every $x \in \Omega$ the function f is affine on the open interval $(I, u(x))$. Since any two intervals of this form intersect and f is affine on each of them, we see that f must be the same affine function on both intervals. This implies that f is an affine function over the entire interval (I, S) , as claimed. The same argument applies if u is constant. It remains to consider the case where neither v nor u are constant. Since v is not constant we can find $x_0 \in \Omega$ such that

$$(51) \quad \begin{cases} v(x_0) < u(x_0) & \text{and } f(t) = a + bt \text{ for all } t \in (v(x_0), u(x_0)), \\ |\{v > v(x_0)\}| > 0, \\ |\{v \leq v(x_0)\}| > 0, \end{cases}$$

where we denoted by $|X|$ the Lebesgue measure of any measurable set $X \subset \Omega$.

Let (α, β) be the largest open interval containing $(v(x_0), u(x_0))$ and such that $f(t) = a + bt$ for all $t \in (\alpha, \beta)$. To conclude the proof it is enough to show that $(I, S) \subset (\alpha, \beta)$. Let us first prove that $I \geq \alpha$. Assume to the contrary that $I < \alpha$, then the latter and (51) give $I < \alpha \leq v(x_0) < \sup_{\Omega} v$ and so we can find an integer $m \geq 1$ such that

$$(52) \quad I \leq \inf_{\omega_m} v < \alpha < \sup_{\omega_m} v$$

where $\{\omega_m, m \geq 1\}$ is a countable family of open connected sets of \mathbb{R}^N such $\omega_m \subset \omega_{m+1} \subset \subset \Omega$ and $\Omega = \bigcup_{m \geq 1} \omega_m$ (such a family of open connected sets exists thanks to the fact that Ω is open, bounded and connected). Recall that $u - v \in H^1_0(\Omega)$ is positive a.e. on Ω and satisfies $-\Delta(u-v) + [f'(v)]^-(u-v) \geq [f'(v)]^+(u-v) \geq 0$ in $\mathcal{D}'(\Omega)$, where $[f'(v)]^- \in L^\infty(\Omega)$, then the strong maximum principle yields

$$(53) \quad u - v \geq c(m) > 0 \quad \text{a.e. on } \omega_m.$$

Now, if we set $\epsilon := \min\{\frac{c(m)}{2}, \alpha - \inf_{\omega_m} v\} > 0$, we have $I \leq \inf_{\omega_m} v < \alpha - \epsilon < \alpha < \sup_{\omega_m} v$ and so

$$(54) \quad \begin{cases} |\{v \leq \alpha - \epsilon\} \cap \omega_m| > 0, \\ |\{v \geq \alpha - \frac{\epsilon}{2}\} \cap \omega_m| > 0. \end{cases}$$

Since ω_m is open and connected we can use the "intermediate value theorem" for functions in a Sobolev space (see e.g. Théorème 1 of [8]) to get $|\{\alpha - \epsilon \leq v \leq \alpha - \frac{\epsilon}{2}\} \cap \omega_m| > 0$. This fact and (53) prove the existence of $x_\epsilon \in \omega_m$ such that $\alpha - \epsilon \leq v(x_\epsilon) \leq \alpha - \frac{\epsilon}{2}$ and $u(x_\epsilon) - v(x_\epsilon) \geq c(m)$, whose combination leads to $u(x_\epsilon) \geq \alpha + \frac{c(m)}{2}$ and $v(x_\epsilon) < \alpha$. Therefore, f must satisfy $f(t) = a + bt$ for all $t \in (v(x_\epsilon), \beta)$, which contradicts the definition of (α, β) . This proves that $I \geq \alpha$. Now we show that $S \leq \beta$. If $\beta = +\infty$ we are done, so we suppose that $\beta \in \mathbb{R}$. In this case we observe that $\inf_{\Omega} u < \beta$ since otherwise we would have $u \geq \beta$ a.e. on Ω and so the strong maximum principle (applied to $u - \beta$) would give $u > \beta$ a.e. on Ω (recall that v is not constant by assumption). Hence there would exist $\bar{x} \in \Omega$ such that $v(\bar{x}) < \beta < u(\bar{x})$. This would imply that $f(t) = a + bt$ for all $t \in (\alpha, u(\bar{x}))$, contradicting the definition of (α, β) . If $S > \beta$

(and since $\inf_{\Omega} u < \beta$) the same argument used to prove $\alpha \leq I$ can be applied to the solution u to get a contradiction (again that f would satisfy $f(t) = a + bt$ for all $t \in (\alpha, u(y))$, for some $y \in \Omega$ such that $u(y) > \beta$). Therefore $S \leq \beta$ and $f(t) = a + bt$ for all $t \in (I, S)$. Hence, u and v are of class $C^2(\Omega)$ by standard elliptic estimates. Finally, $u - v \in H_0^1(\Omega)$ solves $-\Delta(u - v) = f(u) - f(v) = b(u - v)$ in Ω and $u - v > 0$ in Ω . Therefore $u - v$ is a positive eigenfunction and so we necessarily have $b = \lambda_1$. This concludes the proof. \square

Theorem 17. *Let Ω be a bounded domain of \mathbb{R}^N , $N \geq 1$ and let $f \in C^1([0; +\infty))$ be a convex function. Assume that $u, v \in H^1(\Omega)$ satisfy $u - v \in H_0^1(\Omega)$, $f(u), f(v) \in L_{loc}^1(\Omega)$ and both u and v are solution to*

$$(55) \quad \begin{cases} -\Delta w = f(w) & \text{in } \mathcal{D}'(\Omega) \\ w \geq 0 & \text{a.e. on } \Omega, \end{cases}$$

If u satisfies $f'(u) \in L_{loc}^1(\Omega)$ and u is stable, then u and v are ordered, namely, one of the following three cases holds true : $u < v$ a.e. in Ω , $u \equiv v$ a.e. in Ω or $u > v$ a.e. in Ω .

Proof. By item ii) of Proposition 15 we have

$$(56) \quad (f(u) - f(v))(u - v)^+ \in L^1(\Omega)$$

In view of the above results we can follow the proof of Theorem 3 to obtain

$$(57) \quad \int_{\Omega} |\nabla(u - v)^+|^2 = \int_{\Omega} (f(u) - f(v))(u - v)^+.$$

Using (40) with $\varphi = (u - v)^+$, (57) and (43) we have

$$\int_{\Omega} f'(u)[(u - v)^+]^2 \leq \int_{\Omega} |\nabla(u - v)^+|^2 = \int_{\Omega} (f(u) - f(v))(u - v)^+ \leq \int_{\Omega} f'(u)[(u - v)^+]^2$$

and so

$$\int_{\Omega} |\nabla(u - v)^+|^2 - \int_{\Omega} f'(u)[(u - v)^+]^2 = 0.$$

The latter and (40) imply that $(u - v)^+$ minimizes the functional $\psi \rightarrow \int_{\Omega} |\nabla \psi|^2 - \int_{\Omega} f'(u)\psi^2$ over $H_0^1(\Omega)$ and therefore $(u - v)^+$ solves

$$(58) \quad -\Delta(u - v)^+ = f'(u)(u - v)^+ \quad \text{in } \mathcal{D}'(\Omega).$$

Then

$$(59) \quad -\Delta(u - v)^+ + [f'(u)]^-(u - v)^+ = [f'(u)]^+(u - v)^+ \geq 0 \quad \text{in } \mathcal{D}'(\Omega)$$

with $[f'(u)]^- \in L^\infty(\Omega)$, since $f'(u) \geq f'(0)$ by convexity of f . By the strong maximum principle, either $(u - v)^+ > 0$ a.e. in Ω or $(u - v)^+ = 0$ a.e. in Ω . That is, either $u > v$ a.e. in Ω or $u \leq v$ a.e. in Ω . In the latter case we have $-\Delta(v - u) = f(v) - f(u) \geq f'(u)(v - u)$ in $\mathcal{D}'(\Omega)$ and so also $-\Delta(v - u) + [f'(u)]^-(v - u) \geq [f'(u)]^+(v - u) \geq 0$ in $\mathcal{D}'(\Omega)$, since $v - u \in H_0^1(\Omega)$ is nonnegative a.e. in Ω . As above, another application of the strong maximum principle yields either $u \equiv v$ or $v > u$ a.e. on Ω . \square

By combining Lemma 14 and Theorem 3 we immediately obtain the following classification result for stable solutions in $H_0^1(\Omega)$.

Theorem 18. *Let Ω be a bounded domain of \mathbb{R}^N , $N \geq 1$ and let $f \in C^1(\mathbb{R})$ be a convex function such that $f(0) = 0$. Assume that $u \in H_0^1(\Omega)$, $f(u) \in L_{loc}^1(\Omega)$ and that u is a stable solution to*

$$(60) \quad \begin{cases} -\Delta u = f(u) & \text{in } \mathcal{D}'(\Omega) \\ u \geq 0 & \text{a.e. on } \Omega. \end{cases}$$

Then, either $u \equiv 0$ or $f(t) = \lambda_1 t$ on $(0, \sup_{\Omega} u)$ and $u \in C^\infty(\Omega) \cap H_0^1(\Omega)$ is a positive first eigenfunction of $-\Delta$ with homogeneous Dirichlet boundary conditions.

Remark 19. If $u \equiv 0$, then necessarily $f'(0) \leq \lambda_1$ by Lemma 14. Also observe that for any $\alpha \leq \lambda_1$ there is a convex function f satisfying $f(0) = 0$, $f'(0) = \alpha$ and such that $u \equiv 0$ is a stable solution to (3). An example is provided by $f(u) = u^2 + \alpha u$.

Proof. $v \equiv 0$ is a solution to (3) since $f(0) = 0$. Then, an application of Theorem 3 provides the desired results. Indeed, since $u \in H_0^1(\Omega)$, we have $a = 0$ (as observed in the footnote to Theorem 3). Therefore $f(t) = \lambda_1 t$ on $(\inf_\Omega v, \sup_\Omega u) = (0, \sup_\Omega u)$. \square

Proposition 20. *Let Ω be a bounded domain of \mathbb{R}^N , $N \geq 1$ and let $f \in C^1([0; +\infty))$ be a convex function. Assume that $u, v \in H_{loc}^1(\Omega)$ satisfy $0 \leq v < u$ a.e. on Ω , $f(u), f(v) \in L_{loc}^1(\Omega)$ and*

$$(61) \quad \begin{cases} -\Delta u \geq f(u) & \text{in } \mathcal{D}'(\Omega), \\ -\Delta v \leq f(v) & \text{in } \mathcal{D}'(\Omega). \end{cases}$$

Then

$$(62) \quad f'(v) \in L_{loc}^1(\Omega), \quad \int_\Omega f'(v) \varphi^2 \leq \int_\Omega |\nabla \varphi|^2 \quad \forall \varphi \in C_c^\infty(\Omega).$$

In particular, if v is a solution to $-\Delta v = f(v)$ in $\mathcal{D}'(\Omega)$, then v is stable.

Proof. Recall that, by convexity of f , we have $f(u) - f(v) \geq f'(v)(u - v)$ and $[f'(v)]^- \leq [f'(0)]^-$ a.e. in Ω . Therefore, $[f'(v)]^- \in L^\infty(\Omega)$ and, using (61), we obtain

$$-\Delta(u - v) + [f'(v)]^-(u - v) \geq f(u) - f(v) + [f'(v)]^-(u - v) \geq [f'(v)]^+(u - v) \geq 0 \quad \text{in } \mathcal{D}'(\Omega).$$

By the strong maximum principle and $u - v > 0$ a.e. in Ω we then get

$$(63) \quad \forall \omega \subset\subset \Omega \quad u - v \geq c(\omega) > 0 \quad \text{a.e. on } \omega,$$

where $c(\omega)$ is a positive constant depending on the open subset ω . The latter implies that $\frac{1}{u-v} \in L_{loc}^\infty(\Omega)$ and so $f'(v) \in L_{loc}^1(\Omega)$, thanks to item iii) of Proposition 15. This proves the first claim of (62). To prove the second one we recall that $f'(v)(u - v) \in L_{loc}^1(\Omega)$ and use once again (61) to get

$$(64) \quad \int_\Omega \nabla(u - v) \nabla \phi \geq \int_\Omega f'(v)(u - v) \phi, \quad \forall \phi \in C_c^\infty(\Omega).$$

As before, a standard approximation argument and Lebesgue's dominated convergence theorem yield that (64) holds true for any $\phi \in H^1(\Omega) \cap L^\infty(\Omega)$ with compact support. Therefore, for every $\varphi \in C_c^\infty(\Omega)$ and recalling (63), we can then take $\phi = \frac{\varphi^2}{u-v}$ in (64) and find

$$(65) \quad \int_\Omega \nabla(u - v) \nabla \left(\frac{\varphi^2}{u-v} \right) \geq \int_\Omega f'(v) \varphi^2, \quad \forall \varphi \in C_c^\infty(\Omega).$$

Hence,

$$\int_\Omega f'(v) \varphi^2 \leq \int_\Omega 2 \frac{\varphi \nabla(u - v)}{u - v} \nabla \varphi - \int_\Omega \frac{\varphi^2}{(u - v)^2} |\nabla(u - v)|^2, \quad \forall \varphi \in C_c^\infty(\Omega).$$

The second conclusion of (62) then follows by applying Young's inequality to the first integral on the r-h-s of the latter inequality. The last claim is a consequence of (62). \square

Proof of Theorem 9.

(1) Since u has finite Morse index, there exists a neighborhood of the boundary of the form $\Omega_\epsilon = \{x \in \Omega : \text{dist}(x, \partial\Omega) < \epsilon\}$ such that u is stable in Ω_ϵ (see point (5) below Definition 1). Thanks to Theorem 5, it suffices to prove (9) in the case where $u \in C^{2,\alpha}(\overline{\Omega}_\epsilon)$. Also, the estimate will follow if we prove that for some $\rho \in (0, \epsilon)$ and for every $x \in \Omega_\rho$, there exists a set I_x such that $|I_x| \geq \gamma$ and $u(x) \leq u(y)$, for all $y \in I_x$.

To this end, we apply the moving-plane method. For $y \in \partial\Omega$, let $n(y)$ denote the unit normal vector to $\partial\Omega$, pointing outwards. Thanks to Lemmas 4.1 and 4.2 in [2], there exists a constant $\lambda_0 \in (0, \epsilon/2)$ depending on Ω only, such that

$$\{x = y - tn(y) : 0 < t < 2\lambda_0, y \in \partial\Omega\} \subset \Omega$$

In addition, in a fixed neighborhood of $\partial\Omega$, every point can be written in the form $x = y - tn(y)$, where $0 < t < \lambda_0$ and y is the unique projection of x on $\partial\Omega$. Fix $x_0 \in \partial\Omega$ and $n = n(x_0)$. By applying the standard moving-plane method in the cap $\Sigma_\lambda := \{x \in \Omega : 0 < -(x - x_0) \cdot n < \lambda\}$, we deduce that

$$(66) \quad \partial_n u < 0 \quad \text{in } \Sigma_\lambda, \quad \text{for every } \lambda \in [0, \lambda_0].$$

Next, since Ω is uniformly convex, there exists a radius $r > 0$ depending on Ω only, such that the geodesic ball $B = B(n(x_0), r) \subset \mathbb{S}^{N-1}$ can be realized as the set of normals at nearby points and so $B \subset n(\partial\Omega)$. To see this, assume without loss of generality that $x_0 = 0$ and that $\partial\Omega$ coincides near x_0 with the graph of some C^2 function $\varphi : \mathbb{R}^{N-1} \rightarrow \mathbb{R}$ such that $\varphi(0) = 0$, $\nabla\varphi(0) = 0$ and $\nabla^2\varphi(0)$ is a diagonal matrix with eigenvalues bounded below by a positive constant (i.e. the directions of principal curvature of $\partial\Omega$ at x_0 coincide with the canonical basis of \mathbb{R}^{N-1}). Then, $n(x_0) = (0, \dots, 0, 1)$ and for $t = (t_1, 0, \dots, 0) \in \mathbb{R}^{N-1}$ small, there holds

$$n(\varphi(t)) = \frac{(-\nabla\varphi(t), 1)}{\sqrt{1 + |\nabla\varphi(t)|^2}} = n(x_0) - t_1 \left(\frac{\partial^2\varphi}{\partial t_1^2}(0), 0, \dots, 0 \right) + o(|t_1|)$$

and so n describes an arc of circle in the x_1 direction as t_1 varies in some small interval $(-r_1, r_1)$. This is also true (uniformly, since Ω is uniformly convex) in any direction $e \in \mathbb{R}^{N-1}$ and so a small geodesic ball $B = B(n(x_0), r) \subset \mathbb{S}^{N-1}$ can indeed be realized as the set of normals at nearby points. This in turn implies that for all $\theta \in B$,

$$\partial_\theta u < 0 \quad \text{in } \Sigma := \left\{ x \in \Omega : \frac{1}{4}\lambda_0 < -(x - x_0) \cdot n(x_0) < \frac{3}{4}\lambda_0 \right\}.$$

Indeed, applying the moving-plane procedure at every point $y \in \partial\Omega$ such that $\theta = n(y)$, $\theta \in B$, we have

$$\partial_\theta u < 0 \quad \text{in } \{x \in \Omega : 0 < -(x - y) \cdot \theta < \lambda_0\}.$$

By taking a smaller ball B if necessary, we may assume that

$$|(x - x_0) \cdot (\theta - n(x_0)) + (x_0 - y) \cdot \theta| < \frac{1}{4}\lambda_0, \quad \text{for all } x \in \Sigma \text{ and } \theta = n(y) \in B.$$

Now, since $-(x - y) \cdot \theta = -(x - x_0) \cdot n(x_0) - (x - x_0) \cdot (\theta - n(x_0)) - (x_0 - y) \cdot \theta$, we have for any $x \in \Sigma$,

$$\lambda_0 = \frac{1}{4}\lambda_0 + \frac{3}{4}\lambda_0 > -(x - y) \cdot \theta > \frac{1}{4}\lambda_0 - \frac{1}{4}\lambda_0 = 0$$

and so, as claimed, for any $x \in \Sigma$, there holds

$$\partial_\theta u(x) < 0.$$

Now take $\rho = \lambda_0/8$. Fix a point $x \in \Omega_\rho = \{x \in \Omega : \text{dist}(x, \partial\Omega) < \rho\}$ and let x_0 denote its projection on $\partial\Omega$. On the one hand, $u(x) \leq u(x_1)$, where $x_1 = x_0 - \rho n(x_0)$. On the other hand, $u(x_1) \leq u(z)$, for all z in the cone $I_x \subset \Sigma$ having vertex at x_1 , opening angle B , and height $\lambda_0/2$ and the proof is complete.

(2) Fix $\epsilon > 2\lambda_0 > 0$ as above. According to Theorem 5, there exists a sequence of functions $u_k \in C^{2,\alpha}(\Omega_\epsilon)$ which are stable solutions of a semilinear elliptic equation in Ω_ϵ and converge a.e. to u in Ω_ϵ . If $N \leq 9$, we may apply the interior estimate Theorem A to deduce that

$$\|u_k\|_{L^\infty(\Omega_{2\lambda_0} \setminus \overline{\Omega_{\lambda_0}})} \leq M\|u_k\|_{L^1(\Omega_\epsilon)} \leq M\|u\|_{L^1(\Omega)},$$

for some constant M depending on Ω . Furthermore, if Ω is convex, we know that u_k is monotone in the normal direction i.e. (66) holds for $u = u_k$, and so the inequality remains true all of $\Omega_{2\lambda_0}$. Passing to the limit $k \rightarrow +\infty$ and using standard elliptic regularity, we deduce that $u \in C^{2,\alpha}(\overline{\Omega_{\lambda_0}})$ in this case. In the case where f is nondecreasing on Ω is C^3 , we can directly apply Theorem 5 combined to Theorem 1.5 in [5].

(3) We write a generic point in \mathbb{R}^N as $(x, y) \in \mathbb{R}^{N-1} \times \mathbb{R}$. Let B' be the unit ball in \mathbb{R}^{N-1} , $N-1 \geq 10$ and $\Omega \subset \mathbb{R}^N$ the open set obtained by gluing the cylinder $B' \times (-1, 1)$ to the unit half-ball centered at $(x, y) = (0, -1)$ and to the unit half-ball centered at $(x, y) = (0, 1)$. Let $\lambda_n : [0, 2] \rightarrow \mathbb{R}_+$ be a C^2 increasing concave function such that $\lambda_n(y) = ny$ for $y \in [0, 1]$ and $\lambda_n(2) = (n + \rho_n)$, where $\rho_n \rightarrow 0$. Extend λ_n as an odd function on $[-2, 2]$. Then, the domain $\Omega^n = \{(x, \lambda_n(y)) : (x, y) \in \Omega\}$ is convex but clearly not uniformly, nor even strictly. We let u_n be the minimal solution to (1) with nonlinearity $f(u) = 2(N-3)e^u$ and domain Ω^n . Since $\underline{u} = 0$ and $\bar{u} = -2\ln(|x|)$ are ordered sub and supersolution to the problem, u_n is well-defined, stable and

$$0 < u_n < -2\ln(|x|) \quad \text{in } \Omega^n.$$

It already readily follows that the average $\frac{1}{|\Omega^n|} \|u_n\|_{L^1(\Omega^n)}$ of u_n remains bounded. In addition, since $-2\ln(|x|)$ is a strict supersolution of the equation, u_n cannot be an extremal solution and so u_n is smooth and strictly stable, i.e. its linearized operator has positive first eigenvalue.

Recall that $\rho_n \rightarrow 0$ and assume by contradiction that $\|u_n\|_{L^\infty((\Omega^n)_{\rho_n})} \leq M$ for some constant $M > 0$. For $(x, y) \in \Omega$, let $v_n(x, y) = u_n(x, \lambda_n(y))$. Then, $v_n = 0$ on $\partial\Omega$ and, letting μ_n denote the inverse function of λ_n ,

$$(67) \quad -(\Delta_x + (\mu'_n)^2 \partial_y^2 + \mu''_n \partial_y) v_n = f(v_n) \quad \text{in } \Omega.$$

We claim that

$$\|\nabla v_n\|_{L^\infty(\Omega)} \leq K,$$

for some constant $K > 0$. To see this, we begin by estimating $|\nabla v_n|$ on $\partial\Omega$. On the flat part of the boundary, we have a natural barrier : since $v_n < -2\ln(|x|)$ in Ω and $v_n = -2\ln(|x|) = 0$ on $\partial B' \times (-1, 1)$, we deduce that $\|\nabla v_n\|_{L^\infty(\partial B' \times (-1, 1))} \leq 2$. The function $\zeta(x, y) = 1 - |(x, y) - (0, 1)|^2$ vanishes on the boundary of the half-ball centered at $(0, 1)$ and satisfies

$$-(\Delta_x + (\mu'_n)^2 \partial_y^2 + \mu''_n \partial_y) \zeta = 2(N-1) + 2(\mu'_n)^2 - \mu''_n \partial_y \zeta \geq 2(N-1) \quad \text{for } y \geq 1.$$

Hence, a constant multiple of ζ can be used as a barrier on the half-ball centered at $(0, 1)$. Working similarly with the other half-ball, we deduce that $\|\nabla v_n\|_{L^\infty(\partial\Omega)} \leq K$ on the whole boundary of Ω . To extend the inequality to the whole of Ω , we observe that any partial derivative $\partial_i v_n$ solves the linearized equation. Since v_n is strictly stable (because this is the case for u_n), the linearized operator at v_n has positive first eigenvalue. It follows that

$$\|\nabla v_n\|_{L^\infty(\Omega)} \leq \|\nabla v_n\|_{L^\infty(\partial\Omega)} \leq K$$

as claimed. Up to extraction, the sequence (v_n) converges uniformly to some lipschitz-continuous function v in $\overline{\Omega}$. In addition, $v = 0$ on $\partial\Omega$, and for any $\varphi \in C_c^\infty(B' \times (-1, 1))$, $\int_\Omega f(v_n) \varphi \, dx \rightarrow \int_\Omega f(v) \varphi \, dx$ while

$$\int_\Omega v_n (-(\Delta_x + (\mu'_n)^2 \partial_y^2 + \mu''_n \partial_y)) \varphi \, dx = \int_\Omega v_n (-\Delta_x + \frac{1}{n^2} \partial_y^2) \varphi \, dx \rightarrow \int_\Omega v (-\Delta_x \varphi) \, dx.$$

In particular, the function $w(x) = v(x, 0) \in H_0^1(B')$ is a weak stable solution to

$$-\Delta w = f(w) \quad \text{in } B'.$$

But so is $\bar{u} = -2\ln(|x|)$. By uniqueness of the extremal solution, we must have $w = \bar{u}$, which is impossible, since w is bounded. Hence, up to extraction, $\|u_n\|_{L^\infty((\Omega^n)_{\rho_n})} \rightarrow +\infty$.

APPENDIX A

Proposition 21. *Assume that $\alpha \in (0, 1)$ and $N \geq 2$. Let Ω be a bounded domain of \mathbb{R}^N and let $f \in C^1([0, +\infty))$ be a nondecreasing and convex function. Assume that $u \in H^1(\Omega)$ is a stable¹⁴ solution of (1) such that $u \geq 0$ a.e. in Ω .*

1) There exists a sequence (ε_n) of real numbers in $[0, 1]$ such that $\varepsilon_n \searrow 0$ and a sequence (u_n) of functions in $H^1(\Omega) \cap C^2(\Omega)$ such that u_n is a stable weak solution¹⁵ to

$$(68) \quad \begin{cases} -\Delta u_n = (1 - \varepsilon_n) f(u_n) & \text{in } \Omega, \\ u_n - u \in H_0^1(\Omega), \\ 0 \leq u_n \leq u & \text{a.e. on } \Omega, \end{cases}$$

and

$$(69) \quad u_n \longrightarrow u \quad \text{in } H^1(\Omega), \quad u_n \longrightarrow u \quad \text{a.e. on } \Omega.$$

Also, if we assume in addition that Ω is of class C^1 , T is a $C^{2,\alpha}$ open portion of $\partial\Omega$ and $u|_T = 0$ (in the sense of the traces), then $u_n \in C^{2,\alpha}(\Omega')$ for any domain $\Omega' \subset \subset \Omega \cup T$ (sufficiently small) and $u_n = 0$ in T' .

14. Note that $f'(u)$ is a nonnegative Lebesgue measurable function (by our assumptions on f) and so the stability inequality (2) has a meaning.

15. That is a function u_n satisfying $\int_\Omega \nabla u_n \nabla \varphi = \int_\Omega (1 - \varepsilon_n) f(u_n) \varphi$, for all $\varphi \in H_0^1(\Omega)$.

2) Assume in addition that $u \in H_0^1(\Omega)$. Then, the sequence (u_n) can be chosen in $H_0^1(\Omega) \cap C^2(\Omega)$. If we assume in addition that Ω is of class C^1 , T is a $C^{2,\alpha}$ open portion of $\partial\Omega$ and $u|_T = 0$ (in the sense of the traces), then $u_n \in C_0^2(\Omega')$ for any domain $\Omega' \subset \subset \Omega \cup T$ (sufficiently small). In particular, if Ω is of class $C^{2,\alpha}$, then $u_n \in C_0^2(\Omega)$.

Proof.

i) We argue as in [3] and in subsection 3.2.2 of [10]. Nevertheless, our approach requires several non standard modifications due to the fact that we work in \mathcal{D}' and that u is merely in $H^1(B)$ (i.e., u does not have "zero boundary value").

Given $\epsilon \in (0, 1)$, define $\Phi_\epsilon : [0, +\infty) \rightarrow [0, +\infty)$ by

$$\int_0^{\Phi_\epsilon(t)} \frac{ds}{f(s)} = (1 - \epsilon) \int_0^t \frac{ds}{f(s)}$$

Since Φ_ϵ solves the initial value problem

$$(70) \quad \begin{cases} \Phi'_\epsilon(t)f(t) = (1 - \epsilon)f(\Phi_\epsilon(t)), & t > 0 \\ \Phi_\epsilon(0) = 0, \end{cases}$$

we see that $\Phi_\epsilon \in C^2([0, +\infty))$ is increasing, concave and satisfies $0 < \Phi'_\epsilon(t) < 1$, $0 \leq \Phi_\epsilon(t) \leq t$ for all $t \geq 0$, $\Phi'_\epsilon(0) = 1 - \epsilon \in (0, 1)$, $\Phi''_\epsilon(0) = -\epsilon(1 - \epsilon)\frac{f'(0)}{f(0)} \leq 0$. Also, using the concavity on $[0, +\infty)$ of the function $h(t) := \int_0^t \frac{ds}{f(s)}$ we get that

$$(71) \quad 0 \leq f(\Phi_\epsilon(t)) \leq \frac{C(f)}{\epsilon}(1 + t) \quad \forall t \geq 0,$$

where $C = C(f) > 0$ is a constant depending only on f .

Since $u \in H^1(\Omega)$ and (71) is in force, we have $U_\epsilon = \Phi_\epsilon(u) \in H^1(\Omega)$ and so U_ϵ is a weak supersolution¹⁶ to

$$(72) \quad \begin{cases} -\Delta u_{\epsilon,1} = (1 - \epsilon)f(u_{\epsilon,1}) & \text{in } \Omega, \\ u_{\epsilon,1} - \Phi_\epsilon(u) \in H_0^1(\Omega), \end{cases}$$

while $v = 0$ is a weak subsolution to (72). In addition, we have $0 < U_\epsilon$ a.e. on Ω , since $f > 0$. Therefore by the (standard) method of sub and supersolution in H^1 we obtain a stable weak solution¹⁷ $u_{\epsilon,1} \in H^1(\Omega)$ of (72) such that $0 < u_{\epsilon,1} \leq U_\epsilon$ a.e. on Ω . Furthermore, from (71) we get

$$(73) \quad 0 \leq f(u_{\epsilon,1}) \leq f(U_\epsilon) \leq \frac{C(f)}{\epsilon}(1 + u)$$

and so $f(u_{\epsilon,1}) \in L^2(\Omega)$. The latter implies $u_{\epsilon,1} \in H_{loc}^2(\Omega)$ by elliptic regularity, hence

$$(74) \quad u_{\epsilon,1} \in L_{loc}^p(\Omega) \quad \forall p < \frac{2N}{N-4} \quad (p \leq \infty \text{ if } N \leq 3, \quad p < \infty \text{ if } N = 4)$$

by Sobolev imbedding.

In what follows, for any integer $j \geq 0$, we shall denote by Φ_ϵ^j the composition of Φ_ϵ with itself j times ($\Phi_\epsilon^0 = Id$).

16. That is it satisfies $\int_\Omega \nabla U_\epsilon \nabla \varphi \geq \int_\Omega (1 - \epsilon)f(U_\epsilon)\varphi$, for all $\varphi \in H_0^1(\Omega)$, $\varphi \geq 0$. Indeed, in view of the above properties of Φ_ϵ we can extend it to a C^2 function on the entire real line \mathbb{R} (still denoted by Φ_ϵ) such that Φ_ϵ is nondecreasing and concave, Φ'_ϵ is nonnegative and bounded on \mathbb{R} . Then we can apply a variant of Kato's inequality (see e.g. Lemma 3.2.1 in [10]) to get that U_ϵ is a supersolution in $\mathcal{D}'(\Omega)$. A standard density argument and Fatou's Lemma then give the desired conclusion.

17. This solution is obtained by using the standard method of monotone iterations in H^1 applied to the sequence $(v_k)_{k \geq 1}$ defined by $-\Delta v_{k+1} = (1 - \epsilon)f(v_k)$ in Ω , $v_{k+1} \in \{v \in H^1(\Omega) : v - \Phi_\epsilon(u) \in H_0^1\} := H_{\Phi_\epsilon(u)}^1$ and starting with $v_1 = \Phi_\epsilon(u) \in H_{\Phi_\epsilon(u)}^1$, the supersolution. Note that the sequence is well-defined in $H_{\Phi_\epsilon(u)}^1$ and satisfies $0 \leq v_{k+1} \leq v_k \leq \Phi_\epsilon(u)$ a.e. on Ω thanks to $f' \geq 0$ and since $f(\Phi_\epsilon(u)) \in L^2(B)$ by (71). Furthermore, the stability of $u_{\epsilon,1}$ comes from the stability of u and the fact that f' is positive and nondecreasing. Indeed, $\forall \varphi \in C_c^1(\Omega)$ we have $\int_\Omega |\nabla \varphi|^2 \geq \int_\Omega f'(u)\varphi^2 \geq \int_\Omega (f'(U_\epsilon)\varphi^2) \geq \int_\Omega (f'(u_{\epsilon,1})\varphi^2) \geq \int_\Omega ((1 - \epsilon)f'(u_{\epsilon,1})\varphi^2)$.

Now we can repeat the same construction to find a stable weak solution $u_{\epsilon,2} \in H^1(\Omega)$ to

$$(75) \quad \begin{cases} -\Delta u_{\epsilon,2} = (1-\epsilon)^2 f(u_{\epsilon,2}) & \text{in } \Omega, \\ u_{\epsilon,2} - \Phi_\epsilon^2(u) \in H_0^1(\Omega), \end{cases}$$

such that $0 < u_{\epsilon,2} \leq \Phi_\epsilon(u_{\epsilon,1}) \leq u_{\epsilon,1} \leq u$ a.e. on Ω . Here we have used $\Phi_\epsilon(u_{\epsilon,1}) \in H^1(\Omega)$ as supersolution and again $v = 0$ as subsolution. Also note that $0 \leq \Phi_\epsilon^2(u) \leq \Phi_\epsilon(u_{\epsilon,1})$ on $\partial\Omega$ in the sense of $H^1(\Omega)$, since $[\Phi_\epsilon^2(u) - \Phi_\epsilon(u_{\epsilon,1})]^+ \in H_0^1(\Omega)$. In particular, by (71), $0 \leq f(u_{\epsilon,2}) \leq \frac{C(f)}{\epsilon}(1+u_{\epsilon,1}) \in L_{loc}^p(\Omega)$ for any p in the range (74), and thus $u_{\epsilon,2} \in L_{loc}^q(\Omega)$ for all $q < \frac{2N}{N-8}$ ($q \leq \infty$ if $N \leq 7$, $q < \infty$ if $N = 8$). Also note that $0 < u_{\epsilon,2} \leq \Phi_\epsilon(u_{\epsilon,1}) \leq \Phi_\epsilon(\Phi_\epsilon(u)) = \Phi_\epsilon^2(u) \leq \Phi_\epsilon(u)$ a.e. on Ω .

By iteration, we find that if $k = \left\lfloor \frac{N}{4} \right\rfloor + 1$, the solution $u_{\epsilon,k} \in H^1(\Omega)$ to

$$(76) \quad \begin{cases} -\Delta u_{\epsilon,k} = (1-\epsilon)^k f(u_{\epsilon,k}) & \text{in } \Omega, \\ u_{\epsilon,k} - \Phi_\epsilon^k(u) \in H_0^1(\Omega), \end{cases}$$

is locally bounded (hence of class C^2 inside Ω) and also satisfies $0 < u_{\epsilon,k} \leq \Phi_\epsilon(u_{\epsilon,k-1}) \leq \Phi_\epsilon^k(u) \leq u$ a.e. on Ω .

Since $\epsilon \in (0,1)$ is arbitrary we have proved that, for every $\delta \in (0,1)$ (choose $\delta = 1 - (1-\epsilon)^k$) there exists a nonnegative stable weak solution $u_\delta \in H^1(\Omega) \cap C^2(\Omega)$ to

$$(77) \quad \begin{cases} -\Delta u_\delta = (1-\delta)f(u_\delta) & \text{in } \Omega, \\ u_\delta - \Phi_\delta^k(u) \in H_0^1(\Omega). \end{cases}$$

Since $0 \leq u_\delta \leq \Phi_\delta^k(u) \leq u$ a.e. on Ω by construction, we get $\|u_\delta\|_{L^2(\Omega)} \leq \|\Phi_\delta^k(u)\|_{L^2(\Omega)} \leq \|u\|_{L^2(\Omega)}$ and also that $\Phi_\delta^k(u) \rightarrow u$ in $L^2(\Omega)$ by the dominated convergence theorem (recall that $\Phi_\delta(t) \rightarrow t$ for all $t \geq 0$ and that Φ_ϵ is a contraction on \mathbb{R}^+). Moreover, by choosing $\Phi_\delta^k(u) - u_\delta \in H_0^1(\Omega)$ as test function in the weak formulation of (77) we obtain

$$\int_{\Omega} \nabla u_\delta \nabla (\Phi_\delta^k(u) - u_\delta) = \int_{\Omega} (1-\delta)f(u_\delta)(\Phi_\delta^k(u) - u_\delta) \geq 0$$

since $\Phi_\delta^k(u) - u_\delta \geq 0$ a.e. on Ω and $f \geq 0$. Therefore we deduce that $\int_{\Omega} |\nabla u_\delta|^2 \leq \int_{\Omega} \nabla u_\delta \nabla \Phi_\delta^k(u)$ which leads to $\|\nabla u_\delta\|_{L^2(B)} \leq \|\nabla \Phi_\delta^k(u)\|_{L^2(B)}$ by Young's inequality. On the other hand $\nabla \Phi_\delta^k(u) = \left(\prod_{j=0}^{k-1} \Phi'_\delta(\Phi_\delta^j(u)) \right) \nabla u$, which entails $\|\nabla u_\delta\|_{L^2(\Omega)} \leq \|\nabla \Phi_\delta^k(u)\|_{L^2(\Omega)} \leq \|\nabla u\|_{L^2(\Omega)}$. Therefore

$$(78) \quad \|u_\delta\|_{H^1(\Omega)} \leq \|\Phi_\delta^k(u)\|_{H^1(\Omega)} \leq \|u\|_{H^1(\Omega)}.$$

In particular the families (u_δ) and $(\Phi_\delta^k(u))$ are bounded in $H^1(\Omega)$ and therefore, we may and do suppose that (up to subsequences) $u_\delta \rightarrow v$ in $H^1(\Omega)$, $u_\delta \rightarrow v$ in $L^2(\Omega)$, $u_\delta \rightarrow v$ a.e. on Ω and $\Phi_\delta^k(u) \rightarrow V$ in $H^1(\Omega)$, $\Phi_\delta^k(u) \rightarrow V$ in $L^2(\Omega)$, $\Phi_\delta^k(u) \rightarrow V$ a.e. on Ω , for some $v, V \in H^1(\Omega)$, as $\delta \rightarrow 0$. From those properties we get $V = u$ (recall that $\Phi_\delta^k(u) \rightarrow u$ in $L^2(\Omega)$) and also that v is a solution of $-\Delta v = f(v)$ in Ω . Also v is stable thanks to $u_\delta \rightarrow v$ a.e. on Ω , the positivity and the continuity of f' and Fatou's Lemma. On the other hand, the weak convergence of (u_δ) and $(\Phi_\delta^k(u))$ in $H^1(\Omega)$ and (77) imply $u_\delta - \Phi_\delta^k(u) \rightarrow v - u$ in $H_0^1(\Omega)$. Finally we get that $u_\delta \rightarrow u$ in $H^1(\Omega)$, since $u_\delta \rightarrow u$ in $H^1(\Omega)$ and $\limsup \|u_\delta\|_{H^1(\Omega)} \leq \|u\|_{H^1(\Omega)}$ by (78). Since both u and v are stable solutions in $H^1(\Omega)$ and $0 \leq v \leq u$ a.e. on Ω , we deduce from Theorem 3 that either $u = v$ in Ω or $u > 0$ and $u \in C^\infty(\Omega)$. In the first case the desired conclusion follows by taking $\epsilon_n = \delta_n$, where (δ_n) is any sequence in $(0,1)$ such that $\delta_n \searrow 0$ and $u_n = u_{\delta_n}$, while in the second one it is enough to take $\epsilon_n = 0$, $u_n = u$ for every $n \geq 1$.

The last claim of item i) then follows by standard elliptic theory.

ii) If $f(0) > 0$, the conclusion follows from item i). If $f(0) = 0$, either $u \equiv 0$ or u is a positive first eigenfunction of $-\Delta$ with homogeneous Dirichlet boundary conditions, by Theorem 18. In both cases the smoothness of u up to (a portion of the) boundary follows from the elliptic regularity, since $\partial\Omega$ is smooth enough. To conclude it is enough to take $\epsilon_n = 0$, $u_n = u$ for every $n \geq 1$. □

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