



## Abstract and Linear Algebra (Chapter 8)

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## Chapter 8: Abstract and Linear Algebra

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### INTRODUCTION

Abstract Algebra teaching and learning appeared recently as a new entry in the Encyclopedia on Mathematics Education (Hausberger, 2018b). This is a sign that University Mathematics Education (UME) research has now reached enough maturity on this topic to account for a substantial body of research-based knowledge but also, as stated in the conclusion of the entry, that “research in this area remains limited and several issues would need to be more systematically investigated”.

Linear Algebra has been the focus of more systematic research (Rasmussen & Wawro, 2017), due partly to its importance in the undergraduate curriculum, from freshman courses on matrices, linear spaces and linear transformations, to more advanced topics like eigentheory or applications of Linear Algebra, for instance to quantum mechanics. By contrast, Abstract Algebra “usually designates an upper division undergraduate course typically required for mathematics majors and centred on the structures of groups, rings and fields. In general, it is encountered by students upon completion of courses in Linear Algebra” (Hausberger, 2018b).

As a mathematical discipline, Abstract Algebra denotes the new image of algebra that emerged at the turn of the twentieth century through the work of German algebraists (in particular, Noether and her school). The shift of paradigm is well illustrated in the following quote by Corry, who points out an inversion in the conceptual hierarchy of algebra:

Groups, fields, rings, and other related concepts became the main focus, based on the implicit realization that all of these concepts were, in fact, instances of a more general, underlying idea: the idea of an algebraic structure. Thus, the main task of algebra became the elucidation of the properties of each of these structures and of the relationships among them [...] The systems of real numbers, rational numbers and polynomials were studied as particular instances of certain algebraic structures; the properties of these systems depended on what was known about the general structures of which they were instances, rather than the other way round (Corry, 2016, paragraph 1).

Linear Algebra also inherited this shift towards a more formal, abstract and conceptual presentation, maybe as the product of an even more complex and tortuous historical process. According to Dorier (2000, p. 59):

the concept of vector space, so elementary in terms of structure, encapsulates, in a very elaborate product, the result of a long and complex process of generalization and unification. Yet the inevitable risk of reduction that masks the complexity of the modelling process in the treatment of linear problems is the price that has to be paid for the efficiency of the axiomatic structural approach. Because of this, a knowledge of the historical development provides the teacher and the researcher with a field of investigation from which they can better understand the students' difficulties and, more generally, from which they can put some “meat” to the “bare bones” of axiomatic approach.

As is evidenced by this short epistemological investigation of the mathematical topics that are the focus of this chapter, educational research in these fields has to address the challenges related to the mathematical formalism and abstraction processes involved in the conceptualisation of algebraic structures. This gives room to various approaches (cognitive, sociocultural, semiotic, etc.) of the teaching-learning phenomena.

In their review of UME research presented at the CERME conferences, Winsløw et al. (2018) made the distinction between *descriptive research* focusing on UME ‘as it is’ and *research on and for innovation* that includes all forms of design-based research aiming at intervention. With regard to the first category, the authors also noted a shift from “developmental studies” inscribed in the Advanced Mathematical Thinking (AMT; Tall, 1991) trend, to “studies that endorse cognitive as well as sociocultural, discursive and anthropological perspectives” (Winsløw et al., 2018, p. 64). While the former use such theoretical constructs as concept image/concept definition (Tall & Vinner, 1981) or well-known binaries such as procedural/conceptual understanding (Hiebert, 1986), the latter may be anchored in the Anthropological Theory of the Didactic (ATD) initiated by Chevallard and his collaborators (see Bosch & Gascón, 2014) or in the Commognitive Theoretical Framework (CTF) developed by Sfard (2008).

Out of the 9 papers that were presented at both INDRUM conferences, 6 contributions belong primarily to the descriptive research category and 3 are discussing intervention principles or accounting for a design-based research. Yet, the dichotomy is not as clear-cut as it sounds: although the emphasis may be put in the paper on analysing UME as it is, the study may be part of a wider project that includes intervention as the next step: for instance, the notion of structuralist praxeology introduced by Hausberger (2016) is used to develop a Study and Research Path (Hausberger, 2018a), thus a form of intervention in the ATD framework. In fact, the majority of papers that focus explicitly on intervention, namely the papers by Trigueros & Biahchini (2016) and by Papadaki & Kourouniotis (2018), use cognitive approaches in the AMT trend, for instance an APOS-based genetic decomposition, to the exception of Bosch. et al. (2016) who present an anthropological approach within ATD. The descriptive studies endorse a plurality of perspectives: 2 of them (Donevska-Todorova, 2016; Wawro et al., 2018) adopt a cognitive lens and focus on

the procedural/conceptual distinction, one paper (Lalaude-Labayle et al., 2018) mixes a systemic approach based on the Theory of Didactical Situations (Artigue et al., 2014) and a semiotic approach, one paper (Ioannou, 2016) uses CTF as theoretical framework (thus a discursive approach), one paper (Hausberger, 2016) ATD, and a last paper (Fleischmann & Biehler, 2018) builds its own local epistemological framework through an *a priori* mathematical and didactical analysis of tasks.

On the level of mathematical topics, a first group of 3 papers deal with Abstract Algebra. Among the 6 papers on Linear Algebra, a second group of 4 papers focus on elementary aspects of Linear Algebra related to linearity: subspaces, linear spans, linear transformations. The last group comprises 2 papers that focus on assessing conceptual understanding in the context of either multi-linear forms (Donevska-Todorova, 2016) or Eigentheory (Wawro et al., 2018).

Our goal is to account for “burning issues” within UME research on Abstract and Linear Algebra, with its related methodological challenges, and to point out current and new avenues for research. We also aim at cross-analysing results and methodologies, thus answering the question: in which respect do these studies complement each other or contrast from each other? What are the main results, open questions, debates among researchers, elements of convergence/divergence within these studies? We thus decided to organize our synthesis according to the three groups of papers mentioned above (merely according to the topic), and for each group of papers to present a cross-analysis of these papers according to main themes that crystallise those burning issues (in terms of epistemological content, methodological issues, results, in fact the main objects of research that seemed to us appropriate for a vivid, illuminating and contrasted account of our data). We end this chapter by summarising what appeared to us as major advances in research on Abstract and Linear Algebra teaching and learning through the work of the INDRUM network as well as the further avenues for research that have been brought to light.

## **INDRUM CONTRIBUTIONS TO ABSTRACT ALGEBRA TEACHING AND LEARNING**

All 3 INDRUM papers related to Abstract Algebra were presented and discussed during the INDRUM 2016 conference in Montpellier. Two of them were later published in an expanded format inside the IJRUME special issue on INDRUM 2016 (Hausberger, 2018a; Bosch et al., 2018). The augmented versions will also be considered in this synthesis.

From the point of view of mathematical topics, both Bosch et al. and Ioannou focus on Group Theory, while Hausberger considers Abstract Algebra at large through the analysis of structuralist thinking and praxeologies, yet with examples taken from the context of Ring Theory. From the point of view of theoretical frameworks, both Bosch et al. and Hausberger anchor their research in the framework of the Anthropological Theory of the Didactic (ATD; see Bosch & Gascón, 2014). Ioannou uses the Commognitive Theoretical Framework (CTF) developed by Sfard (2008).

Both frameworks can be considered as adopting a sociocultural perspective, while ATD may be described as anthropological or institutional and CTF as discursive. This common sociocultural ground should facilitate the networking (Bikner-Ahsbahs & Prediger, 2014) of both theories as two languages of description of educational practices and two lenses to analyse data and produce results. Discussions in this spirit took place during the working group sessions and will be accounted for in the sequel.

Ioannou focuses on students' difficulties with regard to formalism and abstraction; he provides insights about the conceptualization of groups viewed through the development of the mathematical discourse. Hausberger starts with an epistemological investigation of algebraic structuralism as a mathematical practice and then raises the ecological question of conditions and constraints for the development of such a praxis (with its corresponding logos, the discourse on the praxis) in an educational context. Finally, Bosch et al. propose an inverse move from didactics to epistemology: the critique of the current teaching paradigm of Abstract Algebra and the didactical quest for a more inquiry-based approach leads to an epistemological inquiry of the rationale of Group Theory. As we may see, this brief account of the content of the three papers already underlines how these contributions complement each other. Indeed, the emphasis lies either on the epistemology of Abstract Algebra, the understanding of students' difficulties, or the development of new instructional approaches.

For each of these three themes, the points of views of the three papers will be analysed and contrasted below.

### **The epistemological question: what is Abstract Algebra about?**

#### *Structuralist thinking and praxeologies (Hausberger)*

The starting point of the methodology followed by Hausberger is an epistemological study of the rationale of algebraic structures, on the basis of the work of historians and philosophers: “the goal is to bring to light the anthropological roots of mathematical structuralism in order to be able to build a praxeological model of reference for the actual mathematical practice in Abstract Algebra” (Hausberger, 2018a, p. 78).

The methodological dimension of structuralist thinking is then put in the fore. Bourbaki for instance, a group of French mathematicians who promoted the views of the German algebraists, spoke of the “axiomatic method” in terms of a “standardization of mathematical techniques”. The application of the method relies on a dialectic between the particular and the general, or in other words between objects and structures. The questions and problems are raised to a higher level of generality in order to apply structuralist concepts (e.g., ideal, principal ideal domain, etc.) and tools (e.g., isomorphism theorems, structure theorems, combinatorial of structures, etc.) according to the motto “generalizing is simplifying”. Abstract Algebra

concepts may thus be called FUGS (formalizing, unifying, generalizing and simplifying) concepts, following Dorier (1995) and Dorier (2000, p.98)<sup>1</sup>.

Anchoring his formalization within ATD, Hausberger (2016, 2018a) calls “structuralist praxeologies” these new praxeologies (combined praxis and logos) that originate from the rewriting of classical algebra in terms of structures. The theoretical construct is illustrated in the context of Ring Theory by an analysis of a thread on an online discussion forum. In order to prove that the ring  $\mathbf{D}$  of decimal numbers is a principal ideal domain, the group of learners searches for a proof of the general statement that any subring of  $\mathbf{Q}$  is principal, then investigates whether principality is transferred from a ring to its subrings. These reasonings transform a type of task  $T$  into a generalisation  $T^g$  that can be treated using structuralist techniques. The objects-structures dialectic is thus visible in the study process of the group and leads to the development of structuralist praxeologies.

#### *The rationale of Group Theory* (Bosch et al.)

By contrast, Bosch et al. focus on a given topic within Abstract Algebra: Group Theory (GT). The epistemological investigation is motivated by the aim to “focus, not only on the official *raison d’être* of GT within university teaching, but also on different possible alternative ones that could motivate or impel the use of GT to solve problematic questions” (Bosch et al. 2018, p. 32). The authors look for external tasks (that is, external to GT) that could lead to the reproduction of a substantial part of GT as a means to ascribe some rational to it.

This quest is close to that of a fundamental situation, in the sense of TDS: the epistemic game relies on a question suitable to generate an inquiry process that bears the necessity of GT concepts. Bosch et al. refute the argument that the so-called FUGS concepts are antagonistic to the elaboration of fundamental situations (see Dorier 1995, p. 180) since FUGS concepts are not specific to higher education (the notion of number is already FUGS). This debate suggests that further epistemological insight is needed in order to clarify the meaning of FUGS concepts and in particular identify characteristics that hinder the elaboration of fundamental situations. Finally, Bosch and al. argue that a counting problem (such as that of symmetries of a square) may be a suitable candidate for a reconstruction of elementary GT. This choice is justified a priori by the links between GT and the notions of symmetry and invariant in the historical development of GT and by the role played by Lagrange’s theorem as a tool to solve the problem.

Nevertheless, Bosch et al. question the feasibility and potential of such an approach: for instance, “is it substantial enough to motivate the study of the isomorphism theorems”? Such a question allows us to draw connections with the structuralist approach conducted by Hausberger: indeed, isomorphism theorems are structuralist

1 The theoretical construct of FUG concept has been introduced in mathematics education research by Robert and Robinet in 1987 (Dorier, 1995, p. 175). The simplifying dimension of Linear Algebra has been later investigated by Dorier (2000, part I and part II, chap. 1) from an epistemological and didactical perspective.

tools elaborated by Noether in an approach transversal to Group and Ring theories that elucidated the relationships between both theories and used the meta-concept of structure as an organizing principle of the mathematical discourse. This epistemological argument unveils a specificity of Abstract Algebra FUGS concepts: if groups encode symmetries, a substantial part of the rationale of GT relies in its relationship with the structuralist methodology. In other words, the question “What is GT good for?” cannot be separated from another more meta-level question: “Why is GT formalised this way?”

*The group structure: a d-object endowed with meta-level aspects (Ioannou)*

The latter question on the form of the mathematical discourse certainly meets the point of view of CTF, which sees mathematical objects as discursive constructs. Ioannou identifies the group concept as a *compound discursive object* (or d-object), since it does not relate primarily to a perceptually accessible entity. The process of construction of the d-object is not elucidated further in terms of CTF theoretical constructs (saming, encapsulating, reifying). According to Sfard, algebra may be described as “metaarithmetic, that is the unification of arithmetic with its own metadiscourse” (Sfard, 2008, p. 120), a discourse on expressions. The new question raised by Hausberger’s account of algebraic structuralism, to be elucidated, would be: how to analyse, in CTF terms, Abstract Algebra as a meta-level discourse on classical algebra? In particular, how to account for the group concept as a meta-level object with respect to instances of groups such that groups of symmetries, modular arithmetic, etc.? Answering these questions would refine the perspective raised by Ioannou that “GT can be considered as a meta-level development of the theory of permutations and symmetries”.

The group axioms are seen by Ioannou as *object-level rules*: they are part of the discourse about groups, as d-objects, and define the regularities in the behaviour of those, for instance in the task “Suppose  $(G, \circ)$  is a group with the property that  $g^2 = e$  for all  $g \in G$ . Prove that for all  $g_1, g_2 \in G$ , we have  $g_1 \circ g_2 = g_2 \circ g_1$  (that is,  $G$  is abelian)” considered by Ioannou. The visual mediators are those of symbolic algebra and the expected proof routine is based on algebraic manipulations, the symbolic computations being interpreted as relations satisfied by elements in a group (the key is to note that  $(g_1 \circ g_2)^{-1} = g_2^{-1} \circ g_1^{-1}$ , every element of order 2 being its own inverse). The task does not show a dialectic between objects and structures (in the sense of Hausberger), thus meta-level aspects of the group structure as narratives (e.g., definitions, propositions) that subsume discourses on lower-level objects (instances of groups). Nevertheless, the group axioms (object-level rules) may be seen as meta-level rules with respect to symbolic algebra.

**Understanding students’ difficulties: from an analysis of local cognitive problems to the questioning of global institutional choices**

*Dealing with object-level rules and meta-rules (Ioannou)*

Ioannou focuses on “Year 2 mathematics students’ conceptual difficulties and the emerging learning and communication aspects in their first encounter with GT”. According to commognitive views, such an encounter should be reflected in the learning process by an expansion of the discourse: both an *exogeneous* type of expansion due to the endorsement of group-theoretic narratives and routines as changes in the meta-rules required by the application of the axiomatic method (*meta-level learning*), and an *endogeneous* expansion as a growth in the complexity of the endorsed narratives (*object-level learning*).

The above epistemological account of the task put in the fore in Ioannou’s paper has shown that the proof routine takes merely the form of an endogeneous discursive expansion inside symbolic algebra. Nevertheless, a complete proof requires that the syntactical transformations be rigorously justified by means of group axioms and extra properties on  $G$ . These narratives may be considered an exogeneous expansion of the discourse, especially in the case of students who have just met the axiomatic definition of a group and may thus experience difficulties in endorsing them as object-level rules due to insufficient objectification of the group concept. The data presented by Ioannou support this analysis; in particular, the associativity of the group law remained totally invisible to several students. Their narratives conflict with those of the instructors whose annotations on the students’ sheet emphasize the new meta-rules (deduction according to the axiomatic method).

This phenomenon is also related to what Hausberger (2018a, p. 81) calls the structuralist dimension of a praxeology: this dimension measures, for instance, the distance between the narrative “ $g^2=e \Rightarrow g=g^{-1} \forall g$ ” produced by a student and a more conceptual narrative such as “any element of order less or equal to 2 in a group is its own inverse”. In CTF terms, the latter narrative is the result of a subsequent endogeneous discursive expansion inside GT, which cannot be observed in the students’ production at this stage of the learning process.

*The access to structuralist thinking: a transition problem* (Hausberger)

Hausberger focuses on students’ difficulties to access structuralist thinking and analyses this issue in terms of a *transition problem* internal to the university curriculum. The phenomenon should be distinguished from the case of the teaching and learning of Linear Algebra: although both topics offer challenges regarding mathematical formalism and abstraction, Abstract Algebra raises the *level of unification* (Hausberger 2018a, p. 75) one step further, which appears as conceptually more demanding. Indeed, algebraic structures are presented in a unified treatment (the same types of questions are raised about different structures and solved using similar tools, highlighting bridges between structures), which goes beyond the unification of different mathematical contexts under a common structure.

Inspired by Winsløw’s praxeological formalisation (Winsløw, 2008) of the concrete to abstract transition in analysis (from calculus tasks to more theoretical tasks involving continuity and differentiability of functions as well as the topology of real

numbers), Hausberger proposes a model for the epistemological transition to Abstract Algebra, in two phases: the first phase is concerned with the transition from  $T$  to  $T^e$  described above and leads to the construction of a structuralist praxeology as a fertile strategy to prove properties of concrete objects; the second phase builds on structuralist praxeologies previously developed in order to introduce more abstract and theoretical types of tasks that only consider classes of objects with their structural properties (e.g.: show that a Noetherian integral domain such that every maximal ideal is principal is a principal ideal domain).

This decomposition brings insights into transition issues potentially created by institutional choices: regarding phase 1, a structuralist praxeologies may be taught with an artificial praxis block disconnected from the origin  $T$ , thus the rationale that motivated the praxeology in history. About phase 2, a transition problem may occur whenever the praxeologies that serve as anchor points to further praxeological development are not available in the praxeological equipment of learners, or the links between praxeologies are too weak. As a methodology, Hausberger suggests conducting praxeological analyses of textbooks and teaching material using this model in order to pinpoint continuities and discontinuities in the mathematical organisations.

#### *A critical view on the standard monumentalist paradigm (Bosch et al.)*

Hausberger's institutional perspective on transition issues in Abstract Algebra meets the critique addressed by Bosch et al. towards what is identified as the current dominant teaching paradigm at university and emblematically called the *paradigm of visiting works* in ATD accounts. This paradigm is characterised by *applicationism* (theory precedes applications) as the dominant epistemology and *monumentalism* (contents are rarely questioned and problematised) as the dominant pedagogy. In GT, typical tasks such as the determination of isomorphism classes of groups of given orders are merely internal to the theory and follow the theoretical exposition rather than motivate its development. In ATD terms, "mathematical techniques and technologies do not evolve urged by mathematical types of tasks". As a consequence, GT "exists for its own sake and is its own *raison d'être*", so that "students risk missing the answers to important questions concerning GT, especially about the motivation of the theory and its use in different domains or disciplines" (Bosch. et al 2018, p. 27). These remarks motivate a further investigation of the rationale of GT as presented above.

#### **Which instructional approach for algebraic structures?**

##### *Two propositions from ATD at different stages of its development (Bosch et al.)*

Theoretical ideas on intervention are the core of the paper by Bosch et al. Two instructional devices are presented, corresponding to two different stages in the development of ATD. The first one, called "practical workshops", is meant to complement traditional lectures and tutorials by lab sessions centred on the *praxis* (problems and techniques) instead of the *logos* (notions, properties and theorems)

which is the primary focus of lectures. The design is based on a praxeological analysis of the mathematical content: the goal is a partial reconstruction of the subject matter into paradigmatic types of tasks and associated main techniques, the study of which is eager to “generate” the subject to be taught. This means that this knowledge will be introduced (by the students or the instructor) in order to fulfil new theoretical needs related to the scope of the techniques, the limits of the types of tasks, etc. For instance, students were given a list of 31 groups, sorted by their order, from 2 to 8, and were asked to determine which ones are isomorphic and which ones are not. This task led to theoretical developments such as the investigation of properties preserved by isomorphism, a classification of groups up to isomorphism for given orders, structural decompositions of groups into simpler ones, etc. This contrasts with applicationism (see above) and contributes to help students make sense of the taught knowledge: the rationale of notions and theorems is made explicit in relation to these key problems. Yet, Bosch et al. argue that the reasons for studying the initial problem remain absent: why do mathematicians consider isomorphism classes of groups? Instructional proposals that tackle this question may be found in the literature: for instance, Larsen (2013) has designed a “local instructional theory” in order to engage students in the “guided reinvention” (RME) of the concepts of groups and group isomorphism.

The quest of Bosch et al. is more global: following Chevallard’s (2015) idea of an inquiry-based counter-paradigm to visiting works, called the *paradigm of questioning the world*, the authors aim for a reorganisation of the curriculum, “not around works to be studied but around questions to be approached”. This is why the question of rationale of GT needs to be investigated (see above). To implement the study, ATD proposes an instructional device called Study and Research Paths (SRP; Chevallard, 2015). Nevertheless, contrary to the first device, the proposal of using generating questions such as counting problems on symmetries within an SRP has not been carried out by the authors and therefore remains hypothetical.

#### *Questioning the world and the objects-structures dialectic* (Hausberger)

Hausberger (2018a) experimented an SRP on arithmetical properties of the ring  $\mathbf{D}$  of decimal numbers. Students worked on a transcript of the thread, extracted from the online forum, which was presented above to illustrate the development of structuralist praxeologies. The theoretical ideas developed by Hausberger in this pilot study meet the project of implementing the paradigm of questioning the world, in the sense that questioning the world “amounts to questioning mathematical objects themselves in such a way that may be developed a fruitful dialectic between objects and structures” (Hausberger, 2018a, p. 91). The quest for external rationale, presented by Bosch et al. as a didactical request, is thus seen by Hausberger as epistemologically grounded in the origin of Abstract Algebra itself as a refoundation of pre-structuralist algebra (thus external to the theory). Many questions about the ecology of such an approach (how to sustain the SRP) are raised by the author. For instance, elementary techniques may hinder the development of more structuralist ones. Therefore,

Hausberger relies on the transcript to enrich the didactical milieu, in other word to foster learning by acculturation through the study and sense-making of the discourse of more advanced learners. This certainly meets commognitive views.

### *Commognitive conflicts and meta-level learning* (Ioannou)

CTF assumes that “any substantial change in individual discourse, one that involves a modification in meta-rules or introduction of whole new mathematical objects, must be mediated by experienced interlocutors” (Sfard 2008, p. 254). A proper instructional approach thus relies on the determination of the conditions for effective mediation, which resonates with the ecological approach of ATD (conditions and constraints that hinder or foster the development of praxeologies). We have seen that the transition to Abstract Algebra requires an exogenous change in discourse. CTF further assumes that such a meta-level change is unlikely to be initiated by the learners themselves, but by the experienced interlocutor (in most cases, the instructor), in the context of a *commognitive conflict*. This is a situation in which the different discursants are acting according to different meta-rules, and such a conflict is seen as a potential trigger of meta-level learning. The stage of the research presented by Ioannou (2016) is restricted to the identification of commognitive conflicts and their root causes. A further study of the students’ rationalisation of the discursive ways of the expert interlocutor in the expected phase of resolution of the conflict would be needed in order to evaluate the effectiveness of the task and mediation. Nevertheless, the presence of commognitive conflicts is already quite informative: it is a marker for the transition issues identified by Hausberger on epistemological and didactical grounds. It also reminds the researcher in the quest for means to create continuities or meaning through task-design that incommensurable discourses are both unavoidable to some extent and potentially fertile as a lever.

## **INDRUM CONTRIBUTIONS TO LINEAR ALGEBRA TEACHING AND LEARNING**

### **The theme of linear spans and linear transformations**

Three of the papers in this section were presented at the INDRUM 2018 conference with one (Trigueros & Biachini, 2016) at the INDRUM 2016 conference. To varying degrees, each of the four papers in this section includes the following components: (1) the creation of innovative tasks, (2) an a priori analysis of the mathematics involved in solving those tasks, (3) an implementation of the tasks in an interactive environment, and (4) an analysis of students’ thinking when solving these tasks.

Mathematically, the papers address issues of linearity, with two papers focused on linear transformations and the other two discussing the linearity of a subspace or a linear span. Beyond this overlap, the linear transformation papers touch on a wide range of issues including whether a transformation is a mapping that is an endomorphism, a function, surjective, and/or injective, and with connections to composition of functions, invertibility and matrix representations. The subspace and

span papers connect to other content such as systems of equations, geometric representations and linear independence.

In terms of student thinking, each paper has a focus on students' exploration of the ideas on their own as well as the importance of interactions with a tutor or a researcher to perturb their ideas for further learning. The papers cite the importance of both informal reasoning, such as work generating examples and hypotheses, and also reasoning that leads to abstraction, generalization and making connections. Each paper also discusses different registers or representations of the mathematical constructs used by students. In what follows we provide an overview of results and framings from each paper within the above topic areas and with an emphasis on synthesizing across the papers.

### *The role of an a priori analysis*

Each of the four papers in this section analyses the mathematical or didactic aspects of tasks prior to using them with students. Lalaude-Labayle et al. (2018) draw on an *a priori analysis* as one component of their use of the Theory of Didactical Situations (TDS; Artigue et al., 2014). This analysis includes a mathematical analysis of the several ways that the linear transformation problem can be solved as well as a didactic analysis of the role of the student in navigating through the different strata of the didactical *milieu* during the solving process and the difficulties a student may have in solving the given problems.

Fleischmann and Biehler (2018) follow Biehler, Kortemeyer, and Schaper (2015) in calling the a priori analysis a *student expert solution* (SES). This includes an idealized student solution based on the knowledge a student would have at this point in the course. In addition, it provides alternative solution methods and other information such as learning objectives. The methodology is applied to discuss the mathematics and potential student thinking involved in a set of problems about vector spaces in  $\mathbb{R}^2$ .

Trigueros and Biahchini (2016) employ a related, but distinct type of a priori analysis that is part of their framing of their work using the Action-Process-Object-Schema (APOS) theory (Arnon et al., 2013). In this theory, researchers develop an a priori *genetic decomposition* (GD) that predicts the mental constructions needed to understand the mathematical concept of the study. In this case, Trigueros and Biahchini draw on a slightly modified version of the GD created by Roa-Fuentes and Oktaç (2010) for the concept of linear transformation. Their GD focuses on the necessary aspects of checking or understanding that the transformation satisfies the linearity properties as well as the relationship between a linear transformation and its associated matrix transformation.

Although the fourth paper in this group, Papadaki and Kourouniotis (2018) does not provide a TDS a priori analysis nor a GD, it does clearly lay out the rationale for the tasks in ways that overlap information included in the a priori analyses of the other papers. The intention of the tasks is to cause the students to encounter a potential

conflict factor that can then be resolved through discussion. To create such a task the authors analyse three aspects of the concept of linear span that their tasks include as well as two conflict factors that may occur in student reasoning. In this way, all four papers include some form of a priori analyses of the mathematics involved in solving the task and potential pedagogical issues.

### *The role of multiple representations*

In analysing students' reasoning, several different symbol systems were prevalent across the four papers. Lalaude-Labayle et al. (2018) frame their discussion of student productions in terms of Pierce's semiotics as well as Duval's (2017) registers of semiotic representations. For the linear transformation questions reported in this study these registers are each symbolic. For example, they ask students to analyse the linear transformation  $\phi(P) = P(X+1) - P(X)$  where  $P$  is a polynomial of degree at most  $n$ , i.e., the domain is  $\mathbf{R}_n[X]$ . The problem can be solved in terms of the polynomials directly or by converting the linear transformation into a matrix in the canonical basis of  $\mathbf{R}_n[X]$ . In each case the student would be working in a different register of semiotic representations, both symbolic.

The other three papers in this group had a stronger focus on connecting graphical representations to the important symbolic representations for the mathematics in their tasks. Trigueros and Biahchini (2016) discussed linear transformations in the context of a cartoonist creating different drawings of a man on a bicycle and needing calculations to convert between them. Students converted between this graphical representation and two different symbolic representations, function notation such as  $T(a+b) = T(a) + T(b)$  and matrix notation.

In the context of subspaces, Fleischmann and Biehler (2018) created questions that had what Gravesen, Grønbaek and Winsløw (2017) call *linkage potential*. In this case, they stated sets in  $\mathbf{R}^2$  in the form of implicitly defined equations of lines and other graphs that they hoped students would connect with their school knowledge about geometric objects and equations. Papadaki and Kourouniotis (2018) expected students to be familiar with geometric representations of 1-, 2-, and 3-dimensional subspaces of  $\mathbf{R}^3$  from earlier in the course. If students had trouble remembering them, the tasks gave the opportunity for discussion. In both cases, students were expected to connect a geometric representation in  $\mathbf{R}^2$  or  $\mathbf{R}^3$  with a symbolic way of representing these sets. In the former paper, sets were given using a relational description, e.g.,  $\{(x_1, x_2) \in \mathbf{R} : x_1 + 2x_2 = 0\}$ , whereas in the latter, the symbolic representation was primarily intended to be linear combinations of vectors.

What is clear from just these few examples is the vast number of relevant representation registers involved in linear algebra problem solving and that different papers on the same mathematical content may focus on different registers. The examples above include systems of linear equations, vector equations, matrix equations, and linear transformation notation, each of which has its own unique geometric representation (Larson and Zandieh, 2013).

### *The role of examples, modelling and heuristic reasoning*

Although each paper indicated the importance of formal deductive reasoning, each paper additionally emphasized the importance of students' use of more informal reasoning to generate problem solving ideas. Fleischmann and Biehler (2018) asked students to determine whether several relationally defined sets were or were not a vector space. Then they asked students to find all subspaces of  $\mathbf{R}^2$ . In doing so, they expected that students would rely at least in part on the examples and non-examples they had generated. In addition, they described this task as one with *research potential* (Gravesen, Grønbaek, & Winsløw, 2017). In other words, it is a research-like activity based on an open question that allows students to form hypotheses and explore examples of subspaces. These generative activities go beyond a more standard task such as requesting students to make a deductive argument by applying the definition of a subspace.

Other papers also expected students to explore sample spaces and use less formal reasoning. Papadaki and Kourouniotis (2018) expected students to reach for vectors in their *example space* (Mason & Watson, 2008) when trying to determine whether a set of three vectors could be linearly dependent even if the third vector is not in the span of the first two.

The Trigueros and Biahchini (2016) paper focused on student work with a modelling problem. They found that the activities caused students to use models to make predictions and to explore those predictions. In this way, the modelling situations allowed students to construct new knowledge.

Lalaude-Labayle et al. (2018) conclude that it would be better for students' learning if the students engaged in a more heuristic approach. By this, Lalaude-Labayle et al. refer to a heuristic milieu that is grounded in the mathematical task and allows students to test and validate or invalidate their conjectures. The student in their study spent more time in the reference milieu, applying learned procedures to the linear transformation tasks, but struggled when requested to explore the heuristic milieu by looking at examples such as calculating  $\phi(P)$  for various values of  $P$ . The observed lack of articulation of the reference milieu and heuristic milieu hindered the student in finding a break-through in resolving the more open-ended aspects of the task.

Across the four papers, there is an expectation that students should use reasoning that is not always deductive and that incorporates exploration and conjecture. This includes creating examples and non-examples.

### *The role of the teacher*

Fleischmann and Biehler (2018) collected data in the form of written work from around 100 students and also video data from 3 groups of 2-4 students each. The students completed the tasks as part of a tutorial session. Although not reported in detail in this paper, the authors were interested in the role of the tutor in students' developing understanding.

Similarly, Papadaki and Kourouniotis (2018) developed tasks intentionally to create conflict factors that would then force a discussion between the students solving the task and others. The interview data that they present does not delve into the role of the interviewer, but they discuss switching these tasks to problem sessions at their institution because of the richness of conversations that they can generate.

Lalaude-Labayle et al. have as a strong component of the TDS framing the notion of *milieu*. Although TDS plans 7 phases, this paper focuses on a heuristic phase which is based on the student's work in problem situation, through a formulation and validation phase, to an institutionalization involving the teacher. In this way, the student's work on the problem is initially adidactic, but later involves a didactical moment in interaction with the teacher. The INDRUM paper of Lalaude-Labayle et al. did not have room to go into depth about the role of the teacher in working with the student, but the examples given involve encouraging the student to engage the problem in a more heuristic way and with alternative registers such as a matrix representation.

Trigueros and Biahchini describe a study that took place over a longer period compared with the other studies (4-5 sessions compared with 1 session). Because of this, there were more opportunities for a variety of roles of the teacher to emerge. As part of their work on the modelling tasks, students were sometimes engaged in mathematical ideas that they had not yet learned in a formal way. An example of this highlights a role of the teacher that was unique to the Trigueros and Biahchini paper among the four discussed in this section. The students had been engaged in modelling tasks where they were asked to create matrix representations for linear transformations that had been presented to them in a graphical and applied setting. As part of this process, students began to notice particular properties that some of the transformations had. For example, one student said, "This one changes the form of the bicycle (shear transformation), but this one (translation) does not, it only changes its position in space" (p. 331). The teacher responded to these discussions by defining isometries and asking students to verify which transformations were isometries. In this way, she was able to build on students' current thinking by introducing formal mathematical terminology after the students had experienced the concept through their own engagement in the mathematics.

Although pedagogy was not the primary focus of any of the four papers, each one acknowledged the important role of the teacher. Trigueros and Biahchini had the most detailed discussion of the teacher's role; each paper noted the role teachers could play in encouraging the student to explore the mathematical ideas.

### **Assessing students' conceptual understanding**

In order to evaluate students' conceptual understanding in two different central topics in Linear Algebra, Wawro, Zandieh, and Watson (2018) and Donevska-Todorova (2016) present two different approaches for the design and development of a set of tasks as an assessment instrument. In the following, we will first provide insight into

the criteria for the classification of students' learning and reasoning. We then proceed with a description of the methods and results of the task design process in both studies. Finally, we will compare the dominant thinking patterns and solution strategies of the students that were identified in the studies.

### *Modes of understanding and modes of description in Linear Algebra*

Both papers focus on the important concepts, processes and modes of description involved in understanding an aspect of linear algebra. Donevska-Todorova discusses this in terms of *procedural* and *conceptual understanding* (Hiebert, 1986), as well as the *modes of description and thinking* used by students when working with (new) mathematical concepts. The latter can be defined, based on Hillel's (2000) modes of descriptions in Linear Algebra (the *abstract*, *algebraic* and *geometric modes*) and Sierpinska's (2000) modes of thinking (*synthetic-geometric*, *analytic-arithmetic* or *analytic-structural*). Donevska-Todorova creates new categories using combinations of these designations to classify the possible solutions of the tasks used in her study, e.g. "abstract/analytic-structural" or "algebraic/arithmetic".

These modes of description/thinking are in fact one of three criteria that she defines to characterize students' work. The other two criteria are *properties of concepts in Linear Algebra* and *subject-specific strategies or solving tools in Linear Algebra*. These thus refine the modes of description by pointing out the main ingredients of the technique used to solve the task, and in particular the conceptual aspects. With respect to the content, she chooses exercises on bi-linear and multi-linear forms, given the lack of studies covering these topics. Her objective is to analyse the students' work with respect to their procedural and conceptual understanding of the subject, based on an attribution of different solution approaches to the three criteria described above.

In Wawro, Zandieh, and Watson (2018), the focus is on eigentheory. This is a rich area for study because of the complexity involved in coordinating the different mathematical concepts (e.g. matrices, vectors, scalars), processes (e.g. matrix multiplication, scalar multiplication) and modes of description that are part of eigentheory.

Both studies have in common that they make efforts to identify the depth of conceptual understanding in terms of the cognitive flexibility needed to deal with abstract mathematical concepts, using homogeneous, written tasks (in contrast to interviews or other individualised assessments of knowledge and skills), that were particularly developed for this purpose. The modes of description as defined by Hillel provide a system for the characterization of thinking processes that are widely applicable within Linear Algebra. Considering central challenge for students' learning of Linear Algebra, both aspects must be taken into account: Firstly, the necessity of understanding of formal, axiomatic definitions of new concepts and their symbol-based transformations, and secondly the translation of these definitions and procedures to a geometric perspective.

### *Assessment in Linear Algebra: Discussion of exemplary tasks*

An explicit objective of the work of Wawro et al. is to develop and test a set of exercises on eigentheory in order to create an assessment instrument on this topic that allows the evaluation of a student's level of knowledge as well as their preferred way of reasoning. This instrument consists of several exercises in the "Multiple Choice Extended" (MCE) format. In this format, each question begins with a multiple-choice element followed by a list of statements, from which all possible justifications for the answer of the first part must be selected. Central for the study's objective is the design of the provided possible justifications. The starting point was a working model for the understanding of eigentheory, in which three main settings for the framing of eigentheory were identified:

- (1) relationships indicated by the eigen-equation  $Ax = \lambda x$ ;
- (2) relationships indicated by the homogeneous form of the eigen-equation  $(A - \lambda I)x = 0$ ;
- (3) relationships indicated by a linear combination of eigenvectors, i.e. relationships between vectors in the same eigenspace.

The working model furthermore is organised in four main interpretations that can be applied when working with the three settings, namely graphical, numeric, symbolic and verbal interpretations. We illustrate in Figure 1 these reflections on one of the tasks that were developed to "elicit student's thinking [...] within and across the settings and interpretations" of the working model.

1. The matrix  $A = \begin{bmatrix} -2 & 4 \\ 2 & 5 \end{bmatrix}$  has  $\lambda = 6$  as one of its eigenvalues. Which of the following vectors is an eigenvector of  $A$  with corresponding eigenvalue  $\lambda = 6$ ?

(a)  $x = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$  ← CORRECT

(b)  $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Because ... (select ALL that could justify your choice)

(i) This vector  $x$  makes  $Ax = 6x$  a true statement.

(ii) This vector  $x$  is the only vector in  $\mathbb{R}^2$  for which  $Ax = 6x$ .

(iii) This vector  $x$  makes  $(A - 6I)x = 0$  a true statement

(iv) Subtracting 6 from the diagonal of  $A$  yields this vector  $x$  as a column vector of the resulting matrix.

(v) The vector  $Ax$  is 6 times the magnitude and in the same direction as this vector  $x$ .

(vi) The matrix  $A$  also has  $\lambda = -3$  as an eigenvalue.

**Figure 1: Excerpt from the MCE format (Wawro et al., 2018, p. 278)**

In this task, a numeric interpretation is used, followed by several justifications where e. g., (i) is a symbolic interpretation in the  $Ax = \lambda x$  setting, (iii) a symbolic interpretation in the  $(A - \lambda I)x = 0$  setting and (v) a geometric interpretation in the  $Ax = \lambda x$  setting. Similarly, other parts of the MCE instrument were designed in order to provide tasks and justifications using different combinations of settings and interpretations from the working model. Parallel to the use of this MCE-instrument, another group of students was asked to solve the same problem, but without a list of possible justifications. These students had to answer an open question and to formulate their own justification for their answer.

Unlike Wawro et al. who contrast closed-ended to open-ended tasks with the same mathematical contents, Donevska-Todorova creates two different types of tasks. Firstly, her study includes (written) open homework exercises, that allow multiple possible solutions (“multiple solution tasks”, MST), where she classifies the various solutions according to the three criteria described in the previous section. For example, the solution of MST 1 (see Figure 2) is described as follows: (1) The mode of description/thinking presented in the given solution is *algebraic/arithmetic*, (2) the property of concepts used here is that *a determinant of an  $n \times n$ -matrix is a sum of determinants of  $n$  sub-matrices of dimension  $(n-1) \times (n-1)$*  and (3) the applied strategy is based on *Laplace (cofactor) expansion*.

MST1: Find the determinant of the matrix

$$M = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Write as many solutions as you can.

Solution 1:

$$\det M = \det \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \stackrel{\text{def}}{=} 2 \cdot 2 \cdot 2 \cdot \det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \stackrel{\text{def}}{=}$$

$$8 \cdot 1 = 8.$$

**Figure 2: Excerpt from the MST worksheet (Donevska-Todorova, 2016, p. 280)**

Similarly, she identifies five other solutions that are possible (based on the previous contents of the lecture) for the students to solve the task, including the use of Sarrus rule, geometric considerations and more.

The second type of exercises is intended for (oral) discussion. The tasks are initially closed, since students are asked to determine the truth value of given statements, but in the study, those questions were opened during the discussion when students were asked to justify their answer. An example for such a question is whether the statement  $\det(A+B) = \det(A) + \det(B)$  holds for arbitrary  $n \times n$ -matrices with real entries.

#### *Reasoning processes and thinking patterns of the students*

As we have seen in the previous examples, both studies choose different approaches to design open and closed tasks that are supposed to allow conclusions concerning the student’s level of conceptual understanding of the given subjects. Regarding the two dimensions of classification, the findings of both studies seem to confirm and complement each other. In both cases, students showed a distinct favour for algebraic reasonings over geometric arguments. Where Wawro et al. argue that this might be caused by the given descriptions in their exercises, Donevska-Todorova deduces that the students have developed mainly procedural understanding and do not show deeper conceptual understanding, which is, in her view, necessary for the transition from algebraic to geometric reasoning.

Both papers emphasize that the geometric mode of description and the related cognitive procedures are likely to be not as intuitive to the students as more algebraic approaches. The influence of the mode of description used to state the task seems to be strong, in particular in cases of open questions. In fact, many students engage in

geometric reasonings whenever tasks are presented geometrically, but they do not provide similar answers themselves if no specific request is stated. Wawro et al. moreover conjecture that students might consider algebraic justifications for mathematical statements more acceptable. This effect of the didactical contract, as perceived by the students, may therefore reduce the possibility of deducing from students' answers their ability to use and move between different modes of description. Wawro et al.'s use of the MCE format is meant in part to mitigate this effect. Nevertheless, both papers provide and discuss a format of innovative tasks to assess learning. Both works emphasize the process of reasoning and justification over the reproduction of calculations. In addition, both papers take into consideration potential limitations of the tasks in terms of the challenge the new format of tasks may incur for both students and researchers.

## CONCLUSION

How has research in Linear Algebra teaching and learning advanced since the publication of Dorier's (2000) synthesis? In his conclusion, Dorier emphasized three main points: the diagnosis of epistemological issues that, for instance, resulted in the elaboration of Hillel's (2000) and Sierpinska's (2000) frameworks; the challenge for students of navigating flexibly through the different modes of description and registers of representation; finally, the challenge for UME research of developing and evaluating long-term intervention plans that, according to him, were necessary in the case of FUGS concepts.

The current INDRUM research on Linear Algebra inherits these epistemological guidelines through Hillel's and Sierpinska's frameworks. Much emphasis is still made on the investigation of student thinking through descriptive studies. Attempts are made to systematize this research and develop assessment instruments (Wawro et al.; Donevska-Todorova). Another current path is the refinement of the semiotic analysis by means of Pierce's semiotics (Lalaude-Labayle et al., 2018).

Most of the studies are mainly local, focusing on a single major concept like linear spans, linear transformations or multi-linearity, and restricted to a single teaching session. These studies point out several levers to foster learning and conceptualisation: Papadaki highlights the role of (cognitive) conflict factors and illustrates how lecturers may design tasks, inspired by their observations of students' misconceptions; Trigueros & Biahchini point out the use of modelling situations and design via APOS genetic decompositions; Fleischmann & Biehler conclude that more guidance and preparation is required for the tutorials since, for instance, students experience difficulties in applying geometric knowledge from high school; finally Lalaude-Labayle et al., in terms of TDS, diagnose a lack of articulation between the *heuristic milieu* and the *reference milieu* which is also an indicator of how to organize the epistemic game better in problem-solving activities.

Long-term didactical engineering was not presented at INDRUM. One may hypothesize that Dorier's argument rather than a papers' page restriction explains this

fact. The high complexity of implementation and evaluation of long-term teaching design due to numerous global and local choices hinders the possibility for this type of research. Therefore, collaboration with university mathematics teachers is more than ever needed in order to make a breakthrough in the scale of studies as well in the dissemination of the approaches that proved to be fruitful in pilot studies.

The number of INDRUM papers on Abstract Algebra (3) was half that of the Linear Algebra papers and all of them were presented at the first INDRUM. Research in this direction remains limited, but there continue to be new avenues for research. The use of ATD as a theoretical framework puts more emphasis on institutional conditions and constraints; it invites us not to take the knowledge to be taught as a given construct but on the contrary to question this knowledge (its rationale) and envisage other potentially very different mathematical organisations. This raises the need for more epistemological input, for example the description of structuralist praxeologies by Hausberger or the quest for alternative non-official and external (to the theory) *raison d'être* for Group Theory. The critique of the current dominant pedagogy at the university and the development of ATD tools for inquiry-based approaches set up the ground for a study by Bosch et al. of a means to offer a complete reorganization of the GT syllabus around questions and problems. Again, close collaboration with mathematicians and further studies will be necessary to implement such a programme.

In a different approach, Ioannou interprets students' thinking and learning in terms of the development of the mathematical discourse. The challenge of structuralist thinking described by Hausberger in epistemological terms translates into potential commognitive conflicts that CTF sees as a potent lever for the learning. In this respect yet with a different theoretical framework, this study meets the spirit of several INDRUM studies on Linear Algebra.

Our synthesis of INDRUM contributions to Linear and Abstract Algebra teaching and learning divided papers according to the topic. This choice reflects the way papers were presented and discussed at the INDRUM conferences. Yet, this separation is certainly arbitrary on the epistemological point of view, as evidenced by Hausberger who considers Abstract Algebra at large through the study of structuralist thinking. This trend based on the axiomatic method encompasses both Abstract and Linear Algebra, although each topic (and subtopic such as GT, etc.) has different historical roots, in other words, a different history of core problems. Therefore, the mainstream organization of the curriculum as Linear Algebra followed by Abstract Algebra may also be questioned, either in terms of a transition: what kinds of continuities/discontinuities may be observed in the teaching and learning of these topics (modes of representations and reasoning, structuralist thinking and praxeologies)? Or in terms of possible alternative reorganizations of the curriculum, could one conceive of a more spiralling approach to both subjects? We are calling for such studies and hope that forthcoming INDRUM conferences will offer opportunities for more interrelations of studies on Abstract and Linear Algebra.

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