

SUPPLEMENTARY MATERIAL
MULTIRESOLUTION-BASED MESH ADAPTATION AND ERROR
CONTROL FOR LATTICE BOLTZMANN METHODS WITH
APPLICATIONS TO HYPERBOLIC CONSERVATION LAWS

THOMAS BELLOTTI*, LOÏC GOUARIN*, BENJAMIN GRAILLE†, AND MARC MASSOT*

The aim of this supplementary material is twofold. On one side, in Section 1, we confirm that the estimates on the details decay hold, from an empirical point of view. This is an important fact since they are used throughout the work, in particular to devise the refinement operator \mathcal{H}_ϵ . On the other side, in Section 2, we present and comment additional plots which were left due to space limitations in the main body of the paper. Moreover, we investigate the possible limitations of using the leaves collision instead of the reconstructed collision, showing that the latter is needed only on manufactured pathological cases.

1. Quality of decay estimates to deduce the magnitude of the details.

In this section, we want to verify by numerical experiences that the inequality:

$$(1.1) \quad |d_{j,k}^i| \lesssim 2^{-j \min(\nu, \mu)} |f^i|_{W_{\infty}^{\min(\nu, \mu)}(\tilde{\Sigma}_{j,k})}.$$

is indeed sharp and can be used to predict the magnitude of details which are not available with a good fidelity. In this inequality, ν is the local Sobolev regularity of f^i , meaning that it belongs to W_{∞}^{ν} is a neighborhood of the cell $I_{j,k}$ and $\mu = 2\gamma + 1$, where $\gamma \geq 0$ is the size of the prediction stencil. Take four test fields, $\gamma = 1$ (thus $\mu = 3$) and a domain $\Omega = [-3, 3]$:

$$(1.2) \quad \begin{aligned} f^0(x) &= e^{-20x^2}, & f^1(x) &= (1+x)\chi_{[-1,0]}(x) + (1-x)\chi_{[0,1]}(x), \\ f^2(x) &= \sqrt{x}\chi_{[0,1]}(x) + \left(\frac{3}{2} - \frac{x}{2}\right)\chi_{[1,3]}, & f^3(x) &= \frac{1+x}{2}\chi_{[-1,1]}(x), \end{aligned}$$

which have different regularities, namely: $f^0 \in W_{\infty}^{\infty}(\Omega)$ (hence $\nu = \infty$), $f^1 \in W_{\infty}^1(\Omega)$ (hence $\nu = 1$), $f^2 \in W_{\infty}^{1/2}(\Omega)$ (hence $\nu = 1/2$) and $f^3 \in W_{\infty}^0(\Omega)$ (hence $\nu = 0$). We consider the detail $d_j^i := \max_k |d_{j,k}^i|$ for the cell where the Sobolev norm is attained, thus maximal, at level j and we look for the ratio with the detail at the finer level $j+1$.

We obtain what is presented in Table 1, showing a very fine agreement with (1.1), meaning that we correctly recover $d_j^i/d_{j+1}^i = 2^{\min(\mu, \nu)}$. We remark that for the most regular function, the size of the details is limited by the choice of prediction operator (μ in this case), whereas for less regular choices, it is the regularity of the function which determines the decay ratio (respectively $\nu = 1, 1/2$ and 0). This confirms the validity of employing (1.1) to devise refinement/coarsening criteria based on multiresolution as we did.

2. Verifications. In this section, concerning the D1Q2 scheme for the solution of a scalar conservation law, we introduce some supplementary figures about the test cases where the leaves collision proved to be effective, namely I, II, III, IV.

*CMAP, CNRS, Ecole polytechnique, Institut Polytechnique de Paris, 91128 Palaiseau Cedex, France. thomas.bellotti@polytechnique.edu

†Institut de Mathématiques d'Orsay, Université Paris-Saclay, 91405 Orsay Cedex, France.

TABLE 1

Empirical detail decay for $\gamma = 1$ for the test cases given by (1.2), measuring the maximum of the detail.

j	$i = 0$		$i = 1$		$i = 2$		$i = 3$	
	d_j^0	d_j^0/d_{j+1}^0	d_j^1	d_j^1/d_{j+1}^1	d_j^2	d_j^2/d_{j+1}^2	d_j^3	d_j^3/d_{j+1}^3
16	4.65e-13	—	3.81e-6	—	4.72e-4	—	1.25e-1	—
15	3.72e-12	8.00	7.63e-6	2.00	6.57e-4	1.39	1.25e-1	1.00
14	2.98e-11	8.00	1.53e-5	2.00	9.23e-4	1.41	1.25e-1	1.00
13	2.38e-10	8.00	3.05e-5	2.00	1.30e-3	1.41	1.25e-1	1.00
12	1.91e-9	8.00	6.10e-5	2.00	1.84e-3	1.41	1.25e-1	1.00
11	1.52e-8	8.00	1.22e-4	2.00	2.60e-3	1.41	1.25e-1	1.00
10	1.22e-7	8.00	2.44e-4	2.00	3.68e-3	1.41	1.25e-1	1.00
9	9.75e-7	8.00	4.88e-4	2.00	5.21e-3	1.41	1.25e-1	1.00
8	7.79e-6	7.99	9.77e-4	2.00	7.37e-3	1.41	1.25e-1	1.00
7	6.22e-5	7.99	1.95e-3	2.00	1.04e-2	1.41	1.25e-1	1.00
6	4.90e-4	7.88	3.91e-3	2.00	1.47e-2	1.41	1.26e-1	1.00
5	3.60e-3	7.35	7.81e-3	2.00	2.08e-2	1.41	1.27e-1	1.01
4	1.96e-2	5.43	1.56e-2	2.00	2.95e-2	1.41	1.29e-1	1.02
3	1.26e-1	6.43	3.13e-2	2.00	4.17e-2	1.41	1.33e-1	1.03
	Theor.	8	Theor.	2	Theor.	$\sqrt{2}$	Theor.	1

Furthermore, we fully describe V, which is constructed in order to warn about the use of the leaves collision.

2.1. D_1Q_2 for a scalar conservation law: advection and Burgers equations.

2.1.1. Tests I, II, III, IV. The information we have not included in the main body of the paper, namely the time behavior of $e^{0,n}$ and of $E^{0,n}/e^{0,n}$ is provided in Figure 1.

It is interesting to observe that the error $e^{0,n}$ accumulates linearly in time as expected. In some cases (especially for IV) some oscillations are present due to the oscillations of the solution close to the shock when using a relaxation parameter $s > 1$. Concerning the ratio $E^{0,n}/e^{0,n}$, one should remark that we have a boundary layer close to the initial time $n = 0$, tending to small values for the regular solutions (I and III) and to very large values ($+\infty$) for the solutions with shocks (II and IV). This can be understood by the fact that at the beginning, when working with Riemann problems, we have added enough security cells around the shock with \mathcal{H}_ϵ and \mathcal{G} in order to make the adaptive MR-LBM scheme “degenerating” to the reference scheme, thus $e^{0,n} \ll E^{0,n}$ for small n . On the other hand, for smooth initial data, there are many unrefined areas where either the approximation made during the stream phase (whatever the collision kernel, linear or non-linear) or the leaves collision phase (for non-linear collision kernels) generate, from the very beginning, an adaptive MR-LBM scheme which is quite different from the reference scheme. Therefore, for small n , we have either $e^{0,n} \sim E^{0,n}$ or maybe also $e^{0,n} \gg E^{0,n}$. Still, even in these cases (I and III), as long as the time grows, we are capable of largely outperforming against the reference scheme, yielding $e^{0,n} \ll E^{0,n}$ for large n .

To go further in the study, we have considered the test case III performing the collision using the reconstructed collision procedure. The result is given in Figure 2. Compared to the leaves collision (third row in Figure 1), the error $e^{0,n}$ is just divided by a factor 2 and the boundary layer close to n for $E^{0,n}/e^{0,n}$ is still present

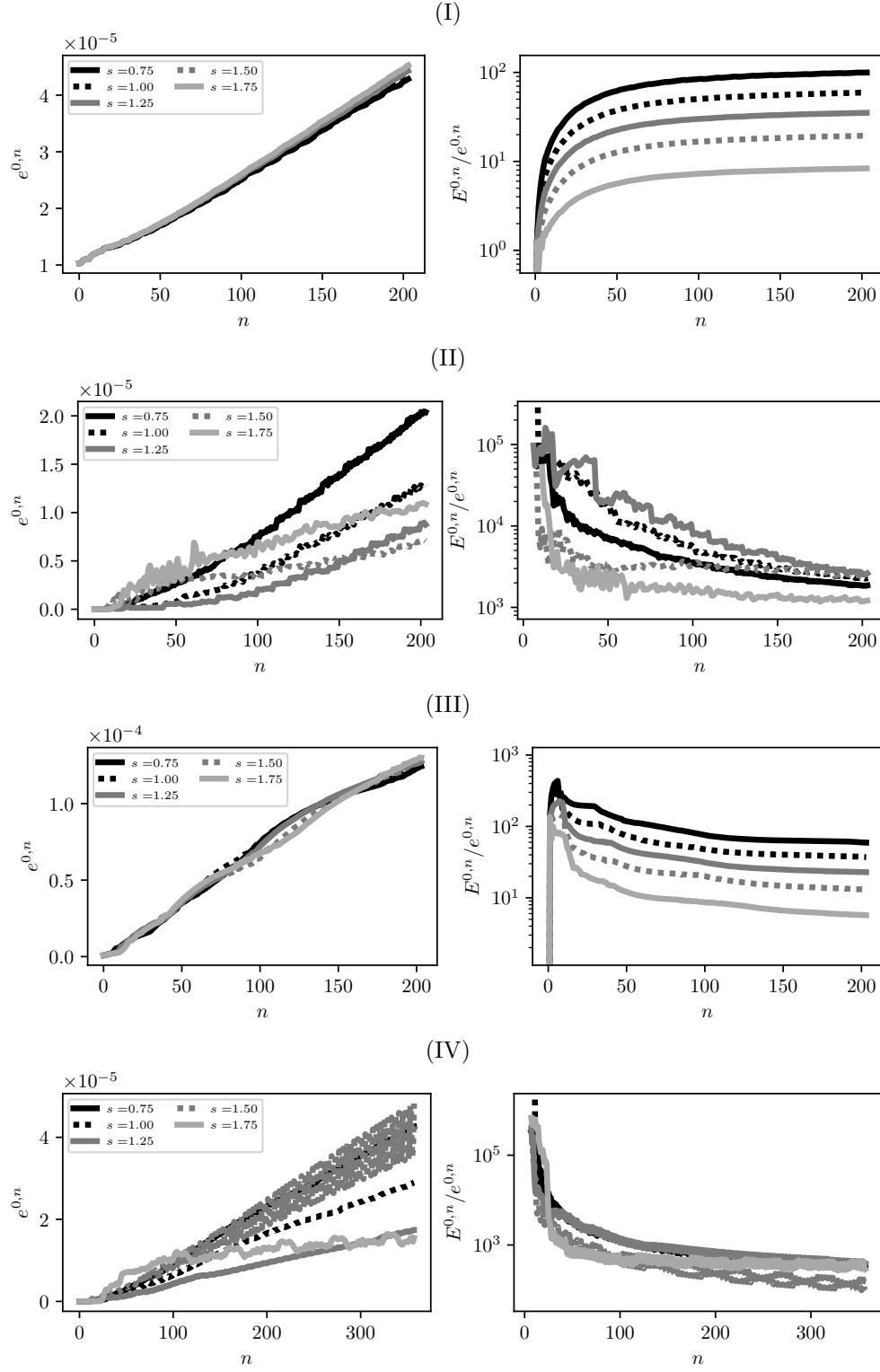


FIG. 1. Behavior of $e^{0,n}$ as function of the time (left) and the ratio $E^{0,n}/e^{0,n}$ as function of the time (right), for test (from top to bottom) I, II, III and IV.

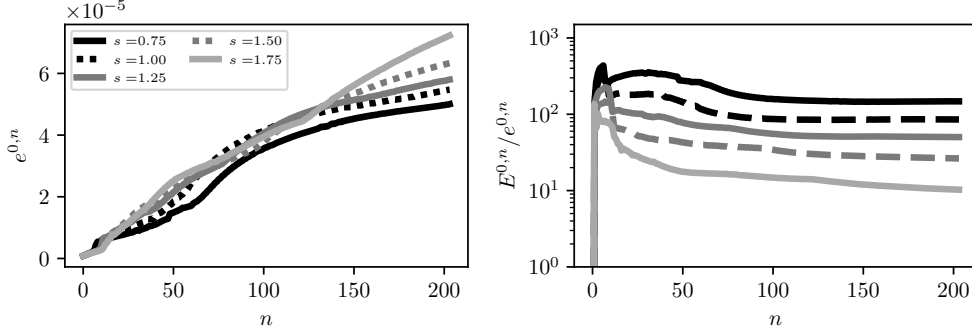


FIG. 2. Behavior of $e^{0,n}$ as function of the time (left) and the ratio $E^{0,n}/e^{0,n}$ as function of the time (right) for test III simulated using the reconstructed collision.

with values close to zero. This shows that this phenomenon is mostly inherent to the approximations made during the reconstruction employed in stream phase and in the collision phase, regardless of the fact that we use a leaves collision or a reconstructed collision. Indeed, we have exactly the same behavior once considering the leaves and the reconstructed collision for a linear collision kernel. The more the initial datum is regular, the more we can imagine this initial boundary layer tending to zero to be important.

The conclusion is that in most of the cases, both for linear and non-linear collision phases, the leaves collision is a reliable and cheaper alternative to the reconstructed collision. It has a minimal impact on the quality of the solution but results in a significant gain in terms of algorithmic efficiency.

2.1.2. Test V. In this section, we consider the Burgers equation with initial datum given by the hat-like function

$$(2.1) \quad \rho_0(x) = (1+x)\chi_{x<0}(x) + (1-x)\chi_{x\geq 0}(x).$$

This test is really interesting because we have constructed it *ad-hoc* to constitute a pathological case where the theoretical bound on the additional error by ϵ are not valid when employing the cheaper “leaves collision” *in lieu* of the “reconstructed collision”. This is due to the fact that the solution is piece-wise linear for every time – especially at initial time – and we know that the prediction operator with $\gamma = 1$ is exact on each linear branch of the solution.

Remark that the weak solution blows up at time $T^* = 1$ and we take $\bar{\mu} = 0$ in order to be sure of correctly capture the jump in the solution after this event. Moreover, the final time is taken to $T = 1.3$ to observe the blowup. The results in term of additional error $e^{0,n}$, ratio of additional and reference error $E^{0,n}/e^{0,n}$ as functions of time; trend of $e^{0,N}$ and the compression factor as function of ϵ are given in Figure 3.

Attentively looking at Figure 3, one remarks three notable facts which shall not be surprising once considering how we fabricated the solution. The first is that the temporal trend of $e^{0,n}$ changes at the blowup time $T^* = 1$. This is coherent with the fact that the solution changes its regularity from W_∞^1 to W_∞^0 (consider the effect of the details which has been fully studied in Section 1), whereas the threshold ϵ to which details are compared whilst applying \mathcal{T}_ϵ and \mathcal{H}_ϵ is kept fixed in time. Second, the ratio $E^{0,n}/e^{0,n}$ shows a time boundary layer close to $n = 0$ tending towards small

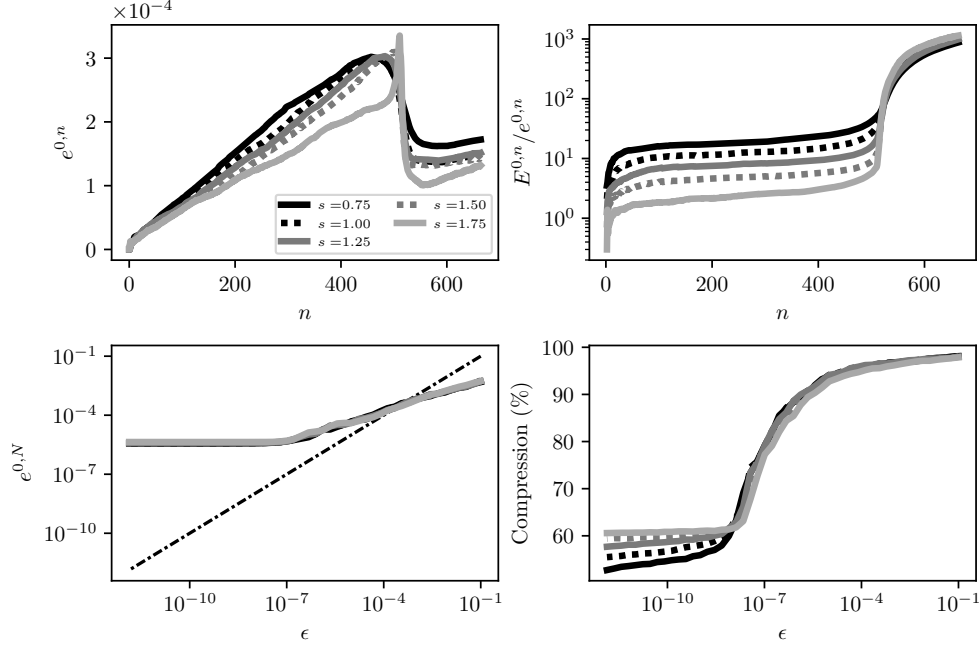


FIG. 3. Test V using the leaves collision as in the rest of the paper. On the top line, we have the additional error $e^{0,n}$ (left) and the ratio of the errors $E^{0,n}/e^{0,n}$ (right) as functions of time. On the bottom line, $e^{0,N}$ (left) and the compression factor at the final time as functions of ϵ . The dot-dashed line gives the reference $e^{0,N} = \epsilon$.

values. This means that at the very beginning of the simulation, the error of the reference scheme is comparable (or smaller) to that of the adaptive MR-LBM scheme, as we already observed for case I and III in the previous section. This fact shall be explained in a moment and we will not come to the same conclusions as in the previous section concerning the dominant causes of this phenomenon. Lastly, we observe that after an initial decrease, $e^{0,N}$ stagnates as ϵ decreases as well as the compression factor. This is in contradiction with the theoretical estimates which give $e^{0,N} \lesssim \epsilon$. However, one should not forget that we have used the “leaves collision” instead of the “reconstructed collision” and this test case has been built on purpose to obtain this.

We now provide a full explanation for these remarks, as well as an additional test. Since the initial solution is piece-wise linear, the multiresolution analysis and the grading of the tree put more and more cells close to the kinks (located at $x = -1, 0, 1$) as ϵ decreases, until reaching a point where the prediction (and thus the reconstruction) is exact and no more cell have to be added. As the reconstruction process pertains to the advection phase, from a certain ϵ and at the beginning of the simulation, the advection is exact. This is false for the collision, because of the non-linearity of the collision operator (generated by the non-linear flux $\varphi(u) = u^2/2$ pertaining to the Burgers equation). Along the sloped sides of the hat (between $[-1, 0]$ and $[0, 1]$), the collision on the leaves adds, at the very beginning of the simulation, an error which is the same for all the ϵ smaller than a certain threshold – because the initial grid is indeed the same – and which remains for the whole simulation, yielding the saturation. We have observed exactly the same saturation as ϵ decreases just compressing the mesh by multiresolution, performing the evaluation of the function

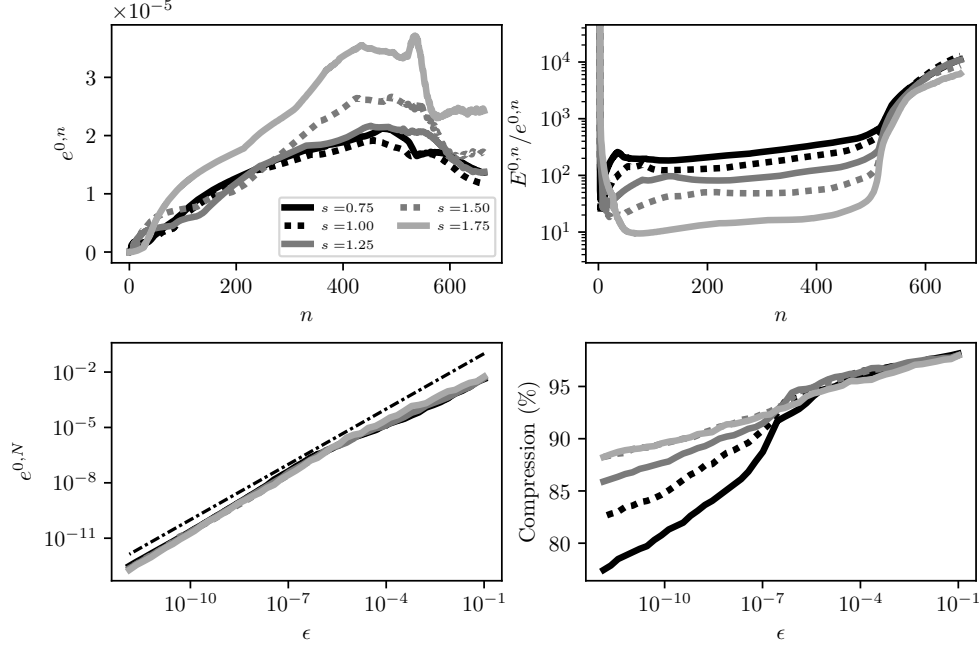


FIG. 4. Test V, using the reconstructed collision, contrarily to the rest of the paper. On the top line, we have the additional error $e^{0,n}$ (left) and the ratio of the errors $E^{0,n}/e^{0,n}$ (right) as functions of time. On the bottom line, $e^{0,N}$ (left) and the compression factor at the final time as functions of ϵ . The dot-dashed line gives the reference $e^{0,N} = \epsilon$.

$\varphi(u)$ on the leaves and measuring the error compared to the evaluation of the function $\varphi(u)$ on the full mesh at the finest level.

To corroborate our observation, we use the reconstructed collision: in this case, we recover the right estimate in ϵ , see Figure 4. This happens because the reconstruction at the finest level is exact on the slopes of the hat and thus the collision has been evaluated at the right resolution. Moreover, the behavior of the initial boundary on the plot concerning $E^{0,n}/e^{0,n}$ has been reversed, yielding large values $E^{0,n}/e^{0,n} \gg 1$ for small n . This is coherent with the other simulations with weak solutions (Figure 1), where at the beginning, the error $e^{0,n}$ is largely negligible compared to $E^{0,n}$ but is different for what happened for the regular test III, where switching from the leaves collision to the reconstructed collision did not change this initial boundary layer. Therefore, we can claim that in the setting of test V, the dominant phenomenon causing the initial boundary layer close to 0 is the leaves collision, and not a combination of stream phase and the collision phase (no matter how it is done) as for test III. Indeed, if we compare the first plot between Figure 3 and Figure 4, we notice that the tangent to the curve in the origin is way less steep in the latter case than in the former. On the other hand, for the test III in Figure 1 and 2, the tangent to $e^{0,n}$ close to $n = 0$ behaves gently both in the case of leaves collision and reconstructed collision.

Coming back to test V, in the case of leaves collision the error $e^{0,n}$ is about one order of magnitude larger than in the case of reconstructed collision. This was not the case in the previous section, where we had only a factor 2 between the errors using the leaves collision and the reconstructed collision.

To conclude, we have devised a particular case where the “reconstructed collision” is needed instead of the “leaves collision” to recover the theoretical estimates. Of course, this does not prevent us from having very interesting ratios $E^{0,n}/e^{0,n}$ far from $n = 0$ for both cases. Moreover, making the comparison between test III and V, between which the only things which changes is the initial datum, it seems clear that the reliability of the leaves collision is not so much a matter of how much the collision kernel is non-linear, but mostly a question of what is the initial datum. In the vast majority of the cases, the leaves collision is largely sufficient and does not prevent one from observing the theoretical behavior.