



HAL
open science

Brahmagupta's apodictic discourse

Satyanad Kichenassamy

► **To cite this version:**

Satyanad Kichenassamy. Brahmagupta's apodictic discourse. *Ganita Bharati (Indian Mathematics): Journal of the Indian Society for History of Mathematics*, 2019, 41 (1-2), pp.93-113. 10.32381/GB.2019.41.1-2.1 . hal-03147427v2

HAL Id: hal-03147427

<https://hal.science/hal-03147427v2>

Submitted on 23 Jan 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

BRAHMAGUPTA'S APODICTIC DISCOURSE

Satyanad Kichenassamy¹

Abstract. We continue our analysis of Brahmagupta's *Brāhmasphuṭasiddhānta* (India, 628), that had shown that each of his sequences of propositions should be read as an *apodictic discourse*: a connected discourse that develops the natural consequences of explicitly stated assumptions, within a particular conceptual framework. As a consequence, we established that Brahmagupta did provide a derivation of his results on the cyclic quadrilateral. We analyze here, on the basis of the same principles, further problematic passages in Brahmagupta's *magnum opus*, regarding number theory and algebra. They make no sense as sets of rules. They become clear as soon as one reads them as an apodictic discourse, so carefully composed that they leave little room for interpretation. In particular, we show that (i) Brahmagupta indicated the principle of the derivation of the solution of linear congruences (the *kuṭṭaka*) at the end of chapter 12 and (ii) his algebra in several variables is the result of the extension of operations on numbers to new types of quantities – negative numbers, surds and “non-manifest” variables.

AMS classification (MSC 2010): 01A32, 01A35, 11-03, 11A05, 51-03.

Keywords: Indian Mathematics; Brahmagupta; linear congruences; *kuṭṭaka*; derivations; algebraic identities; algebra in several variables, discourse analysis.

1. Introduction

The *Brāhmasphuṭasiddhānta*² (India, 628), the *magnum opus* of Brahmagupta, son of Jiṣṇu (*Jiṣṇu-suta-brahmagupta*), is paradoxical in two ways. First, while Brahmagupta seems to record in it major results for the first time, he appears at first sight to have omitted the conditions of validity of his own results. Second, he appears to have included utterly useless or redundant propositions, while being elliptic about difficult results. In the case of his propositions on the cyclic quadrilateral, we have shown that the missing information: definitions, conditions and steps of derivation, are encoded in the discursive structure³. Brahmagupta's derivations of his results differ from all those that have been proposed after him, which indicates a partial break of continuity of tradition shortly after him: his results were transmitted and his name honored, but part of his work was reinterpreted while other schools

¹ Université de Reims Champagne-Ardenne, Laboratoire de Mathématiques de Reims (LMR, CNRS, UMR 9008), B.P. 1039, F-51687 Reims Cedex 2, France. *E-mail* : satyanad.kichenassamy@univ-reims.fr *Web* : <https://www.normalesup.org/~kichenassamy/>

² Henceforth abbreviated as *BSS*. We follow H. T. Colebrooke's numbering (*Algebra, with Arithmetic and Mensuration, from the Sanskrit of Brahmegupta and Bhascara*, London: J. Murray, 1817). The text is from Sudhākara Dvivedin (*Brāhmasphuṭa Siddhānta and Dhyānagrahopadeśādhyāya of Brahmagupta*, Benares, 1902), see also R. S. Sharma (Chief Ed.), *Brāhma-Sphuṭa Siddhānta, with Vāsanā, Vijñāna and Hindi Commentaries* (New Delhi: Indian Institute of Astronomical and Sanskrit Research, 1966).

³ S. Kichenassamy, “Brahmagupta's derivation of the area of a cyclic quadrilateral,” *Historia Mathematica*, **37**(1), (2010), 28-61. <https://dx.doi.org/10.1016/j.hm.2009.08.004> “Brahmagupta's propositions on the perpendiculars of cyclic quadrilaterals,” *Historia Mathematica*, **39**(4) (2012), 387-404. <http://dx.doi.org/10.1016/j.hm.2012.07.004> “L'analyse littéraire au service de l'Histoire des Mathématiques : Critique interne de la Géométrie de Brahmagupta,” *Comptes-Rendus des Séances de l'Académie des Inscriptions et Belles-Lettres*, **CRAI 2012**, II (avril-juin) (2012) 781-796. https://www.persee.fr/doc/crai_0065-0536_2012_num_156_2_93569 “Le ‘triquadrilatère’ de Brahmagupta : Analyse d'un texte mathématique,” in *Comptes Rendus du Séminaire d'Histoire des Mathématiques*, **1** (2015) 47-66. <https://perso.numericable.fr/patrperrin/doc/crvo1ar3.pdf>

became dominant. A similar phenomenon occurred in Renaissance mathematics⁴. Thus, both paradoxes rely on the assumption that the text is a list of results, procedures and problems, rather than a discourse.

We analyze here in the same spirit further problematic passages in *BSS*, namely Prop. 12.58-61, 18.37, 18.38 and 18.42-43. We show in particular that Brahmagupta gave an original derivation of the method of solution of congruences, and that he introduced several-variable algebra through the gradual extension of operations to wider and wider classes of mathematical objects. While it is likely that the material found in *BSS*, and in no other source, is due to Brahmagupta or his father Jiṣṇu, Brahmagupta, like most major authors, does not claim originality. But he speaks in the first person: the discourse is his, all the more reason to heed its structure and to examine Brahmagupta's discursive strategies.

We first propose a typology of mathematical discourse in Section 2, and outline the characteristic features of apodictic discourse. We then analyze Propositions 12.58-61, that belong to the concluding section of Chapter 12 that, Brahmagupta intimates in Prop. 12.66, is in part a preparation for the solution of congruence problems (*kuṭṭākāra*⁵) in chapter 18. We first present the text and translation of the four propositions in Section 3, with a gloss, explicate next the procedure suggested by the text in Section 4, and show that it gives the complete and rigorous solution of equation $ax = by$ when a/b is a given irreducible fraction. In Section 5, we briefly analyze Prop. 18.37, 18.38 and 18.42-43, that illustrate other aspects of Brahmagupta's apodictic discourse. We then summarize the conclusions.

A word about Brahmagupta's mathematical tools may be in order. The basic operation underlying the solution of congruences is *division with remainder*: if a and b are positive quantities, not necessarily integral, with $a > b$, there is a unique nonnegative integer q (the *quotient*) and a unique *remainder* r such that $a = bq + r$ and $0 \leq r < b$. Division is *exact* when there is no remainder. Division with remainder is sometimes called "Euclidean division", although it is not attested in Euclid's *Elements*. All one finds⁶ is the *Euclidean algorithm* to find the greatest common measure of a and b by repeated subtractions. The quotient of division, which plays an essential role in congruence problems in India, never appears. The Euclidean algorithm is a mutual subtraction algorithm, not a mutual division algorithm.

Also, as we shall see, Brahmagupta's results imply what is known in modern mathematics, especially in France, as "Gauss' lemma"⁷: if $ax = by$, where all quantities are integers, and a and b have no common factor, then a divides y . Brahmagupta's result is slightly more general, since his result is stated so as to remain valid for fractions. The relation of 12.58-61 to the *kuṭṭaka* is easy to explain in modern terms, if we take for granted the results

⁴ S. Kichenassamy, "Continued proportions and Tartaglia's solution of cubic equations." *Historia Mathematica*, 42 (4) (Nov. 2015), 407-435. <http://dx.doi.org/10.1016/j.hm.2015.03.004>

⁵ The problem of finding, given a, b and r_1, r_2 , all N such that there exist x and y integral satisfying $N = ax + r_1 = by + r_2$. It seems that this was initially a division or distribution problem. Indeed, *Dhātupāṭha* 10.23 states: *kuṭṭacchedanabhartsnayoh* "kuṭṭa means both *bhartsna* and *chedana*". Now, *bhartsna* means "threat, reproach," while *cheda* "that which cuts" is a standard term for divisor or denominator in mathematics. It seems that *kuṭṭākāra* for Brahmagupta represents the method, while *kuṭṭaka* is the value of the multiplier x . He calls *sthira-kuṭṭaka* the value of x when the difference $r_2 - r_1$ of the remainders is one (18.11-13).

⁶ At the beginning of Book VII of the *Elements*.

⁷ See *Disquisitiones Arithmeticae*, Section II, especially article 19. See also the weaker, but closely related Prop. VII.30 in Euclid's *Elements*. Gauss obtains it from the factorization of an integer as a product prime numbers (art. 16). The notion of prime number is entirely absent from Brahmagupta's development: he focuses on irreducible fractions. An integer is merely a fraction with denominator unity (18.61). From this perspective, a prime number would be a number p such that the fraction a/p is irreducible unless p divides a : this notion does not arise naturally. Mesopotamian Mathematics also does not seem to have considered the notion of prime number, for different reasons.

of Chapter 18: if a and b are coprime, by Brahmagupta's Prop. 18.11-13, one can find a *sthira kuṭṭaka* ξ , and an integer η , such that $a\xi - b\eta = 1$. Therefore, if $ax = by$, we have $y = (a\xi - b\eta)y = a\xi y - (by)\eta = a\xi y - (ax)\eta = a(y\xi - x\eta)$, so that, indeed, a divides y . As we shall see, Brahmagupta's discourse goes in the opposite direction: his analysis of equation $ax = by$ in chapter 12 leads to a derivation of his results on the *kuṭṭaka*.

While the literature on congruence problems in India is extensive, and cannot be reviewed here, it appears that the Propositions considered here have never been studied earlier. In fact, Brahmagupta's treatment seems to be seldom considered independently⁸, probably because Prop. 18.3-6 appear to be a mere elaboration of Āryabhaṭa's Prop. II.32-33.

2. Forms of mathematical discourse.

2.1. Apodictic and dogmatic discourse.

Mathematical activity is recorded and transmitted through the production of discourses, aimed at a variety of audiences: lectures for students, articles or seminar talks for colleagues, or even mathematical diaries for oneself, such as those Gauss used to write. The basic tension between *dogmatic* and *apodictic* elements in discourse corresponds roughly to the tension between teaching and research; *dialogue forms* hold an intermediate position. Typically, a textbook is an example of the dogmatic type, and a research article, of the apodictic type.

Dogmatic discourse, as the etymology indicates, is adapted to teaching, to the imparting of predefined knowledge or skills to a given population. Motivation is minimal, material may not follow the historical order, and artificial examples may be cooked up. Dogmatic discourse suppresses the coherence of the subject and replaces it by another one, suggested by the simplified form that is being taught. At the same time, it must give the student a feeling that the methods being taught are reliable, that they will work in all practical cases, for a plausible reason. Dogmatic discourse is therefore conclusive, but not necessarily coherent.

Dialogues are often found in Indian philosophy: when a student expresses doubts to the master. Here, motivation and derivations may be clarified, but the master's answers in this limited situation are not always conclusive: objections that the student did not think of will not be dealt with. Seminars fall in the same category, especially if they are followed by a substantial discussion. Dialogues are suggestive, often coherent, but not always conclusive.

Apodictic discourse is typically encountered in research, in papers and in some seminars, most often when the writer needs to introduce a new idea, and the audience is fully aware of the state of the art. Here, the problem must be stated without transposition, and all logical objections must be addressed. At the same time, it must remain understandable. The author must therefore compose a discourse that is meant to convince experts but, for that very reason, must not clutter the exposition with elementary material, and produce an apodictic discourse, meant to be both coherent and conclusive.

⁸ An exception is: Pradip Kumar Majumdar ("Rationale of Brahmagupta's method of solving $ax + c = by$," *Indian Journal of History of Science*, 16 (2), (1981), 111-117), who points out correctly that it is unlikely that Brahmagupta directly worked with continued fractions, even if the sequence of operations may be rephrased in terms of them. For background information, without aiming at completeness, one may suggest the well-known treatise by Datta and Singh (*A History of Hindu Mathematics*, Lahore: Motilal Banarsidass, 1938) and, for the literature on Āryabhaṭa's treatment, A. Keller's *Expounding the Mathematical Seed : Bhāskara I on the Mathematical Chapter of the Āryabhaṭīya* (two volumes, Birkhäuser, Berlin, 2006).

It may happen that a research work adopts the dogmatic style. Wallis⁹ wrote that Archimedes seems “as if of set purpose to have covered up the traces of his investigation, as if he had grudged posterity the secret of his method of inquiry, while he wished to extort from them assent to his results”.

There are many mixed types, such as courses that take into account the historical evolution of the subject, or spell out the role of famous problems in the emergence of mathematical concepts. Diaries and mathematical correspondence are often of the apodictic type, because the distortions or transposition due to the limited background of the audience, and typical of the dogmatic style, are absent. We also leave aside memory-aids, lists of exercises with answers, codes, lists, tables or charts. They are not discourses: they only make sense in terms of an implied, or suppressed discourse that may sometimes be inferred from the very sequence of items, or their organization¹⁰. We only deal here with spelled-out discourses.

The production of apodictic and dogmatic discourses is one part of mathematical activity. Another is the analysis of other people’s discourses, especially those received from one’s own master. In the standard *guru-siṣya* relationship, the student knows that the words or the master are meaningful, and that examining them closely is part of student duties. The master, or other advanced students, may be available to dispel doubts if the student’s understanding is incorrect. However, the master may refrain from spelling out the full import of the text if the student is not ready to receive them.¹¹

Brahmagupta’s discourse is an example of an apodictic discourse for a sophisticated public. He uses identifiable discursive strategies that we describe next. A similar analysis also applies to Baudhāyana’s *Śulvasūtra*¹² and, no doubt, to many major scientific texts in India or elsewhere.

2.2. Brahmagupta’s discursive strategies.

His chapters usually open with a statement of purpose, or list of topics. Relations between sections are indicated when they are not immediately obvious from the sequence of propositions. Thus, even though the *kuṭṭaka* appears at first sight to be only treated in chapter 18, Brahmagupta’s discourse on it really begins in chapter 12, of which the final section closes with the mention: “this is only the general direction: I will say more in the *kuṭṭaka* and the *vyotpatti*” (12.66). Here, *kuṭṭaka* is a transparent allusion to chapter 18, where Brahmagupta stresses its importance at the outset (in 18.1), and then expounds its solution in great detail (18.2-30). The *vyotpatti* or “generation of sines” is a part of chapter 21 (Prop. 21.17-23), it describes a method to find sines recursively, that requires the extraction of square roots of quantities having a fractional sexagesimal part. This is precisely the topic of the second part of this final section (12.62-65). Therefore, the first part of the concluding section of Chapter 12 prepares the ground for the *kuṭṭakāra*, and the second part, for the generation of sines.

If needed, a change of subtopic within a section may be indicated by stylistic devices; for instance, the beginning and end of the derivation of the area of a cyclic quadrilateral are

⁹ As quoted by Heath (*A History of Greek Mathematics*, vol. II, Oxford, 1921. Reprint: Dover, 1981, see p. 20).

¹⁰ An example may be found on p.199 of *Ramanujan’s Notebooks, Part III* (B. Berndt, Springer-Verlag, New York, 1991), in which we find a rule for solving $x^3 + y^3 = u^3 + v^3$ in integers, followed by a list of examples. As the editor observes, the last example is not a special case of the rule. Thus, Ramanujan is implying that he is aware that the rule does not provide all solutions.

¹¹ As in *Chāndogya Upaniṣad* VIII.7 sqq.; see also *ibid.*, VII.1 where the master adapts teaching to the student’s current level.

¹² Satyanad Kichenassamy, 2006. “Baudhāyana’s rule for the quadrature of the circle. *Historia Mathematica*,” 33(2), (2006), 149-183. <http://dx.doi.org/10.1016/j.hm.2005.05.001> and “Textual analysis of ancient Indian Mathematics,” *Gaṇita Bhāratī*, 33 (1–2), (2011), 15–28.

marked by the recurrence of the technical term for it, *tricuturbhuja*. Here we shall see that the beginning and end of an argument are indicated respectively by the statement of the problem and by the spelling out of the assumptions. The reader is expected to understand that the section is not complete until the statements are mathematically complete.

Sequential reading is implied by default. Brahmagupta also indicates jumps from one part of the text to another when needed. Mentions such as “as before” are clear enough. Motivation is often suggested by contrast. For instance, in Prop. 12.21, two incompatible formulae for the area of a cyclic quadrilateral are given; one depends on the order of sides, the other does not. This suggests that the fact that the area must be a symmetric expression in the sides of the quadrilateral plays a role in the derivation – and perhaps was even the motivation for seeking such an expression in the first place. Assumptions are often stated together with the result, as the condition for cyclicity in Prop. 12.21, 12.27, or the perpendicularity condition in Prop. 12.30-31. Also, the text may be read at several levels. Taking again 12.21 as an example, one of the formulae is “gross”, the other “subtle”: this implies a process of thought, from one to the other, and possibly beyond. Finally, redundancy is suspect, as in 12.24, that seems to state three times the property of the ‘diagonal’¹³ of a right triangle.¹⁴

Thus, Brahmagupta is giving out information in a progressive, thought-out format, and giving the reader or listener the means to follow his meaning. To understand Brahmagupta, we need to act as one of his students would have, to try and understand why his discourse is conclusive and coherent, why it is apodictic rather than dogmatic.

3. Propositions 12.58-61.

We give in this section the text and translation of each of these four propositions, followed by a gloss. A more detailed analysis of 12.58 will be given in the next section.

12.58 *guṇyaśchedaphalavadho guṇakahrto guṇyabhājito guṇakaḥ /
chedoddhṛtaḥ phalaṃ guṇyaguṇavadhaḥ phalahṛtaśchedaḥ ||*

The multiplicand is the product of quotient and divisor divided by the multiplier.

Divided out by the multiplicand, it is the multiplier.

The quotient is [obtained] by dividing off by the divisor the product of multiplicand and factor.

Dividing [this product] by the quotient, [we obtain] the divisor.

Gloss. We deal here with four quantities, called: multiplicand, quotient, multiplier and divisor. Calling them respectively a, b, x, y , the four parts of this proposition would appear to state that $a = \frac{by}{x}$, $x = \frac{by}{a}$, $b = \frac{ax}{y}$ and $y = \frac{ax}{b}$. This interpretation is not satisfactory because it is redundant. However, in Indian Mathematics, such names may represent places on a board, that may be filled with values that could change in the course of an operation, some being erased and replaced by others, as we shall see in the very next proposition. It is therefore by no means necessary that the four names should represent the same values in the four operations described. A first indication is that there are two names for the multiplier x (*guṇaka* and *guṇa*, rendered by ‘multiplier’ and ‘factor’). The fully symmetric formulation shows that one may, in some

¹³ In Sanskrit, *karṇa*. Recall that the right triangle should be conceived as a half-rectangle. The Śulvas call it *akṣṇayā* (*rajju*).

¹⁴ For a discussion of the possible meanings of this proposition, see Kichenassamy (2010) (reference in note 3 above), section 4.4, pp. 45-47.

calculations, divide by this “multiplier”, or multiply by the “divisor.” Similar remarks apply to all four quantities. All this suggests that the names of these quantities are merely labels for variables. We shall see in the next section that indeed, the value of x in the third and fourth relations is not the same as in the first two ones.

12.59 *guṇyaguṇakārayośchedalabdhayoryadi dvayordvayor nāśah /
teṣāṃ drśyau vyastau kṛtvā tatsthānayośceṣṭau ||*

If one of the pairs, formed with the multiplicand or multiplier,
And the divisor or quotient, is destroyed,
The two among these [quantities that are still] visible are exchanged, and
One sets two [times the same] arbitrary [quantity] in their two places.

Gloss. Brahmagupta now deals with the situation in which one of the pairs (a, b) , (a, y) , (x, b) , (x, y) of which the first term is the multiplier or the multiplicand, and the second is the divisor or quotient, is missing (has been “destroyed”). This describes four problems of the same type; for instance, if x and y are missing, we are dealing with the equation $au = bv$, with unknowns u and v . If a and y are missing, the problem reads $xu = bv$, and so forth. Let us consider $au = bv$ for definiteness. Brahmagupta directs us to put a and b in the reverse order, in the missing places for x and y , and to put an arbitrary k in the places left vacant by this moving around of a and b . The arrangement suggested is similar to the following, where the asterisk (*) denotes a vacant place. The first arrow corresponds to the “destruction”, the second to the transposition, and the third to the filling-in of the vacant places.

$$\begin{array}{ccccccc} a & x & \rightarrow & a & * & \rightarrow & * & b & \rightarrow & k & b \\ b & y & \rightarrow & b & * & \rightarrow & * & a & \rightarrow & k & a \end{array}$$

Multiplication being implied¹⁵, since $ax = by$, the solution is $x = kb$, $y = ka$. This is the general solution of $ax = by$ if a and b are integers without common factors, but not otherwise. For instance, if we consider what Brahmagupta calls *karaṇī* (irrational surds, see section 5.2 below), it is easy to find counterexamples using the results he gives in chapter 18: since $(\sqrt{10} - 2)(\sqrt{10} + 2) = 6 = 2 \times 3$, equation $(\sqrt{10} - 2)x = 2y$ does not imply that x is an integral multiple of 2. This is already false for fractions. For instance, $\frac{25}{6}x = 2y$ holds with $y = 1$ and $x = \frac{12}{25}$, and the latter is not an even integer. It is however true that the most general solution of $25x = 12y$ is $(12k, 25k)$.¹⁶ Therefore, the reader is expected to understand that Brahmagupta’s argument is not complete, because essential assumptions have not been stated yet.

One cannot take the missing places to be, say, b and y , or divisor and quotient, since this would lead to the problem $ax = uv$, namely: “find all possible divisors of ax .” This problem is quite complicated for large numbers¹⁷ such as occur in Indian astronomy, and Brahmagupta does not seem to have given here any procedure to solve it.

¹⁵ One could equally have put a and c in a column. The symmetric wording of 12.58 also implies that the material arrangement is irrelevant.

¹⁶ If $x = \frac{x_1}{N}$, $y = \frac{y_1}{N}$, with integral numerators, the equation becomes $25x_1 = 12y_1$, hence $25 \mid y_1$. The result follows.

¹⁷ This complexity makes some modern cryptographic methods possible: if you transmit the value of ax over an insecure channel, it is computationally difficult to recover the possible values of a from it.

The next two propositions are unproblematic as far as their meaning is concerned. The issue is to understand why they have been stated at this point of the discourse. Indeed, they appear to contain no new results. We show that they contain essential assumptions.

12.60. *guṇyaṃ guṇakāraṃ vā guṇayecchedena bhāgahārasya |*
guṇyaguṇakārarāśyośchedaguṇo bhāgahāraśca ||

The multiplicand or the multiplier
 Should be multiplied by the denominator of the divisor,
 And the divisor, multiplied by the denominators
 Of the [fractional] quantities that are multiplicand and multiplier.

Gloss. The meaning of this proposition is clear: an expression of the form ax/b must be transformed by multiplying the numerator of a or x by the denominator of b , and the denominator of the resulting fraction, by the product of the denominators of a and x . In symbols, assuming the denominator of b multiplies the numerator of a for definiteness,

$$\frac{\left[\frac{a_1}{a_2} \times \frac{x_1}{x_2}\right]}{\frac{b_1}{b_2}} \rightarrow \frac{\left[\frac{a_1 b_2}{a_2} \times \frac{x_1}{x_2}\right]}{b_1} \rightarrow \frac{a_1 b_2 x_1}{b_1 a_2 x_2}.$$

We multiplied here the numerator of a by b_2 but, as Brahmagupta points out, we could just as well have multiplied the numerator of x instead.

This appears to be again redundant, since it is an immediate consequence of the results on the reduction of fractions to the standard form u/v , given at the beginning of Chapter 12 (see 12.8-9). It therefore has another import. It brings in an essential piece of information: since the multiplicand and multiplier have a denominator *they must be fractional*. Thus, Brahmagupta's quantities in this problem are fractional unless otherwise specified.

12.61. *acchedasya cchedaṃ rūpaṃ kṛtvā 'nyaduktavat sarvam |*
apavartyau chedaguṇau tulyeneṣṭena guṇyau vā ||

Making unity the divisor of a divisorless quantity,
 Everything else [goes through] as has been said.
 Divisor and multiplier, resp. multiplicand, should be abridged
 By the same arbitrary [quantity].

Gloss. Since all quantities are all fractional by default, integers must be treated as fractions with unit denominator. This makes it possible to state results in a uniform fashion, where some of the quantities could be fractional, and others integral. It is also here that the condition that a/b is irreducible is spelled out. With this restriction, we have reached a statement that is neither redundant nor unacceptable from Brahmagupta's perspective: if a/b is irreducible, all fractions with the same value may be written $\frac{ka}{kb}$ with k an arbitrary fraction. Taking the example given above, if $\frac{25}{6}x = 2y$, then $x = \frac{12}{25}y$, where $25/12$ is irreducible. Therefore, $(x, y) = (12k, 25k)$, with k arbitrary. For instance, if $k = 1/25$, we recover the solution $y = 1$ and $x = \frac{12}{25}$. It remains to analyze Brahmagupta's suggested derivation of his results, encoded in Prop. 12.58.

4. Analysis of 12.58.

Let us examine the relations between the four divisions, referred to in the sequel as (A), (B), (C) and (D), in the light of Brahmagupta’s conceptual framework, without assuming that the four quadruplets involved are the same. Recall that this is an introduction to the solution of problems of the form $ax + r_1 = by + r_2$, and that Brahmagupta already knows Āryabhaṭa’s solution of it. We show that the four divisions are steps in an argument that becomes natural if one keeps Brahmagupta’s results in mind, and forgets everything else. Formulae are written in modern notation for the convenience of the modern reader only.

4.1. Step 1. Using 12.57. Let us start from the first division

$$a = by/x. \tag{A}$$

All four quantities are nonnegative by default. Let us apply the immediately preceding proposition, Prop. 12.57, that expresses that

$$\frac{a}{b} = \frac{a}{b \pm \varepsilon} \pm \frac{a}{b \pm \varepsilon} \times \frac{\varepsilon}{b}.$$

To this end, we define q and x' by $x = qy + x'$, with $0 \leq x' < y$. The fraction by/x thus appears as a modification of by/qy , in which simplification by q is possible. Applying 12.57 to (A), we obtain

$$a = \frac{by}{x} = \frac{by}{qy + x'} = \frac{by}{qy} - \frac{by}{qy} \times \frac{x'}{x} = \frac{b}{q} - \frac{bx'}{qx}.$$

Hence $\frac{b}{q} - a = \frac{bx'}{qx} \geq 0$. Define $b' = b - qa \geq 0$. It follows that $\frac{b}{q} - a = \frac{b'}{q}$. Therefore,

$$b' = \frac{bx'}{x}. \tag{A*}$$

Now, the question arises: Is the quotient q , in the division with remainder of x by y , also equal to the quotient of b by a ? Since $b' = b - qa$, this is true if $0 \leq b' < a$. But so far, Brahmagupta only knows the first inequality. The second follows from Step 2.

4.2. Step 2. Using (B) and 12.9 to derive (C). At this point, Brahmagupta tells us how to proceed by stating division (B):

$$x = \frac{by}{a}, \tag{B}$$

which is a consequence of (A). This suggests replacing x by this value in (A*):

$$b' = \frac{bx'}{x} = \frac{bx'}{\frac{by}{a}} = \frac{abx'}{by} = \frac{ax'}{y}.$$

Here, 12.9 has been used for the third equality. Hence,

$$b' = \frac{ax'}{y}. \tag{C}$$

Since we know that $0 \leq x' < y$, we now know that $b' < a$. Brahmagupta has therefore proved that

- If $ax = by$, then the quotient of the division with remainder of b by a is equal to the quotient of the division with remainder of x by y .
- In addition, by (A*), $\frac{x}{b} = \frac{x'}{b'}$. Hence, both divisions terminate at the same time: $x' = 0$ if and only if $b' = 0$.

Here, the quotient q is an integer, but the other quantities need not be. Equality (A) is the starting-point of the argument, (B) is a hint that guides the derivation, and (C) is the outcome of the second part of the argument.

4.3. Steps 3 and 4. Using (C), 12.57 and 12.9 to derive (D).

Since we have not made use of (D) yet, the argument is not complete. It is natural to apply to (C) the method we used to transform (A), taking (D) as a hint analogous to (B). The argument is entirely similar to the first two steps. Start from (C), that yields $b'y = ax'$, and define q' and y' by $y = q'x' + y'$, with $0 \leq y' < x'$. Apply 12.57 to (C), and obtain

$$b' = \frac{ax'}{y} = \frac{ax'}{q'x' + y'} = \frac{ax'}{q'x'} - \frac{ax'}{q'x'} \times \frac{y'}{y} = \frac{a}{q'} - \frac{ay'}{q'y'}$$

Hence $\frac{a}{q'} - b' = \frac{ay'}{q'y'} \geq 0$. Define $a' = a - q'b' \geq 0$. We then have $\frac{a}{q'} - b' = \frac{a'}{q'}$. Hence,

$$a' = \frac{ay'}{y}. \quad (C^*)$$

To go further, we take into account Brahmagupta's fourth division, which follows from (C):

$$y = \frac{ax'}{b'}. \quad (D)$$

Substituting (D) into (C*) and using 12.9 again, we obtain

$$a' = \frac{ay'}{y} = \frac{ay'}{\frac{ax'}{b'}} = \frac{b'y'}{x'}.$$

Hence,

$$a' = \frac{b'y'}{x'}. \quad (A')$$

Since, from the definition of y' , we know that $0 \leq y' < x'$, we conclude from (A') that $a' < b'$. Therefore, $0 \leq a' < b'$, which implies that q' is not only the quotient, with remainder, of y by x' , but also the quotient of a by b' .

Brahmagupta has therefore established the following.

- The quotient, with remainder, of y by x' is equal to the quotient of a by b' .
- $\frac{y}{a} = \frac{y'}{a'}$, so that both divisions terminate together.

We now have a relation (A') of the form (A), except that all the variables have been replaced by their primed counterparts. This suggests an iteration, leading to two sequences $(b, a, b', a', b'', a'', \dots)$ and $(x, y, x', y', x'', y'', \dots)$.

To sum up, Brahmagupta has obtained the following results: if $ax = by$, or $\frac{y}{a} = \frac{x}{b}$, or $\frac{a}{b} = \frac{y}{x}$ etc.¹⁸

- The quotients of the mutual division of b by a are equal to those of the division of x by y .
- Both divisions terminate at the same step.
- The remainders in these divisions are related at every step by the relations

$$\frac{x}{b} = \frac{x'}{b'} = \frac{x''}{b''} = \dots = \frac{y}{a} = \frac{y'}{a'} = \frac{y''}{a''} = \dots.$$

4.4. Solution of $ax = by$ in integers.

This problem is the congruence problem without remainders: “Find $N = ax = by$ that leaves zero remainder when divided by a by b .” If a/b is an irreducible fraction, with a and b multiples of the *rūpa* (unity), the remainders a', b', \dots form a decreasing sequence of multiples of the unit, so that the mutual division must end when the last remainder is unity. Indeed, this last remainder divides both a and b .¹⁹ Since all quantities are integers, the common value of $\frac{x}{b}$ and $\frac{y}{a}$ has the form $\frac{k}{1}$, where k is an integer. Therefore, $x = kb$ and $y = ka$. It follows that $ax = by = kab$. This establishes that the *yuga* in this situation is the product of the *yugas*, as Brahmagupta states in 18.6. If x and y are fractional, $x = \frac{x_1}{N}$, $y = \frac{y_1}{N}$, with integral numerators, and we are led to the problem $ax_1 = by_1$, to be solved with integer unknowns. The solution is therefore also valid if the unknowns are fractions.

4.5. Solution of $ax + r_1 = by + r_2$.

The solution of congruence problems is now immediate. Divide b by a : $b = qa + b'$. Define $x' = x - qy$, and subtract qay from both sides of the equation to be solved: $ax + r_1 - qay = by + r_2 - qay$, hence $a(x - qy) + r_1 = (b - qa)y + r_2$. Therefore,

$$ax' + r_1 = b'y + r_2.$$

We are left with a problem with the same remainders, and smaller divisors. From x' and y , we recover $x = qy + x'$. The argument may be iterated and leads to the usual construction involving the sequence of quotients in mutual division²⁰. The solution is determined up to a multiple of ab if a/b is irreducible. As we pointed out earlier, this result is not obvious.

4.6. Conclusions from the discussion of 12.58-61.

The four divisions in 12.58 are steps in an argument: (A) is the starting point of the analysis of $\frac{a}{b} = \frac{y}{x}$, and indicates to what expression the previous proposition 12.57 should be applied. (B) is a hint indicating how to continue the argument. (C) is an intermediate result. Since it is similar to (A), the parallel formulation of the two halves of 12.58 suggests applying

¹⁸ Again, this multiplicity of interpretations is implied by the complete symmetry in Brahmagupta's formulation.

¹⁹ This observation is also the basis of Prop. 18.9-11 on the *sthira-kuttaka*.

²⁰ Described in Prop. 18.3-6.

the same argument to it. (D) is a hint, like (B). After (A) has been transformed into (C), two of the quantities have changed: b and x have become b' and x' . After the second round of transformations, from (C) to (A'), all four quantities have been replaced. The completely symmetric formulation and apparent redundancy suggest the structure of the argument. The discursive structure indicates the results, their assumptions, as well as the elements of derivations, and the very precise formulation guides the reader.

5. Other problematic propositions of Brahmagupta's.

We analyze a few other passages from *BSS* that also do not make sense as mere rules, but become significant if their discursive function in an apodictic discourse is examined.

5.1. Extension of operations, and the concept of identity.

Consider the following Proposition, that immediately follows the rules that extend operations to negative quantities.

18.37. *yogo 'ntarayutahīno dvihṛtaḥ saṅkramaṇamantaravibhaktam vā |*
vargāntaramantarayutahīnaṃ dvihṛtaṃ viṣamakarma ||

Combining the sum and the difference, or depriving it from the latter,
 And dividing [the result] by two is the *saṅkramaṇa*; one may also [find the same quantities by]
 Dividing the difference of the squares by the difference, adding or subtracting the difference
 And dividing [the result] by two, [which is] is the *viṣamakarma*.

Gloss. This statement expresses that we may recover two different quantities a and b from their sum and difference, or from their difference and the difference of their squares, through the following:

$$a, b = \frac{1}{2} [(a + b) \pm (a - b)] = \frac{1}{2} \left[\frac{a^2 - b^2}{a - b} \pm (a - b) \right].$$

These are among the basic tools of Indian Mathematics, and their systematic use is extremely fruitful. However, they are hardly new. Brahmagupta already made casual use of a similar result (12.22)²¹ and Āryabhaṭa (II.24) had already given a closely related result²²:

$$a, b = \frac{1}{2} \left[\sqrt{4ab + (a - b)^2} \pm (a - b) \right].$$

These relations are consequences of the properties of multiplication and addition or subtraction, that Brahmagupta used in chapter 12 for numbers, fractions, and lengths, and that he just extended to negative numbers. Coming right after the section introducing negative numbers, his proposition expresses that these relations remain valid for negative numbers as well. This is an extension of an identity to a wider set of quantities.

²¹ In (12.22), two sides of a trilateral are found from the difference of their squares, and their sum.

²² Also given by Brahmagupta in Prop. 18.100, in very similar terms.

5.2. How to give a constructive definition.

The next proposition introduces a new development on combinations of surds. Brahmagupta begins with a result on the simplification of sums of surds and, characteristically, includes a constructive definition embedded within the statement of his result:

18.38. *karaṇī lambastadkṛtiriṣṭahrteṣṭonasamṃyutā 'lpā bhūḥ |*
adhiko dvihṛto bāhuḥ samkṣepyo yadvadho vargaḥ ||

The “maker” is the perpendicular. Its square is divided by an arbitrary,
 And decreased or increased by this arbitrary. The smaller [result] is the base,
 And the larger, divided by two, is the side.
 Those [“makers”] whose product is a square should be lumped together.

Gloss. Brahmagupta redefines here the phrase “*karaṇī* (maker) of a ,” that referred in the *Śulvasūtra* to the measure of the side of a square of area a , and restricts it to surds: unlike numbers, surds cannot be added indiscriminately. Brahmagupta extends the operations to sums and differences of such surds, and shows here that they can be combined in some cases.

Prop. 18.38 is a geometric definition of quadratic surds: if the surd is written \sqrt{N} , its square is N . It is indeed the “ N -maker”, that may be constructed in infinitely many ways: for any “arbitrary” k , Brahmagupta states that the surd \sqrt{N} is the perpendicular of an (isosceles) triangle with base $\left(\frac{N}{k} - k\right)$ (or its opposite), and side $\frac{1}{2}\left(\frac{N}{k} + k\right)$. The triangle is isosceles since only one side is mentioned. While he deals with quadratic irrationals, Brahmagupta, who had discussed the extraction of cube roots of integers (12.7), does not deal with irrational cube roots, possibly because he does not have a construction of them.

It remains to explain how to lump together the roots of two quantities (\sqrt{a} and \sqrt{b}) when their product is a square ($ab = c^2$). Since these quantities have a geometric interpretation, so do their sums and products. Let us consider the line $\sqrt{a} + \sqrt{b}$. Its square is $a + b + 2\sqrt{ab} = a + b + 2c$. Therefore, $\sqrt{a} + \sqrt{b} = \sqrt{a + b + 2c}$, so that the two surds can indeed be lumped together. For instance,²³ $\sqrt{28} + \sqrt{63} = \sqrt{175}$.

Brahmagupta proceeds to show how to add, subtract, multiply, divide and extract square roots of expressions involving quadratic surds. Despite their interest, we cannot dwell upon these results here. This section (18.38-41) is immediately followed by the introduction of algebra of the “non-manifest” (*avyakta*).

5.3. The introduction of literal algebra (18.42-43).

18.42. *avyaktavargaghanavargavargapañcagataṣaḍgatādīnām |*
tulyānām saṅkalitavyavakalite prthagatulyānām ||

Of the non-manifest, the square, cube, square of square,
 “Reached [the power] five,” “reached the [the power] six,” and so forth,
 Are added or subtracted when they are of the same [nature] ;
 [But left] separate if they are not of the same [nature].

²³ For Brahmagupta, the standard reduced form is a pure surd: he does not consider, say, $2\sqrt{7}$ to be the standard form of $\sqrt{28}$. Traces of this point of view are still found in the Renaissance. See *op.cit.* note 4, p. 422, for an example where the failure to perform the simplification advocated by Brahmagupta sheds light on the *modus operandi* of the time.

Gloss. Brahmagupta introduces here one indeterminate whose value is “non-manifest”²⁴, and its successive powers. The powers can be arbitrarily high, and are labeled by their exponents from the fifth power onwards. Their combinations – polynomials in one variable –, are supposed to be written in a standard form, in which all the terms proportional to a given power are collected, but different powers are left separate. There is a progression from numbers and fractions to negative numbers and surds, and then to polynomials in one indeterminate.

18.43. *sadrśadvivadho vargastryādivadhastadgato ’nyajātivadhah |*
anyonyavarṇaghāto bhāvitakah pūrvavaccheṣam ||

The product of two identicals is the square.

The product of three or more has “reached that [power]”. The product of different species, Or of [two] letters by one another [yields] the *bhāvitaka*.

The rest [goes] as before.

Gloss. This represents an extension and modification of the previous proposition, and introduces several variables. The square, left undefined in 18.42, is defined here as a product of “two identicals” rather than an area; higher powers are defined similarly, which rules out a direct geometrical interpretation. The mention of “identicals” suggests that there may be “non-identical”, or different kinds of variables. The powers here are those of all these variables. The product of two letters²⁵ is introduced with a special name. Brahmagupta therefore introduces an algebra with several variables. Equations with one variable are discussed in 18.44-51. Those with two variables or more are discussed in 18.52-60, those involving the *bhāvita* or *bhāvitaka* in 18.61-64 and the *vargaprakṛti*²⁶ – also an equation involving two variables (or four, if the coefficient and the additive are left unspecified) – are discussed afterwards.

Powers higher than the third are again labeled by their exponents, but now, already starting from the third power. By contrast, higher powers in Greek, Arabic or in Italian texts are typically represented by combinations of squares and cubes, with occasional special names for fifth, seventh or other powers²⁷. It is quite remarkable that the fourth power is not viewed in 18.43 by Brahmagupta as the square of the square, while it is in 18.42. Both propositions seem to represent two stages of algebra: one in which there is only one unknown, and a several-variable algebra, allowing for products of unknowns. Brahmagupta, by including both, is stressing that the new emerges from the old, and that some historical perspective is necessary to understand innovation.

The idea that mental constructs cannot be viewed directly as objects of senses, and that reasoning must allow for inference in addition to sense-perception is widely accepted by most Indian schools with the exception of the Lokāyata²⁸, a school that all other systems dismiss as holding unreasonable theses. In this context, it is natural that Brahmagupta should allow for the

²⁴ The tension between manifest and non-manifest is a common one in Indian philosophy.

²⁵ Recall that the term *varṇa* means both “letter” and “color”, being derived from the verbal roots *vr-* “to choose” (ninth conjugational class) and *vr-* “to cover (fifth class)” and that the variables in Indian algebra are represented by single letters of the Sanskrit language, that may or may not be initials of names of colors. The attestations are too numerous to be listed here.

²⁶ Equation $Nx^2 + k = y^2$.

²⁷ Such as *primo relato* in Italian for the fifth power.

²⁸ They seem to have considered that all means of knowledge, including inferential knowledge, are ultimately reducible to sensory perception.

use in reasoning of non-manifest mathematical objects that exist even if they cannot be perceived directly, such as the value of an unknown before one has determined it.

6. Conclusions.

The analysis of Brahmagupta's text has led to the following results:

- 12.58 does not state four times the same division, but describes the steps in the transformation of a set of four quantities. It motivates the introduction of mutual division – the main tool in the Indian theory of congruences.
- The names “multiplier”, “divisor”, etc. in 12.58 do not refer to the same set of four quantities in the four divisions that it describes.
- Brahmagupta gives in 12.58-61 the general solution of equation $ax = by$, in a way that suggests its derivation. The essential assumption, that a/b should be in lowest terms, is stated in 12.60-61.
- This yields the derivation of the *kuṭṭākāra* method for solving congruences, and proves rigorously its general validity.
- The non-trivial fact that the solution of the *kuṭṭaka* problem is determined modulo ab (18.6) is justified by Prop. 12.58-61, that imply what is known in modern Mathematics as “Gauss’ lemma.”
- Brahmagupta has extended in chapter 18 the validity of operations on fractions to negative numbers, sums of surds, and finally general “non-manifest” quantities.
- He describes, possibly for the first time²⁹, a literal algebra in several variables.
- Fractions are, for Brahmagupta, more basic objects than integers.

Thus, the problematic passages in Brahmagupta's *BSS* may be accounted for by reading *BSS* as an apodictic discourse: it contains within itself its own justification, and succeeds in conveying the coherence of the subject as the author sees it. It is aimed at an active reader with a strong background. Apodictic discourse may be contrasted with dogmatic discourse, that aims at convincing a passive, general reader with minimal background, as well as with other intermediate types of discourse. Precise terminology and taut discursive structure were transparent for scholars of his time, but only partially so for casual readers or for proponents of rival schools.

²⁹ The relation between the theory of equations in India and in China also needs to be investigated. It appears that there is information about Indian mathematics available in Chinese sources, that cannot be obtained from Indian sources at this time (see S. Kichenassamy, *Gaṇita Bhāratī* 42(2) (2018), 181-190). The concepts and discursive strategies of Chinese Mathematics are just beginning to be analyzed (see K. Chemla and Guo, Shuchun, *Les neuf chapitres : Le Classique mathématique de la Chine ancienne et ses commentaires*, Paris: Dunod, 2004). This “Classic” 經, the *Jiuzhang Suanshu* 九章算術, does not deal with congruence problems. However, its eighth chapter is devoted to *fangcheng* 方程 (of disputed meaning, see K. Chemla's discussion on p. 922) and, according to Zhu Yiwen 朱一文, *fangcheng* was considered as the source of the solution of congruence problems in China, before the solution method called *dayan* 大衍 was recorded in the 13th c. (*Studies in the History of Natural Sciences*, 30 (2) (2011) 193-206 and *Zhouyi Yanjiu* (2019) (2) 81-92 (both in Chinese)). This confirms that it is futile to discuss transmission issues before the contents of the texts has been analyzed. For India and China, this task has not been completed.