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AN ALGORITHM TO DETERMINE REGULAR SINGULAR MAHLER SYSTEMS

COLIN FAVERJON AND MARINA POULET

ABSTRACT. This paper is devoted to the study of the analytic properties of Mahler systems at 0. We give an effective characterisation of Mahler systems that are regular singular at 0, that is, systems which are equivalent to constant ones. Similar characterisations already exist for differential and (q -)difference systems but they do not apply in the Mahler case. This work fill in the gap by giving an algorithm which decides whether or not a Mahler system is regular singular at 0.

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1. INTRODUCTION

Let \mathbf{K} be the field of Puiseux series with algebraic coefficients *i.e.* the field

$$\mathbf{K} := \bigcup_{d \in \mathbb{N}^*} \overline{\mathbb{Q}} \left(\left(z^{1/d} \right) \right).$$

For an integer $p \geq 2$ we define the operator

$$\phi_p : \begin{array}{ccc} \mathbf{K} & \rightarrow & \mathbf{K} \\ f(z) & \mapsto & f(z^p) \end{array}.$$

The map ϕ_p naturally extends to matrices with entries in \mathbf{K} . A p -Mahler system or, for short, a Mahler system is a system of the form

$$(1) \quad \phi_p(Y) = AY, \quad A \in \mathrm{GL}_m(\overline{\mathbb{Q}}(z)).$$

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The study of Mahler systems began with the work of Mahler in 1929 [Mah29, Mah30a, Mah30b]. Nowadays, there is an increased interest in their study because they are related to many areas such as automata theory or divide-and-conquer algorithm (see for example [Cob68, MF80, Dum93, Nis97, Phi15, AF17, AF20] for a non-exhaustive bibliography). In this paper, we focus on the singularity at 0 of Mahler systems. Similarly to the case of differential or (q -)difference systems, a singularity at 0 of a Mahler system can be regular. In that case, we say that the Mahler system is *regular singular* at 0. Otherwise, the system is irregular at 0. Precisely, we have the following definition.

Definition 1. A p -Mahler system (1) is *regular singular* at 0, or for short *regular singular*, if there exists $\Psi \in \mathrm{GL}_m(\mathbf{K})$ such that $\phi_p(\Psi)^{-1}A\Psi$ is a constant matrix.

The singularities of differential or (q -)difference systems have been widely studied and algorithms have been given. One of the main interests in studying the regular singular systems is the good analytic properties of their solutions. Linear differential systems have been some of the first ones to be studied. A linear differential system is regular singular at $z = 0$ if and only if all of its solutions have moderate growth at $z = 0$, that is, at most a polynomial growth (see for example [vdPS03, theorem 5.4]). Some criteria and algorithms have been given for a linear differential system to have a regular singularity, see for example [Bir13, Mos59, HW86, Hil87, Bar95]. Then, algorithms have been given for other systems such as difference systems and q -difference systems (see for instance [Pra83, Bar89, BP96, BBP08]). In [BBP08], the authors give a general algorithm for recognizing the regular singularity of linear functional equations satisfying some general properties. This algorithm applies to many systems such as differential systems and (q -)difference systems. However, this general algorithm does not apply to the study of the regular singularity at 0 of Mahler systems because the Mahler operator ϕ_p does not preserve the valuation at 0. The aim of this paper is to fill in this gap and to present an algorithm which decides whether or not a Mahler system is regular singular at 0.

In general, a Mahler system does not admit a fundamental matrix of solutions in $\mathrm{GL}_m(\mathbf{K})$. To find such a matrix, one has to consider some extensions of \mathbf{K} . Let \mathcal{H} denote the field of Hahn power series. One can extend the operator ϕ_p to \mathcal{H} . In [Roq20], Roques proved that for every p -Mahler system there exists a matrix $\Psi \in \mathrm{GL}_m(\mathcal{H})$ such that $\phi_p(\Psi)^{-1}A\Psi$ is a constant matrix. In the mean time, for any constant Mahler system one can build a fundamental matrix of solutions using the functions $\log \log(z)$ and $\log^a(z)$, $a \in \mathbb{C}$ (see [Roq18]). Thus, any Mahler system has a fundamental matrix of solutions of the form $\Psi\Theta$, where Ψ is matrix with entries in \mathcal{H} and Θ is a fundamental matrix of solutions of a constant system. Among them, the regular singular systems are those for which the matrix Ψ belongs to $\mathrm{GL}_m(\mathbf{K})$. The restriction to the subfield \mathbf{K} of \mathcal{H} is essential to preserve the analytic properties of the system. In particular, if $\mathbf{f} \in \mathbf{K}^m$ is a column vector, solution of a Mahler system, it follows from Randé's Theorem [Ran92, BCR13] that the entries of \mathbf{f} are ramified meromorphic functions inside the unit disk.

Definition 2. Let $p \geq 2$ be an integer, and let $A, B \in \mathrm{GL}_m(\overline{\mathbb{Q}}(z))$. Let $\mathbf{k} \subset \mathcal{H}$ be a field. The p -Mahler systems

$$\phi_p(Y) = AY \quad \text{and} \quad \phi_p(Y) = BY$$

are said to be \mathbf{k} -equivalent if there exists a matrix $\Psi \in \mathrm{GL}_m(\mathbf{k})$ such that

$$\phi_p(\Psi)B = A\Psi.$$

In that case, the matrix Ψ is called the *associated gauge transformation*.

This choice for the equivalence class ensures that if Y is such that $\phi_p(Y) = AY$ then $\phi_p(\Psi^{-1}Y) = B(\Psi^{-1}Y)$. With this definition, the regular singular systems are the ones that are \mathbf{K} -equivalent to a constant system.

A Mahler system is said to be *Fuchsian at 0* if the entries of A are analytic functions at 0 and $A(0) \in \mathrm{GL}_m(\overline{\mathbb{Q}})$. To say it differently, a system is *Fuchsian at 0* if 0 is not a singularity of this system. It follows from [Roq18, Prop. 34] that systems which are Fuchsian at 0 are regular singular at 0. Since the multiplication by a rational function of a system does not modify its equivalence class, there exist systems which are regular singular at 0 but are not Fuchsian at 0. Note that not all Mahler systems are regular singular at 0. For example, the system

$$\phi_2(Y) = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ \frac{1}{z} & \frac{-1}{z} \end{pmatrix} Y$$

associated with the generating series of the *Rudin-Shapiro* sequence is not regular singular at 0 (see Section 6). The main result of this paper reads as follow.

Theorem 1. *Let $A \in \mathrm{GL}_m(\overline{\mathbb{Q}}(z))$ and $p \geq 2$. There exists an algorithm which determines whether or not the Mahler system (1) is regular singular at 0. This is done by computing the dimension of an explicit \mathbb{Q} -vector space. If the system is regular singular at 0 the algorithm computes a constant matrix to which the system is equivalent and a truncation at an arbitrary order of the Puiseux development of the associated gauge transformation.*

In [CDDM18], the authors built an algorithm to decide whether or not a Mahler system has a fundamental matrix of solutions in $\mathrm{GL}_m(\mathbf{K})$. In that case, the system is regular singular at 0 and \mathbf{K} -equivalent to the identity matrix. From this point of view, Theorem 1 can be seen as a generalisation of the results of [CDDM18].

Remark 1. It is also interesting to look at Mahler systems around other fixed points of ϕ_p such as 1 and ∞ . Using the change of variable $z = e^u$ one can test the regular singularity at 1 using the theory of q -difference linear systems. Furthermore, one can know if a system is regular singular at ∞ by applying Theorem 1 to the system $A(1/z)$. In particular, when a Mahler system is regular singular at 0, 1 and ∞ , the second author [Pou20] has proved a density theorem for the *Galois group* of the system.

The paper is organised as follows. In section 2 we state Theorem 2, which refines the first part of Theorem 1. We define a vector space whose dimension gives a necessary and sufficient condition for a Mahler system to be regular singular at 0. Assuming that the system is regular singular at 0, we determine in section 3 an upper bound for the degree of ramification of the gauge transformation Ψ and a lower bound for the valuation of its entries. Our proof relies on the Cyclic Vector Lemma for Mahler systems for which we provide a simple proof together with an algorithm. Section 4 is then devoted to the proof of Theorem 2. We build an isomorphism between the \mathbb{Q} -vector space spanned by the columns of Ψ and the vector space described in Theorem 2. The algorithm of Theorem 1 is described in section 5 and a bound for its complexity

is given. Section 6 is devoted to the study of some examples. Eventually, in Section 7 we discuss some open problems.

Notation. We let $\overline{\mathbb{Q}}$ denote the algebraic closure of \mathbb{Q} in \mathbb{C} and $\overline{\mathbb{Q}}^* = \overline{\mathbb{Q}} \setminus \{0\}$. We let $v_0 : \overline{\mathbb{Q}}[[z]] \mapsto \overline{\mathbb{Q}}$ denote the valuation at $z = 0$: for $f \in \overline{\mathbb{Q}}[[z]]$, $v_0(f)$ is the greatest integer v such that f belongs to the ideal $z^v \overline{\mathbb{Q}}[[z]]$. It extends uniquely to \mathbf{K} . We also extend it to the set of matrices with entries in \mathbf{K} where $v_0(U)$ denotes the minimum of the valuations at 0 of the entries of a matrix U . For a matrix $U \in \mathcal{M}_{m_1, m_2}(\mathbf{K})$ we write

$$U = \sum_{n \geq v_0(U)} U_n z^{n/d} \quad \text{with} \quad U_n \in \mathcal{M}_{m_1, m_2}(\overline{\mathbb{Q}}), \quad d \in \mathbb{N}^*.$$

and we let U_n denote the zero matrix of size $m_1 \times m_2$, when $n < v_0(U)$. Let $R = P/Q$ be a rational function, $P, Q \in \overline{\mathbb{Q}}[z]$ relatively prime. We let $\deg(R)$ denote the maximum of the degrees of P and Q . This notation extends to matrices with entries in $\overline{\mathbb{Q}}(z)$, taking the maximum of the degrees of the entries.

Our bounds for the complexity of the algorithms presented here are given in terms of arithmetical operations in $\overline{\mathbb{Q}}$. Given $f, g : \mathbb{N} \mapsto \mathbb{R}_{\geq 0}$ we use the classical Landau notation $f(n) = \mathcal{O}(g(n))$ if there exists a positive real number κ such that $f(n) \leq \kappa g(n)$ for every integer n large enough. Given an integer n , we let $M(n)$ denote the complexity of the multiplication of two polynomials of degree at most n , and $MM(n)$ denote the complexity of the multiplication of two matrices of size n .

For the sake of clarity, we shall denote by roman capital letters A, B, C, \dots matrices whose coefficients are effectively known and by Greek capital letters $\Psi, \Theta, \Lambda, \dots$ the other matrices. While matrices are denoted by capital letters Ψ, Θ, U, \dots , the columns of these matrices should be denoted by bold lowercase letters $\boldsymbol{\psi}, \boldsymbol{\theta}, \mathbf{u}, \dots$.

2. A CHARACTERISATION OF REGULAR SINGULARITY AT 0

Let $d \geq 1$ be an integer. We set $B(d) := \phi_d(A)^{-1}$ and we denote by

$$B(d) := \sum_{n \geq dv_0(A^{-1})} B_n(d) z^n$$

the development of $B(d)$ in Laurent power series. Set

$$(2) \quad \mu_d := \lceil -dv_0(A^{-1}) / (p-1) \rceil, \quad \text{and} \quad \nu_d := \lceil dv_0(A) / (p-1) \rceil.$$

Since $AA^{-1} = I_m$, we have $v_0(A) + v_0(A^{-1}) \leq 0$ so

$$dv_0(A) / (p-1) \leq -dv_0(A^{-1}) / (p-1)$$

and eventually, $\nu_d \leq \mu_d$. We let M_d and N_d denote the block matrices

$$M_d := (B_{i-pj}(d))_{\nu_d \leq i, j \leq \mu_d}$$

and

$$N_d := \begin{cases} (B_{i-pj}(d))_{dv_0(A^{-1}) + p\nu_d \leq i \leq \nu_d - 1, \nu_d \leq j \leq \mu_d} & \text{if } \nu_d < \mu_d \\ \mathbf{0} \in \mathcal{M}_{1, m(\mu_d - \nu_d + 1)}(\overline{\mathbb{Q}}) & \text{if } \nu_d = \mu_d \end{cases}$$

Remark 2. The condition $\nu_d < \mu_d$ ensures that the matrix is well-defined because it is equivalent to

$$(3) \quad dv_0(A^{-1}) + p\nu_d \leq \nu_d - 1.$$

Indeed, if ν_d satisfies (3), then

$$\nu_d \leq \frac{-dv_0(A^{-1})}{p-1} - \frac{1}{p-1} < \mu_d.$$

Conversely, if $\nu_d < \mu_d$, then

$$\nu_d \leq \mu_d - 1 = \left\lceil \frac{-dv_0(A^{-1})}{p-1} \right\rceil - 1 \leq \frac{-dv_0(A^{-1})}{p-1} - \frac{1}{p-1},$$

which is equivalent to (3). The last inequality comes from the fact that the number $\left\lceil \frac{-dv_0(A^{-1})}{p-1} \right\rceil - \frac{-dv_0(A^{-1})}{p-1}$ is of the form $b/(p-1)$ with $b \in \{0, \dots, p-2\}$.

Let $\ker(N_d)$ denote the (right) kernel of N_d in $\overline{\mathbb{Q}}^{m(\mu_d - \nu_d + 1)}$. When $\nu_d = \mu_d$ we have $\ker(N_d) = \overline{\mathbb{Q}}^{m(\mu_d - \nu_d + 1)}$. For an integer $n \in \mathbb{N}$ we set

$$M_d^{-n} \cdot \ker(N_d) = \ker(N_d M_d^n).$$

We also let

$$(4) \quad \mathcal{D} = \{d \in \mathbb{N}, 1 \leq d \leq p^m - 1 \mid \gcd(d, p) = 1\} \subset \mathbb{N}$$

The following result gives a necessary and sufficient condition for a system to be regular singular at 0.

Theorem 2. *The system (1) is regular singular at 0 if and only if there exists an integer $d \in \mathcal{D}$ such that the dimension of the $\overline{\mathbb{Q}}$ -vector space*

$$\mathfrak{X}_d := \bigcap_{n \in \mathbb{Z}} M_d^n \cdot \ker(N_d)$$

is greater than or equal to m . In that case, it is equal to m and the system (1) is $\overline{\mathbb{Q}}((z^{1/d}))$ -equivalent to a constant system. Furthermore, there is an algorithm to compute this integer d .

Remark 3. We show in Lemma 16 that

$$\mathfrak{X}_d = \bigcap_{-c_d \leq n \leq c_d} M_d^n \cdot \ker(N_d)$$

where $c_d := m(\mu_d - \nu_d + 1)$.

3. INDEX OF RAMIFICATION AND VALUATION AT 0 OF A GAUGE TRANSFORMATION

Assume that the system is regular singular at 0. The entries of the associated gauge transform belong to $z^v \overline{\mathbb{Q}}[[z^{1/d}]]$ for some integers $v \in \mathbb{Z}$ and $d \geq 1$. The aim of this section is to provide an upper bound for the ramification index d and a lower bound for the valuation v .

3.1. The Cyclic Vector Lemma. For the sake of completeness, we develop here a proof of a result called the Cyclic Vector Lemma. Any Mahler system is associated with an homogeneous Mahler equation, that is an equation of the form

$$q_0 y + q_1 \phi_p(y) + q_2 \phi_p^2(y) + \dots + q_{m-1} \phi_p^{m-1}(y) - \phi_p^m(y) = 0,$$

with $q_0, \dots, q_{m-1} \in \overline{\mathbb{Q}}(z)$. This result is known as the Cyclic Vector Lemma. We provide a proof of this result here, together with an algorithm to realize it.

Theorem 3 (Cyclic Vector Lemma). *Every Mahler system (1) is $\overline{\mathbb{Q}}(z)$ -equivalent to a companion matrix system, i.e., there exist $P \in \mathrm{GL}_m(\overline{\mathbb{Q}}(z))$ and $q_0, \dots, q_{m-1} \in \overline{\mathbb{Q}}(z)$ such that $\phi_p(P)AP^{-1} = A_{\mathrm{comp}}$ where*

$$(5) \quad A_{\mathrm{comp}} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ q_0 & \cdots & \cdots & \cdots & q_{m-1} \end{pmatrix}.$$

Proof. We adapt the proof of Birkhoff given in [Bir30, §1.] and of Sauloy given in [Sau00, Annexe B.2]. In order to build such a matrix P , we build its rows $\mathbf{r}_1, \dots, \mathbf{r}_m$. These rows must be linearly independent and must satisfy

$$\phi_p(\mathbf{r}_i)A = \mathbf{r}_{i+1} \quad \text{for } 1 \leq i \leq m-1.$$

Therefore, we are looking for a vector $\mathbf{r} \in \overline{\mathbb{Q}}(z)^m$ such that the vectors $\mathbf{r}_1 := \mathbf{r}$, $\mathbf{r}_{i+1} := \phi_p(\mathbf{r}_i)A$, $1 \leq i \leq m-1$ form a basis of $\overline{\mathbb{Q}}(z)^m$. For this purpose, we choose a $z_0 \in \overline{\mathbb{Q}}^\star$ not a root of unity such that $A(z_0), \dots, A(z_0^{p^{m-2}}) \in \mathrm{GL}_m(\overline{\mathbb{Q}})$ (such a z_0 exists because the matrix A has finitely many singularities). Now, by interpolation, since $z_0, z_0^p, \dots, z_0^{p^{m-1}}$ are all different, we may choose $\mathbf{r} \in \overline{\mathbb{Q}}(z)^m$ such that

$$(6) \quad \begin{cases} \mathbf{r}(z_0) = \mathbf{e}_1 \\ \mathbf{r}(z_0^p) = \mathbf{e}_2 A(z_0)^{-1} \\ \vdots \\ \mathbf{r}(z_0^{p^{m-1}}) = \mathbf{e}_m A(z_0)^{-1} \cdots A(z_0^{p^{m-2}})^{-1} \end{cases}$$

where $\mathbf{e}_1, \dots, \mathbf{e}_m$ is the canonical basis of $\overline{\mathbb{Q}}(z)^m$. Set $\mathbf{r}_1 := \mathbf{r}$ and define recursively $\mathbf{r}_{i+1} := \phi_p(\mathbf{r}_i)A$, $1 \leq i \leq m-1$. By construction,

$$\mathbf{r}_i(z_0) = \mathbf{e}_i.$$

The matrix P whose rows are $\mathbf{r}_1, \dots, \mathbf{r}_m$ satisfies $P(z_0) = I_m$. Thus, $P \in \mathrm{GL}_m(\overline{\mathbb{Q}}(z))$. Set

$$(q_0, \dots, q_{m-1}) := \phi_p(\mathbf{r}_m)AP^{-1}.$$

Then $A_{\mathrm{comp}} = \phi_p(P)AP^{-1}$ is a companion matrix of the form (5). \square

Recall that, from the Lagrange Theorem, the roots of a polynomial

$$p := p_0 + p_1 z + p_2 z^2 + \cdots + p_h z^h \quad \text{with } p_0, \dots, p_{h-1} \in \mathbb{C}, p_h \in \mathbb{C}^\star$$

have a module strictly less than 1 plus the max of $\frac{|p_k|}{|p_h|}$, $0 \leq k \leq h-1$. We thus set

$$\|p\| := 1 + \max \left\{ \frac{|p_k|}{|p_h|}, 0 \leq k \leq h-1 \right\}.$$

The following algorithm takes a Mahler system as input and returns a matrix P such that $\phi_p(P)AP^{-1}$ is a companion matrix together with the last row of $\phi_p(P)AP^{-1}$.

Algorithm 1: Find a *cyclic vector* associated with the Mahler system (1)

Input: A, p .

Output: The last row of a companion matrix $\overline{\mathbb{Q}}(z)$ -equivalent to A and its associated gauge transform.

Set m the size of A .

Set f the numerator of $\det(A)$ and $z_0 := \|f\|$.

for $1 \leq i, j \leq m$ **do**

 Let $g \in \overline{\mathbb{Q}}[z]$ stand for the denominator of the entry (i, j) of A .

if $\|g\| \geq z_0$ **then**

$z_0 := \|g\|$.

end

end

Let $\mathbf{r} \in \overline{\mathbb{Q}}[z]^m$ satisfy the interpolation (6).

Set P the matrix with rows $\mathbf{r}_1 := \mathbf{r}$, $\mathbf{r}_{i+1} := \phi_p(\mathbf{r}_i)A$, $1 \leq i \leq m-1$.

return P and $\phi_p(\mathbf{r}_m)AP^{-1}$.

Proposition 4. *Algorithm 1 has complexity*

$$\mathcal{O}(\text{MM}(m)\text{M}(p^m(\deg(A) + m))).$$

Proof. The computation of $\det(A)$ has complexity $\mathcal{O}(\text{MM}(m)\text{M}(\deg(A)))$. Given a polynomial P , the complexity of computing $\|P\|$ is $\mathcal{O}(\deg(P))$. Thus, the computation of z_0 has complexity

$$\mathcal{O}(m^2 \deg(A) + \text{MM}(m)\text{M}(\deg(A))).$$

To obtain \mathbf{r} we need to compute the matrices

$$\left(A(z_0) A(z_0^p) A(z_0^{p^2}) \dots A(z_0^{p^k}) \right)^{-1}, \quad 1 \leq k \leq m-2.$$

This has complexity $\mathcal{O}(m^3 \deg(A) + m\text{MM}(m))$. Then, we must interpolate at m points, for each of the m entries of \mathbf{r} , which has complexity $\mathcal{O}(m\text{M}(m)\log(m))$. Since \mathbf{r} has degree $m-1$ the computation of P has complexity $\mathcal{O}(m^2\text{M}(p^{m-1}(\deg(A) + m)))$. Then we need to compute the inverse of P which takes

$$\mathcal{O}(\text{MM}(m)\text{M}(p^{m-1}(\deg(A) + m))).$$

Thus, the computation of $\phi_p(\mathbf{r}_m)AP^{-1}$ has complexity $\mathcal{O}(\text{MM}(m)\text{M}(p^m(\deg(A) + m)))$. Eventually, Algorithm 1 has complexity

$$\mathcal{O}(\text{MM}(m)\text{M}(p^m(\deg(A) + m))).$$

□

3.2. On the possible ramification indexes of a gauge transformation. Recall that, from (4), \mathcal{D} is the set of $d \in \{1, \dots, p^m - 1\}$ such that p and d are relatively prime. The aim of this subsection is to prove the following result.

Proposition 5. *Assume that the system (1) is regular singular at 0 with an associated gauge transformation $\Psi \in \text{GL}_m(\mathbf{K})$. Then, the matrix Ψ belongs to $\text{GL}_m(\overline{\mathbb{Q}}((z^{1/d})))$ for some $d \in \mathcal{D}$. Furthermore, Algorithm 2 below computes such an integer d .*

Let

$$(7) \quad \Lambda := \phi_p(\Psi)^{-1} A \Psi \in \mathrm{GL}_m(\overline{\mathbb{Q}})$$

denote the constant matrix which is \mathbf{K} -equivalent to A . Without loss of generality, we assume that Λ has a Jordan normal form that is

$$\Lambda := \begin{pmatrix} J_{s_1}(\lambda_1) & & & \\ & J_{s_2}(\lambda_2) & & \\ & & \ddots & \\ & & & J_{s_r}(\lambda_r) \end{pmatrix}$$

where $\lambda_1, \dots, \lambda_r$ are algebraic numbers and $J_{s_i}(\lambda_i)$ is the Jordan block of size s_i associated to the eigenvalue λ_i . Let

$$\psi_{1,1}, \dots, \psi_{1,s_1}, \psi_{2,1}, \dots, \psi_{2,s_2}, \dots, \psi_{r,1}, \dots, \psi_{r,s_r}$$

denote the columns of Ψ indexed according to the Jordan normal form of Λ . From (7), for any $i, j \in \mathbb{N}$ such that $1 \leq i \leq r$, $2 \leq j \leq s_i$ one has

$$(8) \quad \lambda_i \phi_p(\psi_{i,1}) = A \psi_{i,1}, \quad \text{and} \quad \lambda_i \phi_p(\psi_{i,j}) + \phi_p(\psi_{i,j-1}) = A \psi_{i,j}.$$

Before proving Proposition 5 we need two lemmas about the solutions of homogeneous and inhomogeneous linear Mahler equations.

Lemma 6. *For any system (1), there exists an integer $d \in \mathcal{D}$ such that for any $\lambda \in \overline{\mathbb{Q}}^\star$ and $\mathbf{f} \in \mathbf{K}^m$ satisfying*

$$(9) \quad \lambda \phi_p(\mathbf{f}) = A \mathbf{f}$$

we have $\mathbf{f} \in \overline{\mathbb{Q}}((z^{1/d}))^m$.

Proof. By Theorem 3 there exists $P \in \mathrm{GL}_m(\overline{\mathbb{Q}}(z))$ such that $A_{\mathrm{comp}} = \phi_p(P) A P^{-1}$ is a companion matrix, that is,

$$(10) \quad A_{\mathrm{comp}} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ q_0 & \cdots & \cdots & \cdots & q_{m-1} \end{pmatrix} \quad \text{where } q_0, \dots, q_{m-1} \in \overline{\mathbb{Q}}(z).$$

We let \mathbf{f}, λ be as in the lemma. We have

$$\lambda \phi_p(P \mathbf{f}) = A_{\mathrm{comp}} P \mathbf{f}.$$

Substituting $P \mathbf{f}$ to \mathbf{f} , we might assume that $A = A_{\mathrm{comp}}$. Let $f_1, \dots, f_m \in \mathbf{K}$ be the entries of \mathbf{f} . Since A is a companion matrix, for every $i \in \{1, \dots, m-1\}$,

$$f_{i+1} = \lambda \phi_p(f_i).$$

Thus, in order to prove that $\mathbf{f} \in \overline{\mathbb{Q}}((z^{1/d}))^m$, it is enough to prove that $f_1 \in \overline{\mathbb{Q}}((z^{1/d}))$. It follows from (9) and (10) that f_1 is a solution of the Mahler equation

$$(11) \quad q_0 y + \lambda q_1 \phi_p(y) + \cdots + \lambda^{m-1} q_{m-1} \phi_p^{m-1}(y) - \lambda^m \phi_p^m(y) = 0.$$

From [CDDM18, Prop. 2.19], there exists an integer $d \in \mathcal{D}$ such that any solution $y \in \mathbf{K}$ of (11) belongs to $\overline{\mathbb{Q}}((z^{1/d}))$. Furthermore, as shown in [CDDM18], this integer d depends only on the valuation of the rational functions which are the coefficients of

the Mahler equation (11). In particular, it does not depend on λ . Precisely, let $\mathcal{H} \in \mathbb{R}^2$ denote the *lower hull* of the set

$$\{(p^i, v_0(q_i))\} \cup \{(p^m, 0)\}.$$

The integer d can be taken to be the least common multiple of the denominators of the slopes of \mathcal{H} which are prime together with p . We fix such an integer d . Therefore, $f_1 \in \overline{\mathbb{Q}}((z^{1/d}))$ and d does not depend on λ . \square

Lemma 7. *Consider a system (1) and let d be the integer obtained in the Lemma 6 for this system. If $\mathbf{g} \in \overline{\mathbb{Q}}((z^{1/d}))^m$, $\lambda \in \overline{\mathbb{Q}}^\times$ and $\mathbf{f} \in \mathbf{K}^m$ satisfy*

$$(12) \quad \phi_p(\mathbf{f})\lambda + \mathbf{g} = A\mathbf{f}$$

then $\mathbf{f} \in \overline{\mathbb{Q}}((z^{1/d}))^m$.

Proof. By Theorem 3, there exists $P \in \text{GL}_m(\overline{\mathbb{Q}}(z))$ such that $A_{\text{comp}} = \phi_p(P)AP^{-1}$ is a companion matrix of the form (10). We let $\mathbf{f}, \mathbf{g}, \lambda$ be as in the lemma. We have

$$\phi_p(P\mathbf{f})\lambda + \phi_p(P)\mathbf{g} = A_{\text{comp}}P\mathbf{f}$$

so up to replace $P\mathbf{f}$ with \mathbf{f} and $\phi_p(P)\mathbf{g}$ with \mathbf{g} , we might assume that $A = A_{\text{comp}}$. Since A is a companion matrix, for every $i \in \{1, \dots, m-1\}$,

$$f_{i+1} = g_i + \lambda\phi_p(f_i).$$

Thus, it is enough to prove that $f_1 \in \overline{\mathbb{Q}}((z^{1/d}))$. It follows from (10) and (12) that

$$(13) \quad q_0f_1 + \lambda q_1\phi_p(f_1) + \dots + \lambda^{m-1}q_{m-1}\phi_p^{m-1}(f_1) - \lambda^m\phi_p^m(f_1) = g_0,$$

where $g_0 \in \overline{\mathbb{Q}}((z^{1/d}))$ is a $\overline{\mathbb{Q}}(z)$ -linear combination of the $\phi_p^j(g_i)$, $i \in \{1, \dots, m\}$, $j \in \{0, \dots, m-1\}$. Assume that f_1 does not belong to $\overline{\mathbb{Q}}((z^{1/d}))$ and let h be the sum of all the monomials in the Puiseux series development of f_1 whose power cannot be written as k/d , with $k \in \mathbb{Z}$. Since $\gcd(d, p) = 1$, none of the monomials of the power series $\phi_p^\ell(h)$, $\ell \in \mathbb{N}$, belong to $\overline{\mathbb{Q}}((z^{1/d}))$. On the other hand, since $f_1 - h \in \overline{\mathbb{Q}}((z^{1/d}))$, $\phi_p^\ell(f_1 - h) \in \overline{\mathbb{Q}}((z^{1/d}))$ for every $\ell \in \mathbb{N}$. It thus follows from (13) that h is a solution of (11), the Mahler homogeneous equation associated with (13). From Lemma 6, $h \in \overline{\mathbb{Q}}((z^{1/d}))$, a contradiction. \square

The following algorithm computes an integer $d \in \mathcal{D}$ satisfying the properties of Lemma 6.

Algorithm 2: The integer d

Input: A, p .

Output: An integer d satisfying the properties of Lemma 6

Compute the last row (q_0, \dots, q_{m-1}) of the matrix A_{comp} with Algorithm 1.

Compute the lower hull \mathcal{H} of the set $\{(p^i, v_0(q_i))\} \cap \{(p^m, 0)\}$.

Compute \mathcal{S} the set of denominators of the slopes of \mathcal{H} which are prime together with p .

return $\text{lcm}(\mathcal{S})$.

We can now prove Proposition 5.

Proof of Proposition 5. Let d denote the integer given by Lemma 6. Fix an integer $i \in \{1, \dots, r\}$. We prove by induction on $j \in \{1, \dots, s_i\}$ that $\psi_{i,j} \in \overline{\mathbb{Q}}((z^{1/d}))^m$ for every $j \in \{1, \dots, s_i\}$. From (8), the vector $\psi_{i,1}$ satisfies

$$\lambda_i \phi_p(\psi_{i,1}) = A\psi_{i,1}.$$

Thus, from Lemma 6, $\psi_{i,1} \in \overline{\mathbb{Q}}((z^{1/d}))^m$. Fix an integer $j \in \{2, \dots, s_i\}$ and assume that $\psi_{i,j-1} \in \overline{\mathbb{Q}}((z^{1/d}))^m$. From (8), $\psi_{i,j}$ satisfies

$$\lambda_i \phi_p(\psi_{i,j}) + \phi_p(\psi_{i,j-1}) = A\psi_{i,j}$$

thus it follows from Lemma 7 that $\psi_{i,j} \in \overline{\mathbb{Q}}((z^{1/d}))^m$, as desired. \square

3.3. A lower bound for the valuation of a gauge transformation. Assume that the system (1) is regular singular at 0. Let $\Psi \in \mathrm{GL}_m(\mathbf{K})$ such that $\phi_p(\Psi)^{-1}A\Psi$ is a constant matrix. We let $d \in \mathcal{D}$ denote an integer such that the entries of Ψ belong to $\overline{\mathbb{Q}}((z^{1/d}))$. Write

$$\Theta(z) := \phi_d(\Psi)(z) = \sum_{n \geq v_0(\Theta)} \Theta_n z^n \quad \text{and} \quad A(z) =: \sum_{n \geq v_0(A)} A_n z^n,$$

where $\Theta_n, A_n \in \mathcal{M}_n(\overline{\mathbb{Q}})$. By definition, $v_0(\Theta) = v_0(\phi_d(\Psi))$. We give a lower bound for $v_0(\Theta)$, the valuation at 0 of Θ . It follows from the identity $\phi_p(\Theta) = \phi_d(A)\Theta\Lambda^{-1}$ that

$$pv_0(\Theta) \geq v_0(\Theta) + dv_0(A).$$

Hence $v_0(\Theta) \geq \frac{dv_0(A)}{p-1}$ thus

$$v_0(\Theta) \geq \nu_d$$

where $\nu_d = \lceil dv_0(A)/(p-1) \rceil$ is defined in (2). It follows that we have a decomposition $\Psi(z) := \sum_{n \geq \nu_d} \Theta_n z^{n/d}$ where Θ_{ν_d} might be a zero matrix. Eventually, we just proved that the valuation at 0 of the entries of $\phi_d(\Psi)$ are greater than ν_d .

4. PROOF OF THEOREM 2

This section is devoted to the proof of Theorem 2. If the system is regular singular at 0, we describe how to compute a gauge transformation Ψ such that $\phi_p(\Psi)^{-1}A\Psi$ is constant with Jordan normal form. Of course, a delicate point in such a construction is that we have to compute the eigenvalues of this Jordan normal form matrix. Moreover, the gauge transformation Ψ appears with coefficients in an algebraic extension of the field spanned by the coefficients of the entries of A , which is not optimal.

4.1. Equations satisfied by the columns of a gauge transformation. We recall that the Mahler system (1) is regular singular at 0 if and only if there exist matrices $\Psi \in \mathrm{GL}_m(\mathbf{K})$ and $\Lambda \in \mathrm{GL}_m(\overline{\mathbb{Q}})$ such that

$$\phi_p(\Psi)\Lambda = A\Psi.$$

Assume that the system (1) is regular singular at 0 with gauge transformation Ψ and a constant matrix Λ . One has

$$(14) \quad \Psi\Lambda^{-1} = A^{-1}\phi_p(\Psi)$$

Then, from Proposition 5, Ψ belongs to $\mathrm{GL}_m(\overline{\mathbb{Q}}((z^{1/d})))$ for some $d \in \mathcal{D}$. We fix such an integer d and we apply ϕ_d to (14):

$$(15) \quad \Theta\Lambda^{-1} = B\phi_p(\Theta)$$

where $\Theta := \phi_d(\Psi)$ and $B := \phi_d(A^{-1})$. These matrices have Laurent series development

$$\Theta(z) = \sum_{n \geq \nu_d} \Theta_n z^n \text{ and } B(z) = \sum_{n \geq \nu_0(B)} B_n z^n,$$

where $\Theta_n, B_n \in \mathcal{M}_m(\overline{\mathbb{Q}})$ and $\nu_d = \lceil dv_0(A)/(p-1) \rceil$.

Remark 4. We infer from (15) that for all $n \in \mathbb{Z}$,

$$(16) \quad \Theta_n \Lambda^{-1} = \sum_{(k,l): k+p\ell=n} B_k \Theta_\ell.$$

If $n > -dv_0(A^{-1})/(p-1)$ then the Θ_ℓ which are taken into account in the right hand side of (16) are the one for which $\ell < n$. Therefore, the sequence $(\Theta_n)_{n \geq \nu_d}$ is uniquely determined by the matrices Θ_ℓ with

$$\nu_d \leq \ell \leq \mu_d,$$

where $\mu_d = \lceil -dv_0(A^{-1})/(p-1) \rceil$ is defined in (2).

Assume that the matrix Λ^{-1} has a Jordan normal form

$$\Lambda^{-1} := \begin{pmatrix} J_{s_1}(\gamma_1) & & & \\ & J_{s_2}(\gamma_2) & & \\ & & \ddots & \\ & & & J_{s_r}(\gamma_r) \end{pmatrix}$$

where $\gamma_1, \gamma_2, \dots, \gamma_r$ are algebraic numbers and $J_{s_i}(\gamma_i)$ is the Jordan block of size s_i associated with the eigenvalue γ_i . Let

$$\theta_{1,1}, \dots, \theta_{1,s_1}, \theta_{2,1}, \dots, \theta_{2,s_2}, \dots, \theta_{r,1}, \dots, \theta_{r,s_r} \in \overline{\mathbb{Q}}((z))^m$$

denote the columns of Θ indexed with respect to the Jordan normal form of Λ^{-1} . We infer from (15) that the columns $\theta_{i,j}$ satisfy

$$(17) \quad \begin{aligned} \gamma_i \theta_{i,1} &= B \phi_p(\theta_{i,1}) \\ \gamma_i \theta_{i,j} + \theta_{i,j-1} &= B \phi_p(\theta_{i,j}) \quad j \geq 2. \end{aligned}$$

4.2. Computing the column's candidates. In this section we do not assume anymore that the system is regular singular at 0. We fix the integer $d \in \mathcal{D}$ given by Algorithm 2. We recall that if the Mahler system (1) is regular singular at 0 then the entries of a gauge transformation Ψ belong to $\overline{\mathbb{Q}}((z^{1/d}))$. Therefore, instead of looking for a gauge transformation with entries in $\overline{\mathbb{Q}}((z^{1/d}))$, we shall apply ϕ_d to the Mahler system and look for a gauge transformation with entries in $\overline{\mathbb{Q}}((z))$. Let

$$B := \phi_d(A^{-1}).$$

We will show how to build a matrix U with entries in $\overline{\mathbb{Q}}((z))$ and with as many linearly independent columns as possible, together with a constant square matrix C such that

$$UC = B \phi_p(U).$$

Our matrix U may not be a square matrix. As we shall see, it will be if and only if the system (1) is regular singular at 0. In that case, setting $\Psi := \phi_{1/d}(U)$ we will have $\phi_p(\Psi)^{-1} A \Psi = C^{-1} \in \text{GL}_m(\overline{\mathbb{Q}})$, as wanted. It follows from (17) that to determine the columns of this matrix U , one has to solve some homogeneous and inhomogeneous Mahler systems. In the mean time, we infer from Remark 4 that this can be done by solving some equations in a finite number of the coefficients of the Puiseux series

development of the columns of U . This section is devoted to the construction of the pair (U, C) .

Let $M := M_d$ and $N := N_d$ be defined as in section 2. We define a map

$$\begin{aligned} \pi_d : \quad \overline{\mathbb{Q}}((z))^m &\rightarrow \overline{\mathbb{Q}}^{m(\mu_d - \nu_d + 1)} \\ \mathbf{g}(z) = \sum_{n \in \mathbb{Z}} \mathbf{g}_n z^n &\mapsto \begin{pmatrix} \mathbf{g}_{\nu_d} \\ \vdots \\ \mathbf{g}_{\mu_d} \end{pmatrix}. \end{aligned}$$

If γ is an eigenvalue of M , for all positive integer j , we consider the $\overline{\mathbb{Q}}$ -vector space

$$V_{\gamma, j} := \ker \left((M - \gamma I)^j \right) \cap \left(\bigcap_{k=0}^{j-1} \ker \left(N(M - \gamma I)^k \right) \right).$$

We set $V_{\gamma, 0} := \{0\}$.

4.2.1. Solving the homogeneous equations.

Lemma 8. *Let $\gamma \in \overline{\mathbb{Q}}$. If the equation*

$$(18) \quad \gamma \mathbf{f} = B\phi_p(\mathbf{f})$$

has a nonzero solution then $\gamma \neq 0$. In that case, the map π_d induces a bijection between the $\overline{\mathbb{Q}}$ vector-space of solutions $\mathbf{f} \in \overline{\mathbb{Q}}((z))^m$ of (18) and $V_{\gamma, 1}$. Furthermore, given an element $\mathbf{y} \in V_{\gamma, 1}$, there is a recurrence relation to determine the coefficients of the Laurent series development of the vector $\mathbf{f} \in \overline{\mathbb{Q}}((z))^m$ such that $\pi_d(\mathbf{f}) = \mathbf{y}$.

Note that there might be no nonzero solution of (18), even when γ is a nonzero eigenvalue of M . More precisely, from this lemma, there exists a nonzero solution of (18) if and only if γ is a nonzero eigenvalue of M which satisfies $\dim(V_{\gamma, 1}) \geq 1$.

Proof. Since the matrix B is nonsingular, we must have $\gamma \neq 0$. It follows from the results in Section 3.3 that any solution in $\overline{\mathbb{Q}}((z))^m$ of (18) has a valuation at 0 greater than ν_d . Let $\mathbf{f} \in \overline{\mathbb{Q}}((z))^m$ with valuation greater than ν_d and set

$$\mathbf{f} = \sum_{n \geq \nu_d} \mathbf{f}_n z^n, \quad \mathbf{f}_n \in \overline{\mathbb{Q}}^m \text{ for } n \geq \nu_d.$$

The function \mathbf{f} satisfies (18) if and only if

$$\forall n \in \mathbb{Z}, \quad \gamma \mathbf{f}_n = \sum_{(k, l) : k + pl = n} B_k \mathbf{f}_l,$$

where we set $\mathbf{f}_n = 0$ if $n < \nu_d$. This is equivalent to

$$\begin{cases} N\pi_d(\mathbf{f}) = 0 & (\text{coming from the cases } n < \nu_d) \\ M\pi_d(\mathbf{f}) = \gamma\pi_d(\mathbf{f}) & (\text{coming from the cases } \nu_d \leq n \leq \mu_d) \\ \forall n \geq \mu_d + 1, \quad \mathbf{f}_n = \frac{1}{\gamma} \left(\sum_{(k, l) : k + pl = n} B_k \mathbf{f}_l \right) \end{cases}.$$

The first and the second equality imply that $\pi_d(\mathbf{f}) \in V_{\gamma, 1}$. As explained in Remark 4, from the third equality in this system, we deduce that the sequence $(\mathbf{f}_n)_{n \geq \nu_d}$ is uniquely determined by the vectors \mathbf{f}_ℓ with $\nu_d \leq \ell \leq \mu_d$, i.e. it is uniquely determined by $\pi_d(\mathbf{f})$. Therefore, \mathbf{f} is a solution of (18) if and only if $\pi_d(\mathbf{f}) \in V_{\gamma, 1}$. Furthermore, if \mathbf{f} and \mathbf{g} are two solutions of (18) such that $\pi_d(\mathbf{f}) = \pi_d(\mathbf{g})$ then $\mathbf{f} = \mathbf{g}$. Since any $\mathbf{y} \in V_{\gamma, 1}$ is the image of some $\mathbf{f} \in \overline{\mathbb{Q}}((z))^m$ of valuation greater than ν_d , the lemma is proved. \square

4.2.2. Solving the inhomogeneous equations.

Lemma 9. Let $\gamma \in \overline{\mathbb{Q}}^*$ and $j \geq 1$ an integer. Let $g \in \overline{\mathbb{Q}}((z))^m$ be a Laurent power series with valuation greater than ν_d and such that $\pi_d(g) \in V_{\gamma,j}$. The map π_d induces a bijection between the affine space of solutions $\mathbf{f} \in \overline{\mathbb{Q}}((z))^m$ of

$$(19) \quad \gamma \mathbf{f} + \mathbf{g} = B\phi_p \mathbf{f},$$

and the affine space of solutions $\mathbf{y} \in V_{\gamma,j+1}$ of

$$(M - \gamma I)\mathbf{y} = \pi_d(\mathbf{g}).$$

Proof. Set

$$\mathbf{g} = \sum_{n \geq \nu_d} \mathbf{g}_n z^n, \quad \mathbf{g}_n \in \overline{\mathbb{Q}}^m,$$

From Section 3.3, any solution of (19) has a valuation greater than ν_d . Let

$$\mathbf{f} = \sum_{n \geq \nu_d} \mathbf{f}_n z^n \in \overline{\mathbb{Q}}((z))^m, \quad \mathbf{f}_n \in \overline{\mathbb{Q}}^m.$$

If $n < \nu_d$, we set $\mathbf{f}_n := 0$ and $\mathbf{g}_n := 0$. The power series \mathbf{f} is a solution of (19) if and only if

$$\forall n \in \mathbb{Z}, \quad \gamma \mathbf{f}_n + \mathbf{g}_n = \sum_{(k,l): k+p\ell=n} B_k \mathbf{f}_\ell.$$

This is equivalent to

$$\begin{cases} N\pi_d(\mathbf{f}) = 0 & (\text{from the cases } n < \nu_d) \\ M\pi_d(\mathbf{f}) = \gamma\pi_d(\mathbf{f}) + \pi_d(\mathbf{g}) & (\text{from the cases } \nu_d \leq n \leq \mu_d) \\ \forall n \geq \mu_d + 1, \quad \mathbf{f}_n = \frac{1}{\gamma} \left(\sum_{(k,l): k+p\ell=n} B_k \mathbf{f}_\ell - \mathbf{g}_n \right) \end{cases}$$

The second equality implies that $(M - \gamma I)\pi_d(\mathbf{f}) \in V_{\gamma,j}$. Hence, $(M - \gamma I)^{j+1}\pi_d(\mathbf{f}) = 0$ and thus $\pi_d(\mathbf{f}) \in \cap_{k=1}^j \ker(N(M - \gamma I)^k)$. Eventually, the first equality implies that $\pi_d(\mathbf{f}) \in \ker(N) = \ker(N(M - \gamma I)^0)$. It follows that $\pi_d(\mathbf{f}) \in V_{\gamma,j+1}$. From Remark 4, a sequence $(\mathbf{f}_n)_{n \geq \nu_d}$ verifying the third equation is uniquely determined by $\pi_d(\mathbf{f})$. Thus \mathbf{f} is a solution of (19) if and only if $\pi_d(\mathbf{f}) \in V_{\gamma,j+1}$ satisfies $(M - \gamma I)\pi_d(\mathbf{f}) = \pi_d(\mathbf{g})$, as wanted. Furthermore, $\pi_d(\mathbf{f})$ is uniquely determined by \mathbf{f} . \square

4.2.3. Computing each column.

We will now select some columns for the matrix U .

Lemma 10. Let $\gamma \in \overline{\mathbb{Q}}^*$. We have a chain of vector spaces

$$V_{\gamma,0} := \{0\} \subset V_{\gamma,1} \subset V_{\gamma,2} \subset \cdots \subset V_{\gamma,j} \subset \cdots \subset \overline{\mathbb{Q}}^{m(\mu_d - \nu_d + 1)}.$$

Set

$$a(\gamma) := \min\{n \in \mathbb{N}^* \mid V_{\gamma,n} = V_{\gamma,n+1}\}.$$

Then, $V_{\gamma,j} = V_{\gamma,a(\gamma)}$ for all $j \geq a(\gamma)$. Moreover, for $i \in \{1, \dots, a(\gamma)\}$, if

$$k_{\gamma,j} := \dim(V_{\gamma,j}) - \dim(V_{\gamma,j-1}).$$

then $0 < k_{\gamma,a(\gamma)} \leq k_{\gamma,a(\gamma)-1} \leq \cdots \leq k_{\gamma,2} \leq k_{\gamma,1}$.

Proof. Let $j \in \mathbb{N}$ and $\mathbf{v} \in V_{\gamma,j}$. Then, $(M - \gamma I)^j \mathbf{v} = 0$. Thus, $\mathbf{v} \in \ker(N(M - \gamma I)^j)$ and $(M - \gamma I)^{j+1} \mathbf{v} = 0$. This proves the first point. The second point is obvious. In order to prove the last point, fix an integer $j \in \{1, \dots, a(\gamma) - 1\}$ and consider the linear map

$$\begin{aligned} \psi_j : V_{\gamma,j+1} &\rightarrow V_{\gamma,j}/V_{\gamma,j-1} \\ x &\mapsto (M - \gamma I)x \end{aligned}$$

Its kernel is $V_{\gamma,j}$ so the induced map $\overline{\psi_j} : V_{\gamma,j+1}/V_{\gamma,j} \rightarrow V_{\gamma,j}/V_{\gamma,j-1}$ is injective. Therefore, taking the dimensions, we have

$$\forall j \in \{1, \dots, a(\gamma) - 1\}, k_{\gamma,j+1} \leq k_{\gamma,j}.$$

□

Lemma 11. *We use the notations of Lemma 10. For all $j \in \{2, \dots, a(\gamma)\}$, there exists a subspace $W_{\gamma,j}$ of $V_{\gamma,j}$ such that*

$$V_{\gamma,j} = V_{\gamma,j-1} \oplus W_{\gamma,j}$$

and such that the map

$$\begin{aligned} W_{\gamma,j} &\rightarrow W_{\gamma,j-1} \\ x &\mapsto (M - \gamma I)x \end{aligned}$$

is injective, where we set $W_{\gamma,1} := V_{\gamma,1}$.

Proof. The proof is similar to a proof of Jordan theorem for nilpotent matrices using Young tableau. We first choose a basis $\mathbf{v}_1^{(a(\gamma))}, \dots, \mathbf{v}_{k_{\gamma,a(\gamma)}}^{(a(\gamma))}$ of the vector space $W_{\gamma,a(\gamma)}$. Let $0 \leq \ell \leq a(\gamma) - 1$. Assume that we have chosen a basis

$$\mathbf{v}_1^{(a(\gamma)-\ell)}, \dots, \mathbf{v}_{k_{\gamma,a(\gamma)-\ell}}^{(a(\gamma)-\ell)}$$

of $W_{\gamma,a(\gamma)-\ell}$. We set $M_\gamma := M - \gamma I$ and

$$\mathbf{v}_1^{(a(\gamma)-\ell-1)} := M_\gamma \mathbf{v}_1^{(a(\gamma)-\ell)}, \dots, \mathbf{v}_{k_{\gamma,a(\gamma)-\ell}}^{(a(\gamma)-\ell-1)} := M_\gamma \mathbf{v}_{k_{\gamma,a(\gamma)-\ell}}^{(a(\gamma)-\ell)},$$

and we complete it with vectors $\mathbf{v}_{k_{\gamma,a(\gamma)-\ell}+1}^{(a(\gamma)-\ell-1)}, \dots, \mathbf{v}_{k_{\gamma,a(\gamma)-\ell-1}}^{(a(\gamma)-\ell-1)}$ to form a basis of the vector space $W_{\gamma,a(\gamma)-\ell-1}$. In the end, we have the following Young tableau, where the j th column is a basis of $W_{\gamma,j}$.

$\mathbf{v}_1^{(1)} := M_\gamma^{a(\gamma)-1} \mathbf{v}_1^{(a(\gamma))}$	\dots	$\mathbf{v}_1^{(a(\gamma)-1)} := M_\gamma \mathbf{v}_1^{(a(\gamma))}$	$\mathbf{v}_1^{(a(\gamma))}$
\vdots	\dots	\vdots	\vdots
$\mathbf{v}_{k_{\gamma,a(\gamma)}}^{(1)} := M_\gamma^{a(\gamma)-1} \mathbf{v}_{k_{\gamma,a(\gamma)}}^{(a(\gamma))}$	\dots	$\mathbf{v}_{k_{\gamma,a(\gamma)}}^{(a(\gamma)-1)} := M_\gamma \mathbf{v}_{k_{\gamma,a(\gamma)}}^{(a(\gamma))}$	$\mathbf{v}_{k_{\gamma,a(\gamma)}}^{(a(\gamma))}$
$\mathbf{v}_{k_{\gamma,a(\gamma)+1}}^{(1)} := M_\gamma^{a(\gamma)-2} \mathbf{v}_{k_{\gamma,a(\gamma)+1}}^{(a(\gamma)-1)}$	\dots	$\mathbf{v}_{k_{\gamma,a(\gamma)+1}}^{(a(\gamma)-1)}$	
\vdots	\dots	\vdots	
$\mathbf{v}_{k_{\gamma,a(\gamma)-1}}^{(1)} := M_\gamma^{a(\gamma)-2} \mathbf{v}_{k_{\gamma,a(\gamma)-1}}^{(a(\gamma)-1)}$	\dots	$\mathbf{v}_{k_{\gamma,a(\gamma)-1}}^{(a(\gamma)-1)}$	
\vdots	\vdots		
$\mathbf{v}_{k_{\gamma,2}+1}^{(1)}$			
\vdots			
$\mathbf{v}_{k_{\gamma,1}}^{(1)}$			

□

Let $\{\gamma_1, \dots, \gamma_t\}$ be the set of all algebraic numbers γ for which (18) has a solution. From Lemma 8, it is the set of nonzero eigenvalue γ of M which satisfy $\dim(V_{\gamma,1}) \geq 1$. For each $i \in \{1, \dots, t\}$, we recall that

$$V_{\gamma_i,j} := \ker \left((M - \gamma_i I)^j \right) \cap \left(\bigcap_{k=0}^{j-1} \ker \left(N(M - \gamma_i I)^k \right) \right).$$

We decompose

$$V_{\gamma_i,a(\gamma_i)} = \bigoplus_{j=1}^{a(\gamma_i)} W_{\gamma_i,j}$$

as in Lemma 11 and for each j we consider the basis $(\mathbf{v}_{\gamma_i,1}^{(j)}, \dots, \mathbf{v}_{\gamma_i,k_{\gamma_i,j}}^{(j)})$ of $W_{\gamma_i,j}$ constructed as in the Young tableau of Lemma 11. Thus,

$$\{\mathbf{v}_{\gamma_i,\ell}^{(j)} : 1 \leq j \leq a(\gamma_i), 1 \leq \ell \leq k_{\gamma_i,j}\}$$

is a basis of $V_{\gamma_i,a(\gamma_i)}$. For every $\ell \in \{1, \dots, k_{\gamma_i,1}\}$ we let $\mathbf{u}_{\gamma_i,\ell}^{(1)} \in \overline{\mathbb{Q}}((z))_{\nu_d}^m$ denote the unique solution of

$$\gamma_i \mathbf{f} = B\phi_p(\mathbf{f})$$

such that $\pi_d(\mathbf{u}_{\gamma_i,\ell}^{(1)}) = \mathbf{v}_{\gamma_i,\ell}^{(1)}$. The existence and unicity of this solution follows from Lemma 8. Now, we define recursively on $j \in \{2, \dots, a(\gamma_i)\}$ vectors $\mathbf{u}_{\gamma_i,\ell}^{(j)} \in \overline{\mathbb{Q}}((z))_{\nu_d}^m$, with $1 \leq \ell \leq k_{\gamma_i,j}$, solutions of

$$\gamma_i \mathbf{f} + \mathbf{u}_{\gamma_i,\ell}^{(j-1)} = B\phi_p(\mathbf{f})$$

and such that $\pi_d(\mathbf{u}_{\gamma_i,\ell}^{(j)}) = \mathbf{v}_{\gamma_i,\ell}^{(j)}$. The existence and unicity of such vectors follows from Lemma 9. *In fine*, we have constructed vectors

$$(20) \quad \mathbf{u}_{\gamma_i,1}^{(1)}, \dots, \mathbf{u}_{\gamma_i,k_{\gamma_i,1}}^{(1)}, \mathbf{u}_{\gamma_i,1}^{(2)}, \dots, \mathbf{u}_{\gamma_i,k_{\gamma_i,2}}^{(2)}, \dots, \mathbf{u}_{\gamma_i,1}^{(a(\gamma_i))}, \dots, \mathbf{u}_{\gamma_i,k_{\gamma_i,a(\gamma_i)}}^{(a(\gamma_i))},$$

such that $\pi_d(\mathbf{u}_{\gamma_i,\ell}^{(j)}) = \mathbf{v}_{\gamma_i,\ell}^{(j)}$ for every $(j, \ell) \in \{1, \dots, a(\gamma_i)\} \times \{1, \dots, k_{\gamma_i,j}\}$. In particular, the family (20) is linearly independent over $\overline{\mathbb{Q}}$.

4.2.4. Building the matrix. Let U_i be the matrix whose columns are the elements in (20) re-ordered using the lexical order on the indices, that is:

$$\mathbf{u}_{\gamma_i,1}^{(1)}, \mathbf{u}_{\gamma_i,1}^{(2)}, \dots, \mathbf{u}_{\gamma_i,2}^{(1)}, \mathbf{u}_{\gamma_i,2}^{(2)}, \dots, \mathbf{u}_{\gamma_i,k_{\gamma_i,1}}^{(1)}, \dots$$

As already noticed, the columns of U_i are linearly independent. Let C_i be the normal Jordan form matrix whose Jordan blocks are

$$J_{\gamma_i}(s_{i,1}), \dots, J_{\gamma_i}(s_{i,k_1})$$

where we let $s_{i,\ell}$ denote the number of column vector of the form $\mathbf{u}_{\gamma_i,\ell}^{(\cdot)}$ (that is the number of columns in the ℓ th row of the above table). For example, $s_{i,1} = a(\gamma_i)$. Then,

$$U_i C_i = B\phi_p(U_i).$$

We let $U := (U_1 \mid \dots \mid U_t)$ denote the matrix whose columns are the ones of the U_i and C denote the Jordan normal form matrix whose blocks are C_1, \dots, C_t that is

$$C := \begin{pmatrix} C_1 & & \\ & \ddots & \\ & & C_t \end{pmatrix}.$$

The matrix U is not necessarily a square matrix but it follows from our construction that

$$(21) \quad UC = B\phi_p(U).$$

Consider a pair of matrices (U', C') such that $U'C' = B\phi_p(U')$. It follows from Lemmas 8 and 9 that U' has less columns than U . Furthermore, since the vector spaces $V_{\gamma_1, a(\gamma_1)}, \dots, V_{\gamma_t, a(\gamma_t)}$ are in direct sum, the columns of U - which are the elements of (20) - are linearly independent over $\overline{\mathbb{Q}}$. It follows from the following lemma that they are actually linearly independent over $\overline{\mathbb{Q}}((z))$

Lemma 12. *Let T be a matrix with entries in $\overline{\mathbb{Q}}((z))$ and D be a constant matrix such that*

$$(22) \quad TD = B\phi_p(T) .$$

If the columns of T are linearly dependent over $\overline{\mathbb{Q}}((z))$ then they are linearly dependent over $\overline{\mathbb{Q}}$.

Proof. We might assume, without loss of generality, that D is upper triangular. Let $a \geq 2$ be the least integer such that the columns $\mathbf{t}_1, \dots, \mathbf{t}_a$ are linearly dependent over $\overline{\mathbb{Q}}((z))$. Then there exists a column vector $\mathbf{g} =: (g_1, \dots, g_{a-1}, 1, 0, \dots, 0)^\perp \in \overline{\mathbb{Q}}((z))^m$ such that

$$(23) \quad T\mathbf{g} = 0 .$$

Mutliplying (22) by $\phi_p(\mathbf{g})$ one obtains

$$(24) \quad TD\phi_p(\mathbf{g}) = B\phi_p(T)\phi_p(\mathbf{g}) = B\phi_p(T\mathbf{g}) = 0.$$

Since D is upper triangular, only the a first coordinates of $D\phi_p(\mathbf{g})$ can be nonzero and its a th coordinate is a some eigenvalue $\eta \in \overline{\mathbb{Q}}$ of D . By minimality of a , we infer from (23) and (24) that

$$D\phi_p(\mathbf{g}) = \eta\mathbf{g} .$$

From [Nis97, Thm. 3.1], $\mathbf{g} \in \overline{\mathbb{Q}}^m$ so (23) gives a $\overline{\mathbb{Q}}$ -linearly dependence of the columns of T , as wanted. \square

4.3. A new characterisation of the regular singularity at 0. The following proposition states that U is a square matrix if and only if the system (1) is $\overline{\mathbb{Q}}((z^d))$ -equivalent to a constant system. This give a characterisation of Mahler systems that are regular singular at 0.

Proposition 13. *The system (1) is $\overline{\mathbb{Q}}((z^d))$ -equivalent to a constant system if and only if the number of columns of U is greater than m . In that case, it is equal to m and, setting $\Psi := \phi_{1/d}(U) \in \mathrm{GL}_m(\overline{\mathbb{Q}}((z^{1/d})))$, one has*

$$(25) \quad \phi_p(\Psi)^{-1}A\Psi = C^{-1} \in \mathrm{GL}_m(\overline{\mathbb{Q}}) .$$

Proof. Since the columns of U are linearly independent over $\overline{\mathbb{Q}}((z))$, its number is less than m . Assume that U is a square matrix. Then, it is nonsingular, and equality (25) follows from (21) and the system (1) is $\overline{\mathbb{Q}}((z^d))$ -equivalent to a constant system.

Assume now that the system (1) is $\overline{\mathbb{Q}}((z^d))$ -equivalent to a constant system with gauge transformation $\Psi \in \mathrm{GL}_m(\overline{\mathbb{Q}}((z^{1/d})))$. Since the columns of Ψ are $\overline{\mathbb{Q}}$ -linearly independent, it follows from Remark 4 that their images under π_d are $\overline{\mathbb{Q}}$ -linearly independent. Thus, from (17) they belong to the vector space

$$V_{\gamma_1, a(\gamma_1)} \oplus \cdots \oplus V_{\gamma_t, a(\gamma_t)}.$$

Thus, the number of columns of U , which is the dimension of this vector space, is greater than m , as wanted. \square

Therefore, Proposition 13 gives a first algorithm to test whether or not a system is regular singular at 0: it can be done by computing the solutions of some homogeneous and inhomogeneous Mahler systems. However, this algorithm is not necessarily very efficient, since one has to find eigenvalues and eigenvectors for M .

4.4. An equality of vector space. We let \mathfrak{U} denote the $\overline{\mathbb{Q}}$ -vector space spanned by the columns of U , that is

$$\mathfrak{U} := \mathrm{span}_{\overline{\mathbb{Q}}} \left\{ \mathbf{u}_{\gamma_i, \ell}^{(j)} : 1 \leq i \leq t, 1 \leq j \leq a(\gamma_i), 1 \leq \ell \leq k_{\gamma_i, j} \right\},$$

and set

$$(26) \quad \mathfrak{V} = V_{\gamma_1, a(\gamma_1)} \oplus \cdots \oplus V_{\gamma_t, a(\gamma_t)}$$

From Lemma 8 and Lemma 9, π_d induces an isomorphism between these two vector spaces. Recall that we set

$$\mathfrak{X} := \mathfrak{X}_d := \bigcap_{n \in \mathbb{Z}} M_d^n \ker(N_d).$$

We have the following equality of vector spaces.

Lemma 14. *We have*

$$\mathfrak{V} = \mathfrak{X}.$$

Proof. We first prove that $\mathfrak{V} \subset \mathfrak{X}$. From (26) it is enough to prove that $V_{\gamma_i, a(\gamma_i)} \subset \mathfrak{X}$ for every $i \in \{1, \dots, t\}$. Fix an integer $i \in \{1, \dots, t\}$. We recall that for all $k \geq a(\gamma_i)$,

$$V_{\gamma_i, a(\gamma_i)} = V_{\gamma_i, k} := \ker \left((M - \gamma_i I)^k \right) \cap \left(\bigcap_{j=0}^{k-1} \ker \left(N (M - \gamma_i I)^j \right) \right).$$

One can check that

$$V_{\gamma_i, k} = \ker \left((M - \gamma_i I)^k \right) \cap \left(\bigcap_{j=0}^{k-1} \ker (N M^j) \right).$$

Therefore, for all $n \in \mathbb{N}$, $M^n V_{\gamma_i, a(\gamma_i)} \subset \ker(N)$. Then, we prove by induction on $n \in \mathbb{N}$ that for any $\mathbf{v} \in V_{\gamma_i, a(\gamma_i)}$, $\mathbf{v} \in M^n \ker(N)$. It is clear for $n = 0$. Assume that $\mathbf{v} \in M^\ell \ker(N)$ for all $\ell \leq n$. Since $\mathbf{v} \in V_{\gamma_i, a(\gamma_i)}$, for all $k \geq a(\gamma_i)$,

$$(M - \gamma_i I)^k \mathbf{v} = 0.$$

Thus \mathbf{v} is a $\overline{\mathbb{Q}}$ -linear combination of $M\mathbf{v}, \dots, M^k \mathbf{v}$. Moreover,

- if $j \in \{1, \dots, n+1\}$ then $M^j \mathbf{v} \in M^{n+1} \ker(N)$ because by the induction hypothesis $\mathbf{v} \in M^{n+1-j} \ker(N)$;
- if $j \geq n+1$, $M^j \mathbf{v} \in M^{n+1} \ker(N)$ because $M^j \mathbf{v} = M^{n+1} \cdot M^{j-(n+1)} \mathbf{v}$ and, previously, we showed that $M^{j-(n+1)} \mathbf{v} \in \ker(N)$.

Therefore, $\mathbf{v} \in M^{n+1} \ker(N)$. It follows that $V_{\gamma_i, a(\gamma_i)} \subset \mathfrak{X}$.

We now prove that $\mathfrak{X} \subset \mathfrak{V}$. Let $\mathbf{e}_1, \dots, \mathbf{e}_n$ denote a basis of \mathfrak{X} and let E be the $m(\mu_d - \nu_d + 1) \times n$ matrix whose columns are the $\mathbf{e}_1, \dots, \mathbf{e}_n$. Since \mathfrak{X} is M -invariant, there exists $R \in \mathcal{M}_n(\overline{\mathbb{Q}})$ such that

$$(27) \quad ME = ER.$$

The matrix R is invertible because $\ker(M) \cap \mathfrak{X} = 0$. Up to change the basis $\mathbf{e}_1, \dots, \mathbf{e}_n$ we might assume that R has a Jordan normal form. Thus, (27) implies that the set of eigenvalues of R is included in the set of eigenvalue of M . Let $J_\gamma(s)$ be the first Jordan block of R . Then, using the fact that the columns of E belong to $\ker(N)$, one can prove by induction on $j \in \{1, \dots, s\}$ that $\mathbf{e}_j \in V_{\gamma, j}$. Doing this for all the Jordan blocks of J , we obtain that the columns of E belong to \mathfrak{V} , as wanted. \square

4.5. End of the proof of Theorem 2. There is not much left to do. Assume that the system (1) is regular singular at 0. Then, it follows from Proposition 5 that there exists a gauge transformation Ψ with entries in $\overline{\mathbb{Q}}((z^d))$ such that $\phi_p(\Psi)^{-1} A \Psi$ is a constant matrix, where $d \in \mathcal{D}$ is given by Algorithm 2. Thus, from Proposition 13, the matrix U has m linearly independent columns, that is, $\dim \mathfrak{U} = \dim \mathfrak{V} = m$. But, from Lemma 14, $\mathfrak{V} = \mathfrak{X}$. Thus \mathfrak{X} has dimension m .

Assume now that $\mathfrak{X} := \mathfrak{X}_d$ has dimension m for some $d \in \mathcal{D}$. Then, from Lemma 14, $\dim \mathfrak{U} = \dim \mathfrak{V} = m$. Thus, the matrix U has m columns and it follows from Proposition 13 that the system is regular singular and that $\Psi = \phi_{1/d}(U) \in \mathrm{GL}_m(\overline{\mathbb{Q}}((z^{1/d})))$ is such that $\phi_p(\Psi)^{-1} A \Psi$ is a constant matrix.

5. THE ALGORITHM OF THEOREM 1

Theorem 2 gives the description of a vector space whose dimension characterises the regular singularity at 0 of a Mahler system. However, if the system is regular singular at 0, this theorem does not tell how to build a matrix Ψ such that $\phi_p(\Psi)^{-1} A \Psi \in \mathrm{GL}_m(\overline{\mathbb{Q}})$. Such a construction has been done with the help of the matrix U in section 4. However, the interest of this construction is more heuristic than effective for it necessitates the extraction of roots of polynomials. In this section we describe an algorithm to compute the matrix Ψ which does not require the determination of eigenvalues. *In fine*, the coefficients of the Puiseux series defining the matrix Ψ belong to the same number field as the coefficients of the rational functions defining the matrix A .

5.1. A direct construction of a gauge transformation. Assume that the Mahler system (1) is regular singular at 0. From Theorem 2, there exists an integer $d \in \mathcal{D}$ such that the dimension of \mathfrak{X}_d is equal to m . Let $\mathbf{e}_1, \dots, \mathbf{e}_m$ be a basis of \mathfrak{X}_d and let E be the $m(\mu_d - \nu_d + 1) \times m$ matrix whose columns are $\mathbf{e}_1, \dots, \mathbf{e}_m$. As in (27), we have

$$M_d E = E R$$

for some matrix $R \in \mathrm{GL}_m(\overline{\mathbb{Q}})$. We make a block decomposition of the columns of E into $\mu_d - \nu_d + 1$ matrices $E_{\nu_d}, \dots, E_{\mu_d} \in \mathcal{M}_m(\overline{\mathbb{Q}})$:

$$E = \begin{pmatrix} E_{\nu_d} \\ \vdots \\ E_{\mu_d} \end{pmatrix}.$$

We then define by induction on $n > \mu_d$ a matrix $E_n \in \mathcal{M}_m(\overline{\mathbb{Q}})$ setting

$$(28) \quad E_n = \left(\sum_{(k,\ell): k+\ell=n} B_k(d) E_\ell \right) R^{-1}$$

where $B(d) = \sum_{n \in \mathbb{Z}} B_n(d) z^n = \phi_d(A)^{-1}$. As seen in Remark 4, when $n > \mu_d$, the matrices E_ℓ contributing to right-hand side of the equality are those for which $\ell < n$. Hence the matrices E_n are well-defined. If $n < \nu_d$, we set $E_n = 0$. We stress that (28) holds for any $n \in \mathbb{Z}$:

- by construction, it holds when $n > \mu_d$;
- when $\nu_d \leq n \leq \mu_d$, it follows from the fact that $ER = M_d E$;
- when $n < \nu_d$, it follows from the fact that $N_d E = 0$, for $\mathfrak{X} \subset \ker(N_d)$.

We eventually set $U := \sum_{n \geq \nu_d} E_n z^n$. It follows from (28) that

$$(29) \quad UR = B(d) \phi_p(U).$$

Since e_1, \dots, e_m is a basis of \mathfrak{X}_d , the columns of U are linearly independent over $\overline{\mathbb{Q}}$. It follows from Lemma 12 that they are linearly independent over $\overline{\mathbb{Q}}((z))$. Thus, the matrix U is nonsingular. Now, set $\Psi = \phi_{1/d}(U)$. It follows from (29) that

$$(30) \quad \phi_p(\Psi)^{-1} A \Psi = R^{-1} \in \mathrm{GL}_m(\overline{\mathbb{Q}}).$$

Thus, Ψ is the gauge transformation we are looking for.

5.2. On the base field of a gauge transformation. The above construction gives information about the field to which the coefficients of the Puiseux series development of the entries of Ψ belong.

Proposition 15. *Consider a Mahler system (1) and let \mathbb{K} denote a number field containing the coefficients of the rational functions defining the matrix A . Assume that (1) is $\overline{\mathbb{Q}}((z^d))$ -equivalent to a constant system, for some integer $d \in \mathbb{N}$. Then the system is $\mathbb{K}((z^{1/d}))$ -equivalent to a constant system.*

Therefore, any Mahler system defined over some number field \mathbb{K} which is regular singular at 0 is equivalent to a constant matrix with entries in \mathbb{K} . The entries of the gauge transformation are Puiseux series with coefficients in \mathbb{K} .

Proof. From Theorem 2, the dimension of the $\overline{\mathbb{Q}}$ -vector space \mathfrak{X}_d equals m . Since M_d and N_d have entries in \mathbb{K} , \mathfrak{X}_d is defined over \mathbb{K} . Hence, the basis e_1, \dots, e_m of \mathfrak{X}_d can be chosen in $\mathbb{K}^{m(\mu_d - \nu_d + 1)}$. It follows that the matrix R and the matrices E_n , $n \geq \nu_d$ have their entries in \mathbb{K} . As a consequence the matrix

$$\Psi(z) = \sum_{n \geq \nu_d} E_n z^{n/d}$$

belongs to $\mathrm{GL}_m(\mathbb{K}((z^{1/d})))$. □

Remark 5. The proof of Theorem 2 and the construction of the gauge transformation in 5.1 do not rely on the base field on which the entries of A are defined. It does not have to be a subfield of $\overline{\mathbb{Q}}$. Actually, this field could even have positive characteristic.

5.3. Description of the algorithm. We now describe precisely the algorithm of Theorem 1. We first present an algorithm to determine whether or not a Mahler system (1) is $\overline{\mathbb{Q}}((z^{1/d}))$ -equivalent to a constant system for some fixed integer $d \in \mathbb{N}$.

Algorithm 3: Test to determine if a Mahler system (1) is $\overline{\mathbb{Q}}((z^{1/d}))$ -equivalent to a constant system.

Input: A , p , and d .

Output: Whether or not the system (1) is $\overline{\mathbb{Q}}((z^{1/d}))$ -equivalent to a constant system. In that case the constant matrix Λ and enough coefficients of the Puiseux development of the associated gauge transformation so that one can compute recursively the whole development.

Set m the size of A .

Compute ν_d, μ_d, M_d, N_d .

Set $\mathfrak{X} := \ker N_d$ and $Y = N_d$.

for i from 0 to $m(\mu_d - \nu_d + 1)$ **do**

$\mathfrak{X} := \mathfrak{X} \cap M_d^i \mathfrak{X}$.

for i from 0 to $m(\mu_d - \nu_d + 1)$ **do**

$Y := Y M_d$;

$\mathfrak{X} := \mathfrak{X} \cap \ker(Y)$.

if $\dim \mathfrak{X} = m$ **then**

 From a basis of \mathfrak{X} , compute R and $E_{\nu_d}, \dots, E_{\mu_d}$ as in section 4.

 Set $\Lambda := R^{-1}$.

return “ True ”, Λ and $E_{\nu_d}, \dots, E_{\mu_d}$.

The following lemma implies that, at the end of Algorithm 3, the vector space \mathfrak{X} is the same as in Theorem 2.

Lemma 16. Set $c_d := m(\mu_d - \nu_d + 1)$. We have

$$\bigcap_{-c_d \leq n \leq c_d} M_d^n \ker(N_d) = \bigcap_{n \in \mathbb{Z}} M_d^n \ker(N_d).$$

Proof. We denote $M := M_d$ and $N := N_d$. Set $\mathfrak{J}_n = \bigcap_{k=0}^n M^{-k} \ker(N)$. It is immediate that if $\mathfrak{J}_n = \mathfrak{J}_{n+1}$ then $\mathfrak{J}_\ell = \mathfrak{J}_n$ for all $\ell \geq n$. Therefore, $\bigcap_{k=0}^\infty M^{-k} \ker(N) = \mathfrak{J}_{c_d}$.

Let $\mathfrak{J}_n = \bigcap_{k=-c_d}^n M^k \ker(N) = \bigcap_{k=-\infty}^n M^k \ker(N)$. Since c_d is the size of the matrix M we have $\ker(M^{c_d}) = \ker(M^\ell)$ for every $\ell \geq c_d$. Furthermore $\dim(\ker(N)) \leq c_d$. We shall prove that $\mathfrak{J}_k = \mathfrak{J}_{k+1}$ for all $k \geq c_d$. The only nontrivial inclusion is $\mathfrak{J}_k \subset \mathfrak{J}_{k+1}$. Fix a integer $k \geq c_d$. We highlight two facts.

- The $\overline{\mathbb{Q}}$ -vector space \mathfrak{J}_k is stable by M .
- $\mathfrak{J}_k \cap \ker(M^k) = \{0\}$. Indeed, if $\mathbf{x} \in \mathfrak{J}_k \cap \ker(M^k)$ then there exists $\mathbf{y} \in \ker(N)$ such that $\mathbf{x} = M\mathbf{y}$ and $M^{k+1}\mathbf{y} = M^k\mathbf{x} = 0$. Thus, $\mathbf{y} \in \ker(M^{k+1}) = \ker(M^k)$ and $\mathbf{x} = M^k\mathbf{y} = 0$.

Let $\mathbf{x} \in \mathfrak{J}_k$ be nonzero. The vectors $\mathbf{x}, M\mathbf{x}, \dots, M^k\mathbf{x} \in \mathfrak{J}_k$ are linearly dependant. Thus, there exist $a_1, \dots, a_k \in \overline{\mathbb{Q}}$ not all zero such that

$$a_0\mathbf{x} + a_1M\mathbf{x} + \dots + a_kM^k\mathbf{x} = 0.$$

Set $i := \min(i \in \{0, \dots, k\} \mid a_i \neq 0)$ and $\mathbf{y} = -\sum_{j=i+1}^k \frac{a_j}{a_i} M^{j-i-1} \mathbf{x} \in \mathfrak{J}_k$ so that $M^i \mathbf{x} = M^{i+1} \mathbf{y}$. We have $\mathbf{x} - M\mathbf{y} \in \mathfrak{J}_k \cap \ker(M^k)$ and it follows that

$$\mathbf{x} = M\mathbf{y} \in M\mathfrak{J}_k \cap \mathfrak{J}_k \subset \mathfrak{J}_{k+1},$$

as wanted. \square

Proposition 17. *Algorithm 3 tests whether or not system (1) is $\overline{\mathbb{Q}}((z^{1/d}))$ -equivalent to a constant system.*

Proof. From Lemma 16, Algorithm 3 computes the vector space

$$\mathfrak{X}_d := \bigcap_{n \in \mathbb{Z}} M_d^n \ker(N_d).$$

From Lemma 14, $\dim \mathfrak{X}_d = \dim \mathfrak{V} = \dim \mathfrak{U}$. Thus, it follows from Proposition 13 that the system (1) is $\overline{\mathbb{Q}}((z^{1/d}))$ -equivalent to a constant system if and only if $\dim \mathfrak{X}_d = m$. This is precisely what Algorithm 3 tests. \square

We can then describe an Algorithm for Theorem 1.

Algorithm 4: Test for the regularity singularity at 0 of a Mahler system

Input: A , p , and $n \geq 0$, the order of truncation.

Output: If the system (1) is regular singular at 0 and in that case the constant matrix Λ to which it is equivalent and a truncation of the associated gauge transformation Ψ at order n .

Compute d with Algorithm 2.

Run Algorithm 3 with that d .

if Algorithm 3 returns “True” **then**

for ℓ from $\mu_d + 1$ to $\max\{\mu_d + 1; dn\}$ **do**

 | Compute E_ℓ from (28).

end

return “The system is regular-singular at 0”, Λ and $\sum_{\ell \geq \nu_d}^{dn} E_\ell z^{\ell/d}$.

end

return “The system is not regular singular at 0”.

Proposition 18. *Algorithm 4 returns whether or not a system (1) is regular singular at 0, and in that case, a constant matrix to which the system is \mathbf{K} -equivalent and a truncation of an associated gauge transformation at an arbitrary order.*

Proof. It follows from Proposition 5 that the system is regular singular at 0 if and only if it is $\overline{\mathbb{Q}}((z^{1/d}))$ -equivalent to a constant system where d is defined by Algorithm 2. Assume that the system (1) is regular singular at 0. Then it follows from Proposition 17 that Algorithm 3 shall return “True”. In that case, $\Psi(z) = \sum_{\ell \geq \nu_d} E_\ell z^{\ell/d}$ satisfies (30).

Now, assume that the Algorithm 4 returns that the system is regular singular at 0. Then Algorithm 3 must have returned “True”. It thus follows from Proposition 17 that the system is indeed regular singular at 0. Thus, from the first part of the proof, $\Psi(z) = \sum_{\ell \geq \nu_d} E_\ell z^{\ell/d}$ satisfies (30). \square

When the system (1) is regular singular at 0, Algorithm 4 computes a \mathbf{K} -equivalent constant matrix. Furthermore, Roques [Roq18, §5.2] described fundamental matrices of solution for constant systems. Precisely, for $c \in \overline{\mathbb{Q}}^\star$ we let e_c denote a function such that $\phi_p(e_c) = ce_c$, and let ℓ denote a function such that $\phi_p(\ell) = \ell + 1$. For example, we can take suitable determinations of $\log(z)^{\log(c)/\log(p)}$ and $\log(\log(z))$. Any constant system has a basis of solution in $\overline{\mathbb{Q}} \left[(e_c)_{c \in \overline{\mathbb{Q}}^\star}, \ell \right]$. Then, we have the following.

Corollary 19. *Consider a system (1) which is regular singular at 0. From Algorithm 4, one can compute a fundamental matrix of solution of (1) with entries in $\mathbf{K} \left[(e_c)_{c \in \overline{\mathbb{Q}}^\star}, \ell \right]$.*

5.4. On the complexity of Algorithm 4. We now propose to discuss the complexity of this algorithm. We start with an observation about the shape of the matrices M_d and N_d .

Definition 3. Let $D = (D_{i,j})_{1 \leq i \leq r, 1 \leq j \leq s}$ be a block matrix with $D_{i,j} \in \mathcal{M}_m(\overline{\mathbb{Q}})$. We say that D is a d -gridded matrix if for all $(i_0, j_0) \in \{1, \dots, r\} \times \{1, \dots, s\}$ such that D_{i_0, j_0} is nonzero, the matrices $D_{i_0, j}$, D_{i, j_0} with $i \not\equiv i_0 \pmod{d}$ and $j \not\equiv j_0 \pmod{d}$ are zero matrices. Let σ be a permutation of the set $\{1, \dots, d\}$. We say that σ is associated with the d -gridded matrix D if $D_{i,j} = 0$ for every $i, j \in \{1, \dots, d\}$ with $j \neq \sigma(i)$.

Lemma 20. *The multiplication on d -gridded matrices and the computation of a kernel can be done with complexity*

$$\mathcal{O}(d\text{MM}(s/d))$$

where s is an upper bound for the number of rows and the number of columns of the considered matrices. Furthermore, the multiplication of two d -gridded matrices with associated permutation σ_1 and σ_2 is a d -gridded matrix with associated permutation $\sigma_2 \circ \sigma_1$.

Proof. We describe the computation for the matrix multiplication. A similar computation can be made for the determination of a kernel. Consider two d -gridded matrices $D = (D_{i,j})_{i,j}$ and $E = (E_{i,j})_{i,j}$ and denote by σ_D and σ_E their associated permutations. For $i \in \{1, \dots, d\}$ we let D_i (resp. E_i) denote the block-matrices $(D_{i+kd, \sigma_D(i)+\ell d})_{k,\ell}$ (resp. $(E_{i+kd, \sigma_E(i)+\ell d})_{k,\ell}$). Let $F_i = D_i E_{\sigma_D(i)}$ and set $F_i =: (F_{i,k,\ell})_{k,\ell}$ its block decomposition. For any integer i , write $i = i_0 + kd$ with $i_0 \in \{1, \dots, d\}$ and

$$G_{i,j} := \begin{cases} F_{i_0,k,\ell} & \text{if } j = \sigma_E \circ \sigma_D(i_0) + \ell d, \\ 0 & \text{else.} \end{cases}$$

Then

$$DE = (G_{i,j})_{i,j}.$$

The matrix $(G_{i,j})_{i,j}$ is itself a d -gridded matrix. Its associated permutation is $\sigma_E \circ \sigma_D$. The computation of the product of two permutations of $\{1, \dots, d\}$ has complexity $\mathcal{O}(d)$. Once $\sigma_E \circ \sigma_D$ is known, the computation of each matrix F_i has complexity $\mathcal{O}(\text{MM}(s/d))$. Thus, the computation of DE has complexity

$$\mathcal{O}(d + d\text{MM}(s/d)) = \mathcal{O}(d\text{MM}(s/d)).$$

□

Lemma 21. *The matrices M_d and N_d are d -gridded matrices.*

Proof. Recall that $M_d := (B_{i-pj}(d))_{\nu_d \leq i, j \leq \mu_d}$ and

$$N_d := \begin{cases} (B_{i-pj}(d))_{\substack{\nu_0(B(d))+p\nu_d \leq i \leq \nu_d-1 \\ \nu_d \leq j \leq \mu_d}} & \text{if } \nu_d < \mu_d \\ 0 \in \mathcal{M}_{1, m(\mu_d - \nu_d + 1)}(\overline{\mathbb{Q}}) & \text{if } \nu_d = \mu_d \end{cases}$$

where $\phi_d(A^{-1}) := \sum_n B_n z^n$. In particular if d does not divide $i - pj$ then $B_{i-pj} = 0$. Thus if $B_{i_0 - pj_0} \neq 0$ then $B_{i - pj_0} = 0$ for all i such that $i \not\equiv i_0 \pmod{d}$. Moreover, since p and d are relatively prime, if $B_{i_0 - pj_0} \neq 0$ then $B_{i_0 - pj} = 0$ for all j such that $j \not\equiv j_0 \pmod{d}$. Associated permutations to these matrices are σ_M and σ_N such that, for every $k \in \{1, \dots, d\}$,

$$\begin{aligned} p\sigma_M(k) &\equiv (p-1)(1-\nu_d) + k \pmod{d} \\ p\sigma_N(k) &\equiv \nu_0(B(d)) + p - 1 + k \pmod{d}. \end{aligned}$$

□

We can now find an upper bound for the complexity of Algorithm 4.

Proposition 22. *Apart from the computation of the Puiseux development of Ψ , the complexity of Algorithm 4 is*

$$\mathcal{O}(\text{MM}(m)\text{M}(p^m(\deg(A) + m)) + p^{2m-1}mv\text{MM}(mv/p)) ,$$

where $v := -(v_0(A) + v_0(A^{-1})) \geq 0$.

Proof. We follow the script of Algorithm 4. From Proposition 4, the complexity of Algorithm 1 is

$$\mathcal{O}(\text{MM}(m)\text{M}(p^m(\deg(A) + m))) .$$

Thus d can be computed from Algorithm 2 with the same complexity. To compute M_d and N_d one needs to compute the inverse of A and the Laurent series expansion of A^{-1} between $v_0(A^{-1})$ and $(\mu_d - p\nu_d)/d$. The computation of the inverse of A may be done in $\mathcal{O}(\text{MM}(m)\text{M}(\deg(A)))$. Computing n terms of the Laurent expansion of a rational function can be done in $\mathcal{O}(\text{M}(n))$. Set $v := -(v_0(A) + v_0(A^{-1})) \geq 0$. One has

$$\frac{\mu_d - p\nu_d}{d} - v_0(A^{-1}) = \mathcal{O}(v) .$$

Once d is fixed, the computation of M_d and N_d can be done with complexity

$$\mathcal{O}(\text{MM}(m)\text{M}(\deg(A)) + \text{M}(v)) .$$

Computing the intersection of two vector spaces given a basis of each is the same as computing a kernel. Since the number of rows and the number of columns of M_d and N_d is at most $\mathcal{O}(mdv)$, it follows from Lemmas 20 and 21 that the computation of

$$\mathfrak{X}_d := \bigcap_{-c_d \leq n \leq c_d} M^n \ker(N) ,$$

necessitates $2c_d + 1$ steps with complexity $\mathcal{O}(d\text{MM}(mv))$, where $c_d := m(\mu_d - \nu_d + 1)$. Thus, the computation of \mathfrak{X}_d can be done with

$$\mathcal{O}(md^2v/p\text{MM}(mv))$$

operations. Putting all this together, since $d \leq p^m$, Algorithm 4 returns if a system is regular singular or not in

$$\mathcal{O}(\text{MM}(m)\text{M}(p^m(\deg(A) + m)) + p^{2m-1}mv\text{MM}(mv)) .$$

□

If one considers the naive bounds $M(n) = \mathcal{O}(n^2)$ and $MM(n) = \mathcal{O}(n^3)$, the complexity of Algorithm 4 is

$$(31) \quad \mathcal{O}(m^5 p^{2m} + m^3 p^{2m} \deg(A)^2 + m^4 v^4 p^{2m-1}).$$

Remark 6. In Algorithm 4, we choose to compute first the integer d thanks to the Cyclic Vector Lemma, Algorithm 1 and Algorithm 2. Then we run Algorithm 3 with this precised d . This is not always the fastest way to proceed, since the computation of d necessitates to work with possibly great numbers (see Algorithm 1). However, the cost of manipulating great numbers is hidden by the fact that we describe the complexity as the number of arithmetic operations in $\overline{\mathbb{Q}}$. Another way to proceed would be to run Algorithm 3 for every $d \in \mathcal{D}$. In that case the complexity is

$$\mathcal{O}(MM(m)M(\deg(A)) + p^{3m-1}mvMM(mv)).$$

Using the naive bounds for $M(n)$ and $MM(n)$ leads to a complexity in

$$\mathcal{O}(m^3 \deg(A)^2 + p^{3m-1}m^4 v^4),$$

which can be less than (31), especially for great $\deg(A)$.

6. EXAMPLES

In this section, we explore the regular singular property of some particular systems.

6.1. Equations of order 1.

6.1.1. *Homogeneous equation of order 1.* An homogeneous equation of order 1 is an equation of the form

$$(32) \quad \phi_p(y) = ay$$

where $a \in \overline{\mathbb{Q}}(z)$, $a \neq 0$.

Proposition 23. *Any homogeneous equation of order 1 (32) is regular singular at 0.*

Proof. Let ν denote the valuation at 0 of a and set $\psi = z^{\nu/(p-1)}$. Then, the system $\phi_p(y) = by$ with $b := \phi_p(\psi)^{-1}a\psi$ is Fuchsian at 0. Thus, the homogeneous equation (32) is $\overline{\mathbb{Q}}((z^{\nu/(p-1)}))$ -equivalent to a Fuchsian equation at 0, which implies that (32) is regular singular at 0 (see [Roq18, Prop. 34]). \square

6.1.2. *Inhomogeneous Mahler equations of order 1.* Consider a *inhomogeneous* Mahler equation of order 1

$$(33) \quad q_{-1} + q_0 y + q_1 \phi_p(y) = 0,$$

with $q_{-1}, q_0, q_1 \in \overline{\mathbb{Q}}[z]$, $q_0 q_1 \neq 0$. The corresponding system has for associated matrix, the matrix

$$A(z) = \begin{pmatrix} -\frac{q_0}{q_1} & -\frac{q_{-1}}{q_1} \\ 0 & 1 \end{pmatrix}$$

Proposition 24. *A sufficient condition for the system to be regular singular at 0 is for (33) to have a solution in \mathbf{K} .*

Proof. Assume that (33) has a solution $f \in \mathbf{K}$. Let $g \in \mathbf{K}$ and $\gamma \in \overline{\mathbb{Q}}^\star$ such that $\phi_p(g)\gamma = -\frac{q_0}{q_1}g$. Then

$$\phi_p \begin{pmatrix} f & g \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \gamma \end{pmatrix} = A \begin{pmatrix} f & g \\ 1 & 0 \end{pmatrix},$$

and the system is regular singular at 0. \square

Note that it is not a necessary condition. For example, take $q_{-1} = q_0 = -1$ and $q_1 = 1$.

6.2. An equation of order 2. Consider the 3-Mahler equation:

$$z^3(1 - z^3 + z^6)(1 - z^7 - z^{10})\phi_3^2(y) - (1 - z^{28} - z^{31} - x^{37} - z^{40})\phi_3(y) + z^6(1 + z)(1 - z^{21} - z^{30})y = 0.$$

The matrix of the 3-Mahler system associated to this equation is

$$A(z) := \begin{pmatrix} 0 & 1 \\ -\frac{z^3(1+z)(1-z^{21}-z^{30})}{(1-z^3+z^6)(1-z^7-z^{10})} & \frac{1-z^{28}-z^{31}-x^{37}-z^{40}}{z^3(1-z^3+z^6)(1-z^7-z^{10})} \end{pmatrix}.$$

We propose to check whether or not the 3-Mahler system associated to this matrix is regular singular at 0. Since we already know an homogeneous linear equation associated with this system, it is not necessary to run Algorithm 1. Algorithm 2 applied to this system returns $d := 2$. We now run Algorithm 3 with $d = 2$. We have $v_0(A) = -3$, $v_0(A^{-1}) = -6$ and thus $\nu_2 = -3$ and $\mu_2 = 6$. We can compute N and M :

$$N := \left(\begin{array}{c|c|c|c|c|c|c|c|c|c} \begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} & \\ \hline & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} \\ \hline \begin{smallmatrix} -1 & 0 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} & \\ \hline & \begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} \\ \hline \begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} & \\ \hline & \begin{smallmatrix} -1 & 0 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} \\ \hline \begin{smallmatrix} -1 & -1 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} & \\ \hline & \begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} \\ \hline \begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} -1 & 0 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} & \\ \hline & \begin{smallmatrix} -1 & -1 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} \\ \hline \begin{smallmatrix} -1 & -1 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} & \\ \hline & \begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} -1 & 0 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} \\ \hline \begin{smallmatrix} 1 & 2 \\ 1 & 0 \end{smallmatrix} & & \begin{smallmatrix} -1 & -1 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} & \\ \hline & \begin{smallmatrix} -1 & -1 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} \\ \hline \begin{smallmatrix} -1 & -2 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} -1 & 0 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} & \\ \hline & \begin{smallmatrix} 1 & 2 \\ 1 & 0 \end{smallmatrix} & & \begin{smallmatrix} -1 & -1 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} \\ \hline \begin{smallmatrix} 1 & 2 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} -1 & -1 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} & \\ \hline & \begin{smallmatrix} -1 & -2 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} -1 & 0 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} \end{array} \right),$$

$$M := \left(\begin{array}{c|c|c|c|c|c|c|c|c|c} \begin{smallmatrix} -1 & -3 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 1 & 2 \\ 1 & 0 \end{smallmatrix} & & \begin{smallmatrix} -1 & -1 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} \\ \hline & \begin{smallmatrix} 1 & 2 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} -1 & -1 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} \\ \hline \begin{smallmatrix} 1 & 4 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 1 & -2 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} -1 & 0 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} & \\ \hline & \begin{smallmatrix} -1 & -3 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 1 & 2 \\ 1 & 0 \end{smallmatrix} & & \begin{smallmatrix} -1 & -1 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} \\ \hline \begin{smallmatrix} -1 & -4 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 1 & 2 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} -1 & -1 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} & \\ \hline & \begin{smallmatrix} 1 & 4 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} -1 & -2 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} -1 & 0 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} \\ \hline \begin{smallmatrix} 1 & 4 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 1 & -3 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 1 & 2 \\ 1 & 0 \end{smallmatrix} & & \begin{smallmatrix} -1 & -1 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix} & \\ \hline & \begin{smallmatrix} -1 & -4 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 1 & 2 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} -1 & -1 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} \\ \hline \begin{smallmatrix} -1 & -4 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 1 & 4 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} -1 & -2 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} -1 & 0 \\ 0 & 0 \end{smallmatrix} & \\ \hline & \begin{smallmatrix} 1 & 4 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} -1 & -3 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 1 & 2 \\ 1 & 0 \end{smallmatrix} & & \begin{smallmatrix} -1 & -1 \\ 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix} \end{array} \right)$$

where we left empty blocks for the ones corresponding to zero-matrices in the 2-gridded block decomposition of M and N . In that case, the vector space \mathfrak{X} is spanned by the transpose of the two linearly independent vectors

$$\begin{aligned} & (0, 1, 0, 0, 1, 0, 0, 0, -1, 0, 0, 0, 1, -1, 0, 0, -1, 0, 0, 0) \\ & (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0) \end{aligned}$$

In particular, \mathfrak{X} has dimension 2 and, from Theorem 2, the system is regular singular at 0. One can check that these vectors are eigenvectors of the matrix M for the eigenvalue 1. Thus the matrix R is the identity matrix of size 2. In particular, the associated gauge transformation Ψ is a fundamental matrix of solution because it satisfies

$$\phi_3(\Psi)^{-1} A \Psi = I_2.$$

From these two vectors, we can compute the first terms of the Puiseux development of Ψ

$$\Psi = \begin{pmatrix} f_1 & f_2 \\ f_3 & 0 \end{pmatrix} + \mathcal{O}(z^{17/2})$$

with

$$\begin{aligned} f_1(z) &= z^{-1/2} - z^{1/2} + z^{3/2} - z^{5/2} + z^{7/2} - z^{9/2} + z^{11/2} - z^{13/2} + z^{15/2}, \\ f_2(z) &= -z^3 + z^4 - z^5 + 2z^6 - 2z^7 + 2z^8, \\ f_3(z) &= z^{-3/2} - z^{3/2} + z^{9/2} - z^{15/2}. \end{aligned}$$

Remark 7. Note that this example is the same as the one that illustrates the paper [CDDM18].

6.3. Systems coming from finite deterministic automata. As mentioned in the introduction, Mahler systems are related with the automata theory. Indeed, the generating function of an *automatic sequence* (see [AS03]) is solution of a Mahler equation. Numerous famous automatic sequences are related to homogeneous or inhomogeneous Mahler equations of order 1. This is the case of the *Thue-Morse* sequence, the *regular paper-folding* sequence, the sequences of power of a given integer, the characteristic sequence of *triadic Cantor integers* - those whose base-3 representation contains no 1. Thus, their associated systems are regular singular at 0.

Among the sequences satisfying equations with a greater order, an important one is the *Baum-Sweet* sequence, which is the characteristic sequence of integers whose binary development have no blocks of consecutive 0 of odd length. The system associated to

this sequence is Fuchsian at 0 and thus regular singular at 0. An other important one is the *Rudin-Shapiro* sequence whose general term is

$$\begin{cases} a_n = 1 & \text{if the number of occurrence of two consecutive 1} \\ & \text{in the binary expansion of } n \text{ is even} \\ a_n = -1 & \text{else.} \end{cases}$$

Its generating power series $f := \sum_{n \in \mathbb{N}} a_n z^n$ satisfies the equation

$$\phi_2 \begin{pmatrix} f(z) \\ f(-z) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ \frac{1}{z} & -\frac{1}{z} \end{pmatrix} \begin{pmatrix} f(z) \\ f(-z) \end{pmatrix}$$

This system is not regular singular. Indeed, running Algorithm 3 with $d = 3$ shows that the dimension of \mathfrak{X} is only 1.

The regular singular property can be seen as “normal” for Mahler systems because a sufficient condition is to be Fuchsian at 0. However, the generating series of an automatic sequence satisfy Mahler systems with a very precise shape : $A^{-1}(0)$ has at most one nonzero entry in each column. Among these systems, the Fuchsian ones appear to be the exceptions. Consider for example the following variation of the Baum-Sweet sequence : $(a_n)_{n \in \mathbb{N}} \in \{0, 1\}^{\mathbb{N}}$, where $a_n = 0$ if and only if the binary development of n contains a block of consecutive 1 with odd length. A Mahler system associated to this sequence is

$$\phi_2(Y) = \begin{pmatrix} 1 & z & 0 \\ z & 0 & 0 \\ 0 & 1 & 1+z \end{pmatrix}^{-1} Y,$$

which is not regular singular at 0.

7. OPEN PROBLEMS

We discuss here some open problems about the regular singularity at 0 of a Mahler system.

7.1. The inverse matrix system. Let $A \in \text{GL}_m(z)$ and $p \geq 2$ an integer. If the p -Mahler system with matrix A is Fuchsian at 0, then the p -Mahler system with matrix A^{-1} is also Fuchsian at 0 (and hence, regular singular at 0). This property does not extend to regular singular systems. For example, if A denotes the matrix of the regular singular system in subsection 6.2, the 3-Mahler system associated with A^{-1} is *not* regular singular at 0. We ask the following question.

Is there a characterisation of matrices A such that the p -Mahler systems associated with both A and A^{-1} are regular singular at 0 ?

7.2. Changing the Mahler operator. Assume that a system is Fuchsian at 0. If we change the integer p then the system remains Fuchsian at 0 (hence regular singular at 0). This property does not extend to regular singular systems. Indeed, the 3-Mahler system of subsection 6.2 is regular singular at 0, while the 2-Mahler system with the same matrix is not. Similarly, the p -Mahler system associated to this matrix is not regular singular when $p \in \{4, \dots, 30\}$ (and probably beyond). Similarly, the companion system associated with the p -Mahler equation

$$(z^{11} + z^{13})\phi_p^2(y) + (-1/z - z - z^6 + z^{10})\phi_p(y) + (1 - z)y = 0,$$

is regular singular at 0 for $p = 2$ and $p = 4$ but not for $p \in \{3, 5, 6, \dots, 100\}$ (and probably beyond). It seems that for a matrix $A \in \text{GL}_m(\overline{\mathbb{Q}}(z))$ the p -Mahler system

associated with A is either regular singular at 0 for every integers or for finitely many (possibly none) integers $p \geq 2$.

Is that true that only these two situations may occur ?

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