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Measure data problems for a class of elliptic equations with mixed absorption-reaction

Marie-Françoise Bidaut-Véron[*],
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Abstract

We study the existence of nonnegative solutions to the Dirichlet problem $\mathcal{L}^M_{p,q}u := -\Delta u + u^p - M|\nabla u|^q = \mu$ in a domain $\Omega \subset \mathbb{R}^N$ where μ is a nonnegative Radon measure, when $p > 1$, $q > 1$ and $M \geq 0$. We also give conditions under which nonnegative solutions of $\mathcal{L}^M_{p,q}u = 0$ in $\Omega \setminus K$ where K is a compact subset of Ω can be extended as a solution of the same equation in Ω.
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1 Introduction

Let Ω be a bounded domain of $\mathbb{R}^N$, $N \geq 2$, and $\mathcal{L}^M_{p,q}$ be the operator

$$u \mapsto \mathcal{L}^M_{p,q} u := -\Delta u + |u|^{p-1}u - M|\nabla u|^q \quad \text{for all } u \in C^2(\Omega) \tag{1.1}$$

where $M \geq 0$ and $p, q > 1$. We first provide an *a priori* estimate for a positive solution of (1.1) and its gradient in the range $1 < q < p$. Then we study under what conditions on the parameters any solution of

$$\mathcal{L}^M_{p,q} u = 0 \qquad \text{in } \Omega \setminus K, \tag{1.2}$$

where K is a compact subset of Ω, can be extended as a solution of the same equation in whole Ω, and if it is the case, whether the solution is bounded or not in Ω. We also consider the Dirichlet problem with measure data

$$\begin{aligned} \mathcal{L}^M_{p,q} u &= \mu \qquad && \text{in } \Omega \\ u &= 0 && \text{in } \partial\Omega, \end{aligned} \tag{1.3}$$

where μ is a nonnegative bounded Radon measure in Ω and exhibit conditions which guarantee the existence of nonnegative solutions to this problem.

If $M = 0$, $\mathcal{L}^M_{p,q}$ reduces to the Emden-Fowler operator

$$u \mapsto \mathcal{L}_p u := -\Delta u + |u|^{p-1}u. \tag{1.4}$$

Singularity problems for solutions of $\mathcal{L}_p u = 0$ have been investigated since fourty years, starting with the work of Brezis and Véron [13] who gave conditions for the removability of an isolated singularity. Later on Baras and Pierre [3] extended the result in [13] to more general removable sets, introducing the good framework. They obtained a necessary and sufficient condition expressed in terms of the Bessel capacities $cap^{\mathbb{R}^N}_{2,p'}$ ($p' = \frac{p}{p-1}$) both for the removability of compact subsets of Ω and the solvability of the associated Dirichlet problem with measure data. Another class of operator strongly related to $\mathcal{L}^M_{p,q}$ is the Riccati operator

$$u \mapsto \mathcal{R}^M_q u := -\Delta u - M|\nabla u|^q. \tag{1.5}$$

The Dirichlet problem with measure data

$$\begin{aligned} -\Delta u - M|\nabla u|^q &= \mu \qquad && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned} \tag{1.6}$$

has been studied by Maz'ya and Verbitsky [19] and Hansson, Maz'ya and Verbitsky [16] when $q > 2$ (and also in $\mathbb{R}^N$ when $q > 1$) and Phuc [23]. Their results necessitate an extensive use of Riesz potentials.

When $M > 0$ there is a balance between the absorption term $|u|^{p-1}u$ and the source term $M|\nabla u|^q$, and this interaction is the origin of many unexpected new effects. In the study of singularity problems the effect of the diffusion can be neglectable compared to the two nonlinear terms. The scale of the two opposed reaction terms depends upon the position of q with respect to $\frac{2p}{p+1}$. This is due to the fact that if $q = \frac{2p}{p+1}$, (1.1) is equivariant with respect to the scaling transformation T_ℓ defined for $\ell > 0$ by

$$T_\ell[u](x) = \ell^{\frac{2}{p-1}} u(\ell x) = \ell^{\frac{2-q}{q-1}} u(\ell x). \tag{1.7}$$

If $q < \frac{2p}{p+1}$, the absorption term is dominant and the behaviour of the singular solutions is modelled by the equation $\mathcal{L}_p u = 0$ studied in [28]. If $q > \frac{2p}{p+1}$, the diffusion is negligible and the behaviour of the singular solutions is modelled by positive separable solutions of $\mathcal{E}^M_{p,q} u = 0$ where $\mathcal{E}^M_{p,q}$ is an eikonal type operator defined by

$$\mathcal{E}^M_{p,q} u = u^p - M|\nabla u|^q. \tag{1.8}$$

This problem is studied in the forthcoming article [8]. If $q = \frac{2p}{p+1}$, the coefficient $M > 0$ plays a fundamental role in the properties of the set of solutions, in particular for the existence of singular solutions and removable singularities; this is not the case when $q \neq \frac{2p}{p+1}$ since by an homothety M can be assumed to be equal to 1.

Brezis and Véron proved in [13] that isolated singularities of solutions of $\mathcal{L}_p u = 0$ are removable when $p \geq \frac{N}{N-2}$. The removability property has been extended to more general sets using a capacity framework in [3]. Using a change of variable inspired by [6] where boundary singularities of solutions of (1.8) are considered we prove a series of removability results for solutions of

$$\mathcal{L}^M_{p,q} u = 0. \tag{1.9}$$

Theorem 1.1 *Assume $0 \in \Omega \subset \mathbb{R}^N$, $N \geq 3$, $M > 0$, $p \geq \frac{N}{N-2}$, $1 < q \leq \frac{2p}{p+1}$ and $(p,q) \neq (\frac{N}{N-2}, \frac{N}{N-1})$. Then any nonnegative solution $u \in C^2(\Omega \setminus \{0\})$ of (1.9) in $\Omega \setminus \{0\}$ belongs to $W^{1,q}_{loc}(\Omega) \cap L^p_{loc}(\Omega)$, and it can be extended as a weak solution of (1.9) in Ω.*

Furthermore, if we assume either

(i) $p \geq \frac{N}{N-2}$ and $1 < q < \frac{2p}{p+1}$, or

(ii) $p > \frac{N}{N-2}$, $q = \frac{2p}{p+1}$ and

$$M < m^* := (p+1)\left(\frac{(N-2)p - N}{2p}\right)^{\frac{p}{p+1}}, \tag{1.10}$$

then $u \in C^2(\Omega)$.

The existence of radial singular solutions when $(p,q) = (\frac{N}{N-2}, \frac{N}{N-1})$ and $M > 0$, or when $p > \frac{N}{N-2}$, $q = \frac{2p}{p+1}$ and $M \geq m^*$ shows the optimality of the statements (see [8]). A series of pointwise *a priori* estimates concerning u and ∇u are presented in the first section. They are obtained by a combination of Keller-Osserman type estimates, rescaling techniques and Bernstein method. They play a key role for analyzing the case $p = \frac{N}{N-2}$ in the previous theorem, and will be of fundamental importance in the forthcoming paper [8].

The method introduced in the proof of Theorem 1.1 combined with the result of [3] yields a more general removability result. For such a task we denote by $cap^{\mathbb{R}^N}_{k,b}$ the Bessel capacity relative to $\mathbb{R}^N$ with order $k > 0$ and power $b \in (1, \infty)$. If $k \in \mathbb{N}^*$ it coincides with the Sobolev capacity associated to the space $W^{k,b}(\mathbb{R}^N)$ by Calderon's theorem (see e.g. [1] for a detailed presentation).

Theorem 1.2 *Let $p > \frac{N}{N-2}$ and $\frac{N}{N-2} < r < p$. Suppose that one of the following conditions is verified:*

(i) either $q = \frac{2p}{p+1}$ *and*

$$0 < M < m^*(r) := (p+1)\left(\frac{p-r}{p(r-1)}\right)^{\frac{p}{p+1}}, \tag{1.11}$$

(ii) or $1 < q < \frac{2p}{p+1}$ *and* $M > 0$.

Then, if K *is a compact subset of* Ω *such that* $cap^{\mathbb{R}^N}_{2,r'}(K) = 0$, *any nonnegative solution* $u \in C^2(\Omega \setminus K)$ *of* (1.9) *in* $\Omega \setminus K$ *can be extended to* Ω *as a solution still denoted by* u *in the sense of distributions in* Ω. *Furthermore, if* $r \leq \frac{2N}{N-2}$, *then* $u \in C^2(\Omega)$.

Next we obtain sufficient conditions on a positive measure in Ω in order (1.3) be solvable. In the sequel we assume that $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a bounded smooth domain. We denote by $\mathfrak{M}(\Omega)$ (resp. $\mathfrak{M}^b(\Omega)$) the set of Radon measures (resp. bounded Radon measures) in Ω and by $\mathfrak{M}_+(\Omega)$ (resp. $\mathfrak{M}^b_+(\Omega)$) its positive cone. The total variation norm of a bounded measure μ is $\|\mu\|_{\mathfrak{M}}$.

Since for any $\mu \in \mathfrak{M}^b_+(\Omega)$ the nonnegative solutions of $\mathcal{L}_p v = \mu$ and $\mathcal{R}^M_q w = \mu$ are respectively a subsolution and a supersolution of equation (1.3) and they satisfy $0 \leq v \leq w$, the construction of v and w is the key-stone for solving (1.3). It is known that these two problems can be solved when the measure μ satisfies some continuity properties with respect to some specific capacities.

Theorem 1.3 *Assume* $p > 1$, $1 < q < 2$. *Let* $\mu \in \mathfrak{M}^b_+(\Omega)$. *If* μ *satisfies*

$$\mu(E) \leq C \min\left\{cap^{\mathbb{R}^N}_{2,p'}(E), cap^{\mathbb{R}^N}_{1,q'}(E)\right\} \quad \text{for all Borel sets } E \subset \Omega, \tag{1.12}$$

there is a constant $c_0 > 0$ *such that for any* $0 \leq c \leq c_0$ *there exists a function* $u \in W^{1,q}_0(\Omega) \cap L^p(\Omega)$, $u \geq 0$, *satisfying*

$$-\int_\Omega u \Delta\zeta dx + \int_\Omega \left(u^p - M|\nabla u|^q\right)\zeta dx = c\int_\Omega \zeta d\mu \quad \text{for all } \zeta \in C^2_c(\overline{\Omega}). \tag{1.13}$$

The condition on the measure is satisfied if $W^{-1,q}(\Omega) \hookrightarrow W^{-2,p}(\Omega)$, and we prove the following:

Corollary 1.4 *Let* $\frac{Np}{N+p} \leq q < 2$ *and* $\mu \in \mathfrak{M}^b_+(\Omega)$ *be such that*

$$\mu(E) \leq C cap^{\mathbb{R}^N}_{1,q'}(E) \quad \text{for all Borel set } E \subset \Omega, \tag{1.14}$$

for some $C > 0$, *then there exists a constant* $c_1 > 0$ *such that for any* $0 \leq c \leq c_1$ *there exists a nonnegative function* $u \in W^{1,q}_0(\Omega) \cap L^p(\Omega)$ *satisfying* (1.13).

By comparison results between capacities we have another type of result:

Corollary 1.5 *Let* $\frac{N}{N-1} \leq q \leq \frac{2p}{p+1}$. *If* $\mu \in \mathfrak{M}^b_+(\Omega)$ *satisfies,*

$$\mu(E) \leq C cap^{\mathbb{R}^N}_{2,p'}(E) \quad \text{for all Borel set } E \subset \Omega, \tag{1.15}$$

for some $C > 0$, *then there exists* $c_2 > 0$ *such that for any* $0 \leq c \leq c_2$ *there exists a nonnegative function* $u \in W^{1,q}_0(\Omega) \cap L^p(\Omega)$ *satisfying* (1.13).

As an application of the previous results, we prove the following

Corollary 1.6 *Let $p > 1$, $1 < q < 2$ and $\mu \in \mathfrak{M}_+^b(\Omega)$. There exists a function $u \in W_0^{1,q}(\Omega) \cap L^p(\Omega)$ solution of* (1.13) *if one of the following conditions is satisfied:*

(i) When $p < \frac{N}{N-2}$ and $q < \frac{N}{N-1}$, if $\|\mu\|_{\mathfrak{M}} \leq c_3$ for some $c_3 > 0$.

(ii) When $p < \frac{N}{N-2}$ and $\frac{N}{N-1} \leq q < 2$, if μ satisfies (1.14) *for some $C > 0$. In that case there exists $c_4 > 0$ such that there must hold $0 < c < c_4$ in problem* (1.12)*.*

(iii) When $p \geq \frac{N}{N-2}$ and $q < \frac{N}{N-1}$, if $\|\mu\|_{\mathfrak{M}} \leq c_4^* M^{-\frac{1}{q-1}}$ for some $c_4^* = c_4^*(N, q, \Omega) > 0$ which can be estimated, and if

$$\mu(E) = 0 \quad \textit{for all Borel set} \subset \Omega \textit{ such that } cap_{2,p'}^{\mathbb{R}^N}(E) = 0. \tag{1.16}$$

In the case (i) we show in a forthcoming article [8] and by a completely different method that there is no restriction on c if $\mu = c\delta_a$ for some $a \in \Omega$. In the above mentioned article we construct many types of local or global singular solutions using methods inherited from dynamical systems.

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2 Removable singularities

Throughout this article we denote by c and C generic constants the value of which may vary from one occurrence to another even within a single string of estimates, and by c_j, $(j = 1, 2, ...)$ some constants which have a more important significance and a more precise dependence with respect to the parameters.

2.1 *A priori* estimates

We give two estimates for positive solutions of (1.1) which differ according to the sign of M. If G is an open subset of $\mathbb{R}^N$ we set $d_G(x) = \text{dist}\,(x, \partial G)$

Proposition 2.1 *Let $G \subset \mathbb{R}^N$ be an open subset, $M > 0$ and $1 < q < p$. If $u \in C^1(G)$ is a nonnegative solution of* (1.1)*, there holds,*

$$u(x) \leq c_5 \max\left\{M^{\frac{1}{p-q}}(d_G(x))^{-\frac{q}{p-q}}, (d_G(x))^{-\frac{2}{p-1}}\right\} \quad \textit{for all } x \in G, \tag{2.1}$$

for some $c_5 = c_5(N, p, q) > 0$.

Proof. Following the method of Keller [17] and Osserman [22], we fix $x \in G$ and $0 < a < d_G(x)$, and introduce $U(z) = \lambda(a^2 - |z - x|^2)^{-b}$ for some $b > 0$. Then putting $r = |x - z|$ and $\tilde{U}(r) = U(z)$, we have in $B_a(x)$

$$\begin{aligned} L\tilde{U} &= -\tilde{U}'' - \frac{N-1}{r}\tilde{U}' - M|\tilde{U}'|^q + \tilde{U}^p \\ &= \lambda(a^2 - r^2)^{-2-b}\left[\lambda^{p-1}(a^2 - r^2)^{2-b(p-1)} + 2b(N - 2(b+1))r^2 - 2Nba^2\right. \\ &\qquad \left. -M2^q b^q \lambda^{q-1} r^q (a^2 - r^2)^{2+b-q(b+1)}\right]. \end{aligned}$$

If $M > 0$, the two necessary conditions on $b > 0$ to be fulfilled is order that $\tilde{U}$ is a supersolution in $B_{|a|}(a)$ are

$$(i) \qquad 2 - b(p-1) \le 0 \Longleftrightarrow b(p-1) \ge 2,$$

$$(ii) \qquad 2 + b - q(b+1) \ge 2 - b(p-1) \Longleftrightarrow b(p-q) \ge q.$$

The above inequalities are satisfied if

$$b = \max\left\{\frac{2}{p-1}, \frac{q}{p-q}\right\}. \tag{2.2}$$

If $q > \frac{2p}{p+1}$ then $b = \frac{q}{p-q}$ and

$$L\tilde{U} \ge \lambda\left(a^2 - r^2\right)^{-\frac{2p-q}{p-q}}\left[\lambda^{q-1}\left(\lambda^{p-q} - M2^q b^q \rho^q\right)\left(a^2 - r^2\right)^{\frac{2p-q(p+1)}{p-q}} - (3b+1)Na^2\right].$$

There exists $c_5^1 > 0$ depending on N, p and q such that if we choose

$$\lambda = c_5^1 \max\left\{M^{\frac{1}{p-q}} a^{\frac{q}{p-q}}, a^{\frac{2p(q-1)}{(p-1)(p-q)}}\right\},$$

there holds

$$L\tilde{U} \ge 0 \quad \text{in } B_a(x). \tag{2.3}$$

Since $\tilde{U}(z) \to \infty$ when $r \to a$, we derive by the maximum principle that $u \le \tilde{U}$ in $B_a(x)$. In particular

$$u(x) \le \tilde{U}(x) = \lambda a^{-\frac{2q}{p-q}} = c_5^1 \max\left\{M^{\frac{1}{p-q}} a^{-\frac{q}{p-q}}, a^{-\frac{2}{p-1}}\right\}. \tag{2.4}$$

If $q \le \frac{2p}{p+1}$ then $b = \frac{2}{p-1}$ and

$$\begin{aligned} L\tilde{U} \ge \lambda\left(|a|^2 - \rho^2\right)^{-\frac{2p}{p-1}} \Bigg[&\lambda^{p-1} + \frac{2}{p-1}\left(N - \frac{2(p+1)}{p-1}\right)\rho^2 - \frac{2N}{p-1}|a|^2 \\ &- M2^q\left(\frac{2}{p-1}\right)^q \lambda^{q-1}\rho^q\left(|a|^2 - \rho^2\right)^{\frac{2p-q(p+1)}{p-1}}\Bigg] \\ \ge \lambda\left(|a|^2 - \rho^2\right)^{-\frac{2p}{p-1}} &\left[\lambda^{p-1} - c_2|a|^2 - c_3\lambda^{q-1}M|a|^{\frac{4p-q(p+3)}{p-1}}\right]. \end{aligned}$$

Hence, if $q = \frac{2p}{p+1}$, (2.3) holds if for some $c_5^2 > 0$ depending on N, p, q,

$$\lambda = c_5^2 \max\left\{M^{\frac{p+1}{p(p-1)}}, 1\right\}|a|^{\frac{2}{p-1}},$$

which yields

$$u(x) \le \tilde{U}(x) = \lambda a^{-\frac{4}{p-1}} = c_5^2 \max\left\{M^{\frac{p+1}{p(p-1)}}, 1\right\} a^{-\frac{2}{p-1}}. \tag{2.5}$$

While if $q < \frac{2p}{p+1}$, we choose

$$\lambda = c_5^3 \max\left\{M^{\frac{1}{p-q}} a^{\frac{4p-q(p+3)}{(p-1)(p-q)}}, a^{\frac{2}{p-1}}\right\},$$

where $c_5^3 > 0 = c_5^3(N,p,q)$, which implies

$$u(x) \le \tilde{U}(x) = \lambda a^{-\frac{4}{p-1}} = c_5^3 \max\left\{M^{\frac{1}{p-q}} a^{-\frac{q}{p-q}}, a^{-\frac{2}{p-1}}\right\}. \tag{2.6}$$

By letting $a \uparrow d_G(x)$ we derive (2.1) with a constant $c_5 = c_5^3$, depending on N,p,q. □

Corollary 2.2 *Under the assumptions of Proposition 2.1 with $G = B_{2R} \setminus \{0\}$, there holds for $x \in B_R \setminus \{0\}$,*

$$u(x) \le c_5 \max\left\{M^{\frac{1}{p-q}} |x|^{-\frac{q}{p-q}}, |x|^{-\frac{2}{p-1}}\right\}. \tag{2.7}$$

We infer from Proposition 2.1 an estimate of the gradient of a positive solution when $M > 0$. We set $\sigma = 2p - q(p+1)$, then $\sigma > 0$ (resp. $\sigma < 0$) according $2p > q(p+1)$ (resp. $2p < q(p+1)$).

Proposition 2.3 *Let $p > q > 1$. For any $M_0 > 0$ and $R > 0$ there exists a constant $c_8 = c_8(N,p,q,M_0R^{\frac{\sigma}{p-1}})$ such that, for $0 < M \le M_0$ there holds:*
(i) If $q \le \frac{2p}{p+1}$ (then $\sigma \ge 0$), any positive solution u of (1.1) in $B_{2R}\setminus\{0\}$ satisfies

$$|\nabla u(x)| \le c_8 \max\left\{M^{\frac{1}{p-q}} |x|^{-\frac{p}{p-q}}, |x|^{-\frac{p+1}{p-1}}\right\}, \tag{2.8}$$

for all $x \in B_R \setminus \{0\}$.
(ii) If $\frac{2p}{p+1} \le q \le 2$ (then $\sigma \le 0$), any positive solution u of (1.1) in $\mathbb{R}^N \setminus \overline{B}_{\frac{R}{2}}$ satisfies (2.1) for all $x \in \mathbb{R}^N \setminus B_R$.

Proof. (i) For $0 < r < 2R$ we set

$$u(x) = r^{-\frac{2}{p-1}} u_r(\tfrac{x}{r}) = r^{-\frac{2}{p-1}} u_r(y) \quad \text{with } y = \tfrac{x}{r}.$$

If $\frac{r}{2} < |x| < 2r$, then $\frac{1}{2} < |y| < 2$ and $u_r > 0$ satisfies

$$-\Delta u_r + u_r^p - M r^{\frac{2p-q(p+1)}{p-1}} |\nabla u_r|^q = 0 \qquad \text{in } B_2 \setminus B_{\frac{1}{2}}. \tag{2.9}$$

Since $0 < M r^{\frac{\sigma}{p-1}} \le M(2R)^{\frac{\sigma}{p-1}} \le M_0(2R)^{\frac{\sigma}{p-1}}$ as $\sigma \ge 0$, it follows that

$$\max\left\{|\nabla u_r(z)| : \tfrac{2}{3} < |z| < \tfrac{3}{2}\right\} \le c \max\left\{|u_r(z)| : \tfrac{1}{2} < |z| < 2\right\}, \tag{2.10}$$

where c depends on N,p,q and $R^{\frac{\sigma}{p-1}} M_0$ (see e.g. [14, Chapter 13]). From Proposition 2.1 there holds

$$\max\left\{|u_r(z)| : \tfrac{1}{2} < |z| \le 2\right\} \le 2^{\frac{2}{p-1}} c_5 \max\left\{M^{\frac{1}{p-q}} r^{\frac{2p-q(p+1)}{(p-1)(p-q)}}, 1\right\}$$

by (2.1). Therefore

$$\begin{aligned} \max\left\{|\nabla u(y)| : \tfrac{r}{2} < |z| < 2r\right\} &\le 2^{\frac{2}{p-1}} c c_5 r^{-\frac{p+1}{p-1}} \max\left\{M^{\frac{1}{p-q}} r^{\frac{2p-q(p+1)}{(p-1)(p-q)}}, 1\right\} \\ &\le c_8 \max\left\{M^{\frac{1}{p-q}} |x|^{-\frac{p}{p-q}}, |x|^{-\frac{p+1}{p-1}}\right\}, \end{aligned} \tag{2.11}$$

which implies (2.8).

(ii) For $r > R$ we define u_r as in (i). It satisfies (2.9) and since $\sigma \leq 0$, we have again $0 < Mr^{\frac{\sigma}{p-1}} \leq MR^{\frac{\sigma}{p-1}} \leq M_0 R^{\frac{\sigma}{p-1}}$ if $r \geq R$. Since $1 < q < 2$, (2.10) holds and we derive (2.8). $\square$

Remark. If $q = \frac{2p}{p+1}$ the constant c_8 depends only on N and p.

The previous estimate necessitates $1 < q \leq 2$. This limitation can be by passed in some cases using the Bernstein approach.

Lemma 2.4 *Assume $p, q > 1$ and $M > 0$. If $u \in C^2(\overline{B}_{2R})$ is a nonnegative solution of* (1.1) *in B_{2R}, there holds*

$$|\nabla u(x)| \leq c_9 \left(|x|^{-\frac{1}{q-1}} + \max_{|z-x| \leq \frac{|x|}{2}} u^{\frac{p}{q}}(z) \right) \qquad \text{for all } x \in B_{\frac{R}{2}}, \tag{2.12}$$

$c_9 > 0$ depends on N, p, q and M.

Proof. Set $z = |\nabla u|^2$, then by a classical computation and the use of Schwarz inequality,

$$-\Delta |\nabla u|^2 + \frac{1}{N}(\Delta u)^2 + \langle \nabla \Delta u, \nabla u \rangle \leq 0.$$

Replacing Δu by its expression from $\mathcal{L}^M_{p,q} u = 0$, we obtain

$$-\Delta z + \frac{2}{N}\left(u^{2p} + M^2 z^q - 2M u^p z^{\frac{q}{2}} \right) + 2p u^{p-1} z \leq q M z^{\frac{q}{2}-1} \langle \nabla z, \nabla u \rangle.$$

We notice that

$$q M z^{\frac{q}{2}-1} \langle \nabla z, \nabla u \rangle \leq q M z^{\frac{q}{2}-1} |\nabla z| \sqrt{z} = q M z^{\frac{q}{2}} \frac{|\nabla z|}{\sqrt{z}} \leq \frac{M^2 z^q}{2N} + \frac{2Nq^2}{M^2} \frac{|\nabla z|^2}{z},$$

and

$$\frac{4M}{N} u^p z^{\frac{q}{2}} \leq \frac{M^2 z^q}{2N} + \frac{8u^{2p}}{NM^2},$$

thus

$$-\Delta z + \frac{M^2 z^q}{N} \leq \frac{2Nq^2}{M^2} \frac{|\nabla z|^2}{z} + \frac{2}{N}\left(\frac{4}{M^2} - 1 \right) u^{2p}.$$

For simplicity we set

$$A = \frac{M^2}{N}\ ,\ B = \frac{2Nq^2}{M^2} \text{ and } C = \frac{2}{N}\left(\frac{4}{M^2} - 1 \right) \max_{+\ |z-x| \leq \frac{|x|}{2}} u^{2p}(z)$$

Then z satisfies

$$-\Delta z + A z^q \leq B \frac{|\nabla z|^2}{z} + C \quad \text{in } B_{\frac{R}{2}}(x)$$

and obtain by [5, Lemma 3.1] (see also a simpler approach in [7, Lemma 2.2]),

$$z(x) \leq c_{10} \left(|x|^{-\frac{2}{q-1}} + C^{\frac{1}{q}} \right) \tag{2.13}$$

where $c_{10} > 0$ depends on N, p, q and M. This yields (2.12). $\square$

Remark. The constants c_9 and c_{10} can be expressed in terms of M, but their stability when $M \to 0$ is not clear since in the limit case of the equation $\mathcal{L}_p u = 0$ the estimate of the gradient obtained by a very different and much simpler method combining the Keller-Osserman estimate and scaling methods.

Using Corollary 2.2 we obtain the new estimate

Corollary 2.5 *Assume $1 < q < p$ and $M > 0$. Then any nonnegative solution $u \in C^2(B_{2R})$ of (1.1) satisfies*

$$|\nabla u(x)| \le c_{11}\left(|x|^{-\frac{1}{q-1}} + \max\left\{M^{\frac{p}{q(p-q}}|x|^{-\frac{p}{p-q}}, |x|^{-\frac{2p}{q(p-1)}}\right\}\right) \quad \text{for all } x \in B_{\frac{R}{2}}, \tag{2.14}$$

where $c_{11} > 0$ depends on N, p, q and M.

Then we can combine this estimate with Proposition 2.1 to complete the cases not treated in Proposition 2.3.

Proposition 2.6 *Let $1 < q < p$. For any $M > 0$ there exists a constant $c_{12} = c_{12}(N, p, q, M) > 0$ such that:*
(i) If $\frac{2p}{p+1} \le q < p$, any positive solution u of (1.1) in $B_{2R} \setminus \{0\}$ with $0 < R \le 1$ satisfies,

$$|\nabla u(x)| \le c_{12} \max\left\{M^{\frac{p}{q(p-q)}}|x|^{-\frac{p}{p-q}}, |x|^{-\frac{2p}{q(p-1)}}\right\} \quad \text{for all } x \in B_R \setminus \{0\}. \tag{2.15}$$

(ii) If $1 < q \le \frac{2p}{p+1}$, any positive solution u of (1.1) in $\mathbb{R}^N \setminus \overline{B}_{\frac{R}{2}}$ with $R \ge 1$ satisfies,

$$|\nabla u(x)| \le c_{12} \max\left\{M^{\frac{p}{q(p-q)}}|x|^{-\frac{p}{p-q}}, |x|^{-\frac{1}{q-1}}\right\} \quad \text{for all } x \in \mathbb{R}^N \setminus B_R. \tag{2.16}$$

Proof. We can compare the different exponents of $|x|$ which appear in the expressions (2.8) and (2.14)

$$\begin{aligned} &(i) \qquad \frac{p}{p-q} < \frac{p+1}{p-1} < \frac{2p}{q(p-1)} < \frac{1}{q-1} \qquad \text{if } 1 < q < \frac{2p}{p+1}, \\ &(ii) \qquad \frac{1}{q-1} < \frac{2p}{q(p-1)} < \frac{p+1}{p-1} < \frac{p}{p-q} \qquad \text{if } \frac{2p}{p+1} < q < p, \end{aligned} \tag{2.17}$$

with equality if $q = \frac{2p}{p+1}$. If $1 < q \le \frac{2p}{p+1}$ (resp. $\frac{2p}{p+1} \le\le 2$), estimate (2.8) is better than (2.14) in $B_R \setminus \{0\}$ (resp. $\mathbb{R}^N \setminus B_{2R}$). Then (2.15) and (2.16) follow from (2.14) and (2.17). □

In the case $M < 0$ an upper estimate on a solution is obtained by combining a result of Lions and the method of Keller and Osserman.

Proposition 2.7 *Let $G \subset \mathbb{R}^N$ be an open subset, $M \le 0$ and $p, q > 1$. If $u \in C^1(G)$ is a nonnegative solution of $\mathcal{L}^M_{p,q}u = 0$, there exists $c_6 = c_6(N, p) > 0$, $c_7 = c_7(N, q) > 0$ and $\delta = \delta(G) > 0$ such that there holds for all $x \in G$ and $\emptyset < \delta \le \delta(G)$,*

$$u(x) \le \min\left\{c_6(d_G(x))^{-\frac{2}{p-1}}, c_7|M|^{-\frac{1}{q-1}}(d_G(x))^{-\frac{2-q}{q-1}} + \max_{d_G(z)=\delta} u(z),\right\}. \tag{2.18}$$

Proof. This estimates follows from the fact that the solutions of $\mathcal{L}^M_{p,q}u = 0$ are subsolutions of $\mathcal{L}_p u = 0$ and $\mathcal{R}^M_q u = 0$. The estimate $u(x) \le c_6(d_G(x))^{-\frac{2}{p-1}}$ corresponds to the Keller-Osserman estimate for solutions of $\mathcal{L}_p u = 0$. The second estimate corresponds to the fact that if u is a positive solution of $\mathcal{R}^M_q u = 0$ in G there holds (see [18, Theorem IV-1])

$$|\nabla u(x)| \le c_7'|M|^{-\frac{1}{q-1}}(d_G(x))^{-\frac{1}{q-1}}.$$

Integrating this inequality yields the second part of the inequality. □

Remark. This estimate can be transformed into the universal estimate

$$u(x) \leq \min\left\{c_6(d_G(x))^{-\frac{2}{p-1}}, c_7|M|^{-\frac{1}{q-1}}(d_G(x))^{-\frac{2-q}{q-1}} + c_6\delta^{-\frac{2}{p-1}},\right\}, \tag{2.19}$$

since $\max\limits_{d_G(z)=\delta} u(z) \leq c_6\delta^{-\frac{2}{p-1}}$ by (2.18).

The gradient estimates are due to Nguyen [20, Proposition 1.1]. Below we recall his result proved by the Bernstein method in a more general framework but which can also be obtained by scaling techniques in the present case.

Proposition 2.8 *Let $p > 1$ and $1 < q < 2$. For any $M < 0$ and $R > 0$ there exists a constant $c'_{12} = c'_{12}(N, p, q, M,) > 0$ such that: if u is a positive solution of* (1.1) *in $B_{2R} \setminus \{0\}$, there holds*

$$u(x) + |x||\nabla u(x)| \leq c'_{12}\max\left\{|x|^{-\frac{2}{p-1}}, |x|^{-\frac{2-q}{q-1}}\right\} \quad \textit{for all } x \in B_R \setminus \{0\}. \tag{2.20}$$

Remark. There are many estimates of positive solutions of (1.1) (or even with u^p replaced by $f(u)$) in a domain which tends to infinity on the boundary (*large solutions*) or of solutions in $\mathbb{R}^N$ (*ground states*). Many of these estimates are obtained by comparison with one dimensional problems and they can be found in [2], [4], [15].

2.2 Proof of Theorem 1.1

Without loss of generality we can assume that $u \in C^2(\overline{\Omega} \setminus \{0\}$ and $\overline{B}_{2R_0} \subset \Omega$ with $2R_0 \leq 1$. If $M \leq 0$, u is a nonnegative subsolution of $-\Delta u + u^p = 0$, hence it is bounded in $\overline{\Omega}$ by [13].

Step 1. We assume $M > 0$ and we prove first that under condition (i) or (ii), $|\nabla u|^q \in L^1(\Omega)$, $u \in L^p(\Omega)$, and then

$$\int_\Omega (-u\Delta\zeta + u^p\zeta - M|\nabla u|^q\zeta)\, dx = 0 \qquad \forall \zeta \in W^{2,\infty}(\Omega) \cap C^1_c(\overline{\Omega}). \tag{2.21}$$

By Proposition 2.3

$$|\nabla u|^q \leq c|x|^{-\frac{(p+1)q}{p-1}} \quad \text{in } B_{R_0},$$

since $q \leq \frac{2p}{p+1}$, and where c depends also on M. By (i) or (ii), $\frac{(p+1)q}{p-1} < N$. Hence $\nabla u \in L^q_{loc}(\Omega)$.

For any $\epsilon > 0$ small enough we denote by ρ_ϵ a nonnegative C^∞-function such that $\text{supp}(\rho_\epsilon) \subset \overline{B}_\epsilon$, $0 \leq \rho_\epsilon \leq 1$, $|\nabla\rho_\epsilon| \leq 2\epsilon^{-1}\chi_{\overline{B}_\epsilon}$ and we set $\eta_\epsilon = 1 - \rho_\epsilon$. Then

$$-\int_{B_{2R_0}} \langle\nabla u, \nabla\rho_\epsilon\rangle dx + \int_{B_{2R_0}} u^p\eta_\epsilon dx + \int_{\partial B_{2R_0}} \frac{\partial u}{\partial \mathbf{n}} dS = M\int_{B_{2R_0}} |\nabla u|^q\eta_\epsilon dx. \tag{2.22}$$

Next

$$\left|\int_{B_{2R_0}} \langle\nabla u, \nabla\rho_\epsilon\rangle dx\right| \leq 2c_N\epsilon^{\frac{N}{q'}-1}\left(\int_{B_\epsilon} |\nabla u|^q dx\right)^{\frac{1}{q}} \to 0 \quad \text{as } \epsilon \to 0, \tag{2.23}$$

since $1 < q \leq \frac{N}{N-1}$. Since $|\nabla u|^q \in L^1(B_{2R_0})$ we deduce by monotone convergence that $u^p \in L^1(B_{2R_0})$. Finally, if $\zeta \in C_0^\infty(\Omega)$ and $\zeta_\epsilon = \zeta\eta_\epsilon$, we have

$$\int_\Omega \langle \nabla u, \nabla \zeta_\epsilon \rangle dx + \int_\Omega u^p \zeta_\epsilon dx - M \int_\Omega |\nabla u|^q \zeta_\epsilon dx = 0.$$

Letting $\epsilon \to 0$ and using (2.23), we infer that u satisfies

$$\int_\Omega \langle \nabla u, \nabla \zeta \rangle dx + \int_\Omega u^p \zeta dx - M \int_\Omega |\nabla u|^q \zeta dx = 0.$$

Hence it is a weak solution of (1.9) in Ω.

Step 2. Let us assume that $p > \frac{N}{N-2}$. If u is nonnegative and not identically zero, it is positive in $\Omega \setminus \{0\}$ by the maximum principle. We set $u = v^b$ with $0 < b \leq 1$. Then

$$-\Delta v - (b-1)\frac{|\nabla v|^2}{v} + \frac{1}{b} v^{1+(p-1)b} - M b^{q-1} v^{(b-1)(q-1)} |\nabla v|^q = 0. \tag{2.24}$$

For $\epsilon > 0$,

$$v^{(b-1)(q-1)} |\nabla v|^q \leq \frac{q\epsilon^{\frac{2}{q}}}{2} \frac{|\nabla v|^2}{v} + \frac{2-q}{2\epsilon^{\frac{2}{2-q}}} v^{1+\frac{2b(q-1)}{2-q}}.$$

Therefore

$$-\Delta v + \left(1 - b - M\frac{qb^{q-1}\epsilon^{\frac{2}{q}}}{2}\right)\frac{|\nabla v|^2}{v} + \frac{1}{b} v^{1+b(p-1)} - M b^{q-1} \frac{2-q}{2\epsilon^{\frac{2}{2-q}}} v^{1+\frac{2b(q-1)}{2-q}} = 0. \tag{2.25}$$

We notice that $1 + \frac{2b(q-1)}{2-q} = 1 + b(p-1) - a$ with $a = b\frac{2p-(p+1)q}{2-q} \geq 0$. We fix b as follows,

$$(p-1)b + 1 = \frac{N}{N-2} \Longleftrightarrow b = \frac{2}{(N-2)(p-1)}, \tag{2.26}$$

hence $p > \frac{N}{N-2}$ if and only if $0 < b < 1$. Next we impose

$$1 - b - M\frac{qb^{q-1}\epsilon^{\frac{2}{q}}}{2} = 0 \Longleftrightarrow \epsilon = \left(\frac{2(1-b)}{Mqb^{q-1}}\right)^{\frac{q}{2}} = \left(\frac{2((N-2)p-N)}{Mqb^{q-1}(N-1)(p-1)}\right)^{\frac{q}{2}}. \tag{2.27}$$

This transforms (2.25) into

$$-\Delta v + \frac{(N-2)(p-1)}{2} v^{\frac{N}{N-2}} - \frac{(2-q)b^{q-1}}{2}\left(\frac{q}{2(1-b)}\right)^{\frac{q}{2-q}} M^{\frac{2}{2-q}} v^{\frac{N}{N-2}-a} \leq 0. \tag{2.28}$$

We first assume that $0 < q < \frac{2p}{p+1}$. Then $a > 0$, hence there exists $A > 0$, depending on M, such that

$$-\Delta v + \frac{(N-2)(p-1)}{4} v^{\frac{N}{N-2}} \leq A. \tag{2.29}$$

Set $\tilde{v} = (v - cA^{\frac{N-2}{N}})_+^{\frac{N}{N-2}}$ with $c = \left(\frac{4}{(N-2)(p-1)}\right)^{\frac{N}{N-2}}$ satisfies

$$-\Delta \tilde{v} + \frac{(N-2)(p-1)}{4} \tilde{v}^{\frac{N}{N-2}} \leq 0. \tag{2.30}$$

By [13], $\tilde{v} \leq \max_{\partial\Omega} \tilde{v}$ which implies $v \leq cA^{\frac{N-2}{N}} + \max_{\partial\Omega} v$ and therefore $u(x) \leq B$ for some $B \geq 0$ in Ω. Furthermore $|\nabla u|^{q-1} \in L^{\frac{q}{q-1}}(\Omega)$ since $\nabla u \in L^q(\Omega)$, and $\frac{q}{q-1} > N$ as we assume $q < \frac{N}{N-1}$. Writing (1.9) under the form

$$-\Delta u + u^p - MC(x)|\nabla u| = 0,$$

with $C(x) = |\nabla u(x)|^{q-1}$, it follows from Serrin's theorem [25, Theorem 10] that the singularity at 0 is removable and u can be extended as a regular solution of (1.9) in Ω. Hence $u \in C^2(\Omega)$.

Then we assume that $q = \frac{2p}{p+1}$. By the choice of b in (2.26), inequality (2.25) becomes

$$-\Delta v + \left(1 - b - \frac{Mpb^{\frac{p-1}{p+1}}\epsilon^{\frac{p+1}{p}}}{p+1}\right)\frac{|\nabla v|^2}{v} + \left(\frac{1}{b} - \frac{Mb^{\frac{p-1}{p+1}}}{(p+1)\epsilon^{p+1}}\right)v^{\frac{N}{N-2}} \leq 0. \quad (2.31)$$

Notice that

$$\frac{1}{b} - \frac{Mb^{\frac{p-1}{p+1}}}{(p+1)\epsilon^{p+1}} = 0 \Longleftrightarrow \epsilon = \left(\frac{M}{p+1}\right)^{\frac{1}{p+1}} b^{\frac{2p}{(p+1)^2}}, \quad (2.32)$$

and therefore

$$1 - b - \frac{Mpb^{\frac{p-1}{p+1}}\epsilon^{\frac{p+1}{p}}}{p+1} = 1 - b - pb\left(\frac{M}{p+1}\right)^{\frac{p+1}{p}}. \quad (2.33)$$

This coefficient vanishes if

$$p\left(\frac{M}{p+1}\right)^{\frac{p+1}{p}} = \frac{p(N-1)-(N+1)}{2}.$$

Therefore, if M satisfies

$$p\left(\frac{M}{p+1}\right)^{\frac{p+1}{p}} = \frac{p(N-2)-N}{2}, \quad (2.34)$$

we can choose $\epsilon > 0$ so that the coefficient of $v^{(p-1)b+1}$ in (2.34) is equal to some $\tau > 0$. Therefore v satisfies

$$-\Delta v + \tau v^{\frac{N}{N-2}} \leq 0 \qquad \text{in } \Omega \setminus \{0\}. \quad (2.35)$$

It follows by [13], $v \leq \max_{\partial\Omega} v$ and the same type of uniform estimate holds for u. This ends the case $p > \frac{N}{N-2}$.

Step 3. Finally we assume $p = \frac{N}{N-2}$ and $1 < q < \frac{2p}{p+1} = \frac{N}{N-1}$. From (2.12),

$$M|\nabla u(x)|^q \leq c_9|x|^{-q\frac{p+1}{p-1}} = c_9|x|^{-q(N-1)} := Q(x).$$

and $Q \in L^1(B_{2R_0})$. Let $\{\sigma_n\} \subset C_0^\infty(\mathbb{R}^N)$ such that $0 \leq \sigma_n \leq 1$

$$\sigma_n(x) = \begin{cases} 1 & \text{if } \frac{2}{n} \leq |x| \leq R_0 \\ 0 & \text{if } |x| \in [0, \frac{1}{n}] \cup [2R_0, \infty), \end{cases}$$

and

$$|\Delta\sigma_n| \leq 2Nn^2\chi_{B_{\frac{2}{n}}\setminus B_{\frac{1}{n}}} + \phi,$$

where ϕ is a smooth nonnegative function with support in $B_{2R_0} \setminus B_{R_0}$. Then

$$-\int_{\{\frac{1}{n}\leq|x|\leq\frac{2}{n}\}} u\Delta\sigma_n dx - \int_{\{R_0\leq|x|\leq 2R_0\}} u\Delta\sigma_n dx + \int_{\frac{1}{n}\leq|x|} u^p\sigma_n dx = M\int_{\frac{1}{n}\leq|x|} |\nabla u|^q\sigma_n dx. \tag{2.36}$$

The right-hand side of (2.36) is bounded since $|\nabla u| \in L^q(B_{2R_0})$, the second term on the left is also uniformly bounded. Using the fact that $|x|^{N-2}u(x)$ is bounded by (2.1), we get

$$\left|\int_{\{\frac{1}{n}\leq|x|\leq\frac{2}{n}\}} u\Delta\sigma_n dx\right| \leq C,$$

for some $C > 0$ independent of n. Letting $n \to \infty$ we infer that $u \in L^p_{loc}(\Omega)$. By the maximum principle

$$u(x) \leq u_1(x) = C\mathbf{G}^{B_{2R_0}}[Q](x) + \max_{|z|=2R_0} u(z), \tag{2.37}$$

where $\mathbf{G}^{B_{2R_0}}$ denotes the Green kernel in B_{2R_0}. Since $Q(x) = C|x|^{-q(N-1)}$, a direct computation shows that $u_1(x) \leq c_N C|x|^{2-q(N-1)} = c_N C|x|^{2-N+\epsilon}$ for some $\epsilon > 0$. We can write (1.1) under the form

$$-\Delta u + c(x)u + d(x)|\nabla u| = 0 \quad \text{in } \Omega \setminus \{0\},$$

with $c(x) = u^{\frac{2}{N-2}}$ and $d(x) = |\nabla u|^{q-1}$. Then $c \in L^{\frac{N}{2}+\epsilon_1}(B_{2R_0})$ and $d \in L^{N+\epsilon_2}(B_{2R_0})$; with $\epsilon_1, \epsilon_2 > 0$. It follows from [25, Theorem 10] that 0 is a removable singularity for u in the sense that it can be extended as a C^2 solution in Ω. □

When the conditions of the theorem are not fulfilled there exist singular solutions. However these singular solutions may exhibit different types of behaviour according $1 < q < \frac{2p}{p+1}$, $\frac{2p}{p+1} < q < 2$ and $q = \frac{2p}{p+1}$. In this case there may exist radial separable solutions of (1.9) under the form $u_X(r) = Xr^{-\frac{2}{p-1}}$. Setting $\alpha = \frac{2}{p-1}$, then X satisfies

$$\Phi_p(X) := X^{p-1} - M\alpha^{\frac{2p}{p+1}}X^{\frac{p-1}{p+1}} + \alpha(N-2-\alpha) = 0 \tag{2.38}$$

The following result is easy to prove by a standard analysis of the function Φ_p.

Proposition 2.9 *Let $p > 1$ and $M \in \mathbb{R}$.*

(i) If M is arbitrary and $1 < p < \frac{N}{N-2}$, or $M > 0$ and $p = \frac{N}{N-2}$, there exists one and only one positive solution X_1 to (2.38).

(ii) If $p > \frac{N}{N-2}$ and $M > m^*$, there exist two positive solutions $X_1 < X_2$ to (2.38).

(iii) If $p > \frac{N}{N-2}$ and $M = m^*$ there exists one positive solution X_1 to (2.38).

(iv) If $p > \frac{N}{N-2}$ and $0 < M < m^*$, or $M \leq 0$ and $p \geq \frac{N}{N-2}$, there exists no positive solution to (2.38).

When $q \neq \frac{2p}{p+1}$ the existence of singular solutions is much involved. It is developed in the subsequent paper [8].

Remark. It is noticeable that in the case $q = \frac{2p}{p+1}$, $p > \frac{N}{N-2}$ and $M \geq m^*$, the equation exhibits a phenomenon which is characteristic of Lane-Emden type equations

$$-\Delta u = u^p \quad \text{in} \;\; B_1 \setminus \{0\}. \tag{2.39}$$

If u is nonnegative then there exists $\alpha \geq 0$ such that

$$-\Delta u = u^p + \alpha\delta_0 \quad \text{in} \;\; \mathcal{D}'(B_1). \tag{2.40}$$

If $1 < p < \frac{N}{N-2}$ then α can be positive, but if $p \geq \frac{N}{N-2}$, then $\alpha = 0$. This means that the singularity cannot be seen in the sense of distributions, however there truly exist singular solutions, e.g. if $p > \frac{N}{N-2}$,

$$u_s(x) = c_{N,p}|x|^{-\frac{2}{p-1}}. \tag{2.41}$$

Here also for $q = \frac{2p}{p+1}$, $p > \frac{N}{N-2}$, the isolated singularities are not seen in the sense of distributions.

2.3 Proof of Theorem 1.2

As in the proof of Theorem 1.1, we distinguish according $1 < q < \frac{2p}{p+1}$ or $q = \frac{2p}{p+1}$. Without loss of generality we can suppose that $u > 0$. We perform the same change of unknown as in the previous theorem putting $u = v^b$, but now we choose b as follows

$$(p-1)b + 1 = r \Longleftrightarrow b = \frac{r-1}{p-1}, \tag{2.42}$$

and we first assume that

$$1 - b - M\frac{qb^{q-1}\epsilon^{\frac{2}{q}}}{2} = 0 \Longleftrightarrow \epsilon = \left(\frac{2(1-b)}{Mqb^{q-1}}\right)^{\frac{q}{2}} = \left(\frac{2(p-r)}{Mq(p-1)b^{q-1}}\right)^{\frac{q}{2}}. \tag{2.43}$$

Hence (2.28) becomes

$$-\Delta v + \frac{p-1}{r-1}v^r - \frac{(2-q)b^{q-1}}{2}\left(\frac{q}{2(1-b)}\right)^{\frac{q}{2-q}} M^{\frac{2}{2-q}} v^{\frac{(2r-p-1)q+2(p-r)}{(p-1)(2-q)}} \leq 0. \tag{2.44}$$

Condition $r \geq \frac{(2r-p-1)q+2(p-r)}{(p-1)(2-q)}$ is equivalent to $2p - q(p+1) \leq r(2p - q(p+1))$, since $1 < r < p$.
Assuming first that $q < \frac{2p}{p+1}$, we obtain from (2.44)

$$-\Delta v + \frac{p-1}{2(r-1)}v^r \leq A. \tag{2.45}$$

for some constant $A \geq 0$. Since $cap^{\mathbb{R}^N}_{2,r'}(K) = 0$ the function v is bounded from [3] and $v \leq cA^{\frac{1}{r}} + \max\limits_{\partial\Omega} v$ for some $c > 0$, hence u is also uniformly upper bounded in Ω by some constant a.

Next we have to show that $\nabla u \in L^q(\Omega)$. Let $\{\rho_n\}$ be a sequence of $C_0^\infty(\Omega)$ non-negative functions such that $0 \le \rho_n \le 1$, $\rho_n = 1$ in a small enough neighborhood of K and $\|\rho_n\|_{W^{2,r'}} \to 0$ when $n \to \infty$, and set $\eta_n = 1 - \rho_n$. Since

$$\int_\Omega u\Delta\rho_n dx - \int_{\partial\Omega} \frac{\partial u}{\partial \mathbf{n}} dS + \int_\Omega u^p \eta_n dx = M \int_\Omega |\nabla u|^q \eta_n dx,$$

and

$$\left| \int_\Omega u\Delta\rho_n dx \right| \le c \|u\|_{L^\infty} \|\rho_n\|_{W^{2,r'}} \to 0 \quad \text{as } n \to \infty,$$

we get

$$\int_\Omega u^p dx - \int_{\partial\Omega} \frac{\partial u}{\partial \mathbf{n}} dS = M \int_\Omega |\nabla u|^q dx.$$

Hence $\nabla u \in L^q(\Omega)$. If $\zeta \in C_0^\infty(\Omega)$ and $\zeta_n = \zeta\eta_n$, there holds

$$-\int_\Omega \eta_n u\Delta\zeta dx + \int_\Omega \zeta u\Delta\rho_n dx + \int_\Omega u^p \zeta_n dx = M \int_\Omega |\nabla u|^q \zeta_n dx.$$

Since the second term on the left-hand side tends to 0 and $\zeta_n \to \zeta$ when $n \to \infty$, we obtain that

$$-\int_\Omega u\Delta\zeta dx + \int_\Omega u^p \zeta dx = M \int_\Omega |\nabla u|^q \zeta dx.$$

Hence u is a solution in the sense of distribution in Ω.
Next we show that $\nabla u \in L^2(\Omega)$. Multiplying (1.9) by $u\eta_n$ and integrating, we obtain

$$\int_\Omega |\nabla u|^2 \eta_n dx - \int_\Omega u\langle \nabla u, \nabla\rho_n\rangle dx - \int_{\partial\Omega} u \frac{\partial u}{\partial \mathbf{n}} dS + \int_\Omega u^{p+1}\eta_n dx = M \int_\Omega u|\nabla u|^q \eta_n dx.$$

As

$$\begin{aligned} \int_\Omega u\langle \nabla u, \nabla\rho_n\rangle dx &= \frac{1}{2}\int_\Omega \langle \nabla u^2, \nabla\rho_n\rangle dx \\ &= -\frac{1}{2}\int_\Omega u^2 \Delta\rho_n dx, \end{aligned}$$

and

$$\left| \int_\Omega u^2 \Delta\rho_n dx \right| \le c \|u\|_{L^\infty}^2 \|\rho_n\|_{W^{2,r'}} = o(1) \quad \text{as } n \to \infty,$$

we infer that

$$\int_\Omega |\nabla u|^2 dx - \int_{\partial\Omega} u \frac{\partial u}{\partial \mathbf{n}} dS + \int_\Omega u^{p+1} dx = M \int_\Omega u|\nabla u|^q dx.$$

Finally if $\zeta \in C_0^\infty(\Omega)$ and $\zeta_n = \zeta\eta_n$, then

$$\int_\Omega \eta_n \langle \nabla u, \nabla\zeta\rangle dx - \int_\Omega \zeta\langle \nabla u, \nabla\rho_n\rangle dx + \int_\Omega u^p \zeta_n dx = M \int_\Omega |\nabla u|^q \zeta_n dx.$$

Since $r \le \frac{2N}{N-2}$ there holds

$$\|\rho_n\|_{W^{1,2}} \le c \|\rho_n\|_{W^{2,r'}} \Longrightarrow \|\rho_n\|_{W^{1,2}} \to 0 \quad \text{as } n \to \infty.$$

Using the fact that $\nabla u \in L^2(\Omega)$ and Hölder's inequality, we derive

$$\int_\Omega \zeta \langle \nabla u, \nabla \rho_n \rangle dx \to 0 \quad \text{as } n \to \infty.$$

Hence

$$\int_\Omega \langle \nabla u, \nabla \zeta \rangle dx + \int_\Omega u^p \zeta dx = M \int_\Omega |\nabla u|^q \zeta dx.$$

This implies that u is a weak solution of (1.9) and it is therefore C^2 in Ω.
Next we assume $q = \frac{2p}{p+1}$. We choose $b = \frac{r-1}{p-1}$ and (2.31) becomes

$$-\Delta v + \left(1 - b - \frac{Mpb^{\frac{p-1}{p+1}} \epsilon^{\frac{p+1}{p}}}{p+1}\right) \frac{|\nabla v|^2}{v} + \left(\frac{1}{b} - \frac{Mb^{\frac{p-1}{p+1}}}{(p+1)\epsilon^{p+1}}\right) v^r \leq 0. \quad (2.46)$$

If (2.32) holds with this choice of b, (2.33) becomes

$$\begin{aligned} 1 - b - \frac{Mpb^{\frac{p-1}{p+1}} \epsilon^{\frac{p+1}{p}}}{p+1} &= 1 - b - pb \left(\frac{M}{p+1}\right)^{\frac{p+1}{p}} \\ &= \frac{1}{p-1} \left(p - r - p(r-1) \left(\frac{M}{p+1}\right)^{\frac{p+1}{p}}\right). \end{aligned} \quad (2.47)$$

If $M < m_r^*$ defined by (1.10), we can choose ϵ such that

$$1 - b - \frac{Mpb^{\frac{p-1}{p+1}} \epsilon^{\frac{p+1}{p}}}{p+1} = 0,$$

and

$$\frac{1}{b} - \frac{Mb^{\frac{p-1}{p+1}}}{(p+1)\epsilon^{p+1}} = \tau := \tau(\epsilon) > 0.$$

Then v satisfies

$$-\Delta v + \tau v^r \leq 0 \qquad \text{in } \Omega \setminus K.$$

Since $cap_{2,r'}^{\mathbb{R}^N}(K) = 0$ it follows from [3] that $v \leq \max_{x \in \partial\Omega} v(x)$. Hence u is bounded.
The different steps of the proof in the first case applies without any modification: first $\nabla u \in L^q(\Omega)$ and the equation holds in the sense of distributions in Ω, then $\nabla u \in L^2(\Omega)$ and since $r \leq \frac{2N}{N-2}$ we infer that u is a weak solution and thus a strong one. □

3 Measure data

Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain with diameter smaller than $2R$. Also any Radon measure in Ω is extended by 0 in Ω^c with the same notation.

3.1 Proof of Theorem 1.3: the case $1 < q < \frac{N}{N-1}$

If $1 < q < \frac{N}{N-1}$ assumption (1.12) with $\mu \geq 0$ reduces to

$$\mu(K) \leq C cap_{2,p'}^{\mathbb{R}^N}(K) \qquad \text{for all compact set } K \subset \Omega. \quad (3.1)$$

The construction of solutions is based upon the following result due to Boccardo-Murat-Puel [10]. It is concerned with a general quasilinear equation in a domain $G \subset \mathbb{R}^N$

$$\mathcal{Q}(u) := -\Delta u + B(., u, \nabla u) = 0 \qquad \text{in } \mathcal{D}'(G), \tag{3.2}$$

where $B \in C(G \times \mathbb{R} \times \mathbb{R}^N)$ satisfies

$$|B(x, r, \xi)| \leq \Gamma(|r|)(1 + |\xi|^2) \quad \text{for all } (x, r, \xi) \in G \times \mathbb{R} \times \mathbb{R}^N, \tag{3.3}$$

for some continuous increasing function Γ from $\mathbb{R}^+$ to $\mathbb{R}^+$.

Theorem 3.1 *Let G be a bounded domain in $\mathbb{R}^N$. If there exists a supersolution ϕ and a subsolution ψ of the equation $\mathcal{Q}v = 0$ belonging to $W^{1,\infty}(G)$ and such that $\psi \leq \phi$, then for any $\chi \in W^{1,\infty}(G)$ satisfying $\psi \leq \chi \leq \phi$ there exists a function $u \in W^{1,2}(G)$ solution of $\mathcal{Q}u = 0$ such that $\psi \leq u \leq \phi$ and $u - \chi \in W_0^{1,2}(G)$.*

The sub and super solutions are linked to the two problems in which p and q are bigger than 1, and μ and ω are Radon measures

$$\begin{aligned} -\Delta v + |v|^{p-1}v &= \mu \qquad && \text{in } \Omega \\ v &= 0 && \text{in } \partial\Omega, \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} -\Delta w - M|\nabla w|^q &= \omega \qquad && \text{in } \Omega \\ w &= 0 && \text{in } \partial\Omega. \end{aligned} \tag{3.5}$$

It is proved in [3, Theorem 4.1] that Problem (3.4) admits a solution, $v \in L^p(\Omega)$, necessarily unique, if and only if μ is absolutely continuous with respect to the Bessel capacity $cap_{2,p'}^{\mathbb{R}^N}$, that is

$$\textit{For any compact set } E \subset \Omega\,,\ cap_{2,p'}^{\mathbb{R}^N}(E) = 0 \Longrightarrow |\mu|(E) = 0. \tag{3.6}$$

Concerning (3.5), from [21, Theorem 1.9] a sufficient condition for solvability is the estimate

$$\textit{For any compact set } E \subset \Omega\,,\ |\omega|(E) \leq C cap_{1,q'}^{\mathbb{R}^N}(E), \tag{3.7}$$

for some $C > 0$. When μ is nonnegative and has compact support in Ω, this condition turns out to be necessary. If (3.7) is satisfied there exists $\epsilon_0 > 0$ such that (3.5) admits a solution with ω replaced by $\epsilon\omega$ with $0 < \epsilon \leq \epsilon_0$. Furthermore $\nabla w \in L^q(\Omega)$ and the following estimates hold [9, Theorem 1.2],

$$|\nabla w(x)| \leq c_{13}\epsilon \mathbf{I}_1^{2R}[|\omega|](x), \tag{3.8}$$

at least if ω has compact support or is a smooth function, and, with no such conditions on μ,

$$|w(x)| \leq c_{14}\epsilon \mathbf{G}^{\Omega}[|\omega|](x), \tag{3.9}$$

with c_{13}, c_{14} depending on N and q, where $\mathbf{I}_1^{2R}$ is the truncated Riesz potential in $\mathbb{R}^N$ defined for any measure μ by

$$\mathbf{I}_1^{2R}[\mu](x) = \int_0^{2R} \frac{\mu(B_\rho(x))}{\rho^{N-1}} \frac{d\rho}{\rho} \quad \text{for all } x \in \mathbb{R}^N, \tag{3.10}$$

and $\mathbf{G}^{\Omega}$ the Green potential in Ω. If $R = \infty$ we denote by $\mathbf{I}_1 := \mathbf{I}_1^{\infty}$ the classical Riesz potential and if $\Omega = \mathbb{R}^N$ the role of $\mathbf{G}^{\Omega}$ is played by the Newtownian potential $\mathbf{I}_2$. We start with the following easy result:

Lemma 3.2 *Let $r > 1$, $k \in \mathbb{N}^*$ and $\mu \in \mathfrak{M}_+(\Omega)$. If $\mu \in W^{-k,r}(\Omega)$ is nonnegative, then there exists $C > 0$ such that*

$$\mu(E) \leq C \left(cap^{\Omega}_{k,r'}(E) \right)^{\frac{1}{r'}} \quad \text{for any compact set } E \subset \Omega. \tag{3.11}$$

where $r' = \frac{r}{r-1}$. Conversely when $k = 1, 2$ and μ satisfies

$$\mu(E) \leq C cap^{\Omega}_{k,r'}(E) \quad \text{for all compact set } E \subset \Omega, \tag{3.12}$$

for some $C > 0$, then,
(i) if $k = 2$ then $\mu \in W^{-2,r}(\Omega)$,
(ii) if $k = 1$ then $\mu \in W^{-1,r}(\Omega)$.

Proof. Assume first that $\mu \in \mathfrak{M}_+(\Omega) \cap W^{-k,r}(\Omega)$. Let $\zeta \in C_0^\infty(\Omega)$ such that $0 \leq \zeta \leq 1$ and $\zeta \geq \chi_K$. Then

$$\mu(E) \leq \int_\Omega \zeta d\mu = \langle \mu, \zeta \rangle \leq \|\zeta\|_{W_0^{k,r'}} \|\mu\|_{W^{-k,r}}.$$

By the definition of capacity

$$\mu(K) \leq \|\mu\|_{W^{-k,r}} \left(cap^{\Omega}_{k,rp'}(K) \right)^{\frac{1}{r'}}.$$

Conversely if (3.12) holds with $k = 2$ there exists $\epsilon_0 > 0$ such that for every $\epsilon \in (0, \epsilon_0]$, there exists $z \in L^r(\Omega)$ satisfying

$$\begin{aligned} -\Delta z &= z^r + \epsilon\mu && \text{in } \Omega \\ z &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{3.13}$$

(see [24, Theorem 2.10, Remark 2.11]). Since $z \geq \mathbf{G}^\Omega[\epsilon\mu]$, it follows that $\mathbf{G}^\Omega[\mu] \in L^r(\Omega)$ and therefore $\mu \in W^{-2,r}(\Omega)$. Since $\mathbf{G}^\Omega$ is an isomorphism from $L^{r'}(\Omega)$ into $W^{2,r'}(\Omega) \cap W_0^{1,r'}(\Omega)$, we infer by duality that $\mathbf{G}^\Omega$ is an isomorphism from $W^{-2,r}(\Omega)$ into $L^r(\Omega)$. Hence $\mu \in W^{-2,r}(\Omega)$.

Finally, if (3.12) holds with $k = 1$, then there exists $\epsilon_0 > 0$ such that for every $\epsilon \in (0, \epsilon_0]$ there exists $z \in W^{1,r}(\Omega)$ satisfying

$$\begin{aligned} -\Delta z &= |\nabla z|^r + \epsilon\mu && \text{in } \Omega \\ z &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{3.14}$$

Then z satisfies $z \geq \mathbf{G}^\Omega[\epsilon\mu]$. Since $z \in L^{r^*}(\Omega)$ by Sobolev imbedding theorem, we have that $\mathbf{G}^\Omega[\mu] \in L^{r^*}(\Omega)$, which implies the claim. □

Proof of the theorem. We put $\mu_n = \mu * \eta_n$ where $\{\eta_n\} \subset C_0^\infty(\mathbb{R}^N)$ is a sequence of mollifiers with $\text{supp}(\eta_n) \subset B_{\frac{1}{n}}$, and we denote by v_n the solution of

$$\begin{aligned} -\Delta v + v^p &= \epsilon\mu_n\chi_\Omega && \text{in } \Omega \\ v &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{3.15}$$

Since μ satisfies (3.12) so does μ_n with the same constant C. Hence $\mu_n \in W^{-2,p}(\Omega)$ and $\mu_n \to \mu$ in $W^{-2,p}(\Omega)$ as $n \to \infty$. We also denote by z_n a nonnegative solution of

$$\begin{aligned} -\Delta z &= z^p + \epsilon\mu_n\chi_\Omega && \text{in } \Omega \\ z &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{3.16}$$

and by w_n a nonnegative solution of

$$\begin{aligned} -\Delta w &= M|\nabla w|^q + \epsilon\mu_n\chi_\Omega && \text{in } \Omega \\ w &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{3.17}$$

Since w_n is C^2, it is unique by the strong maximum principle. Then there holds by (3.8), (3.9),

$$\begin{aligned} &(i) && v_n \le \epsilon\mathbf{G}^\Omega[\mu_n] \le w_n \le c_{14}\epsilon\mathbf{G}^\Omega[\mu_n] \le c_{14}\epsilon\mathbf{I}_2[\mu_n]\lfloor_\Omega, \\ &(ii) && |\nabla w_n| \le c_{13}\epsilon\mathbf{I}_1^{2R}[\mu_n]. \end{aligned} \tag{3.18}$$

Since v_n and w_n are respectively a subsolution and a supersolution of

$$\begin{aligned} -\Delta u + u^p &= M|\nabla u|^q + \epsilon\mu_n && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{3.19}$$

it follows by Theorem 3.1 that there exists $u = u_n \in W_0^{1,\infty}(\Omega)$ satisfying (3.19) in the sense that for any $\zeta \in C_c^2(\overline{\Omega})$ there holds

$$-\int_\Omega u_n\Delta\zeta dx + \int_\Omega \left(u_n^p - M|\nabla u_n|^q\right)\zeta dx = \epsilon\int_\Omega \zeta d\mu_n. \tag{3.20}$$

It is unique by the strong maximum principle and it satisfies

$$v_n \le u_n \le w_n \le c_{14}\epsilon\mathbf{I}_2[\mu_n]\lfloor_\Omega. \tag{3.21}$$

Since $\mathbf{I}_2[\mu_n]\lfloor_\Omega$ is uniformly bounded in $L^p(\Omega)$, the sequence of functions $\{u_n\}$ shares this property. If $\eta = \mathbf{G}^\Omega[1]$, there holds

$$\int_\Omega \left(u_n + \eta u_n^p\right) dx = M\int_\Omega |\nabla u_n|^q\eta dx + \epsilon\int_\Omega \eta d\mu_n. \tag{3.22}$$

Hence $|\nabla u_n|$ is uniformly bounded in $L^q_{d_\Omega}(\Omega)$ where $d_\Omega(x) = \text{dist}\,(x, \partial\Omega)$. By (3.8),

$$\begin{aligned} |\nabla u_n| &\le c_{13}\epsilon\mathbf{I}_1^{2R}[|\mu_n - u_n^p|] \le c_{13}\left(\epsilon\mathbf{I}_1^{2R}[\mu_n] + \mathbf{I}_1^{2R}[u_n^p]\right) \\ &\le c_{13}\left(\epsilon\mathbf{I}_1^{2R}[\mu_n] + c_{14}^p\epsilon^p\mathbf{I}_1^{2R}[(\mathbf{I}_2[\mu_n\lfloor_\Omega])^p]\right). \end{aligned} \tag{3.23}$$

Using [16, Lemma 4.2] we have equivalence between

$$\mathbf{I}_1[(\mathbf{I}_2[\mu_n\lfloor_\Omega])^p] \le c_{15}\mathbf{I}_1[\mu_n\lfloor_\Omega], \tag{3.24}$$

and

$$\mathbf{I}_2[(\mathbf{I}_2[\mu_n\lfloor_\Omega])^p] \le c_{17}\mathbf{I}_2[\mu_n\lfloor_\Omega], \tag{3.25}$$

and $c_{17} \le c_{15} \le C(N,p)c_{17}$. Moreover, since $\text{diam}(\Omega) < 2R$,

$$\left(\mathbf{I}_1^{2R}[(\mathbf{I}_2[\mu])^p]\right)^q \le c_{21}^q(\mathbf{I}_1[\mu])^q \le c^q c_{21}^q(\mathbf{I}_1^{2R}[\mu])^q. \tag{3.26}$$

for some $c = c(N,R) > 0$. Next, it is quoted in [16, Theorem 1.1] that the inequality (3.25) is equivalent to the main assumption of Theorem 1.3,

$$\mu_n\lfloor_\Omega(E) \le C cap_{2,p'}^{\mathbb{R}^N}(E) \quad \text{for all compact set } E \subset \Omega, \tag{3.27}$$

for some $C > 0$. Actually this equivalence is proved in [19]. By [1, Theorem 3.14-(a)] there exists $A = A(N) > 0$ such that

$$\left|\{x \in \mathbb{R}^N : |\mathbf{I}_1[\mu_n\lfloor_\Omega](x)| > \lambda\}\right| \leq A\lambda^{-\frac{N}{N-1}} \|\mu_n\lfloor_\Omega\|_{L^1}^{\frac{N}{N-1}}. \tag{3.28}$$

Clearly the above inequality holds if $\mathbf{I}_1$ is replaced by $\mathbf{I}_1^{2R}$ and $\mathbb{R}^N$ by Ω. This is an estimate of $\mathbf{I}_1^{2R}[\mu_n\lfloor_\Omega]$ in the Lorentz space $L^{\frac{N}{N-1},\infty}(\Omega)$ (or Marcinkiewicz space). Clearly

$$\|\mu_n\lfloor_\Omega\|_{L^1} \leq \|\mu\|_{\mathfrak{M}}.$$

Therefore (3.23) implies that $|\nabla u_n|$ is bounded in $L^{\frac{N}{N-1},\infty}(\Omega)$, hence equi-integrable in $L^q(\Omega)$ since $q < \frac{N}{N-1}$. By Lemma 3.2-(ii) and classical harmonic analysis results, $\mathbf{I}_2[\mu_n\lfloor_\Omega \to \mathbf{I}_2[\mu\lfloor_\Omega$ in $L^p(\Omega)$ (see e.g. [26]). It follows from (3.21) that u_n is equi-integrable in $L^p(\Omega)$. By standard results on elliptic equations and measure theory [11, Corollary IV], the sequences $\{u_n\}$ and $\{|\nabla u_n|\}$ are relatively compact in $L^1(\Omega)$. Hence there exist a subsequence $\{n_j\}$, converging to ∞ and a function $u \in W_0^{1,q}(\Omega) \cap L^p(\Omega)$ such that $u_{n_j} \to u$ in $W^{1,1}(\Omega)$ and a.e. in Ω. Since $\{u_n\}$ and $\{|\nabla u_n|\}$ are also equi-integrable in $L^p(\Omega)$ and $L^q(\Omega)$ respectively, we infer that $u_{n_j} \to u$ in $W_0^{1,q}(\Omega) \cap L^p(\Omega)$. It follows from Vitali's convergence theorem that $u_{n_j} \to u$ in $L^p(\Omega)$ and $\nabla u_{n_j} \to \nabla u$ in $L^q(\Omega)$. Letting $n_j \to \infty$ in (3.20) we conclude that the identity

$$-\int_\Omega u\Delta\zeta dx + \int_\Omega (u^p - M|\nabla u|^q)\,\zeta dx = \epsilon\int_\Omega \zeta d\mu_n, \tag{3.29}$$

holds for any $\zeta \in C_c^2(\overline{\Omega})$. Clearly

$$v \leq u \leq w \leq C\epsilon\mathbf{I}_2[\mu]\lfloor_\Omega, \tag{3.30}$$

where v and w are respectively the solution of (3.4) and the minimal solution of (3.5) with μ replaced by $\epsilon\mu$. □

3.2 Proof of Theorem 1.3: the general case

The approach with super and sub solutions does not work directly and we follow the method developed for proving [21, Theorem 1.9] which is a delicate extension of the one in the subcritical case. The fact that a sequence of approximation $\{u_n\}$ is bounded in $W_0^{1,q}(\Omega) \cap L^p(\Omega)$ does not imply the uniform integrability of $\{\nabla u_n\}$ in $L^q(\Omega)$ for $q \geq \frac{N}{N-1}$.

Definition 3.3 *If $r > 1$ and $k \in \mathbb{N}^*$ we denote by $\mathbf{M}^{k,r}(\Omega)$ the set of bounded measures μ in Ω which satisfy, for some $C > 0$,*

$$|\mu|(E)| \leq C cap_{k,r}^\Omega(E) \quad \textit{for all compact sets } E \subset \Omega. \tag{3.31}$$

If Ω is replaced by $\mathbb{R}^N$, the set is denoted by $\mathbf{M}^{k,r}$. The smallest constant C such that (3.31) holds is denoted by $[\mu]_{M^{k,r}}$.

For $T > 1$ we denote by $E_1(T,\mu)$ the subset of functions $\zeta \in W_0^{1,q}(\Omega)$ such that

$$\int_\Omega |\nabla\zeta|^q w dx \leq T\epsilon^q \int_\Omega (\mathbf{I}_1^{2R}[|\mu|])^q w dx \quad \text{for all } w \in \mathbf{A}_1 \cap L^\infty. \tag{3.32}$$

The following estimate is obtained in [21, Lemma 5.2] in a more general context.

Lemma 3.4 *Let $q > 1$ and $\mu \in \mathbf{M}^{1,q'}(\Omega)$. Then there exists $c_{17} = c_{17}(N, q) > 0$ such that for any $\zeta \in E_1(T, \mu)$*

$$\mathbf{I}_1^{2R}[|\nabla\zeta|^q\chi_\Omega](x) \le c_{17}T\epsilon^q[\mu]_{M^{1,q'}}^{q-1}\mathbf{I}_1^{2R}[|\mu|](x) \tag{3.33}$$

a.e. in Ω.

Lemma 3.5 *Let $q > 1$, $\zeta \in E_1(T, \mu)$ where $\mu \in \mathbf{M}^{1,q'}(\Omega)$, and $S_1(\zeta)$ be the solution of*

$$\begin{aligned} -\Delta\phi + \phi^p &= M|\nabla\zeta|^q + \epsilon\mu && \text{in } \Omega \\ \phi &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{3.34}$$

Then there exists $c_{18} = c_{18}(N, q) > 0$ such that

$$v \le S_1(\zeta) \le S(\zeta) \le (\epsilon + c_{18}\epsilon^q T)I_2[\mu], \tag{3.35}$$

where $S(\zeta)$ is the solution of

$$\begin{aligned} -\Delta\phi &= M|\nabla v|^q + \epsilon\mu && \text{in } \Omega \\ \phi &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{3.36}$$

and v is the solution of (3.4) with μ replaced by $\epsilon\mu$.

Proof. There holds

$$S_1(\zeta) = \epsilon\mathbf{G}^\Omega[\mu] + \mathbf{G}^\Omega[|\nabla\zeta|^q] \le \epsilon\mathbf{I}_2[\mu] + \mathbf{I}_2[|\nabla\zeta|^q] = \epsilon\mathbf{I}_2[\mu] + \mathbf{I}_1[\mathbf{I}_1[|\nabla\zeta|^q]].$$

By Lemma 3.4 with $R = \infty$, $\mathbf{I}_1[|\nabla\zeta|^q\chi_\Omega] \le c_{17}T\epsilon^q[\mu]_{M^{1,q'}}^{q-1}\mathbf{I}_1[\mu]$, hence

$$S_1(\zeta) \le \left(\epsilon + c_{17}T\epsilon^q[\mu]_{M^{1,q'}}^{q-1}\right)\mathbf{I}_2[\mu] := (\epsilon + c_{18}\epsilon^q T)\mathbf{I}_2[\mu], \tag{3.37}$$

since $T \ge 1$ and $\epsilon \in (0, 1]$. □

Lemma 3.6 *There exists $T_1 > 0$ such that for any $T > T_1$, there exists $\epsilon_T > 0$ such that for all $\epsilon \in (0, \epsilon_T]$, S_1 maps $E(T, \mu)$ into itself.*

Proof. By [21, Theorem 1.4], and using the proof of [21, Theorem 1.9], there exist constants $c_{19}, c_{20}, c_{21} > 0$ depending on N, q, R and m such that for any $w \in \mathbf{A}_1 \cap L^\infty$ such that $[w]_{A_1} \le m$

$$\begin{aligned} \int_\Omega |\nabla S_1(\zeta)|^q w dx &\le c_{19}\int_\Omega \left(\mathbf{I}_1^{2R}[|\nabla\zeta|^q\chi_\Omega] + \epsilon\mathbf{I}_1^{2R}[\mu] + \mathbf{I}_1^{2R}[S_1^p(\zeta)]\right)^q w dx \\ &\le c_{20}\int_\Omega \Big(\left(\mathbf{I}_1^{2R}[|\nabla\zeta|^q\chi_\Omega]\right)^q + \epsilon^q(\mathbf{I}_1^{2R}[\mu])^q \\ &\qquad + (\epsilon^{pq} + c_{21}\epsilon^{pq^2}T^{pq})\left(\mathbf{I}_1^{2R}[(\mathbf{I}_2[\mu])^p]\right)^q\Big) w dx. \end{aligned} \tag{3.38}$$

Using again the equivalence between (3.24), (3.25), (3.26) and (3.27), inequality (3.38) is transformed in

$$\int_\Omega |\nabla S_1(\zeta)|^q w dx \le c_{22}\left(\epsilon^q + c_{24}qT^q\epsilon^{q^2} + c_{25}^q(\epsilon^{pq} + c_{20}\epsilon^{pq^2}T^{pq})\right)\int_\Omega (\mathbf{I}_1^{2R}[\mu])^q w dx. \tag{3.39}$$

Finally for any $T > c_{23}$ there exists ϵ_0 such that for $0 < \epsilon \leq \epsilon_0$, there exist positive constants c_{24}, c_{25} such that

$$c_{22}\left(\epsilon^q + c_{24}qT^q\epsilon^{q^2} + c_{25}^q(\epsilon^{pq} + c_{20}\epsilon^{pq^2}T^{pq})\right) \leq T\epsilon^q,$$

which implies the claim. □

Lemma 3.7 *Under the assumptions of Lemma 3.6 with $T > c_{23}$ and $0 < \epsilon \leq \epsilon_T$, the mapping S_1 is compact from $E(T,\mu)$ into itself.*

Proof. We prove first the continuity. Let $\{\zeta_j\} \subset E(T,\mu)$ such that $\zeta_j \to \zeta$ in $W_0^{1,q}(\Omega)$. Using monotonicity as in [12, Theorem 8] we obtain that for any $r \in [1, \frac{N}{N-1})$, there exists $\alpha = \alpha(N,R,r) > 0$ such that

$$\alpha \left\|S_1(\zeta_j) - S_1(\zeta_\ell)\right\|_{W_0^{1,r}} + \left\|S_1^p(\zeta_j) - S_1^p(\zeta_\ell)\right\|_{L^1} \leq M \left\||\nabla\zeta_j|^q - |\nabla\zeta_\ell|^q\right\|_{L^1}. \tag{3.40}$$

Since $\{\nabla\zeta_j\}$ is a Cauchy sequence in $L^q(\Omega)$ it follows that $S_1(\zeta_j) \to S_1(\zeta)$ in $W_0^{1,r}(\Omega) \cap L^p(\Omega)$. Hence

$$\begin{aligned} -\Delta S_1(\zeta) + S_1^p(\zeta) &= M|\nabla\zeta|^q + \mu && \text{in } \Omega \\ S_1(\zeta) &= 0 && \text{in } \partial\Omega. \end{aligned} \tag{3.41}$$

If we take $\nu := \nu_j = M|\nabla\zeta_j|^q + \mu + S_1^p(\zeta_j)$. By (3.26)

$$\mathbf{I}_1^{2R}[S_1^p(\zeta_j)] \leq \mathbf{I}_1^{2R}[(\mathbf{I}_2[\mu])^p] \leq C_6\mathbf{I}_1^{2R}[\mu].$$

Combined with (3.33) we infer that

$$\mathbf{I}_1^{2R}[M|\nabla\zeta_j|^q + S_1^p(\zeta_j) + \mu] \leq c_{28}\mathbf{I}_1^{2R}[\mu]. \tag{3.42}$$

Let $\mathbf{M}_1$ be the first order fractional maximal function defined by

$$\mathbf{M}_1(\mu)(x) := \sup_{\rho>0} \frac{\omega(B_\rho(x))}{\rho^{N-1}} \qquad \text{for all } \ x \in \mathbb{R}^N. \tag{3.43}$$

It is classical that for all $\nu \in \mathfrak{M}_b(\Omega)$,

$$\mathbf{M}_1[|\nu|](x) \leq m_{_N}\mathbf{I}_1^{2R}[|\nu|])(x) \qquad \text{for a.e. } \ x \in \mathbb{R}^N. \tag{3.44}$$

Hence, if we set $c_{29} = m_{_N}(c_{28}+1)$, there holds.

$$\mathbf{M}_1[M|\nabla\zeta_j|^q + \mu + S_1^p(\zeta_j)](x) \leq c_{29}\mathbf{I}_1^{2R}[\mu]. \tag{3.45}$$

Since $\mathbf{I}_1^{2R}[\mu] \in L^q(\Omega)$ we deduce that the left-hand side of (3.45) is bounded and equi-integrable in $L^q(\Omega)$. Then we apply [21, Corollary 1.7], with $w = 1$ and deduce that there exists a subsequence $\{\nabla S_1(\zeta_{j_n})\}$ which converges in $L^q(\Omega)$. By uniqueness the limit is $\{\nabla S_1(\zeta)\}$ and the whole sequence $\{\nabla S_1(\zeta_j)\}$ converges. Therefore S_1 is continuous.

The proof of the compactness follows the same ideas: If $\{\zeta_j\}$ is a bounded sequence in $E(T,\mu)$, then $\{\nabla S_1(\zeta_j)\}$ is bounded in $L_w^q(\Omega)$ by (3.38), hence (3.40) holds. By (3.25), $\{S_1(\zeta_j)\}$ is bounded in $L^p(\Omega)$ and equi-integrable since $\mathbf{I}_2[\mu]$ belongs to $L^p(\Omega)$ (it is a consequence of (3.27)) and Lemma 3.2. By [11] the sequence $\{S_1(\zeta_j)\}$ is relatively compact in $W_0^{1,1}(\Omega)$. Hence, there exist

$\zeta \in E(T,\mu)$ and a subsequence $\{\zeta_{j_n}\}$ such that $\zeta_{j_n} \to \zeta$ weakly in $W_0^{1,q}(\Omega)$ and $\{S_1(\zeta_{j_n})\}$ converges to some $S \in E(T,\mu)$ in $W_0^{1,1}(\Omega) \cap L^p(\Omega)$, a.e. in Ω and weakly in $W_0^{1,q}(\Omega)$. As above we derive from [21, Corollary 1.7] that the sequence $\{\mathbf{M}_1[M|\nabla\zeta_{j_n}|^q+\mu+S_1^p(\zeta_j)]\}$ is bounded and equi-integrable in $L^q(\Omega)$, hence $\{\nabla S_1(\zeta_{j_n})\}$ converges to ∇S in $L^q(\Omega)$. Therefore S is a solution of

$$\begin{array}{ll} -\Delta S + S^p = M|\nabla\zeta|^q + \mu & \text{in } \Omega \\ S = 0 & \text{on } \partial\Omega. \end{array} \tag{3.46}$$

This implies that $S = S_1(\zeta)$ and the mapping S_1 is compact. □

End of the proof of Theorem 1.3. It follows from Lemma 3.7 that S_1 is a compact mapping from $E(T,\mu)$ into itself. Hence it admits a fixed point u by Schauder's theorem and $u \in W_0^{1,q}(\Omega) \cap L^p(\Omega)$ is nonnegative and satisfies (1.13). □

3.3 Proof of the Corollaries

Proof of Corollary 1.4. If q satisfies $\frac{Np}{N+p} \le q < 2$, then $1 < p < \frac{2N}{N-2}$. By Sobolev imbedding theorem there holds

$$\|\phi\|_{W^{1,q'}} \le c_{27}\,\|\phi\|_{W^{2,p'}} \quad \text{for all } \phi \in C^2(\overline{\Omega}),$$

where c_{27} depends on p, q and $|\Omega|$, provided

$$\frac{1}{q'} \ge \frac{1}{p'} - \frac{1}{N} \Longleftrightarrow q \ge \frac{Np}{N+p}.$$

The condition $q < 2$ necessitates that $\frac{Np}{N+p} < 2$, equivalently $p < \frac{2N}{N+2}$. □

Proof of Corollary 1.5. We recall [1, Theorem 5.5.1] (a)-(b). If $2p' \le q' < N$, then

$$\left(cap_{2,p'}^{\mathbb{R}^N}(E)\right)^{\frac{1}{N-2p'}} \le A\left(cap_{1,q'}^{\mathbb{R}^N}(E)\right)^{\frac{1}{N-q'}} \qquad \text{for all Borel set } E \subset \Omega. \tag{3.47}$$

Since $\frac{N-2p'}{N-q'} \ge 1$ we deduce

$$cap_{2,p'}^{\mathbb{R}^N}(E) \le A' cap_{1,q'}^{\mathbb{R}^N}(E) \qquad \text{for all Borel set } E \subset \Omega. \tag{3.48}$$

Hence (1.15) implies (1.12) and existence follows from Theorem 1.3. The assumption $2p' \le q' < N$ is equivalent to $\frac{N}{N-1} < q \le \frac{2p}{p+1}$. Note that this implies $p > \frac{N}{N-2}$. □

Proof of Corollary 1.6. -(i) Since p and q are subcritical, (1.12) is verified as soon as there holds for every Borel set $E \subset \Omega$,

$$\mu(E) \le \frac{\mu(\Omega)}{\min\left\{cap_{2,p'}^{\mathbb{R}^N}(\{0\}), cap_{1,q'}^{\mathbb{R}^N}(\{0\})\right\}} \min\left\{cap_{2,p'}^{\mathbb{R}^N}(E), cap_{1,q'}^{\mathbb{R}^N}(E)\right\}.$$

-(ii) If p is subcritical and q supercritical, (1.14) implies (1.12).

-(iii) If q is subcritical and μ satisfies (1.15) then (1.12) holds and the existence follows by Theorem 1.3. However the condition that μ vanishes on Borel sets

with $cap_{2,p'}^{\mathbb{R}^N}$-capacity is weaker. Let $T > 0$ and B_T be the set of $\zeta \in W_0^{1,q}(\Omega)$ such that $\|M|\nabla\zeta|^q\|_{L^1(\Omega)} \leq T$. If $\zeta \in B_T$ there exists a unique solution $S(\zeta) \in W_0^{1,1}(\Omega) \cap L^p(\Omega)$ to

$$\begin{aligned} -\Delta\phi + \phi^p &= M|\nabla\zeta|^q + \mu && \text{in } \Omega \\ \phi &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{3.49}$$

This follows from [3] since the right-hand side of the equation is a measure absolutely continuous with respect to the $cap_{2,p'}^{\mathbb{R}^N}$-capacity and the solution is nonnegative. The following estimate holds [12, Theorem 8]

$$\begin{aligned} &(i) \qquad \int_\Omega S^p(\zeta)dx \leq \mu(\Omega) + M\int_\Omega |\nabla\zeta|^q dx \\ &(ii) \qquad \alpha\,\|S(\zeta)\|_{W_0^{1,r}} \leq 2\mu(\Omega) + 2M\int_\Omega |\nabla\zeta|^q dx, \end{aligned} \tag{3.50}$$

for any $1 < r < \frac{N}{N-1}$. This implies in the case $q = r$,

$$\left(\int_\Omega |\nabla S(\zeta)|^q dx\right)^{\frac{1}{q}} \leq A\mu(\Omega) + B\int_\Omega |\nabla\zeta|^q dx,$$

with

$$A = \frac{2}{\alpha} \quad \text{and} \quad B = \frac{2M}{\alpha}.$$

For $X > 0$ set $F(X) = BX^q - X + A\mu(\Omega)$. Then $F'(X) = 0$ iff $X = X_0 := (qB)^{-\frac{1}{q-1}}$ and

$$F(X_0) = A\mu(\Omega) - \frac{q-1}{q}(qB)^{-\frac{1}{q-1}}.$$

If we assume that

$$F(X_0) < 0 \Longleftrightarrow \mu(\Omega) < \frac{q-1}{qA}(qB)^{-\frac{1}{q-1}} = (q-1)\left(\frac{\alpha}{2}\right)^{\frac{q}{q-1}} M^{-\frac{1}{q-1}}, \tag{3.51}$$

then $\min F(X) < 0$, therefore F admits two positive roots $X_2 < X_0 < X_1$. Hence the inequality

$$X_2 < \left(\int_\Omega |\nabla\zeta|^q dx\right)^{\frac{1}{q}} \leq X_1, \tag{3.52}$$

implies

$$\left(\int_\Omega |\nabla S(\zeta)|^q dx\right)^{\frac{1}{q}} \leq B\int_\Omega |\nabla\zeta|^q dx + A\mu(\Omega) \leq \left(\int_\Omega |\nabla\zeta|^q dx\right)^{\frac{1}{q}} \leq X_1, \tag{3.53}$$

and if

$$\left(\int_\Omega |\nabla\zeta|^q dx\right)^{\frac{1}{q}} \leq X_2, \tag{3.54}$$

then

$$\left(\int_\Omega |\nabla S(\zeta)|^q dx\right)^{\frac{1}{q}} \leq A\mu(\Omega) + BX_2^q \leq A\mu(\Omega) + BX_1^q = X_1. \tag{3.55}$$

Therefore the mapping S sends B_{X_1} into itself. It is continuous since

$$\begin{aligned} &(i) \qquad \int_\Omega |S^p(\zeta_1) - S^p(\zeta_2)| \leq M \int_\Omega ||\nabla\zeta_1|^q - |\nabla\zeta_2|^q| \, dx \\ &(ii) \qquad \alpha \left\| S(\zeta_1) - S(\zeta_2) \right\|_{W_0^{1,r}} \leq 2M \int_\Omega ||\nabla\zeta_1|^q - |\nabla\zeta_2|^q| \, dx. \end{aligned} \tag{3.56}$$

The fact that this operator is compact follows from the *a priori* estimate (3.50) and Lemma 3.7. The conclusion is a consequence of Schauder Theorem. □

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