



**HAL**  
open science

## Rate of escape of conditioned Brownian motion

Orphée Collin, Francis Comets

► **To cite this version:**

Orphée Collin, Francis Comets. Rate of escape of conditioned Brownian motion. 2021. hal-03146283v2

**HAL Id: hal-03146283**

**<https://hal.science/hal-03146283v2>**

Preprint submitted on 4 Apr 2021 (v2), last revised 27 Oct 2021 (v3)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Rate of escape of conditioned Brownian motion

Orphée Collin <sup>1</sup>, Francis Comets <sup>2</sup>

April 4, 2021

<sup>1</sup> École Normale Supérieure, DMA, 45 Rue d’Ulm, 75005 Paris, France

e-mail: [orphee.collin@ens.psl.eu](mailto:orphee.collin@ens.psl.eu)

<sup>2</sup> Université de Paris and LPSM, Mathématiques, case 7012, F-75205 Paris Cedex 13, France

e-mail: [comets@lpsm.paris](mailto:comets@lpsm.paris)

## Abstract

We study the norm of the two-dimensional Brownian motion conditioned to stay outside the unit disk at all times. By conditioning the process is changed from barely recurrent to slightly transient. We obtain sharp results on the rate of escape to infinity of the process of future minima. For this, we introduce a renewal structure attached to record times and values. Additional results are given for the long time behavior of the norm.

**Keywords:** Brownian motion, Bessel process, conditioning, transience, Wiener moustache, regeneration, upper-class and lower-class, random difference equation

**AMS 2020 subject classifications:** 60K35, 60J60, 60J65, 60G17

## Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
<b>2</b>	<b>Main results</b>	<b>4</b>
2.1	Results for the future minimum . . . . .	5
2.2	Long time behavior of $R(t)$ . . . . .	5
<b>3</b>	<b>Regenerative structure</b>	<b>6</b>
3.1	Renewal times . . . . .	6
3.2	Description of a cycle . . . . .	7
3.3	Tail estimates for $T$ . . . . .	9
3.4	Tail estimate for $U$ . . . . .	12
<b>4</b>	<b>Proofs for section 2.2</b>	<b>13</b>
<b>5</b>	<b>Proof of Theorem 2.1</b>	<b>16</b>

<b>6 Proof of Theorem 2.2</b>	<b>19</b>
6.1 Proof of (10).	20
6.2 Proof of (11)	23

# 1 Introduction

This paper is devoted to the planar Brownian motion conditioned to stay outside the unit ball  $\mathbf{B}(0, 1)$  at all times. Besides its own appeal from its fundamental character, this process has attracted a keen interest as being the elementary brick of the two-dimensional Brownian random interlacement recently introduced in [9]. By rotational symmetry, the norm  $R$  of the conditioned Brownian motion follows a stochastic differential equation in  $[1, \infty)$ ,

$$dR(t) = \left( \frac{1}{R(t) \ln R(t)} + \frac{1}{2R(t)} \right) dt + dB(t) \tag{1}$$

with  $B$  a standard Brownian motion in  $\mathbb{R}$ , and we can – and we will – restrict the study of the conditioned process to that of  $R$  itself since the angle obeys a diffusion subordinated to it. The two-dimensional Brownian motion is critically recurrent, but conditioning it outside the unit ball turns it into (delicately) transient. A natural question is the rate at which  $R(t)$  tends to  $\infty$  as  $t \rightarrow \infty$ , this is the object of the present paper. A measure of the reluctance of  $R$  to tend to infinity is given by the *future minima process*

$$M(t) = \inf\{R(s); s \geq t\} \tag{2}$$

which is non-decreasing to  $\infty$  a.s. The corresponding model in the discrete case, the two-dimensional simple random walk conditioned to avoid the origin at all times, has motivated many recent papers. Estimates on the future minimum distance to the origin have been obtained in [23], we will use them as benchmarks. It is also shown that two independent conditioned walkers meet infinitely often although they are transient. The range of the walk, i.e. the set of visited sites, is studied in [12]: if a finite  $A \subset \mathbb{Z}^2 \setminus \{0\}$  is "big enough and well distributed in space", then the proportion of visited sites is approximately uniformly distributed on  $[0, 1]$ . In [21] the explicit formula for the Green function is obtained, and a survey is given in Chapter 4 of [22].

For dimensions  $d \geq 3$ , the random interlacement model has been introduced in [28] to describe the local picture of the visited set by a random walk at large times on a large  $d$ -dimensional torus, and similarly in [29], the Brownian random interlacement to describe the Wiener sausage around the Brownian motion on a  $d$ -dimensional torus. For dimension  $d = 2$ , the random interlacement model is the local limit of the visited set by the random walk *around a point which has not been visited so far* [7], and analogously, the Brownian random interlacement is the local limit of the Wiener sausage on the two-dimensional torus *around a point which is outside the sausage* [9]. Formally, the two-dimensional Brownian random interlacement is defined as a Poisson process of bi-infinite paths, which are rescaled instances of the so-called "Wiener moustache". The Wiener moustache is obtained by gluing two instances (for positive and negative times, see Figure 1 in [9]) of planar Brownian motion conditioned to stay outside the unit ball, which are independent except that they share the same starting

point (see Lemma 3.9 in [9]). Hence, the process we consider in this paper is the building brick of Brownian random interlacement in the plane. We also recall that the complement of the sausage around the interlacement has an interesting phase transition, changing from a.s. unbounded to a.s. bounded as the Poisson intensity is increased, see Th. 2.13 in [9] and [8] for the discrete case.

With a slight abuse of terminology, we say  $f(t) \leq g(t)$  *i.o.* (infinitely often) if the set  $\{t \geq 0 : f(t) \leq g(t)\}$  is unbounded, and  $f(t) \leq g(t)$  *ev.* (eventually) if the set  $\{t \geq 0 : f(t) \leq g(t)\}$  is a neighborhood of  $\infty$  in  $\mathbb{R}_+$ .

We now give a short overview of some of our results on the rate of escape of  $R$  to infinity. They are consequences of the results in section 2.1.

**Theorem 1.1.** *For  $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  non-increasing such that  $(\ln t)g(\ln \ln t)$  is non-decreasing,*

$$\mathbb{P}\left(M(t) \leq e^{(\ln t)g(\ln \ln t)} \text{ i.o.}\right) = \begin{cases} 0 \\ 1 \end{cases} \quad \text{according to} \quad \int^\infty g(u)du \begin{cases} < \infty \\ = \infty \end{cases} .$$

This result with an integral condition has a flavor of Kolmogorov's test (see, e.g., sect. 4.12 in [15]).

**Theorem 1.2.** *The limit*

$$K^* = \limsup_{t \rightarrow \infty} \frac{M(t)}{\sqrt{t \ln \ln t}}$$

*is a.s. constant, and  $0 < K^* < \infty$ .*

Though we do not know the actual value of  $K^*$  we can see that both theorems are much finer than the corresponding Theorem 1.2 of [23]. These two theorems together yield a precise version of the observation from [21] that the pathwise divergence of  $R$  to infinity occurs in a highly irregular way. The future minima process has been considered earlier, e.g. [17] and [18] for Bessel processes and for random walks, and [20] for positive self-similar Markov processes. Let us recall the similar result for transient Bessel processes. Denote by  $\text{BES}^d$  the  $d$ -dimensional Bessel process, i.e. the solution of

$$dX(t) = \frac{d-1}{2X(t)} dt + dB(t) , \tag{3}$$

that is the norm of the standard Brownian motion in  $\mathbb{R}^d$  when  $d$  integer : then, by Th. 4.1 in [17],

$$\text{for } d > 2, \quad \limsup_{t \rightarrow \infty} \frac{\min\{\text{BES}^d(s); s \geq t\}}{\sqrt{2t \ln t}} = 1. \tag{4}$$

An important (and beautiful) finding of our work is a renewal structure in Section 3 which allows sharp estimates. To illustrate that, let's mention that we will find a sequence of relevant random variables  $S_n > 0$  solving a random difference equation

$$S_n = \alpha_n S_{n-1} + \beta_n , \quad n \geq 1, \tag{5}$$

where the sequence  $(\alpha_n, \beta_n)_n$  is i.i.d. with positive coefficients,  $\alpha_n < 1$  and  $\beta_n$  with logarithmic tails,  $\mathbb{P}(\beta_1 > t) \sim c/\ln t$  for large  $t$ . Although autoregressive processes of the type (5) are

usually addressed with exponential or power-law tail for  $\beta_n$  [5], the case of logarithmic tail has been also considered, see [16], [32], [3], and also [1] for a recent account. Interestingly, our model is critical in the perspective of the Markov chain  $S_n$ , in the sense that the actual value of the constant  $c$  is precisely the transition from recurrence to transience for the chain.

The paper is organized as follows. We give the main results in the next section. The regeneration structure is defined in Section 3, together with the basic estimates, and ending with Remark 3.8 on the above random difference equation. In the next section we prove some results showing that  $R$  somewhat behaves at large times like the two-dimensional Bessel process. In Sections 5 and 6 we prove the two above theorems.

## 2 Main results

We first collect a few properties of the involved processes.

We start with some notations. Consider  $W$  a two-dimensional standard Brownian motion and denote by  $\mathbf{P}_x$  the law of  $W$  starting at  $x$ ,  $\widehat{W}$  a Brownian motion conditioned to stay outside the unit ball, and denote by  $\widehat{\mathbf{P}}_x$  its law starting at  $x$ , and  $R = |\widehat{W}|$  its Euclidean norm with  $P_r$  the corresponding law ( $r = |x|$ ). In this paper we are mainly interested in  $\mathbb{P} = P_1$ . The construction of the process starting from  $R(0) > 1$  is standard from taboo process theory, and the one starting from  $R(0) = 1$  is given in definition 2.2 of [9].

For a closed subset  $B$  of the state space of a process  $Y$ , we denote the entrance time  $\tau(Y; B) = \inf\{t \geq 0 : Y(t) \in B\}$ , and write for short  $\tau(Y; r) = \tau(Y; \partial\mathbf{B}(0, r))$  and also  $\tau(r) = \tau(R; r)$  when  $Y = R$ . The function  $h(x) = \ln|x|$  is harmonic in  $\mathbb{R}^2 \setminus \{0\}$ , positive on  $\mathbb{R}^2 \setminus \mathbf{B}(0, 1)$  and vanishes on the unit circle. Then, the law  $\widehat{\mathbf{P}}_x$  of the planar Brownian motion  $W$  conditioned outside  $\mathbf{B}(0, 1)$  is given by Doob's  $h$ -transform of  $\mathbf{P}_x$ . By definition, for  $A \subset \mathcal{C}(\mathbb{R}^+, \mathbb{R})$  which is  $\mathcal{F}_{\tau(r_1)}$ -measurable ( $1 < |x| = r < r_1$ )

$$\begin{aligned} P_r(R \in A) &= \mathbf{P}_x(|W| \in A | \tau(W, r_1) < \tau(W, 1)) \\ &= \mathbf{P}_x(|W| \in A, \tau(W, r_1) < \tau(W, 1)) \times \frac{\ln r_1}{\ln |x|} \end{aligned} \tag{6}$$

recalling that  $\mathbf{P}_x(\tau(W, r_1) < \tau(W, 1)) = \frac{\ln|x|}{\ln r_1}$  since  $\ln|x|$  is harmonic in  $\mathbb{R}^2 \setminus \{0\}$ .

Another remarkable property is Remark 3.8 in [9] : For all  $x \notin \mathbf{B}(1)$ ,  $\rho > 0$ , we have

$$\widehat{\mathbf{P}}_x[\tau(\widehat{W}; \mathbf{B}(y, \rho)) < \infty] \rightarrow \frac{1}{2} \quad \text{as } |y| \rightarrow \infty.$$

The scale function for the process  $R$  is  $S(r) = \frac{-1}{\ln r}$ . Then, for  $1 < a < r < b$ ,

$$P_r[\tau(b) < \tau(a)] = \frac{\ln(r/a) \times \ln b}{\ln(b/a) \times \ln r}. \tag{7}$$

We refer to section 2.1 in [9] for more details on the many interesting properties of  $\widehat{W}$  and  $R$ .

## 2.1 Results for the future minimum

With  $L(t) = \ln(t \vee 1)$  and  $\ln(\cdot)$  the natural logarithm, define  $\ln_1(t) = L(t)$ , and for  $k \geq 2$ ,  $\ln_k(t) = L(\ln_{k-1}(t))$  so that  $\ln_k(t) = (\ln \circ \dots \circ \ln)(t)$  for  $t$  large.

**Theorem 2.1.** *For  $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  non-increasing such that  $(\ln t)g(\ln_2 t)$  is non-decreasing, we have:*

$$\int^{\infty} g(u)du < \infty \implies \text{a.s.}, M(t) \geq e^{(\ln t)g(\ln_2 t)} \text{ eventually,} \quad (8)$$

and

$$\int^{\infty} g(u)du = \infty \implies \text{a.s.}, M(t) \leq e^{(\ln t)g(\ln_2 t)} \text{ infinitely often.} \quad (9)$$

(Note that the second assumption is quite natural in view of the monotonicity of  $M_t$ .) Theorem 1.1 is a direct consequence of the above theorem. This result with an integral condition is reminiscent of Kolmogorov's test (see, e.g., sect. 4.12 in [15]), but the process  $M$  here is not Markov.

These estimates are stronger than the corresponding ones in Th. 1.2 of [23]. So are the following ones:

**Theorem 2.2.** *There exist  $0 < K' < K < \infty$  such that, almost surely,*

$$M(t) \leq K\sqrt{t \ln_3 t} \text{ eventually,} \quad (10)$$

and

$$M(t) \geq K'\sqrt{t \ln_3 t} \text{ infinitely often.} \quad (11)$$

Theorem 1.2 is essentially a reformulation of Theorem 2.2, it will be proved below Remark 6.2.

We recall the similar result (4) for transient Bessel processes: a.s. for all  $a < \sqrt{2} < b$ , the future minima process  $\min\{\text{BES}^d(s); s \geq t\}$  is eventually smaller than  $b\sqrt{t \ln_2 t}$  and infinitely often larger than  $a\sqrt{t \ln_2 t}$ .

Finally we mention that, for  $d > 2$ ,  $\min\{\text{BES}^d(s); s \geq t\} \leq \varepsilon\sqrt{2t \ln_2 t}$  i.o., a.s. for all  $\varepsilon > 0$ . (See [17], P.349).

## 2.2 Long time behavior of $R(t)$

At large times the process  $R$  behaves like  $\text{BES}^2$ . We emphasize that this is for the marginal law, but not for the future minimum. We formulate here precise statements of these facts.

It is well known that the random variable  $t^{-1/2}\text{BES}^2(t)$  converges to the Rayleigh distribution

$$d\nu(x) = xe^{-x^2/2}\mathbf{1}_{(0,\infty)}(x)dx \quad (12)$$

as  $t \rightarrow \infty$ . Similarly for  $R$ , we have

**Theorem 2.3.** *Let  $Z \sim \nu$ . As  $t \rightarrow \infty$ ,*

$$\frac{R(t)}{\sqrt{t}} \xrightarrow{\text{law}} Z .$$

**Theorem 2.4** (Pointwise ergodic theorem). *For all bounded continuous function  $f$  on  $(0, \infty)$ , as  $t \rightarrow \infty$ ,*

$$\frac{1}{t} \int_0^{e^t-1} f\left(\frac{R(u)}{\sqrt{1+u}}\right) \frac{1}{1+u} du \longrightarrow \int_{\mathbb{R}} f d\nu \quad a.s. \quad (13)$$

### 3 Regenerative structure

We fix a parameter  $r > 1$ . We construct a regenerative structure associated with the process  $R$  starting from  $R(0) = 1$ .

#### 3.1 Renewal times

We define a random sequence  $(H_n, A_n, T_n)_{n \geq 0}$  by  $H_0, T_0 = 0, A_0 = 1$ , then

$$\begin{cases} H_1 &= \inf\{t > T_0 : R(t) = r\} \\ A_1 &= \inf\{R(t); t \geq H_1\} \\ T_1 &= \inf\{t \geq H_1 : R(t) = A_1\} \end{cases}$$

and for  $n \geq 1$ ,

$$\begin{cases} H_{n+1} &= \inf\{t > T_n : R(t) = rA_n\} \\ A_{n+1} &= \inf\{R(t); t \geq H_{n+1}\} \\ T_{n+1} &= \inf\{t \geq H_{n+1} : R(t) = A_{n+1}\} \end{cases} \quad (14)$$

Since  $R$  is a continuous function with  $\lim_{t \rightarrow \infty} R(t) = \infty$  a.s., we see by induction that  $T_n < \infty$  a.s, with  $T_n < T_{n+1}$  and  $\lim_{n \rightarrow \infty} T_n = \infty$  a.s. The  $T_n$  are not stopping times, but they are called *renewal times* for the following reasons.

**Proposition 3.1.** *Let  $\mathcal{G}_1 = \sigma(T_1, (R(t)\mathbf{1}\{t < T_1\}; t \geq 0))$ . Then,*

$$\left(\frac{R(T_1 + A_1^2 t)}{A_1}; t \geq 0\right) \quad \text{has same law as } R \text{ and is independent of } \mathcal{G}_1.$$

This proposition is the building brick of the

**Theorem 3.2.** *[Renewal structure] The sequence*

$$\left(\frac{R(T_n + A_n^2 t)}{A_n}; t \in \left[0, \frac{T_{n+1} - T_n}{A_n^2}\right]\right)_{n \geq 0}$$

*is independent and identically distributed with the law of  $(R(t); t \in [0, T_1])$ .*

In particular, since  $R(T_{n+1}) = A_{n+1}$ , the sequence

$$\left(\frac{T_{n+1} - T_n}{A_n^2}, \frac{A_{n+1}}{A_n}\right)_{n \geq 0}$$

is i.i.d. and distributed as  $(T_1, A_1)$ . Therefore  $(T_n, A_n)$  can be written using i.i.d.r.v.'s, which will be used repeatedly all through.

*Proof.* Proposition 3.1. Recall that  $P_r$  denotes the law of the process  $R$  with  $R(0) = r$ . Observe that  $H_1$  is a stopping time, and denote by  $\mathcal{F}_{H_1}$  the sigma-field of events that occur before time  $H_1$ . By the strong Markov property,

*under  $P_1$ ,  $(R(t + H_1))_{t \geq 0}$  is independent of  $\mathcal{F}_{H_1}$  and has the law  $P_r$ .*

Moreover, by Theorem 2.4 in [31] (see also the proof of Lemma 3.9 in [9]), conditionnally on  $T_1, (R(t); t \in [H_1, T_1])$  and  $A_1 = a, (R(T_1 + t); t \geq 0)$  has the same law as  $R$  starting from  $a$  and conditioned to  $R(t) \geq a, \forall t \geq 0$ . By Remark 2.5 in [9], the latter law is equal to that of  $aR(\cdot/a^2)$  under  $P_1$ . Since  $\mathcal{G}_1 = \sigma(\mathcal{F}_{H_1}; (R(t); t \in [H_1, T_1]))$  up to nul events, we obtain the desired statement.  $\square$

*Proof.* Theorem 3.2. By induction, Proposition 3.1 implies that for all  $n$ , the process  $\left(\frac{R(T_n + A_n^2 t)}{A_n}; t \geq 0\right)$  is independent of  $\mathcal{G}_n = \sigma(T_n, (R(t); t < T_n))$  with the law of  $R$ . Then, the claim follows.  $\square$

As a direct consequence we have discovered a simple representation of crucial times and points of the process.

**Corollary 3.3.** *Define*

$$A'_{n+1} = \frac{A_{n+1}}{A_n}, \quad T'_{n+1} = \frac{T_{n+1} - T_n}{A_n^2}, \quad n \geq 0.$$

*Then,  $(A'_n, T'_n)_{n \geq 1}$  is an i.i.d. sequence with the same law as  $(A_1, T_1)$ , and we have the representation*

$$\begin{cases} T_n &= T'_1 + A_1'^2 T'_2 + \dots + (A'_1 \dots A'_{n-1})^2 T'_n \\ A_n &= A'_1 \dots A'_n \end{cases} \quad n \geq 1. \quad (15)$$

## 3.2 Description of a cycle

Recall  $r > 1$  is fixed. We will shorten the notations:  $(H, A, T) = (H_1, A_1, T_1)$ . Recall that  $R$  starts from  $R(0) = 1$ , hits  $r$  at  $H$  for the first time, and reaches its future minimum  $A \in (1, r)$  at time  $T$ . We also introduce its maximum  $B > r$  on  $[H, T]$ , as well as their logarithms  $U, V$ :

$$\begin{cases} A &= r^U &= \min\{R(t); t \geq H\} \\ B &= r^V &= \max\{R(t); t \in [H, T]\} \end{cases}$$

see figure 1. It was shown in [9] that  $U$  is uniform on  $[0, 1]$ , but we can even compute the joint law of  $U, V$ . For  $1 < a - h < a < r < b$ , we have by the strong Markov property

$$\begin{aligned} \mathbb{P}(A \in [a - h, a], B > b) &= P_r(\tau(b) < \tau(a)) \times P_b(\min\{R(t); t \geq 0\} \in [a - h, a]) + o(h) \\ &= \frac{\ln(r/a) \ln b}{\ln(b/a) \ln r} \times \left( \frac{1}{a \ln b} h + o(h) \right) \end{aligned}$$

using (2.16) in [9] and that, for  $R$  started at  $b$ ,  $\min\{R(t); t \geq 0\}$  has density  $(a \ln b)^{-1}$  on  $(1, b)$ . Hence  $(A, B)$  has a density given by the negative of the  $b$ -derivative of the dominant term as  $h \searrow 0$ , i.e.,

$$p_{A,B}(a, b) = \frac{1}{ab \ln r} \frac{\ln(r/a)}{\ln^2(b/a)}, \quad 1 < a < r < b.$$



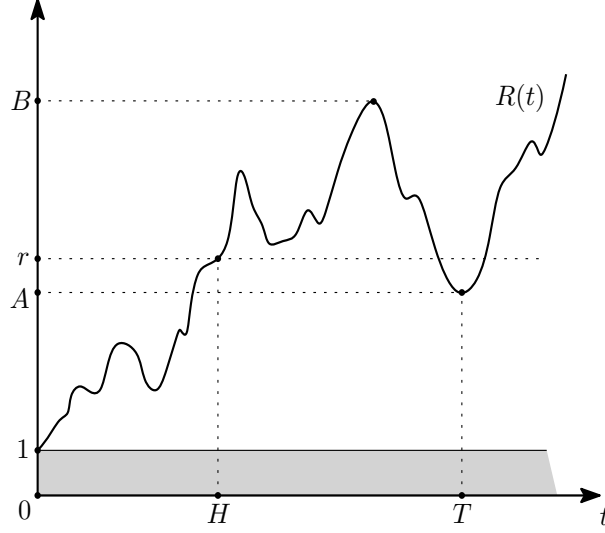


Figure 1: First cycle:  $A = r^U$ ,  $B = r^V$

By changing variables, it follows that  $(U, V)$  has density

$$p_{U,V}(u, v) = \frac{1-u}{(v-u)^2} \mathbf{1}\{0 < u < 1 < v\} \quad (16)$$

We recover that  $U$  is uniform on  $(0,1)$  and that  $V$  has density

$$p_V(v) = -\ln(1 - 1/v) - 1/v, \quad v > 1.$$

It follows that for  $v \geq 1$ ,

$$\mathbb{P}(V > v) = \sum_{n=1}^{\infty} \frac{1}{n(n+1)v^n}, \quad (17)$$

and then  $\mathbb{P}(V > v) \sim 1/(2v)$  as  $v \rightarrow \infty$ .

We also need information on the cycle length  $T$ . For any  $s \geq 1$  we consider the hitting time by  $R$  starting at  $s$  of its absolute minimum, and denote by  $\mu_s$  a r.v. with the same law:

$$\mu_s \sim P_s(\arg \min\{R(t); t \geq 0\} \in \cdot)$$

Recall that, under  $\mathbb{P}$ ,  $R(0) = 1$ .

**Proposition 3.4.** (i) We have

$$T = H + (T - H),$$

where  $H$  and  $(T - H)$  are independent with  $T - H \stackrel{\text{law}}{=} \mu_r$ .

(ii) For  $u \in (0, 1)$ , the conditional law of  $T$  given  $U \geq u$  is equal to the law of an independent sum  $H + r^{2u} \mu_{(r^{1-u})}$ .

*Proof.* (i) directly follows from the strong Markov property for the Markov process  $R$  and the stopping time  $H$ .

For (ii), we recall Remark 2.5 in [9]: for  $c > 1$ , denoting by  $R^c$  the diffusion  $R$  conditioned to stay outside  $(1, c]$ , we have

$$R^c(\cdot) = cR(\cdot/c^2)$$

(Alternatively, this follows from  $R$  being the norm of conditioned Brownian motion (6) and from Brownian scaling.) Hence, for  $s \in \mathbb{R}$ , again from the strong Markov property,

$$\begin{aligned} E_1[e^{isT} | U \geq u] &= E_1[e^{is(T-H+H)} | U \geq u] \\ &= E_1[e^{isH}] \times E_r[e^{is(T-H)} | U \geq u] \\ &= E_1[e^{isH}] \times E_r[e^{is \times \arg \min\{R(t); t \geq 0\}} | \min\{R(t); t \geq 0\} \geq r^u] \\ &= E_1[e^{isH}] \times E[e^{isr^{2u} \mu_{(r^{1-u})}}] \end{aligned}$$

which proves the result.  $\square$

### 3.3 Tail estimates for $T$

We need some estimates of the upper and lower tails of  $T$ , that we derive in this section. But first we state elementary comparisons of  $R$  and Bessel processes, see (3), that will be used all through the paper.

**Proposition 3.5.** (i) *There exists a coupling of the processes  $R$  and  $\text{BES}^2$  starting at 1 such that*

$$\forall t \geq 0, \quad R(t) \geq \text{BES}^2(t).$$

(ii) *For  $\delta > 0$  there exists a coupling of the processes  $R$  and  $\text{BES}^{2+\delta}$  starting at 1 such that for  $\sigma = \sup\{t \geq 0; R(t) \leq e^{2/\delta}\}$ ,*

$$\forall s \geq 0, \quad R(\sigma + s) \leq \text{BES}^{2+\delta}(\sigma + s) - \text{BES}^{2+\delta}(\sigma) + e^{2/\delta}.$$

*Proof.* It is well known [6] that the stochastic differential equation (3) has a strong solution, so we can couple the processes  $R$  and  $\text{BES}^2, \text{BES}^{2+\delta}$  by driving equations (1) and (3) by the same Brownian motion  $B$ . Then, with  $x^+ = \max\{x, 0\}$  for  $x$  real, we have for all  $t > 0$  and all realization of  $B$ ,

$$\begin{aligned} d(\text{BES}^2(t) - R(t))^+ &= \mathbf{1}_{\{\text{BES}^2(t) \geq R(t)\}} \left( \frac{1}{2\text{BES}^2(t)} - \frac{1}{2R(t)} - \frac{1}{R(t) \ln R(t)} \right) dt \\ &\leq 0 \end{aligned}$$

which implies (i) by integration. Similarly for (ii) we write the differential

$$\begin{aligned} d(R(t) - \text{BES}^{2+\delta}(t))^+ &= \mathbf{1}_{\{\text{BES}^{2+\delta}(t) \leq R(t)\}} \left( \frac{1}{2R(t)} + \frac{1}{R(t) \ln R(t)} - \frac{1+\delta}{2\text{BES}^{2+\delta}(t)} \right) dt \\ &\leq 0 \quad \text{for } t \geq \sigma. \end{aligned}$$

Integrating on  $t \in [\sigma, \sigma + s]$  we obtain (ii).  $\square$

We are now ready to start with the upper tail of  $T$ .

**Proposition 3.6.** *As  $t \rightarrow \infty$ ,*

$$\mathbb{P}(T \geq t) \sim \frac{\ln r}{\ln t}. \quad (18)$$

*More precisely, there exists constants  $t_0$  and  $C$  such that for all  $t \geq t_0$ ,*

$$\left(1 - \frac{\ln_3 t + C}{\ln t}\right) \frac{\ln r}{\ln t} \leq \mathbb{P}[T \geq t] \leq \left(1 + \frac{\ln_3 t + C}{\ln t}\right) \frac{\ln r}{\ln t}. \quad (19)$$

*Proof.* We first obtain two preliminary estimates.

*Upper bound:* for  $0 < \varepsilon < 1$ ,

$$\begin{aligned} \mathbb{P}(T \geq t) &= \mathbb{P}\left(T \geq t, V \geq \frac{\ln t}{2(1+\varepsilon)\ln r}\right) + \mathbb{P}\left(T \geq t, V < \frac{\ln t}{2(1+\varepsilon)\ln r}\right) \\ &\leq \mathbb{P}\left(V \geq \frac{\ln t}{2(1+\varepsilon)\ln r}\right) + \mathbb{P}\left(R(s) \leq t^{\frac{1}{2(1+\varepsilon)}}, s \in [0, t]\right) \\ &\leq \frac{(1+\varepsilon)\ln r}{\ln t} + \frac{1}{5} \left(\frac{2(1+\varepsilon)\ln r}{\ln t}\right)^2 + C_0 \exp(-C_1 t^{\varepsilon/(1+\varepsilon)}) \end{aligned} \quad (20)$$

for  $t \geq t_1$  with  $t_1 > 0$  not depending on  $\varepsilon \in (0, 1)$ . Indeed, to obtain the first term we have used (17) in the form of  $\mathbb{P}(V \geq v) \leq (1/2v) + (1/5v^2)$  for large  $v$ . In order to obtain the second one, we first bound  $R(\cdot) \geq \text{BES}^2(\cdot)$ , with  $\text{BES}^2$  started at 0 using Proposition 3.5, and finally that there exist positive  $C_0, C_1$  such that

$$\forall t > 0, \forall \rho > 0, \quad \mathbb{P}(\text{BES}^2(s) \leq \rho, s \in [0, t]) \leq C_0 \exp\left(-C_1 \frac{t}{\rho^2}\right), \quad (21)$$

see e.g. exercise 1 p.106 in [27].

*Lower bound:* for  $0 < \varepsilon < 1/2$ ,

$$\begin{aligned} \mathbb{P}(T \geq t) &\geq \mathbb{P}\left(T - H \geq t, V \geq \frac{\ln t}{2(1-\varepsilon)\ln r}\right) \\ &= \mathbb{P}\left(V > \frac{\ln t}{2(1-\varepsilon)\ln r}\right) - \mathbb{P}\left(T - H \leq t, V \geq \frac{\ln t}{2(1-\varepsilon)\ln r}\right) \\ &\geq \mathbb{P}\left(V > \frac{\ln t}{2(1-\varepsilon)\ln r}\right) - P_r\left(\tau(R, t^{\frac{1}{2(1-\varepsilon)}}) \leq t\right) \\ &\geq \frac{(1-\varepsilon)\ln r}{\ln t} - C_2 \exp(-C_3 t^{\varepsilon/(1-\varepsilon)}) \end{aligned} \quad (22)$$

for  $t \geq t_2$ , with  $t_2 > 0$  not depending on  $\varepsilon \in (0, \frac{1}{2})$ . In (22) we have used (17) for the first term, and we give details for the second one: for  $|x| = r > 1$  by (6), we get for all  $t > 1$ ,

$$P_r(\tau(R, t^{\frac{1}{2(1-\varepsilon)}}) \leq t) = \mathbb{P}_x\left(\tau(|W|, t^{\frac{1}{2(1-\varepsilon)}}) \leq t \mid \tau(|W|, t^{\frac{1}{2(1-\varepsilon)}}) < \tau(|W|, 1)\right)$$

$$\begin{aligned}
&\leq \mathbb{P}_x \left( \tau(|W|, t^{\frac{1}{2(1-\varepsilon)}}) \leq t \right) \times \frac{\ln t}{2(1-\varepsilon) \ln r} \\
&\leq C_2 \exp \left( -C_3 t^{\varepsilon/(1-\varepsilon)} \right)
\end{aligned}$$

for some constant  $C_2, C_3 > 0$  by the moderate deviation principle for Brownian motion.

For both the upper and lower bounds, we now choose

$$\varepsilon = \varepsilon_t = \frac{\ln_3 t + C_4}{\ln t}$$

with a constant  $C_4$ . Provided the constant  $C_4$  is large enough, the terms  $C_0 \exp(-C_1 t^{\varepsilon_t/(1+\varepsilon_t)})$  and  $C_2 \exp(-C_3 t^{\varepsilon_t/(1-\varepsilon_t)})$  are dominated by  $(\ln t)^{-2}$ . We then get (19) from (20) and (22), taking any  $C > C_4 + \frac{4 \ln r}{5}$ .

Finally, (18) is a direct consequence of (19). The proof is complete.  $\square$

We also need to control the lower tail of  $T$ .

**Proposition 3.7.** (i) For all  $\varepsilon \in (0, r-1)$ , there exists  $t_0 > 0$  such that for  $t \leq t_0$ ,

$$\mathbb{P}[T \leq t] \leq \exp \left( -\frac{(r-1-\varepsilon)^2}{2t} \right). \quad (23)$$

(ii) For all  $\varepsilon > 0$ , there exists  $t_1 > 0$  such that for  $t \leq t_1$ , and all  $u \in [0, 1)$ ,

$$\mathbb{P}[T \leq t | U \geq u] \geq \exp \left( -\frac{(r-1+\varepsilon)^2}{2t} \right). \quad (24)$$

*Proof.* (i) Setting  $a = 1 + \varepsilon/2 \in (1, r)$  and using the strong Markov property for the hitting time of  $a$  by  $R$ , we obtain

$$\begin{aligned}
\mathbb{P}(T \leq t) &\leq P_1(\tau(r) - \tau(a) \leq t) \\
&= P_a(\tau(r) \leq t) \\
&\stackrel{(6)}{=} P_{(a,0)}(\tau(|W|, r) \leq t | \tau(|W|; r) < \tau(|W|, 1)) \\
&\leq P_{(a,0)}(\tau(|W|, r) \leq t) \times \frac{\ln r}{\ln a}.
\end{aligned}$$

Recalling large deviation results for Brownian motion in small time, e.g. section 6.8 of Ch. 5 in [2],

$$\lim_{t \rightarrow 0} t \ln P_{(a,0)}(\tau(|W|, r) \leq t) = -\frac{(r-a)^2}{2}, \quad (25)$$

we see that the above upper bound implies (i).

(ii) Let  $t \leq 1$ . By Proposition 3.4-(ii), and by comparing  $R$  and  $\text{BES}^2$  from Proposition 3.5 (i), we obtain

$$\begin{aligned}
\mathbb{P}(T \leq t | U \geq u) &\geq \mathbb{P}(H \leq t - t^2) \times \mathbb{P}(r^{2u} \mu_{(r^{1-u})} \leq t^2) \\
&\geq \mathbb{P}(\text{BES}^2(t - t^2) \geq r) \times \mathbb{P} \left( \mu_{(r^{1-u})} \leq \frac{t^2}{r^{2u}} \right)
\end{aligned}$$

$$\begin{aligned}
&= \mathbb{P}_{(1,0)} (|W(t-t^2)| \geq r) \times \\
&\quad P_{r^{1-u}} \left( \arg \min \{R(s); s \geq 0\} \leq \theta \right), \tag{26}
\end{aligned}$$

with  $\theta = \frac{t^2}{r^{2u}}$ . We estimate the first term using again large deviation for Brownian motion in small time [2]: for  $|x| < r$ ,

$$\lim_{t \rightarrow 0} t \ln \mathbb{P}_x (|W|(t) \geq r) = -\frac{(r-|x|)^2}{2}. \tag{27}$$

To estimate the second term in (26), note that  $R(\theta) \geq r^{1-u} + \sqrt{\theta}$  and  $R(s) \geq r^{1-u}$  for all  $s \geq \theta$  implies that,  $P_{r^{1-u}}$ -a.s.,  $R$  achieves its minimum before time  $\theta$ . Hence, by Markov property and (7),

$$\begin{aligned}
P_{r^{1-u}} \left( \arg \min \{R(s); s \geq 0\} \leq \theta \right) &\geq P_{r^{1-u}} \left( R(\theta) \geq r^{1-u} + \sqrt{\theta} \right) \times \left( 1 - \frac{\ln r^{1-u}}{\ln(r^{1-u} + \sqrt{\theta})} \right) \\
&\geq \mathbb{P} \left( B(\theta) \geq \sqrt{\theta} \right) \times \left( 1 - \frac{\ln r^{1-u}}{\ln(r^{1-u} + \sqrt{\theta})} \right) \\
&\geq \mathbb{P}(B(1) \geq 1) \times \frac{t}{2r \ln r} \quad \text{for large } t,
\end{aligned}$$

arguing on the second line that  $R$  dominates Brownian motion by comparing the drift. Combined with (26) and (27), this completes the proof of (ii).  $\square$

### 3.4 Tail estimate for $U$

Recall Hoeffding's inequality [14], or Th. 2.8 in [4]: for  $b < 1$ ,  $c > 1$  and  $i \geq 1$ ,

$$\mathbb{P}[2(U_1 + \dots + U_i) \geq c.i] \leq \exp\left(-\frac{i}{2}(c-1)^2\right), \tag{28}$$

and

$$\mathbb{P}[2(U_1 + \dots + U_i) \leq b.i] \leq \exp\left(-\frac{i}{2}(1-b)^2\right). \tag{29}$$

**Remark 3.8** (The random difference equation (5)). *Introduce the sequence*

$$S_n = \frac{T_n}{A_n^2}$$

*which is key in Section 6. In view of (15), we see that it solves the recursion*

$$S_{n+1} = \alpha_{n+1} S_n + \beta_{n+1}$$

*(i.e., (5) above), with*

$$\alpha_n = (A'_n)^{-2}, \quad \beta_n = \frac{T'_n}{(A'_n)^2}.$$

The bi-dimensional sequence  $(\alpha_n, \beta_n), n \geq 1$ , is i.i.d., and falls into the usual setup of random difference equation. In our case, the following quantities exist

$$a := \mathbb{E}[\ln \alpha_1], \quad b := \lim_{t \rightarrow \infty} \mathbb{P}[\beta_1 > t] \times \ln t,$$

and satisfy  $a < 0$  (contractive case),  $0 < b < \infty$  (very heavy tail). Following [1], this prevents the Markov chain  $S_n$  to be positive recurrent: though the contraction brings stability to the process, yet occasional large values of  $\beta_n$  overcompensate this behavior so that positive recurrence fails to hold. In our case, we easily check from (18) that

$$b = -a \quad (= \ln r)$$

in which case the Markov chain  $S_n$  is null recurrent, but in a critical manner: the chain is transient if  $b > -a$  and null recurrent if  $b \leq -a$ .

## 4 Proofs for section 2.2

We consider the process  $R$  from (1) on a geometric scale,

$$X(t) = e^{-t/2} R(e^t - 1) \tag{30}$$

and we observe that

$$\beta(t) = \int_0^{e^t - 1} \frac{1}{\sqrt{1+s}} dB(s)$$

is a standard Brownian motion by Paul Lévy's characterization. We claim that  $X$  solves the SDE

$$\begin{cases} dX(t) &= \left( \frac{1}{2X(t)} - \frac{X(t)}{2} + \frac{1}{X(t) \ln[e^{t/2} X(t)]} \right) dt + d\beta(t) \\ X(0) &= R(0). \end{cases} \tag{31}$$

Indeed,

$$\begin{aligned} X(t) &= e^{-t/2} X(0) + e^{-t/2} \int_0^{e^t - 1} \left( \frac{1}{2R} + \frac{1}{R \ln R} \right)(s) ds + e^{-t/2} B(e^t - 1) \\ &= J(t) + K(t) + L(t), \end{aligned}$$

with  $dJ(t) = -\frac{1}{2}J(t)dt$ , and

$$\begin{aligned} \frac{dK(t)}{dt} &= -\frac{1}{2}K(t) + \frac{1}{2X(t)} + \frac{1}{X(t) \ln[e^{t/2} X(t)]}, \\ dL(t) &= -\frac{1}{2}L(t)dt + e^{-t/2} dB(e^t - 1). \end{aligned}$$

Moreover, we easily check the equality

$$\int_0^t e^{-s/2} dB(e^s - 1) = \int_0^{e^t - 1} \frac{1}{\sqrt{1+u}} dB(u)$$

in the Gaussian space generated by  $B$ . Adding up terms, we see that  $X$  solves the SDE (31). Denote by  $b_t$ , resp.  $b_\infty$  the drift coefficient and its limit, given for  $x \in (0, \infty)$  by

$$b_t(x) = \frac{1}{2x} - \frac{x}{2} + \frac{1}{x(\ln x + t/2)}, \quad b_\infty(x) = \frac{1}{2x} - \frac{x}{2},$$

and by  $X^{(\infty)}$  the homogeneous diffusion

$$dX^{(\infty)}(t) = \left( \frac{1}{2X^{(\infty)}(t)} - \frac{X^{(\infty)}(t)}{2} \right) dt + d\beta(t). \quad (32)$$

Following the approach of Takeyama [30], we state the following

**Lemma 4.1.** *The diffusion  $X(t) = e^{-t/2}R(e^t - 1)$  is asymptotically homogeneous with homogeneous limit  $X^{(\infty)}$ , i.e, for all continuous  $f$  with compact support in  $(0, \infty)$  and all  $t > 0$ ,*

$$E[f(X(t+s))|X(s) = x] \longrightarrow E_x[f(X^{(\infty)}(t))] \quad \text{as } s \rightarrow \infty$$

uniformly on compact subsets of  $(0, \infty)$ .

*Proof.* It is easier to consider  $\widehat{X}(t) = X(t) - e^{-t/2}$  which takes values in the fixed interval  $(0, \infty)$ , and  $\widehat{X}^{(s)}(t) = \widehat{X}(s+t)$ . Then, the coefficients of the diffusion  $\widehat{X}^{(s)}$  converge to those of  $X^{(\infty)}$ , uniformly on compact subsets of  $(0, \infty)$ , and the corresponding martingale problems have a unique solution. Thus, Theorem 11.1.4 in [26] yields the desired result.  $\square$

The process  $X^{(\infty)}$  is the transform  $X^{(\infty)}(t) = X^{(\infty,2)}(t) = e^{-t/2}\text{BES}^2(e^t - 1)$  of  $\text{BES}^2$  by the rescaling and deterministic time-change (30). It is recurrent and ergodic on  $(0, \infty)$  with the Rayleigh law as invariant probability measure,

$$d\nu(x) = xe^{-x^2/2}\mathbf{1}_{(0,\infty)}(x)dx$$

A first consequence is that  $R$  marginally behaves like  $\text{BES}^2$ .

**Corollary 4.2** (Convergence in law). *Let  $Z \sim \nu$ . As  $t \rightarrow \infty$ ,*

$$\frac{R(t)}{\sqrt{t}} \xrightarrow{\text{law}} Z.$$

*Proof.* Denote by  $P_{s,t}, P_{s,t}^{(\infty)}$  ( $0 \leq s \leq t$ ) the Markov semi-groups associated to  $X$  and  $X^{(\infty)}$ ,

$$(P_{s,t}f)(x) = E[f(X(t))|X(s) = x], \quad (P_{s,t}^{(\infty)}f)(x) = E[f(X^{(\infty)}(t))|X^{(\infty)}(s) = x],$$

so that  $P_{s,t}^{(\infty)} = P_{0,t-s}^{(\infty)}$ . For a bounded continuous  $f : (0, \infty) \rightarrow \mathbb{R}$  we write

$$\begin{aligned} P_{0,t+s}f(x) - \int f d\nu &= P_{0,s}(P_{s,s+t}f)(x) - \int f d\nu \\ &= P_{0,s} \left( P_{s,s+t}f - P_{s,s+t}^{(\infty)}f \right) (x) + P_{0,s} \left( P_{s,s+t}^{(\infty)}f - \int f d\nu \right) (x), \end{aligned}$$

where both terms vanish as  $s, t \rightarrow \infty$ , which is our claim. Indeed, by convergence of  $X^{(\infty)}$  to equilibrium,  $P_{s,s+t}^{(\infty)} f - \int f d\nu = P_{0,t}^{(\infty)} f - \int f d\nu \rightarrow 0$  uniformly on compact subsets of  $(0, \infty)$  as  $t \rightarrow \infty$  and Lemma 4.1 implies that  $P_{s,s+t} f - P_{s,s+t}^{(\infty)} f \rightarrow 0$  uniformly on compact as  $s \rightarrow \infty$ : thus, we only need to prove tightness, i.e. that for all positive  $x$ ,

$$\inf \{ P_{0,s}(\mathbf{1}_{[\varepsilon, 1/\varepsilon]})(x); s \geq 1 \} \rightarrow 1 \quad \text{as } \varepsilon \rightarrow 0. \quad (33)$$

But this follows from the next two bounds

- $R \geq \text{BES}^2$  (see Proposition 3.5 (i)) which implies that  $X \geq X^{(\infty)}$ ,
- $\sup_{s \geq 1} \mathbb{E}X(s)^2 \leq \sup_{s \geq 1} s^{-1} \mathbb{E}R(s)^2 < \infty$  that we explain now.

First recall from [9] that  $\frac{1}{\ln R}$  is a martingale, and so, for all  $r > 1$ ,

$$E_r \left[ \frac{1}{\ln R(t)} \right] = \frac{1}{\ln r}. \quad (34)$$

By Itô's formula,

$$d(R^2) = 2 \left( 1 + \frac{1}{\ln R(t)} \right) dt + 2R(t) dB(t). \quad (35)$$

Thus, for all  $r > 1$ ,

$$E_r[R(t)^2] = r^2 + 2t \left( 1 + \frac{1}{\ln r} \right).$$

We now consider the process starting from  $R(0) = 1$ . Integrating (35), we get

$$\begin{aligned} E_1[(R(t)^2 - r^2) \mathbf{1}_{\tau(r) < t}] &= 2E_1 \left[ \int_0^t \mathbf{1}_{\tau(r) < s} \left( 1 + \frac{1}{\ln R(s)} \right) ds + \int_0^t \mathbf{1}_{\tau(r) < s} R(s) dB(s) \right] \\ &\stackrel{\text{Markov}}{=} 2 \int_0^t E_1 \left[ \mathbf{1}_{\tau(r) < s} E_r \left( 1 + \frac{1}{\ln R(\cdot)} \right)_{\cdot = s - \tau(r)} \right] ds + 0 \\ &= 2 \left( 1 + \frac{1}{\ln r} \right) E_1[(t - \tau(r))^+] \end{aligned}$$

by (34). Finally we obtain that

$$\mathbb{E}R(t)^2 \leq r^2 + 2t \left( 1 + \frac{1}{\ln r} \right)$$

for all  $r > 1$ . The corollary is proved.  $\square$

**Corollary 4.3** (Pointwise ergodic theorem). *For all bounded continuous  $f$  on  $(0, \infty)$ , as  $t \rightarrow \infty$ ,*

$$\frac{1}{t} \int_0^t f(X(s)) ds \longrightarrow \int_{\mathbb{R}} f d\nu \quad \text{a.s.},$$

or, equivalently,

$$\frac{1}{t} \int_0^{e^t - 1} f \left( \frac{R(u)}{\sqrt{1+u}} \right) \frac{1}{1+u} du \longrightarrow \int_{\mathbb{R}} f d\nu \quad \text{a.s.}$$



*Proof.* It is easy to check that, w.l.o.g., we can assume that  $f : (0, \infty) \rightarrow \mathbb{R}$  is non-decreasing. By the comparison principles of Proposition 3.5, we can couple the processes  $R, \text{BES}^2, \text{BES}^{2+\delta}$  ( $\delta > 0$ ) starting at 1 such, a.s., for all  $t \geq \sigma = \sup\{s > 0 : R(s) \leq e^{2/\delta}\} < \infty$ , we have

$$X^{(\infty,2)}(t) \leq X(t) \leq X^{(\infty,2+\delta)}(t) - \frac{\text{BES}^{2+\delta}(\sigma) - e^{2/\delta}}{\sqrt{t}}.$$

By the pointwise ergodic theorem for  $X^{(\infty,2)}$  and  $X^{(\infty,2+\delta)}$  and monotonicity of  $f$ , we derive

$$\int f d\nu \leq \liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t f(X(s)) ds \leq \limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t f(X(s)) ds \leq \int f d\nu_\delta,$$

where  $d\nu_\delta(x) = c_\delta x^{1+\delta/2} e^{-x^2/2} \mathbf{1}_{(0,\infty)}(x) dx$  is the invariant law of  $X^{(\infty,2+\delta)}$ . As  $\delta$  vanishes, the two extreme members coincide, ending the proof of the first statement. The second one follows by changing variables.  $\square$

## 5 Proof of Theorem 2.1

Recall the representation (15) from Corollary 3.3,

$$T_k = T'_1 + A_1'^2 T'_2 + \dots + (A'_1 \dots A'_{k-1})^2 T'_k, \quad A_k = A'_1 \dots A'_k$$

with  $(T'_k, A'_k)_{k \geq 1}$  an i.i.d. sequence with the same law as  $(T_1, A_1)$ .

Fix  $r_\pm$  with  $1 < r_- < r < r^+ < \infty$ . By Cramér's theorem [10], with probability one there exists some finite random  $k_0$  such that for all  $k \geq k_0$

$$r_-^{k/2} \leq A'_1 \dots A'_k = r^{U_1 + \dots + U_k} \leq r_+^{k/2}.$$

In what follows we will use the rough bounds

$$\max_{i=1,\dots,k} T'_i \leq T_k \leq T_{k_0} + (k - k_0) \max_{i=1,\dots,k} r_+^{i-1} T'_i. \quad (36)$$

**Lemma 5.1.** *There exists a constant  $c$  such that for all sequence  $(\delta(k))_k$  tending to 0, we have*

$$\mathbb{P}\left[k \max_{i=1,\dots,k} r_+^{i-1} T'_i \geq e^{k/\delta(k)}\right] \leq c\delta(k)$$

*eventually.*

*Proof.* Fix  $a$  with  $1 < a < e$ . Letting  $v_k = a^{\frac{k}{\delta(k)}}$  and  $t_k = kr_+^k v_k$ , we note that  $e^{\frac{k}{\delta(k)}} \geq t_k$  eventually since  $\delta$  vanishes, and we have by independence

$$\mathbb{P}[k \max_{i=1,\dots,k} r_+^{i-1} T'_i < t_k] = \prod_{i=1,\dots,k} \mathbb{P}[T'_i < r_+^{k-i+1} v_k]$$

From Proposition 3.6 there exists  $c_1 > 0$  such that for all  $t > 1$

$$\mathbb{P}(T_1 \geq t) \leq \frac{c_1}{\ln t}$$

and since  $v_k \rightarrow \infty$  as  $k \rightarrow \infty$ , we have for all large enough  $k$ ,

$$\begin{aligned}
\mathbb{P}[k \max_{i=1, \dots, k} r_+^{i-1} T'_i < t_k] &\geq \prod_{i=1}^k \left( 1 - \frac{c_1}{\ln(r_+^{k-i+1} v_k)} \right) \\
&= \prod_{i=1}^k \left( 1 - \frac{c_1}{\ln(r_+^i v_k)} \right) \\
&\geq \exp \left( -2c_1 \sum_{i=1}^k \frac{1}{i \ln r_+ + \ln v_k} \right) \\
&\geq \exp \left( -\frac{2c_1}{\ln r_+} \ln \left( \frac{k \ln r_+ + \ln v_k}{\ln v_k} \right) \right) \\
&= \exp \left( -\frac{2c_1}{\ln r_+} \ln \left( 1 + \frac{\ln r_+}{\ln a} \delta(k) \right) \right) \\
&\geq 1 - c\delta(k)
\end{aligned}$$

with  $c = 2c_1/\ln a$  for all large  $k$ , since  $\delta$  vanishes at  $\infty$ . This ends the proof.  $\square$

*Proof. Theorem 2.1, claim (8).* Let

$$\delta(t) = g(\ln t), \quad \kappa(i) = 2^i, i \geq 1, \quad \mathcal{K} = \{\kappa(i) : i \geq 1\}.$$

Define, for  $x \geq 2$ ,  $\lfloor x \rfloor_{\mathcal{K}} = \max\{k \in \mathcal{K} : k \leq x\} = 2^{\lfloor (\ln x)/(\ln 2) \rfloor}$ . Note that

$$x \geq \lfloor x \rfloor_{\mathcal{K}} \geq x/2. \quad (37)$$

First, since  $g$  is non-increasing,

$$\begin{aligned}
\sum_{k \in \mathcal{K}} \delta(k) &= \sum_{i \geq 1} \delta(k(i)) \\
&= \sum_{i \geq 1} g(\ln k(i)) \\
&= \sum_{i \geq 1} g(i \ln 2) \\
&\leq \frac{1}{\ln 2} \sum_{i \geq 1} \int_{(i-1) \ln 2}^{i \ln 2} g(t) dt \\
&= \frac{1}{\ln 2} \int_0^{\infty} g(t) dt < \infty
\end{aligned} \quad (38)$$

Fix a constant  $c_2 > 0$  to be chosen later and  $c_3 = c_2^{-1}$ . Combining Borel-Cantelli's lemma and Lemma 5.1, we have a.s.

$$k \max_{i=1, \dots, k} r_+^{i-1} T'_i < e^{c_2 k / \delta(k)} \quad \text{for all } k \in \mathcal{K} \text{ large enough,}$$

and, in addition to (36), we have for large  $k \in \mathcal{K}$ ,

$$T_k \leq T_{k_0} + \frac{k - k_0}{k} e^{c_2 k / \delta(k)} \leq e^{c_2 k / \delta(k)} \quad (39)$$

since  $g$  is non-increasing. By integrability,  $g$  is vanishing at infinity, so the function

$$f(t) = c_3(\ln t) g(\ln_2 t)$$

is such that  $f(t) \leq \ln t$  eventually, and also  $g(\ln_2 t) \leq g(\ln f(t))$  by monotonicity. Thus, for large  $k$  and  $t$ 's,

$$k \leq c_3(\ln t)\delta(\ln t) = f(t) \quad \text{implies that} \quad (40)$$

$$\frac{k}{\delta(k)} = \frac{k}{g(\ln k)} \leq \frac{f(t)}{g(\ln f(t))} = \frac{c_3(\ln t)g(\ln_2 t)}{g(\ln f(t))} \leq c_3 \ln t.$$

Now, define random integers  $k(t) = \max\{k \in \mathcal{K}; T_k \leq t\}$ , and note from (39) that a.s., for large  $t$  we have  $k(t) \geq \max\{k \in \mathcal{K}; e^{c_2 \frac{k}{\delta(k)}} \leq t\}$ . Then, a.s., for all large enough  $t$ ,

$$\begin{aligned} M_t &\geq M_{T_{k(t)}} = A_{k(t)} \geq r_-^{\frac{k(t)}{2}} \geq r_-^{\frac{1}{2} \max\{k \in \mathcal{K}; e^{c_2 \frac{k}{\delta(k)}} \leq t\}} \\ &= r_-^{\frac{1}{2} \max\{k \in \mathcal{K}; \frac{k}{\delta(k)} \leq c_3 \ln t\}} \quad (\text{using } c_3 = c_2^{-1}) \\ &\geq r_-^{\frac{1}{2} \max\{k \in \mathcal{K}; k \leq f(t)\}} \quad (\text{by (40)}) \\ &= r_-^{\frac{1}{2} \lfloor c_3(\ln t)\delta(\ln t) \rfloor_{\mathcal{K}}} \\ &\geq r_-^{\frac{c_3}{4}(\ln t)\delta(\ln t)} \quad (\text{by (37)}) \end{aligned}$$

Taking  $c_3 = c_2^{-1} > 4/\ln r_-$ , we conclude that a.s.,  $M(t) \geq e^{(\ln t)g(\ln_2 t)}$  eventually, ending the proof of (8).  $\square$

We now turn to the proof of claim (9) of Theorem 2.1. We start with a lemma:

**Lemma 5.2.** *Let  $(n_k)_{k \geq 0}$  be a non-decreasing sequence of integers and  $(t_k)_{k \geq 0}$  be a sequence with  $t_k > 1$ . Then,*

$$\sum_{k \geq 0} \frac{n_{k+1} - n_k}{\ln t_{k+1}} = \infty \quad \implies \quad \text{a.s., } T_{n_k} \geq t_k \text{ infinitely often.}$$

*Proof.* The events  $E_k = \{\max_{i=n_k+1, \dots, n_{k+1}} T'_i \geq t_{k+1}\}, k \geq 0$  are independent with  $E_k \subset \{T_{n_{k+1}} \geq t_{k+1}\}$ . Hence the conclusion holds as soon as these events occurs infinitely often a.s. By the second Borel-Cantelli lemma, it suffices to show that the assumption implies  $\sum_{k \geq 0} \mathbb{P}(E_k) = \infty$ . We use Proposition 3.6 and independence. The case when  $t_k$  does not tend to infinity is easily considered, so we assume from now on that  $k$  is large enough so that  $\mathbb{P}(T \geq t_{k+1}) \geq c/\ln t_{k+1}$  for some fixed constant  $c \in (0, \ln r)$ . Then, we can bound

$$\begin{aligned} \mathbb{P}(E_k) &= 1 - \mathbb{P}(T \leq t_{k+1})^{n_{k+1} - n_k} \\ &\geq 1 - \left(1 - \frac{c}{\ln t_{k+1}}\right)^{n_{k+1} - n_k} \\ &\geq 1 - \exp\left(-\frac{c(n_{k+1} - n_k)}{\ln t_{k+1}}\right) \end{aligned}$$

which is the general term of a divergent series.  $\square$

*Proof.* *Theorem 2.1, claim (9).* Let us consider

$$t_k = e^{e^k}, \quad n_k = \lfloor f(t_k) \rfloor, \quad f(t) = c_3(\ln t)g(\ln_2 t)$$

with  $c_3 > 0$  to be fixed later. Note that  $f$  is non-decreasing by assumption. We have

$$\begin{aligned} \sum_{k \geq 0} \frac{n_{k+1} - n_k}{\ln t_{k+1}} &= \sum_{k \geq 0} \frac{\lfloor f(t_{k+1}) \rfloor - \lfloor f(t_k) \rfloor}{\ln(t_{k+1})} = \sum_{k \geq 0} \frac{f(t_{k+1}) - f(t_k)}{\ln(t_{k+1})} + c_4 \\ &= c_3 \sum_{k \geq 0} g(k+1) - \frac{1}{e}g(k) + c_4 \end{aligned}$$

with a constant  $c_4$  which is finite since  $t_k$  is increasing fast and the truncation error is bounded. As in (38),  $\sum_{k \geq 0} g(k) \geq \int_0^\infty g(t)dt = \infty$ , and

$$\sum_{k=0}^n g(k+1) - \frac{1}{e}g(k) = g(n+1) - \frac{1}{e}g(0) + \left(1 - \frac{1}{e}\right) \sum_{k=1}^n g(k).$$

Therefore  $\sum_{k \geq 0} \frac{n_{k+1} - n_k}{\ln t_{k+1}} = \infty$ . From Lemma 5.2 we obtain that a.s.,  $T_{n_k} \geq t_k$  i.o., which shows that

$$M_{t_k} \leq M_{T_{n_k}} = A_{n_k} \leq r_+^{n_k} \leq r_+^{f(t_k)}.$$

Taking  $c_3 < 1/\ln r_+$ , we obtain the desired claim.  $\square$

## 6 Proof of Theorem 2.2

We study the sequence

$$S_n = \frac{T_n}{A_n^2} = \sum_{i=1}^n \frac{T_i A_{i-1}^2}{A_n^2} = \sum_{i=1}^n \frac{T_i}{r^{2(U_i + \dots + U_n)}},$$

which can be written in the form

$$S_m = \frac{S_n}{r^{2(U_{n+1} + \dots + U_m)}} + S_{n+1}^m, \tag{41}$$

where, for  $1 \leq n < m$ ,

$$S_{n+1}^m = \sum_{i=n+1}^m \frac{T_i}{r^{2(U_i + \dots + U_m)}}.$$

The point is that, in (41),  $S_n$  and  $S_{n+1}^m$  are independent, with  $S_{n+1}^m$  equal to  $S_{m-n}$  in law.

We study the convergence/divergence of the series  $\sum_{n \geq 1} \mathbb{P}[S_n \leq t_n]$ , with  $t_n$  of the form

$$t_n = \frac{\beta}{\ln_2 n} \wedge 1 \tag{42}$$

for some  $\beta > 0$ .

## 6.1 Proof of (10).

Let  $(i^{(n)})_{i \geq 1}$  be a sequence of integers such that  $1 \leq i^{(n)} \leq n$  and  $(c_i^{(n)})_{i=i^{(n)}+1, \dots, n, n \geq 1}$  be a doubly-indexed sequence of real parameters with  $c_i^{(n)} > 1$ , to be fixed later on.

### Upper bound:

From (41) we have

$$\begin{aligned}
\mathbb{P}[S_n \leq t_n] &\leq \mathbb{P}\left[\frac{T'_1}{r^{2(U_1 + \dots + U_n)}} \leq t_n, S_2^n \leq t_n\right] \\
&\leq \mathbb{P}\left[\frac{T'_1}{r^{2(U_1 + \dots + U_n)}} \leq t_n, S_2^n \leq t_n, 2(U_1 + \dots + U_n) \leq c_n^{(n)} \cdot n\right] \\
&\quad + \mathbb{P}[2(U_1 + \dots + U_n) > c_n^{(n)} \cdot n] \\
&\leq \mathbb{P}[T'_1 \leq t_n r^{c_n^{(n)} \cdot n}, S_2^n \leq t_n] + \mathbb{P}[2(U_1 + \dots + U_n) > c_n^{(n)} \cdot n] \\
&\leq \mathbb{P}[T \leq t_n r^{c_n^{(n)} \cdot n}] \times \mathbb{P}[S_{n-1} \leq t_n] + \mathbb{P}[2(U_1 + \dots + U_n) > c_n^{(n)} \cdot n].
\end{aligned}$$

Iterating the estimate,

$$\mathbb{P}[S_{n-1} \leq t_n] \leq \mathbb{P}[T \leq t_n r^{c_{n-1}^{(n)} \cdot (n-1)}] \times \mathbb{P}[S_{n-2} \leq t_n] + \mathbb{P}[2(U_1 + \dots + U_{n-1}) > c_{n-1}^{(n)} \cdot (n-1)],$$

and so on down to  $i^{(n)} + 1$ , we obtain

$$\begin{aligned}
\mathbb{P}[S_n \leq t_n] &\leq \left( \prod_{i=i^{(n)}+1}^n \mathbb{P}[T \leq t_n r^{c_i^{(n)} \cdot i}] \right) \times \mathbb{P}[S_{i^{(n)}} \leq t_n] \\
&\quad + \sum_{i=i^{(n)}+1}^n \left( \prod_{j=i+1}^n \mathbb{P}[T \leq t_n r^{c_j^{(n)} \cdot j}] \right) \times \mathbb{P}[2(U_1 + \dots + U_i) > c_i^{(n)} \cdot i].
\end{aligned} \tag{43}$$

### Choice of $i^{(n)}$ and the $c_i^{(n)}$

Let  $i^{(n)} = \lfloor \ln_2 n \rfloor$  and for  $i^{(n)} + 1 \leq i \leq n$ ,

$$c_i^{(n)} = 1 + \sqrt{\frac{8}{i} (\ln i + \ln_2 n)}. \tag{44}$$

By (42), we have for  $i^{(n)} + 1 \leq i \leq n$  and large  $n$ ,

$$\begin{aligned}
\ln \mathbb{P}[T \leq t_n r^{c_i^{(n)} \cdot i}] &\leq \ln \mathbb{P}[T \leq r^{c_i^{(n)} \cdot i}] \\
&\leq -\mathbb{P}[T \geq r^{c_i^{(n)} \cdot i}] \\
&\leq -\frac{1}{c_i^{(n)} \cdot i} + \varepsilon_{n,i,1} \quad (\text{by (19)})
\end{aligned}$$

$$\leq -\frac{1}{i} + \varepsilon_{n,i,2} \quad (\text{by (44)}),$$

with error terms

$$\varepsilon_{n,i,1} = \frac{\ln_2 \left( c_i^{(n)} \cdot i \ln r \right) + C}{\left( c_i^{(n)} \cdot i \right)^2 \ln r}, \quad \varepsilon_{n,i,2} = \varepsilon_{n,i,1} + \sqrt{\frac{8}{i^3} (\ln i + \ln_2 n)}.$$

One can check that  $\sup_n \sum_{i=i^{(n)}+1}^n \varepsilon_{n,i,2} < \infty$ , so for some positive constant  $D$ , for  $n$  large and  $i^{(n)} \leq i \leq n$ ,

$$\begin{aligned} \prod_{j=i+1}^n \mathbb{P}[T \leq t_n r^{c_j^{(n)} \cdot j}] &\leq \exp \left( - \sum_{j=i+1}^n \frac{1}{j} + \sum_{j=i+1}^n \varepsilon_{n,j,2} \right) \\ &\leq D \exp \left( - \ln \left( \frac{n}{i} \right) \right) \\ &\leq D \frac{i}{n}. \end{aligned} \quad (45)$$

Combining this with (28), we get for  $n$  large and  $i^{(n)} + 1 \leq i \leq n$ ,

$$\begin{aligned} \left( \prod_{j=i+1}^n \mathbb{P}[T \leq t_n r^{c_j^{(n)} \cdot j}] \right) \times \mathbb{P}[2(U_1 + \dots + U_i) > c_i^{(n)} \cdot i] &\leq D \frac{i}{n} \exp(-4(\ln i + \ln_2 n)) \\ &= \frac{D}{i^3 n (\ln n)^4}. \end{aligned}$$

Thus, the series  $\sum a_n$ , with

$$a_n = \sum_{i=i^{(n)}+1}^n \left( \prod_{j=i+1}^n \mathbb{P}[T \leq t_n r^{c_j^{(n)} \cdot j}] \right) \times \mathbb{P}[2(U_1 + \dots + U_i) > c_i^{(n)} \cdot i],$$

is convergent.

### Choice of $t_n$

To conclude, we need to take care of the first term in the right-hand side of (43). Recall  $t_n$  from (42) (we will assume  $n$  large so that  $\ln_2 n \geq \beta$ ), and fix an integer  $i_1 \geq 1$ . For  $1 \leq i \leq i_1$ , applying (23) we get as  $n \rightarrow \infty$ , for any  $\epsilon \in (0, r - 1)$ ,

$$\mathbb{P}[T \leq t_n r^{2i}] \leq \exp \left( - \frac{(r - 1 - \epsilon)^2}{2\beta r^{2i}} \ln_2 n \right),$$

and then, for  $n$  large,

$$\begin{aligned} \mathbb{P}[S_{i^{(n)}} \leq t_n] &\leq \mathbb{P}[T'_i \leq t_n r^{2i}, i = 1, \dots, i_1] \\ &= \prod_{i=1}^{i_1} \mathbb{P}[T \leq t_n r^{2i}] \end{aligned}$$

$$\begin{aligned}
&\leq \exp\left(-\sum_{i=1}^{i_1} \frac{(r-1-\epsilon)^2}{2\beta r^{2i}} \ln_2 n\right) \\
&\leq \exp\left(-\frac{(r-1-\epsilon)^2}{2\beta} \frac{1}{r^2} \frac{1 - \left(\frac{1}{r^2}\right)^{i_1}}{1 - \frac{1}{r^2}} \ln_2 n\right) \\
&\leq (\ln n)^{-\frac{(r-1-\epsilon)^2}{2\beta(r^2-1)} \left(1 - \left(\frac{1}{r^2}\right)^{i_1}\right)}.
\end{aligned}$$

Using (45) we will bound

$$\left(\prod_{i=i^{(n)+1}}^n \mathbb{P}[T \leq t_n r^{c_i^{(n)} \cdot i}]\right) \times \mathbb{P}[S_{i^{(n)}} \leq t_n] \leq D \frac{i^{(n)}}{n} (\ln n)^{-\frac{(r-1-\epsilon)^2}{2\beta(r^2-1)} \left(1 - \left(\frac{1}{r^2}\right)^{i_1}\right)},$$

where  $i^{(n)} = \lfloor \ln_2 n \rfloor$ . As soon as  $\beta < \frac{(r-1)}{2(r+1)}$ , there exists some integer  $i_1$  and some  $\epsilon \in (0, r-1)$  such that

$$\frac{(r-1-\epsilon)^2}{2\beta(r^2-1)} \left(1 - \left(\frac{1}{r^2}\right)^{i_1}\right) > 1,$$

and combining (43) with  $\sum_n a_n < \infty$ , we obtain  $\sum \mathbb{P}(S_n \leq t_n) < \infty$ . i.e.,

$$\sum_{n \geq 1} \mathbb{P}[T_n \leq A_n^2 t_n] < \infty.$$

## Conclusion

Let  $\beta < \frac{(r-1)}{2(r+1)}$ . It follows from Borel-Cantelli's lemma that a.s., eventually

$$T_n \geq \frac{\beta A_n^2}{\ln_2 n}.$$

Now, for  $T_n \leq t \leq T_{n+1}$ , if  $n$  is large enough,

$$M_t \leq M_{T_{n+1}} = A_{n+1} \leq r A_n \leq r \sqrt{\beta^{-1} T_n \ln_2 n} \leq r \sqrt{\beta^{-1} t \ln_2 n},$$

and since we have  $T_n \geq \frac{\beta A_n^2}{\ln_2 n} \geq r_-^{\frac{n}{2}}$  for  $n$  large enough, we have  $t \geq r_-^{\frac{n}{2}}$ , and  $n \leq \frac{2 \ln t}{\ln r_-}$ . Finally,

$$M_t \leq r \sqrt{\beta^{-1} t \ln_2 \left(\frac{2 \ln t}{\ln r_-}\right)}.$$

Hence, we have proved (10) with any  $K > r \sqrt{\frac{2(r+1)}{(r-1)}}$ . Optimizing on  $r > 1$ , we get the result for all  $K > \sqrt{11 + 5\sqrt{5}} \approx 4.7096$ .

## 6.2 Proof of (11)

We start by proving that it suffices to show divergence of the series introduced above (42):

**Lemma 6.1.** *Let  $\beta_0 = \inf\{\beta > 0 : \sum_n \mathbb{P}(S_n \leq \frac{\beta}{\ln_2 n}) = \infty\}$ . Then*

$$\liminf_n S_n \ln_2 n = \beta_0 \quad a.s.$$

*Proof.* For all  $\beta < \beta_0$ , we have  $\sum_n \mathbb{P}(S_n \leq \frac{\beta}{\ln_2 n}) < \infty$  and the first Borel-Cantelli's lemma shows that  $\liminf_n S_n \ln_2 n \geq \beta_0$ . To prove the reverse inequality we proceed by steps:

- *First step:* For any non-increasing sequence  $(t_n)_n$ ,

$$\sum_{n \geq 1} \mathbb{P}[S_n \leq t_n] = \infty \implies \mathbb{P}(S_n \leq t_n \text{ i.o.}) \geq \frac{1}{4}.$$

Indeed, for  $1 \leq n \leq m$ ,

$$\begin{aligned} \mathbb{P}[S_n \leq t_n, S_m \leq t_m] &\leq \mathbb{P}[S_n \leq t_n, S_{n+1}^m \leq t_m] \\ &= \mathbb{P}[S_n \leq t_n] \times \mathbb{P}[S_{n+1}^m \leq t_m] \\ &= \mathbb{P}[S_n \leq t_n] \times \mathbb{P}[S_{m-n} \leq t_m] \\ &\leq \mathbb{P}[S_n \leq t_n] \times \mathbb{P}[S_{m-n} \leq t_{m-n}], \end{aligned}$$

since  $t_m \leq t_{m-n}$ . Now, for  $k \geq 1$ ,

$$\begin{aligned} \sum_{1 \leq n < m \leq k} \mathbb{P}[S_n \leq t_n, S_m \leq t_m] &\leq \sum_{1 \leq n < m \leq k} \mathbb{P}[S_n \leq t_n] \times \mathbb{P}[S_{m-n} \leq t_{m-n}] \\ &\leq \sum_{1 \leq n, m \leq k} \mathbb{P}[S_n \leq t_n] \times \mathbb{P}[S_m \leq t_m]. \end{aligned}$$

For all  $k$  large enough we have  $\sum_{n=1}^k \mathbb{P}[S_n \leq t_n] \geq 2$ , and then for all  $1 \leq n \leq k$ ,

$$\sum_{1 \leq m \leq k, m \neq n} \mathbb{P}[S_m \leq t_m] \geq 2 - \mathbb{P}[S_n \leq t_n] \geq \mathbb{P}[S_n \leq t_n].$$

Therefore,

$$\begin{aligned} \sum_{1 \leq n, m \leq k} \mathbb{P}[S_n \leq t_n] \times \mathbb{P}[S_m \leq t_m] &\leq 2 \sum_{1 \leq n, m \leq k, n \neq m} \mathbb{P}[S_n \leq t_n] \times \mathbb{P}[S_m \leq t_m] \\ &= 4 \sum_{1 \leq n < m \leq k} \mathbb{P}[S_n \leq t_n] \times \mathbb{P}[S_m \leq t_m] \end{aligned}$$

Cochen-Stone's theorem [19] – a variant of Borel-Cantelli's lemma – yields

$$\mathbb{P}[S_n \leq t_n \text{ i.o.}] \geq \limsup_{k \geq 1} \frac{\sum_{1 \leq n < m \leq k} \mathbb{P}[S_n \leq t_n] \times \mathbb{P}[S_m \leq t_m]}{\sum_{1 \leq n < m \leq k} \mathbb{P}[S_n \leq t_n, S_m \leq t_m]} \geq \frac{1}{4},$$



which concludes this step.

- *Second step:* Let's introduce the  $\sigma$ -fields

$$\mathcal{A}_k = \sigma((A'_n, T'_n); n \geq k), k = 1, 2, \dots, \quad \mathcal{T} = \bigcap_{k \geq 1} \mathcal{A}_k.$$

By Kolmogorov 0–1 law and independence of the sequence  $((A'_n, T'_n); n \geq 1)$ , every element  $A$  of the tail field  $\mathcal{T}$  has  $\mathbb{P}(A) \in \{0, 1\}$ . Fix  $\beta \geq 0$  and introduce the events

$$E = \{\liminf_n S_n \ln_2 n \leq \beta\}, \quad E_k = \{\liminf_n S_{k+1}^{n+k} \ln_2 n \leq \beta\},$$

and

$$\Omega_0 = \left\{ \lim_{n \rightarrow \infty} \frac{\ln_2 n}{r^{2(U_1 + \dots + U_n)}} = 0 \right\}.$$

Note that  $E = E_0$  and that  $\mathbb{P}(\Omega_0) = 1$ . Since, by definition,

$$S_{k+1}^{n+k+1} = \frac{T'_{k+1}}{r^{2(U_{k+1} + \dots + U_{n+k+1})}} + S_{k+2}^{n+k+1},$$

we see that the two sets  $E_k$  and  $E_{k+1}$  coincide on  $\Omega_0$ , for all  $k \geq 0$ . Denoting the common intersection by

$$\widehat{E} = E \bigcap \Omega_0 = E_k \bigcap \Omega_0,$$

we see that  $\widehat{E}$  belongs to  $\mathcal{T}$  and then has probability equal to 0 or 1. The similar 0–1 law holds for  $E$  which is equal to  $\widehat{E}$  up to a negligible set.

- *Final step:* For any  $\beta > \beta_0$ , the series  $\sum_n \mathbb{P}(S_n \leq t_n)$  with  $t_n = \beta / \ln_2 n$  is diverging. By the first step, the probability  $\mathbb{P}[S_n \leq t_n \text{ i.o.}] \geq 1/4$ , and by the second one is equal to 1. Thus  $\liminf_n S_n \ln_2 n \leq \beta$  a.s., for all such  $\beta$ 's. The lemma is proved.  $\square$

**Remark 6.2.** *We have followed the approach of the renewal structure to get the 0–1 law, with the advantage to keep the paper self-contained. A tempting alternative would be to show that the tail  $\sigma$ -field of  $R$  is trivial; we mention the illuminating survey [24] on the tail  $\sigma$ -field of a diffusion.*

Anticipating on the proof of (11) we now give a short proof of Theorem 1.2.

*Proof.* It is not difficult to check the criteria of [11] or [25] for triviality of the tail  $\sigma$ -field of one-dimensional diffusion (see Theorem 3 in [24]). Then,  $K^* = \limsup_{t \rightarrow \infty} \frac{M(t)}{\sqrt{t \ln_3 t}}$  is a.s. constant. Then, (10) and (11) show that  $K^*$  is positive and finite.  $\square$

To continue the proof of (11) we need an intermediate result.

**Lemma 6.3.** *For all  $\alpha_0 > 0$ , there exists  $\beta > 0$  such that, for all  $n$  large enough,*

$$\mathbb{P}[S_{\lfloor \alpha_0 \ln_2 n \rfloor} \leq \frac{\beta}{\ln_2 n}] \geq \frac{1}{\ln n}.$$

*Proof.* Clearly, it suffices to prove that for  $v > 0$ , there exists  $u > 0$  such that, for all large  $n$  we have,

$$\mathbb{P}[S_n \leq \frac{u}{n}] \geq \frac{1}{e^{vn}}. \quad (46)$$

Indeed, substituting  $v, n$  in (46) by  $\alpha_0^{-1}, \lfloor \alpha_0 \ln_2 n \rfloor$  shows that any  $\beta > u/\alpha_0$  fulfills the statement of the lemma.

To show (46), we fix some  $b \in (0, 1)$  ( $b$  will be chosen small later on), and we note that:

$$U_i \geq b \text{ and } T'_i \leq \frac{u}{n}(r^b - 1)r^{b(n-i+1)} \text{ for all } i = 1, \dots, n$$

imply that

$$S_n = \sum_{i=1}^n \frac{T'_i}{r^{2(U_i + \dots + U_n)}} \leq \sum_{i=1}^n \frac{\frac{u}{n}(r^b - 1)r^{b(n-i+1)}}{r^{2b(n-i+1)}} \leq \frac{u}{n}.$$

Then,

$$\begin{aligned} \mathbb{P}[S_n \leq \frac{u}{n}] &\geq \prod_{i=1}^n \mathbb{P}[U_i \geq b, T'_i \leq \frac{u}{n}(r^b - 1)r^{b(n-i+1)}] \\ &= (1-b)^n \prod_{i=1}^n \mathbb{P}[T'_i \leq \frac{u}{n}(r^b - 1)r^{b(n-i+1)} | U_i \geq b] \\ &= (1-b)^n \prod_{i=1}^n \mathbb{P}[T \leq \frac{u}{n}(r^b - 1)r^{bi} | U \geq b] \end{aligned} \quad (47)$$

By Proposition 3.7, we can find  $t_0 > 0$  and  $\rho > 0$  such that, for  $t \leq t_0$ ,

$$\mathbb{P}[T \leq t | U \geq b] \geq \exp(-\frac{\rho}{t}).$$

Now, we fix some  $t_1 > t_0$ , we will bound the factors in (47) as follows:

For  $\frac{\ln(t_1 \frac{n}{u(r^b-1)})}{b \ln r} \leq i \leq n$  :

$$\mathbb{P}[T \leq \frac{u}{n}(r^b - 1)r^{bi} | U \geq b] \geq \mathbb{P}[T \leq t_1 | U \geq b],$$

for  $\frac{\ln(t_0 \frac{n}{u(r^b-1)})}{b \ln r} \leq i \leq \frac{\ln(t_1 \frac{n}{u(r^b-1)})}{b \ln r}$  :

$$\mathbb{P}[T \leq \frac{u}{n}(r^b - 1)r^{bi} | U \geq b] \geq \mathbb{P}[T \leq t_0 | U \geq b],$$

and for  $1 \leq i \leq \frac{\ln(t_0 \frac{n}{u(r^b-1)})}{b \ln r}$  :

$$\mathbb{P}[T \leq \frac{u}{n}(r^b - 1)r^{bi} | U \geq b] \geq \exp\left(-\rho \frac{n}{u(r^b - 1)} \frac{1}{r^{bi}}\right).$$

With this choice, the estimate (47) becomes

$$\mathbb{P}[S_n \leq \frac{u}{n}] \geq (1-b)^n \times \mathbb{P}[T \leq t_1 | U \geq b]^n \times \mathbb{P}[T \leq t_0 | U \geq b]^{\frac{\ln(\frac{t_1}{t_0})}{b \ln r} + 1}$$

$$\begin{aligned}
& \times \prod_{i=1}^{\lfloor \frac{\ln(t_0 \frac{n}{u(r^b-1)})}{b \ln r} \rfloor} \exp\left(-\rho \frac{n}{u(r^b-1) r^{bi}}\right) \\
& \geq (1-b)^n \times \mathbb{P}[T \leq t_1 | U \geq b]^n \times \mathbb{P}[T \leq t_0 | U \geq b]^{\frac{\ln(\frac{t_1}{t_0})}{b \ln r} + 1} \\
& \times \exp\left(-\rho \frac{n}{u(r^b-1)^2}\right)
\end{aligned}$$

From this we derive the claim (46) by taking  $b$  small,  $u$  and  $t_1$  large. This ends the proof of the lemma.  $\square$

*Proof. Theorem 2.2, claim (11).* Similarly to the proof of (10), we let  $t_n = \frac{\beta}{\ln_2 n} \wedge 1$ ,  $(i^{(n)})_{n \geq 1}$  be a sequence of integers, and  $(b_i^{(n)})_{i=i^{(n)+1}, \dots, n, n \geq 1}$  be a doubly-indexed sequence with  $0 < b_i^{(n)} < 1$ , given by

$$b_i^{(n)} = 1 - \sqrt{\frac{8}{i} (\ln i + \ln_2 n)}, \quad \text{for } i^{(n)} + 1 \leq i \leq n, \quad i^{(n)} = \lfloor \alpha_0 \ln_2 n \rfloor$$

with  $\alpha_0$  large (take  $\alpha_0 > 8$  so that  $b_i^{(n)} > 0$  for  $n$  large).

This time, we need an extra doubly-indexed, positive sequence  $(s_i^{(n)})_{i=i^{(n)+1}, \dots, n, n \geq 1}$  such that for  $n$  large

$$\sum_{i=i^{(n)+1}^n s_i^{(n)} \leq t_n.$$

(Note that this implies  $s_i^{(n)} \leq 1$ .) Similarly, using (41) we estimate

$$\begin{aligned}
\mathbb{P}[S_n \leq t_n] & \geq \mathbb{P}\left[\frac{T'_1}{r^{2(U_1 + \dots + U_n)}} \leq s_n^{(n)}, S_2^n \leq t_n - s_n^{(n)}\right] \\
& \geq \mathbb{P}\left[\frac{T'_1}{r^{2(U_1 + \dots + U_n)}} \leq s_n^{(n)}, S_2^n \leq t_n - s_n^{(n)}, 2(U_1 + \dots + U_n) \geq b_n^{(n)} \cdot n\right] \\
& \geq \mathbb{P}\left[T'_1 \leq s_n^{(n)} r^{b_n^{(n)} \cdot n}, S_2^n \leq t_n - s_n^{(n)}, 2(U_1 + \dots + U_n) \geq b_n^{(n)} \cdot n\right] \\
& \geq \mathbb{P}\left[T'_1 \leq s_n^{(n)} r^{b_n^{(n)} \cdot n}, S_2^n \leq t_n - s_n^{(n)}\right] - \mathbb{P}[2(U_1 + \dots + U_n) < b_n^{(n)} \cdot n] \\
& \geq \mathbb{P}\left[T \leq s_n^{(n)} r^{b_n^{(n)} \cdot n}\right] \times \mathbb{P}[S_{n-1} \leq t_n - s_n^{(n)}] - \mathbb{P}[2(U_1 + \dots + U_n) < b_n^{(n)} \cdot n].
\end{aligned}$$

We iterate the procedure,

$$\begin{aligned}
\mathbb{P}[S_{n-1} \leq t_n - s_n^{(n)}] & \geq \mathbb{P}\left[T \leq s_{n-1}^{(n)} r^{b_{n-1}^{(n)} \cdot (n-1)}\right] \times \mathbb{P}\left[S_{n-2} \leq t_n - s_n^{(n)} - s_{n-1}^{(n)}\right] \\
& \quad - \mathbb{P}[2(U_1 + \dots + U_{n-1}) < b_{n-1}^{(n)} \cdot (n-1)],
\end{aligned}$$

and so on down to  $i^{(n)}$ . We obtain

$$\mathbb{P}[S_n \leq t_n] \geq \left( \prod_{i=i^{(n)+1}^n \mathbb{P}\left[T \leq s_i^{(n)} r^{b_i^{(n)} \cdot i}\right] \right) \times \mathbb{P}\left[S_{i^{(n)}} \leq t_n - \sum_{i=i^{(n)+1}^n s_i^{(n)}\right]$$

$$- \sum_{i=i^{(n)}+1}^n \left( \prod_{j=i+1}^n \mathbb{P} \left[ T \leq s_j^{(n)} r^{b_j^{(n)} \cdot j} \right] \right) \times \mathbb{P}[2(U_1 + \dots + U_i) < b_i^{(n)} \cdot i]. \quad (48)$$

Using  $s_i^{(n)} \leq 1$  and  $b_i^{(n)} < 1$ , we have, for  $n$  large and  $i^{(n)} + 1 \leq i \leq n$  :

$$\begin{aligned} \prod_{j=i+1}^n \mathbb{P}[T \leq s_j^{(n)} r^{b_j^{(n)} \cdot j}] &\leq \prod_{j=i+1}^n \mathbb{P}[T \leq r^j] \\ &\leq \exp \left( - \sum_{j=i+1}^n \mathbb{P}[T \geq r^j] \right) \\ &\leq \exp \left( - \sum_{j=i+1}^n \left( \frac{1}{j} - \frac{\ln_2(j \ln r) + C}{j^2 \ln r} \right) \right) \quad (\text{by (19)}) \\ &\leq D' \frac{i}{n}, \end{aligned}$$

for some positive constant  $D'$ .

As we did for the series  $\sum_n a_n$ , cf. below (45) except for using (29) instead of (28), we easily see that the series  $\sum_n a'_n$ , with

$$a'_n = \sum_{i=i^{(n)}+1}^n \left( \prod_{j=i+1}^n \mathbb{P}[T \leq s_j^{(n)} r^{b_j^{(n)} \cdot j}] \right) \times \mathbb{P}[2(U_1 + \dots + U_i) < b_i^{(n)} \cdot i],$$

is finite. Now, we choose

$$s_i^{(n)} = \frac{1}{i^3},$$

and we start to bound from below the product

$$\prod_{i=i^{(n)}+1}^n \mathbb{P}[T \leq s_i^{(n)} r^{b_i^{(n)} \cdot i}] = \exp \left( \sum_{i=i^{(n)}+1}^n \ln(1 - \mathbb{P}[T \geq s_i^{(n)} r^{b_i^{(n)} \cdot i}]) \right)$$

Observe that, by taking  $\alpha_0 > 16$ , we have  $b_i^{(n)} \in (1/2, 1)$  for all large  $n$  and  $i \in [i^{(n)} + 1, n]$ , and also that

$$\inf \{ s_i^{(n)} r^{b_i^{(n)} \cdot i}; i^{(n)} \leq i \leq n \} \geq r^{\frac{\alpha_0}{2} \ln_2 n} \quad \text{for large } n, \quad (49)$$

which tends to  $\infty$  as  $n \rightarrow \infty$ . For  $i^{(n)} + 1 \leq i \leq n$  and  $n$  large, in view of (49) we have (using  $-\ln(1-u) \leq u + u^2$  for small  $u > 0$  and  $\frac{1}{1-u} \leq 1 + 2u$  for  $0 < u < \frac{1}{2}$ )

$$\begin{aligned} -\ln(1 - \mathbb{P}[T \geq s_i^{(n)} r^{b_i^{(n)} \cdot i}]) &\leq \mathbb{P}[T \geq s_i^{(n)} r^{b_i^{(n)} \cdot i}] + \varepsilon'_{n,i,1} \\ &\leq \frac{\ln r}{\ln \left( s_i^{(n)} r^{b_i^{(n)} \cdot i} \right)} + \varepsilon'_{n,i,2} \quad (\text{by (19)}) \\ &= \frac{1}{b_i^{(n)} \cdot i + \frac{\ln s_i^{(n)}}{\ln r}} + \varepsilon'_{n,i,2} \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{b_i^{(n)}.i} + \varepsilon'_{n,i,3} \\
&\leq \frac{1}{i} + \varepsilon'_{n,i,4},
\end{aligned}$$

with error terms

$$\begin{aligned}
\varepsilon'_{n,i,1} &= \mathbb{P} \left[ T \geq s_i^{(n)} r^{b_i^{(n)}.i} \right]^2, \quad \varepsilon'_{n,i,2} = \varepsilon'_{n,i,1} + \frac{1}{\ln r} \times \frac{\ln_3 \left( s_i^{(n)} r^{b_i^{(n)}.i} \right) + C}{\left( b_i^{(n)}.i + \frac{\ln s_i^{(n)}}{\ln r} \right)^2}, \\
\varepsilon'_{n,i,3} &= \varepsilon'_{n,i,2} - 2 \frac{\ln s_i^{(n)}}{\left( b_i^{(n)}.i \right)^2 \ln r}, \quad \varepsilon'_{n,i,4} = \varepsilon'_{n,i,3} + 2 \sqrt{\frac{8}{i^3} (\ln i + \ln_2 n)}.
\end{aligned}$$

One can check that  $\sup_n \sum_{i=i^{(n)}+1}^n \varepsilon'_{n,i,4} < \infty$ , so for some positive constant  $D''$ , for large  $n$ ,

$$\begin{aligned}
\prod_{i=i^{(n)}+1}^n \mathbb{P}[T \leq s_i^{(n)} r^{b_i^{(n)}.i}] &\geq \exp \left( - \sum_{i=i^{(n)}+1}^n \left( \frac{1}{i} + \varepsilon'_{n,i,4} \right) \right) \\
&\geq D'' \frac{i^{(n)}}{n}.
\end{aligned} \tag{50}$$

Finally, consider the term

$$\mathbb{P} \left[ S_{i^{(n)}} \leq t_n - \sum_{i=i^{(n)}+1}^n s_i^{(n)} \right].$$

Note that  $t_n - \sum_{i=i^{(n)}+1}^n s_i^{(n)} = \frac{\beta}{\ln_2 n} - \sum_{i=i^{(n)}+1}^n \frac{1}{i^3} \geq \frac{\beta}{\ln_2 n} - \frac{1}{2i^{(n)2}}$ , which implies that for all  $\beta' < \beta$ ,  $t_n - \sum_{i=i^{(n)}+1}^n s_i^{(n)} \geq \frac{\beta'}{\ln_2 n}$  for large  $n$ , and then

$$\mathbb{P} \left[ S_{i^{(n)}} \leq t_n - \sum_{i=i^{(n)}+1}^n s_i^{(n)} \right] \geq \mathbb{P} \left[ S_{i^{(n)}} \leq \frac{\beta'}{\ln_2 n} \right].$$

Now, we are ready to conclude the proof: Fix  $\alpha_0 > 16$ , and let  $\beta'$  be associated to  $\alpha_0$  by Lemma 6.3. Then,

$$\mathbb{P} \left[ S_{i^{(n)}} \leq \frac{\beta'}{\ln_2 n} \right] \geq \frac{1}{\ln n},$$

and for  $t_n = (\beta/\ln_2 n) \wedge 1$  with  $\beta > \beta'$ , using (50),

$$\left( \prod_{i=i^{(n)}+1}^n \mathbb{P}[T \leq s_i^{(n)} r^{b_i^{(n)}.i}] \right) \times \mathbb{P} \left[ S_{i^{(n)}} \leq t_n - \sum_{i=i^{(n)}+1}^n s_i^{(n)} \right] \geq D'' \frac{i^{(n)}}{n} \times \frac{1}{\ln n}.$$

Using now (48) and  $\sum_n a'_n < \infty$  we obtain  $\sum_{n \geq 1} \mathbb{P}[S_n \leq t_n] = \infty$ . By Lemma 6.1 we have a.s.,

$$T_n \leq \frac{\beta A_n^2}{\ln_2 n} \quad i.o.$$

i.e.,  $A_n \geq \sqrt{\beta^{-1} T_n \ln_2 n}$ . Since, for all large  $n$ ,  $\frac{\beta A_n^2}{\ln_2 n} \leq r_+^n$ , we see that  $T_n \leq r_+^n$ , so  $n \geq \frac{\ln T_n}{\ln r_+}$ , and also

$$M_{T_n} = A_n \geq \sqrt{\beta^{-1} T_n \ln_2 \left( \frac{\ln T_n}{\ln r_+} \right)}.$$

Finally, for some (small)  $K' > 0$ , with probability one,  $M_t \geq K' \sqrt{t \ln_3 t}$  i.o. The proof of (11) is complete.  $\square$

**Acknowledgements:** FC is partially supported by ANR SWIWS.

## References

- [1] G. ALSMEYER, D. BURACZEWSKI, A. IKSANOV (2017) Null recurrence and transience of random difference equations in the contractive case. *J. Appl. Probab.* **54**, 1089–1110.
- [2] R. AZENCOTT (1980) *Grandes déviations et applications* École d’été de Probabilités de Saint-Flour 1978, Lect. Notes Math. **774** Springer.
- [3] M. BABILLOT, P. BOUGEROL, L. ELIE (1997) The random difference equation  $X_n = A_n X_{n-1} + B_n$  in the critical case. *Ann. Probab.* **25**, 478–493.
- [4] S. BOUCHERON, G. LUGOSI, P. MASSART (2013) *Concentration inequalities. A nonasymptotic theory of independence*. Oxford University Press, Oxford.
- [5] D. BURACZEWSKI, E. DAMEK, T. MIKOSCH (2016) *Stochastic models with power-law tails. The equation  $X = AX + B$* . Cham: Springer
- [6] A. CHERNY (2000) On the strong and weak solutions of stochastic differential equations governing Bessel processes. *Stochastics Stochastic Rep.* **70**, 213–219
- [7] F. COMETS, S. POPOV, M. VACHKOVSKAIA (2016) Two-dimensional random interacements and late points for random walks. *Commun. Math. Phys.* **343**, 129–164.
- [8] F. COMETS, S. POPOV (2017) The vacant set of two-dimensional critical random interlacement is infinite. *Ann. Probab.* **45**, 4752–4785.
- [9] F. COMETS, S. POPOV (2020) Two-dimensional Brownian random interlacement. *Potential Analysis* **53**, 727–771
- [10] A. DEMBO, O. ZEITOUNI (1998) *Large Deviation Techniques and Applications*, 2nd Ed. Springer Verlag,

- [11] B. FRISTEDT, S. OREY (1978) The tail  $\sigma$ -field of one-dimensional diffusions. *Stochastic analysis* (Proc. Internat. Conf., Northwestern Univ., Evanston, 1978), pp. 127–138, Academic Press, New York-London.
- [12] N. GANTERT, S. POPOV, M. VACHKOVSKAIA (2019) On the range of a two-dimensional conditioned simple random walk. *Ann. H. Lebesgue* **2**, 349–368.
- [13] M. GRADINARU, Y. OFFRET (2013) Existence and asymptotic behaviour of some time-inhomogeneous diffusions. *Ann. Inst. Henri Poincaré Probab. Stat.* **49**, 182–207.
- [14] W. HOEFFDING (1963) Probability inequalities for sums of bounded random variables. *J. Amer. Statist. Assoc.* **58**, 13–30.
- [15] K. ITÔ, H. P. MCKEAN (1974) *Diffusion processes and their sample paths*. (2nd. ed.) Springer-Verlag, Berlin-New York.
- [16] H.G. KELLERER (1992) Ergodic behaviour of affine recursions I: Criteria for recurrence and transience. Techn. report, Univ. München. Available at <http://www.mathematik.uni-muenchen.de/~kellerer/I.pdf>
- [17] D. KHOSHNEVISAN, T. LEWIS, W. LI (1994) On the future infima of some transient processes. *Probab. Theory Related Fields* **99**, 337–360.
- [18] D. KHOSHNEVISAN, T. LEWIS, Z. SHI (1996) On a problem of Erdős and Taylor. *Ann. Probab.* **24** 761–787.
- [19] S. KOCHEN, C. STONE (1964) A note on the Borel-Cantelli lemma. *Illinois J. Math.* **8**, 248–251.
- [20] J.C. PARDO (2006) On the future infimum of positive self-similar Markov processes. *Stochastics* **78**, 123–155
- [21] S. POPOV (2019) Conditioned two-dimensional simple random walk: Green’s function and harmonic measure. *Journal of Theoretical Probability* **34**, 418–437
- [22] S. POPOV (2020) *Two-dimensional Random Walk: From Path Counting to Random Interlacements*. Cambridge University Press.
- [23] S. POPOV, L. ROLLA, D. UNGARETTI, (2020) Transience of conditioned walks on the plane: encounters and speed of escape. *Electron. J. Probab.* **25**, Paper No. 52, 23 pp
- [24] L.C.G. ROGERS (1988) Coupling and the tail  $\sigma$ -field of a one-dimensional diffusion. *Stochastic calculus in application* (Cambridge, 1987), 78–88, *Pitman Res. Notes Math. Ser.* **197**, Longman Sci. Tech., Harlow.
- [25] U. RÖSLER (1979) The tail  $\sigma$ -field of time-homogeneous one-dimensional diffusion processes. *Ann. Probab.* **7**, 847–857
- [26] D. STROOCK, S. VARADHAN (2006) *Multidimensional diffusion processes*. Springer-Verlag, Berlin.

- [27] A.-S. SZNITMAN (1998) *Brownian motion, obstacles and random media*. Springer Monographs in Mathematics. Berlin: Springer. xvi, 353 p.
- [28] A.-S. SZNITMAN (2010) Vacant set of random interacements and percolation. *Ann. Math. (2)*, **171** (3), 2039–2087.
- [29] A.S. SZNITMAN (2013) On scaling limits and Brownian interacements. *Bull. Braz. Math. Soc.* **44**, 555–592
- [30] O. TAKEYAMA (1985) Asymptotic properties of asymptotically homogeneous diffusion processes on a compact manifold. *J. Math. Soc. Japan* **37**, 63–650
- [31] D. WILLIAMS (1974) Path decomposition and continuity of local time for one-dimensional diffusions. I. *Proc. London Math. Soc.* **28** (3), 738–768.
- [32] A. ZEEVI, P. GLYNN (2004) Recurrence Properties of Autoregressive Processes with Super-Heavy-Tailed Innovations. *J. Appl. Probab.* **41** 639–653