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# Hamilton’s principle of stationary action in multiphase flow modeling

Cosmin Burtea\*, Sergey Gavriluk†and Charlotte Perrin‡

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## Abstract

These lecture notes are concerned with the derivation of the fluid mechanics equations via Hamilton’s principle of stationary action. We recall the main conceptual tools of this variational principle which originally applies to classical finite-degrees-of-freedom mechanics and we explain how these tools can be adapted in a continuous framework, in particular for the derivation of the well-known Euler equations describing the motion of inviscid fluids. The core of these notes is the application of Hamilton’s principle to multiphase flows. We present a new Lagrangian point of view for the derivation of two-phase flow equations.

## 1 Introduction

Hamilton’s principle of stationary action is a variational principle that allows one to obtain the equations of motion (Euler-Lagrange equations) for a given mechanical system. The action is the integral over a finite time interval of the corresponding Lagrangian which is the difference between the kinetic and potential energy of the system. The corresponding Euler-Lagrange equations can be seen as ‘Newton’s laws’ governing the mechanical system. In a nutshell, this principle states that the real trajectory of a system between an initial and final configuration in a specified time is found by selecting from all possible trajectories the one for which the first variation of the action vanishes. In some cases, additional constraints (geometrical and physical) may also be imposed for the class of variations admissible variations. The main advantages of the variational point of view are :

- the whole physics is contained in the definition of a scalar function – the Lagrangian of the system.
- due to the Noether theorem, the fulfillment of the basic physical conservation laws is guaranteed by this approach.
- numerical methods and homogenization techniques can be effectively developed for the corresponding Euler-Lagrange equations which is useful for engineering problems.

The main purpose of these lecture notes is to present some recent developments of such a method used in order to derive equations governing multiphase flows.

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**Organization** The present notes are divided into four parts: in the first part dedicated to standard discrete mechanical systems, we present the formalism of Hamilton's principle yielding the well-known Euler-Lagrange equations; the second part of the notes is devoted to the extension of the previous concepts to the continuous (infinite dimensional) systems. This allows us to recover in the third part the classical (compressible and incompressible) Euler equations for a single fluid phase. Finally, the last section presents the application of Hamilton's principle to the derivation of fluid equations describing the dynamics of multiphase flows.

These notes are intended to be an introduction to multiphase flow modelling via variational principles and are addressed to wide audience: graduate/post graduate students, applied mathematicians, physicists etc.

Before introducing Hamilton's principle, let us recall some notations and formulas that we shall use in the rest of the notes.

## Notations and useful formulas

In the computations that follow we will use the Einstein convention of summation over repeated indices.

**Matrices** For any  $k, n \in \mathbb{N}^*$ , we denote by  $\mathcal{M}_{k \times n}(\mathbb{R})$  the space of matrices (or second order tensors) with  $k$  lines and  $n$  columns considered in an appropriate basis. For the discussion below, let us fix a matrix  $A = (a_{ij})_{i,j \in \llbracket 1, n \rrbracket} \in \mathcal{M}_{n \times n}(\mathbb{R})$ . The trace of  $A$  is the sum of the diagonal elements of  $A$  and we denote it by

$$\text{trace } A = a_{ii}.$$

For any  $i, j \in \llbracket 1, n \rrbracket$ , we denote by  $M_{ij}(A)$  the determinant of the  $(n-1) \times (n-1)$ -type matrix obtained by removing the  $i$ -th line and the  $j$ -th column of  $A$ . This quantity is referred to as the  $(i, j)$ -minor of  $A$ . We recall Laplace's formulae

$$\begin{cases} a_{i_0 j} (-1)^{i_0 + j} M_{i_0 j} = a_{i j_0} (-1)^{i + j_0} M_{i j_0} = \det A, \\ a_{i_0 j} (-1)^{i_0 + j} M_{i_1 j} = a_{i j_0} (-1)^{i + j_1} M_{i j_1} = 0. \end{cases} \quad (1.1)$$

which hold for all  $i_0, i_1, j_0, j_1 \in \llbracket 1, n \rrbracket$  with  $i_0 \neq i_1$  and  $j_0 \neq j_1$ .

The adjugate matrix of  $A$ , denoted  $\text{adj}(A) \in \mathcal{M}_{n \times n}(\mathbb{R})$  is given by

$$(\text{adj}(A))_{ij} = (-1)^{i+j} M_{ji}.$$

Recall that any  $A \in \mathcal{M}_{n \times n}(\mathbb{R})$  with  $\det A \neq 0$  is invertible and Laplace's formulas give us

$$A^{-1} = \frac{1}{\det A} \text{adj}(A).$$

**Differential calculus** Let  $n, k \in \mathbb{N}^*$  and  $\Omega$  be a smooth (say  $\mathcal{C}^1$ ) bounded domain of  $\mathbb{R}^n$ . For any  $\phi : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^k$  the differential at  $x \in \Omega$  is the matrix  $D\phi|_x \in \mathcal{M}_{k \times n}(\mathbb{R})$  given by

$$(D\phi|_x)_{ij} = \partial_j \phi^i(x)$$

for all  $i \in \llbracket 1, k \rrbracket$  and  $j \in \llbracket 1, n \rrbracket$ . Note that, whenever it is not ambiguous, we shall remove the  $x$  from the notation of the differential. We denote the gradient by

$$\nabla \phi = (D\phi)^T.$$

Let  $\Omega_0$  be a smooth bounded domain of  $\mathbb{R}^m$ ,  $m \geq 1$ , and  $f : \Omega_0 \rightarrow \Omega$ . Then  $D(\phi \circ f) \in \mathcal{M}_{k \times m}(\mathbb{R})$  and for all  $x \in \Omega_0$  we have the chain rule

$$D(\phi \circ f)(x) = D\phi|_{f(x)} Df|_x. \quad (1.2)$$

From now on,  $m = n$  and we assume that there exists  $X \in \mathcal{C}^1(\Omega_0, \mathbb{R}^n) \cap \mathcal{C}^0(\overline{\Omega}_0; \mathbb{R}^n)$ , which is a bijection between  $\overline{\Omega}_0$  and  $\overline{\Omega}$  such that  $X^{-1} \in \mathcal{C}^1(\Omega; \mathbb{R}^n) \cap \mathcal{C}^0(\overline{\Omega}; \mathbb{R}^n)$ . Of course such an  $X$  is also a homeomorphism between the boundaries of the two domains and obviously

$$\partial\Omega = X(\partial\Omega_0).$$

For any function  $\phi : \Omega \rightarrow \mathbb{R}^k$  we denote by  $\tilde{\phi} : \Omega_0 \rightarrow \mathbb{R}^k$  the function

$$\tilde{\phi} = \phi \circ X. \quad (1.3)$$

Using the chain rule (1.2), we have that

$$DX|_{X(x)}^{-1} DX|_x = I_n \quad \forall x \in \Omega_0,$$

which rewrites using the convention (1.3)

$$(DX)^{-1} = \widetilde{DX}^{-1}. \quad (1.4)$$

Again, owing to (1.2) we have

$$D\tilde{\phi} = \widetilde{D\phi} DX,$$

and consequently

$$\widetilde{D\phi} = D\tilde{\phi} (DX)^{-1}. \quad (1.5)$$

**Remark 1.1** *Anticipating Section 3, the diffeomorphism  $X$  will represent a fluid particle trajectory and the notation  $\tilde{\cdot}$  will correspond to a Lagrangian description of the dynamics, that is a description which is attached to the particle trajectories.*

**Useful formula** In particular, formula (1.5) applied to a vector field  $B : \Omega \rightarrow \mathbb{R}^n$  yields

$$\widetilde{\operatorname{div} B} = \widetilde{\partial_i B^i} = \operatorname{trace}(\widetilde{DB}) = \operatorname{trace}(D\tilde{B} (DX)^{-1}) = \partial_k \tilde{B}^i ((DX)^{-1})_{ki}. \quad (1.6)$$

For a real-valued function  $b : \Omega \rightarrow \mathbb{R}$ , we are interested in obtaining a formula for  $\widetilde{\nabla b}$ . On the one hand, Equation (1.5) holds. On the other hand, fixing  $\det DX > 0$ , we have for any test function  $\psi \in \mathcal{C}_c^1(\Omega; \mathbb{R}^n)$

$$\begin{aligned} \int_{\Omega} \partial_i b(y) \psi^i(y) dy &= - \int_{\Omega} b(y) \partial_i \psi^i(y) dy \\ &= - \int_{\Omega_0} \tilde{b}(x) \widetilde{\operatorname{div} \psi}(x) \det DX(x) dx \\ &= - \int_{\Omega_0} \tilde{b}(x) \partial_k \tilde{\psi}^i(x) ((DX)^{-1})_{ki} \det DX(x) dx \\ &= \int_{\Omega_0} \partial_k \left( ((DX)^{-1})_{ki} \tilde{b} \det DX \right) (x) \tilde{\psi}^i(x) dx. \end{aligned}$$

Since, by a simple change of variable,

$$\int_{\Omega} \partial_i b(y) \psi^i(y) dy = \int_{\Omega_0} \widetilde{\partial_i b}(x) \tilde{\psi}^i(x) \det DX(x) dx$$

we deduce the following ‘‘conservative formula’’

$$\widetilde{\partial_i b} \det DX = \partial_k \left( ((DX)^{-1})_{ki} \tilde{b} \det DX \right) = \operatorname{div} \left( (DX)^{-1} e_i \tilde{b} \det DX \right) \quad \forall i \in \llbracket 1, n \rrbracket, \quad (1.7)$$

where  $e_i$  is the  $i$ -th vector of the canonical base and where we have defined the divergence of the second order tensor by taking the divergence of each column (sometimes in the literature one uses an opposite definition by taking the divergence of each line).

Identifying  $\mathcal{M}_{n \times n}(\mathbb{R})$  with  $\mathbb{R}^{n^2}$  one can see the determinant as a function from  $\mathbb{R}^{n^2}$  to  $\mathbb{R}$ . Laplace's formula allows us to compute its differential. More precisely, using (1.1) we see that for all  $i_0, j_0 \in \llbracket 1, n \rrbracket$

$$\frac{\partial \det}{\partial a_{i_0 j_0}}(A) = (-1)^{i_0 + j_0} M_{i_0 j_0}.$$

Moreover, if  $A = A(s) \in \mathcal{C}^1(\mathbb{R}; \mathcal{M}_{n \times n}(\mathbb{R}))$ , then we have that

$$\begin{aligned} \frac{d \det A}{ds}(s) &= \frac{\partial \det}{\partial a_{ij}}(A(s)) \frac{\partial a_{ij}}{\partial s}(s) \\ &= (-1)^{i+j} M_{ij}(A(s)) \frac{\partial a_{ij}}{\partial s}(s) = \det A(s) (A^{-1})_{ji} \frac{\partial a_{ij}}{\partial s}(s) \\ &= (-1)^{i+j} M_{ji}(A(s)) \frac{\partial a_{ji}}{\partial s}(s) = \left( \text{adj } A(s) \frac{\partial A}{\partial s}(s) \right)_{ii} \\ &= \text{trace} \left( \text{adj } A(s) \frac{\partial A}{\partial s}(s) \right). \end{aligned}$$

In particular, if  $\det A(s) \neq 0$  then we may write that

$$\frac{d \det A}{ds}(s) = \det A(s) \text{ trace} \left( A^{-1}(s) \frac{\partial A}{\partial s}(s) \right). \quad (1.8)$$

## 2 The classical formulation of the principle of stationary action

Consider a system of  $N$  interacting particles which move in a three-dimensional space. We assume that the evolution of the system can be completely characterized by the knowledge of:

- generalized coordinates  $X(t) = (X_1(t), X_2(t), \dots, X_N(t)) \in \mathbb{R}^{3N}$  where, for  $i \in \llbracket 1, n \rrbracket$ ,  $X_i(t) = (X_i^1(t), X_i^2(t), X_i^3(t)) \in \mathbb{R}^3$  represents the position of the  $i$ -th particle at time  $t$ .
- generalized velocities  $\dot{X}(t) = (\dot{X}_1(t), \dot{X}_2(t), \dots, \dot{X}_N(t)) \in \mathbb{R}^{3N}$  where, for each  $i \in \llbracket 1, n \rrbracket$ ,  $\dot{X}_i(t) = (\dot{X}_i^1(t), \dot{X}_i^2(t), \dot{X}_i^3(t)) \in \mathbb{R}^3$  is the time derivative of  $X_i(t)$  and it stands for the velocity of the  $i$ -th particle at time  $t$ .

The evolution equations of the system will be encoded in a given function  $\mathcal{L} = \mathcal{L}(t, X(t), \dot{X}(t))$  called *Lagrangian*. We fix two times  $t_0 < t_1 \in \mathbb{R}_+$  and define the associated *Hamiltonian action between  $t_0$  and  $t_1$*  by

$$\mathcal{A}_{t_0, t_1}[X] := \int_{t_0}^{t_1} \mathcal{L}(t, X(t), \dot{X}(t)) dt. \quad (2.1)$$

We will denote for each  $i \in \llbracket 1, N \rrbracket$

$$\frac{\partial \mathcal{L}}{\partial X_i} = \left( \frac{\partial \mathcal{L}}{\partial X_i^1}, \frac{\partial \mathcal{L}}{\partial X_i^2}, \frac{\partial \mathcal{L}}{\partial X_i^3} \right), \quad \frac{\partial \mathcal{L}}{\partial \dot{X}_i} = \left( \frac{\partial \mathcal{L}}{\partial \dot{X}_i^1}, \frac{\partial \mathcal{L}}{\partial \dot{X}_i^2}, \frac{\partial \mathcal{L}}{\partial \dot{X}_i^3} \right).$$

Suppose that we know the initial and final configurations of the system, i.e.  $X(t_0) = X_0$  and  $X(t_1) = X_1$  for some given  $X_0, X_1 \in \mathbb{R}^{3N}$ . We say that  $\{\mathbb{X}(\cdot; \mu)\}_{\mu \in (-\mu_0, \mu_0)}$  is a family of *virtual motions* (or *virtual displacements*) if

- $\mathbb{X}(t; 0) = X(t)$  for all  $t$ ;
- $\mathbb{X}(t_0; \mu) = X_0$  and  $\mathbb{X}(t_1; \mu) = X_1$ .

Therefore,  $\mathbb{X}(\cdot, \mu)$  can be seen as a perturbation of  $X$  that has the same initial and final configurations as  $X$  (see Figure 1).

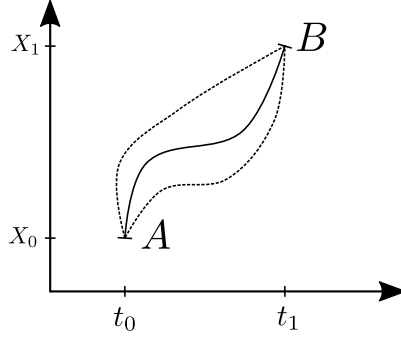


Figure 1: Plain line: the real path  $X$ , dashed lines: virtual paths  $\mathbb{X}(\cdot; \mu)$ .

The variation of  $\mathbb{X}$  is defined as

$$\delta\mathbb{X} = (\delta\mathbb{X}_1, \delta\mathbb{X}_2, \dots, \delta\mathbb{X}_N) := \left[ \frac{\partial\mathbb{X}}{\partial\mu}(\cdot; \mu) \right]_{|\mu=0} \in \mathbb{R}^{3N}.$$

For each value of  $\mu \in (-\mu_0, \mu_0)$ , we may compute  $\mathcal{A}_{t_0, t_1}[\mathbb{X}(\cdot; \mu)]$  by replacing  $X(t)$  in (2.1) with  $\mathbb{X}(t; \mu)$ . The variation of the Hamiltonian action with respect to  $\mathbb{X}$  between  $t_0$  and  $t_1$  is then defined by

$$\delta_{\mathbb{X}}\mathcal{A}_{t_0, t_1} := \left[ \frac{d}{d\mu} \mathcal{A}_{t_0, t_1}[\mathbb{X}(\cdot; \mu)] \right]_{|\mu=0}.$$

## 2.1 Principle of stationary action.

The principle of stationary (or least) action states that the real motion  $X$  is a critical point of the Hamiltonian action, that is

$$\delta_{\mathbb{X}}\mathcal{A}_{t_0, t_1} = 0.$$

The whole evolution of the system is actually encoded in the above equality. Indeed, let us observe that

$$\begin{aligned} \delta_{\mathbb{X}}\mathcal{A}_{t_0, t_1} &= \int_{t_0}^{t_1} \sum_{i=1}^N \left\{ \frac{\partial\mathcal{L}}{\partial X_i}(t, X, \dot{X}) \delta\mathbb{X}_i + \frac{\partial\mathcal{L}}{\partial \dot{X}_i}(t, X, \dot{X}) \frac{d\delta\mathbb{X}_i}{dt} \right\} dt \\ &= \int_{t_0}^{t_1} \sum_{i=1}^N \left\{ \frac{\partial\mathcal{L}}{\partial X_i}(t, X, \dot{X}) - \frac{\partial}{\partial t} \left\{ \frac{\partial\mathcal{L}}{\partial \dot{X}_i}(t, X, \dot{X}) \right\} \right\} \delta\mathbb{X}_i(t) dt. \end{aligned}$$

Since  $\mathbb{X}(\cdot, \mu)$  is arbitrary and  $\delta\mathbb{X}_i(t)$  could be non-zero at any  $t \in (t_0, t_1)$ , the integrand must vanish and we get the so-called *Euler-Lagrange equations*:

$$\frac{\partial}{\partial t} \left\{ \frac{\partial\mathcal{L}}{\partial \dot{X}_i}(t, X, \dot{X}) \right\} - \frac{\partial\mathcal{L}}{\partial X_i}(t, X, \dot{X}) = 0 \quad \text{for all } i \in \llbracket 1, N \rrbracket. \quad (2.2)$$

**Time homogeneity and energy conservation.** Let us assume that the Lagrangian is independent of time, i.e.  $\mathcal{L} = \mathcal{L}(X, \dot{X})$ . Taking the scalar product of (2.2) with  $\dot{X}_i$  and summing over  $i \in \llbracket 1, N \rrbracket$ , we find that

$$\frac{d}{dt} \left\{ \dot{X}_i \frac{\partial \mathcal{L}}{\partial \dot{X}_i} (X, \dot{X}) - \mathcal{L} (X, \dot{X}) \right\} = 0. \quad (2.3)$$

The above relation is known as the *energy conservation equation*, where the total energy of the system would be defined as

$$E (X, \dot{X}) := \dot{X}_i \frac{\partial \mathcal{L}}{\partial \dot{X}_i} (X, \dot{X}) - \mathcal{L} (X, \dot{X}). \quad (2.4)$$

In general, the Lagrangian is the difference between the kinetic energy  $T$  and the potential energy  $W$ , that is

$$\mathcal{L} = T - W.$$

Usually, the kinetic energy is quadratic with respect to  $\dot{X}$ , i.e.

$$T = T (X, \dot{X}) = \frac{1}{2} \dot{X}^T A(X) \dot{X},$$

where  $A(X)$  is a positive definite matrix and  $W$  depends only on the generalized coordinates  $W = W(X)$ . Replacing these expressions into (2.4), one obtains  $E = T + W$  which is the total energy of the system.

## 2.2 Case of constrained systems

From the physical point of view, it may be crucial to take into account additional constraints in the system:

- either a geometric constraint  $F(t, X, \dot{X}) = 0$  like the search of geodesics on a surface, or the double pendulum problem with two particles connected by rigid rods;
- or an integral constraint (also called isoperimetric constraints)  $\int_{t_0}^{t_1} F(t, X, \dot{X}) dt = 0$  like the famous queen Dido's problem (find the extremal area for a given perimeter) or the catenary (hanging chain) problem.

In that case, we introduce a Lagrange multiplier  $\lambda$  associated to the constraint and replace in the previous calculations  $\mathcal{L}$  respectively by

- $\mathcal{L} + \lambda(t)F$ ;
- or  $\mathcal{L} + \lambda F$  with  $\lambda$  independent of time.

For more details on these subjects one may consult classical references on calculus of variations [Sagan, 1992, Gelfand and Fomin, 1963].

## 2.3 Newton's equations of movement

As a simple illustration of the previous considerations, let us show how one can recover Newton's second law. Consider a system of  $N$  particles animated by the Lagrangian:

$$\mathcal{L} = \sum_{i=1}^N \frac{1}{2} m_i \left| \dot{X}_i \right|^2 - W (X_1, X_2, \dots, X_N), \quad (2.5)$$

where the potential  $W$  is a real valued function. Of course this special form of the Lagrangian is obtained as a consequence of physical considerations: existence of an inertial frame, Galileo's relativity principle, time homogeneity etc., see the first chapter of [Landau and Lifschitz, 1976] for more details. In this case we obtain the following Euler-Lagrange equations:

$$m_i \ddot{X}_i = -\frac{\partial W}{\partial X_i} \quad \text{for } i \in \llbracket 1, N \rrbracket. \quad (2.6)$$

For any  $i \in \llbracket 1, N \rrbracket$ , the quantity  $-\frac{\partial W}{\partial X_i}$  can be interpreted as the force that acts on the  $i^{\text{th}}$  particle and the Euler-Lagrange equation (2.5) is nothing else but Newton's second law.

### 3 Hamilton's principle for continuous deformable media

In this section we propose a formulation of the principle of stationary action in the context of continuous media and derive the corresponding Euler-Lagrange equations. All the notions (virtual displacement, action and variation of the action) introduced in the previous section are generalized to the continuous-media case. Although we use the same notations  $X$  and  $\mathbb{X}$  as in Section 2, we warn the reader that the mathematical objects they describe are of a different nature. This choice is motivated by the fact that in this way, the reader may easily draw parallels and compare the classical particle-setting with the continuous setting.

#### 3.1 Continuous deformable media

From a mathematical point of view, a continuous deformable medium (*abv.* CDM below) is a pair  $(\Omega_0, X)$  such that

- $\Omega_0 \subset \mathbb{R}^n$  is a given open set which represents the initial configuration of the medium;
- $X : [0, \infty) \times \overline{\Omega}_0 \rightarrow \mathbb{R}^n$  is a continuous function satisfying
  - for all  $x \in \overline{\Omega}_0$ ,  $X(0, x) = x$ ;
  - for all  $t \geq 0$ ,  $X_t = X(t, \cdot) : \overline{\Omega}_0 \rightarrow X(t, \overline{\Omega}_0)$  is a homeomorphism while  $X_t = X(t, \cdot) : \Omega_0 \rightarrow X(t, \Omega_0)$  is a diffeomorphism;
  - for all  $t \geq 0$ ,  $\det DX(t, x) > 0$ .

Physically speaking, we are given a material occupying the initial configuration  $\overline{\Omega}_0 \subset \mathbb{R}^n$ , each point  $x \in \overline{\Omega}_0$  being occupied by a particle of the medium which will change its position as time evolves. The value  $X(t, x)$  represents the position at time  $t$  of the particle that was located in  $x$  at time 0. The image of  $\overline{\Omega}_0$  through  $X(t, \cdot)$ ,  $\overline{\Omega}_t := X(t, \overline{\Omega}_0) = \{X(t, x) : x \in \overline{\Omega}_0\}$  represents the configuration of the medium at time  $t \geq 0$  (see Figure 2).

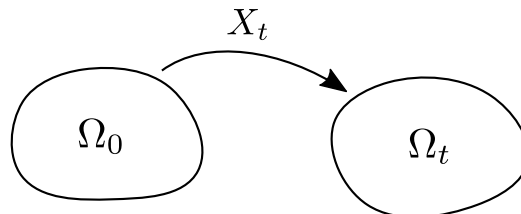


Figure 2: Formal representation of the application  $X_t$  and its image. The domain  $\Omega_0$  is seen as the reference space for the motion.



**Lagrangian and Eulerian standpoints.** There are two different ways of thinking any physical quantity  $B$  that characterizes the deformable medium  $(\Omega_0, X)$ :

- the Lagrangian point of view:  $B$  is attached to the particle motion. In this setup  $B(t, x)$  represents the measure at time  $t$  of the physical quantity associated to the particle that leaves  $x$  at time 0. We will emphasize the Lagrangian character explicitly by an  $\ell$  upper script:  $B^\ell$ . It should be understood automatically that the domain of definition of  $B^\ell$  is  $\mathbb{R}_+ \times \bar{\Omega}_0$ ;
- the Eulerian point of view:  $B$  is measured as time evolves, say between  $t_1, t_2$  in a geometrical point  $y \in \bigcap_{t \in [t_1, t_2]} X(t, \bar{\Omega}_0)$ . We will explicitly emphasize the Eulerian character by an  $e$  upper script:  $B^e$ . It should be understood automatically that the domain of definition of  $B^e$  is  $\bigcup_{t \geq 0} \{t\} \times X(t, \bar{\Omega}_0)$ ;

The two points of view are equivalent in the sense that any ‘‘Lagrangian physical quantity’’ can be regarded as an ‘‘Eulerian physical quantity’’ via the family of diffeomorphisms  $(X(t, \cdot))_{t \geq 0}$ . More precisely, given  $B^\ell$  we have

$$B^e(t, y) = B^\ell(t, X^{-1}(t, y)) \text{ for all } t \geq 0 \text{ and } y \in X(t, \bar{\Omega}_0), \quad (3.1)$$

and conversely

$$B^\ell(t, x) = B^e(t, X(t, x)) \text{ for all } t \geq 0 \text{ and } x \in \bar{\Omega}_0. \quad (3.2)$$

As it will become clearer quite soon, the Lagrangian point of view is appropriate in order to formulate a Hamilton-type principle for CDMs that is close to the classical case briefly presented in Section 2. Thus, we anticipate that we will first obtain equations verified by the Lagrangian physical quantities characterizing the CDM. As we are also interested in getting the equivalent Eulerian equations, we will need some formulae to pass from one formulation to the other. On this point we refer the reader to the introductory section of these notes.

Using the conventions established by relations (3.1) and (3.2) and taking into consideration (1.6) along with (1.7) we have the following relations

$$\text{trace}(D\psi^\ell(DX)^{-1}) = (D\psi^\ell)_{ij} ((DX)^{-1})_{ji} = ((DX)^{-1})_{ji} \partial_j (\psi^\ell)^i = (\text{div } \psi^e)^\ell, \quad (3.3)$$

$$\frac{1}{\det(DX)} \text{div} \left( (DX)^{-1} B^\ell \det DX \right) = (\nabla B^e)^\ell, \quad (3.4)$$

In particular, taking  $B^\ell = 1$  we find the so-called *Piola identity*

$$\text{div} \left( (DX)^{-1} \det DX \right) = 0. \quad (3.5)$$

**Particle velocity and material derivative** In accordance with classical mechanics, we introduce the velocity  $v$  such that

$$\dot{X}(t, x) = v^\ell(t, x) = v^e(t, X(t, x))$$

which represents the velocity of the particle located at  $X(t, x)$ . Thus, we have that

$$\begin{cases} \dot{X}(t, x) = v^e(t, X(t, x)), \\ X(0, x) = x. \end{cases}$$

We are now in the position of stating the time derivative law:

$$\partial_t B^\ell = (\partial_t B^e + v^e \cdot \nabla B^e)^\ell. \quad (3.6)$$

By a material domain we understand the following collection of subsets of  $\mathbb{R}^n$  that is indexed by time:  $\{X(t, \omega)\}_{t \geq 0}$ . Physically, of course,  $X(t, \omega)$  represents the region of the space occupied by the particles that were occupying  $\omega$  at time  $t = 0$ .

For a given  $\omega \subset \overline{\Omega_0}$  it is of physical interest to see how an Eulerian physical quantity evolves in a material domain  $\{X(t, \omega)\}_{t \geq 0}$ . First, we observe that

$$\text{Vol}(X(t, \omega)) = \int_{X(t, \omega)} dy = \int_{\omega} \det DX(t, x) dx. \quad (3.7)$$

Hence,  $\det DX$  is related to the local volume change.

More generally, consider now  $f^e : \bigcup_{t \geq 0} \{t\} \times X(t, \overline{\Omega_0}) \rightarrow \mathbb{R}$  and  $\omega \subset \overline{\Omega_0}$  and using (3.6) let us first write that

$$\int_{X(t, \omega)} f^e(t, y) dy = \int_{\omega} f^l(t, x) \det DX(t, x) dx$$

from which we infer

$$\begin{aligned} \frac{d}{dt} \int_{X(t, \omega)} f^e dy &= \int_{\omega} \{ \partial_t f^l \det DX + f^l \partial_t \det DX \} dx \\ &= \int_{\omega} \left\{ \partial_t f^l \det DX + f^l \det DX \left( (DX)^{-1} \right)_{ji} \partial_t (DX)_{ij} \right\} dx \\ &= \int_{\omega} \left\{ \partial_t f^l \det DX + f^l \det DX \left( (DX)^{-1} \right)_{ji} (Dv^l)_{ij} \right\} dx \\ &= \int_{\omega} \{ \partial_t f^l \det DX + f^l (\text{div } v^e)^l \det DX \} dx \end{aligned}$$

where we have used (3.3) to get the last equality. Hence, using (3.6), we obtain

$$\begin{aligned} \frac{d}{dt} \int_{X(t, \omega)} f^e dy &= \int_{\omega} \left\{ (\partial_t f^e + \text{div}(f^e v^e))^l \det DX \right\} dx \\ &= \int_{X(t, \omega)} \{ \partial_t f^e + \text{div}(f^e v^e) \} dy. \end{aligned}$$

In particular for  $f \equiv 1$ , we infer that

$$\frac{d}{dt} \int_{X(t, \omega)} dy = \int_{\omega} \partial_t \det DX = \int_{X(t, \omega)} \text{div } v^e dy = \int_{\omega} (\text{div } v^e)^l \det DX dx.$$

Since  $\omega$  is arbitrarily, we obtain that

$$\partial_t \det DX = (\text{div } v^e)^l \det DX \quad (3.8)$$

and thus we recover the famous formula for the Wronskian:

$$\det DX = (\det DX)|_{t=0} \exp \left( \int_0^t (\text{div } v^e)^l dt \right).$$

## 3.2 Hamilton's principle

The main difference between continuous mechanics and classical finite-degrees-of-freedom mechanics is that the potential energy of a continuous deformable medium is due to the deformation of the medium [Berdichevsky, 2009, Godunov, 1978, Serrin, 1959]. We use this observation in order to postulate a form of the Lagrangian associated to a CDM.

Similarly to the discrete case, we define the action of the CDM  $(\Omega_0, X)$  between the times  $t_0$  and  $t_1$  as

$$\mathcal{A}_{t_0, t_1}[X] := \int_{t_0}^{t_1} \int_{\Omega_0} \mathcal{L} dx dt, \quad (3.9)$$

where the Lagrangian  $\mathcal{L}$  characterizing the CDM  $(\Omega_0, X)$  is a given function of the velocity  $\dot{X}$ , deformation gradient  $DX$ , Lagrange coordinates  $x$  and time  $t$ . For the description of fluids, it is sufficient to take into account only the dependence on  $d = \det DX$  characterizing the local volume change (see (3.7)) :  $\mathcal{L} = \mathcal{L}(\dot{X}, d, t, x)$ . In the general case, it depends on the whole right Cauchy-Green deformation tensor  $C = (DX)^T DX$  (in the case of isotropic solids, it depends only on the invariants of  $C$ ) [Godunov, 1978, Berdichevsky, 2009]. Also, the dependence on the rate of deformation  $\partial_t d$  [Miles and Salmon, 1985], [Salmon, 1988], [Salmon, 1998], [Gavrilyuk, 1994], [Gavrilyuk and Gouin, 1999], [Gavrilyuk and Teshukov, 2001], [Gavrilyuk, 2011] and its gradients  $\nabla d$  [Van der Waals, 1979, Truskinovskii, 1982], [Gavrilyuk and Serre, 1995], [Casal and Gouin, 1985], [Gavrilyuk and Shugrin, 1996], [Dell'Isola et al., 2009], [Madelung, 1927], [Carles et al., 2012], [Bresch et al., 2019] can be included. In the present notes, we shall restrict ourselves to the most simple situation of classical fluids or a mixture of classical fluids.

**Hamilton's principle of stationary action for CDM** Suppose that for any time  $t \in [t_0, t_1]$ ,  $\Omega_t \subset \mathbb{R}^n$  represents the configuration at time  $t$  of the CDM  $(\Omega_0, X)$ . We consider a family of virtual motions associated to  $X(t, \cdot)$ , that is a family  $\{\mathbb{X}(t, \cdot, \mu)\}_{\mu \in (-\mu_0, \mu_0)}$  verifying the following the properties:

- for all  $t \in [t_0, t_1]$  and  $\mu \in (-\mu_0, \mu_0)$ ,  $\mathbb{X}(t, \cdot, \mu) : \bar{\Omega}_0 \rightarrow \mathbb{R}^n$  is a homeomorphism on its image such that  $\mathbb{X}(t, \cdot; \mu)|_{\Omega_0}$  is a diffeomorphism on its image;
- for all  $t \in [t_0, t_1]$  and  $x \in \bar{\Omega}_0$  we have that  $\mathbb{X}(t, x; 0) = X(t, x)$ ;
- for all  $x \in \Omega_0$ ,  $\mathbb{X}(t_0, x; \mu) = X(t_0, x)$  and  $\mathbb{X}(t_1, x; \mu) = X(t_1, x)$  for all  $\mu \in (-\mu_0, \mu_0)$ ;
- for all  $t \in [t_0, t_1]$ ,  $\mu \in (-\mu_0, \mu_0)$  and  $x \in \partial\Omega_0$ ,  $\mathbb{X}(t, x; \mu) = X(t, x)$ .

The variation of the family of virtual motions  $\mathbb{X}$  is defined as

$$\forall t \in [t_0, t_1] \text{ and } \forall x \in \bar{\Omega}_0 : \quad \delta\mathbb{X}(t, x) := \frac{\partial}{\partial \mu} [\mathbb{X}(t, x; \mu)]|_{\mu=0}. \quad (3.10)$$

In particular, the variation of the Hamiltonian action  $\mathcal{A}$  between  $t_0$  and  $t_1$  (defined in (3.9)) with respect to  $\mathbb{X}$  is

$$\delta_{\mathbb{X}} \mathcal{A}_{t_0, t_1} := \frac{d}{d\mu} \left[ \mathcal{A}_{t_0, t_1}[\mathbb{X}] \right]_{|\mu=0}. \quad (3.11)$$

Then  $X$  is a critical point of the Hamilton action (3.9) if, for any  $\mathbb{X}$  verifying the above properties,

$$\delta_{\mathbb{X}} \mathcal{A} = 0. \quad (3.12)$$

**Remark 3.1** *The reader will notice the analogy with the classical finite-degrees-of-freedom case presented in Section 2 in the definition of the family of virtual motions: in the classical case, we have fixed the initial point and the endpoint for all curves describing the particles trajectories, while in the case of a continuum we fix the points at the boundary of the whole 4D time-space domain  $[t_0, t_1] \times \bar{\Omega}_0$ .*

**Remark 3.2** *Compared to other definitions of Hamilton's action that can be found in the literature (see for instance [Gavrilyuk, 2011]), it is important to note that adopt here a fully Lagrangian standpoint: we integrate the Lagrangian over the domain  $\Omega_0$  and consider the Lagrangian variations of the action.*

Let us see how we can use the stationary action principle to obtain the corresponding Euler-Lagrange equations. Writing that

$$\begin{aligned}
& \frac{d}{d\mu} [\mathcal{A}_{t_0, t_1}[\mathbb{X}]] \\
&= \int_{t_0}^{t_1} \int_{\Omega_0} \left\{ \frac{\partial \mathcal{L}}{\partial \dot{X}} \partial_t \left( \frac{\partial \mathbb{X}}{\partial \mu} \right) + \frac{\partial \mathcal{L}}{\partial d} \frac{\partial}{\partial \mu} (\det DX) \right\} dx dt \\
&= \int_{t_0}^{t_1} \int_{\Omega_0} \left\{ -\frac{\partial}{\partial t} \left( \frac{\partial \mathcal{L}}{\partial \dot{X}} \right) \frac{\partial \mathbb{X}}{\partial \mu} + \frac{\partial \mathcal{L}}{\partial d} \det DX \operatorname{trace} \left( (DX)^{-1} D \left( \frac{\partial \mathbb{X}}{\partial \mu} \right) \right) \right\} dx dt \\
&= - \int_{t_0}^{t_1} \int_{\Omega_0} \left\{ \frac{\partial}{\partial t} \left( \frac{\partial \mathcal{L}}{\partial \dot{X}} \right) + \operatorname{div} \left( (DX)^{-1} \frac{\partial \mathcal{L}}{\partial d} \det DX \right) \right\} \frac{\partial \mathbb{X}}{\partial \mu} dx dt \tag{3.13}
\end{aligned}$$

and using that  $X$  is a critical point of the Hamilton action, we obtain the following Euler-Lagrange equation in Lagrangian coordinates:

$$\frac{\partial}{\partial t} \left( \frac{\partial \mathcal{L}}{\partial \dot{X}} \right) + \operatorname{div} \left( (DX)^{-1} \det DX \frac{\partial \mathcal{L}}{\partial d} \right) = 0. \tag{3.14}$$

Similarly to the discrete case (cf (2.3)), when the Lagrangian does not depend explicitly on time,  $\mathcal{L} = \mathcal{L}(\dot{X}, d, x)$ , one derives an energy conservation equation by combining (3.3), (3.8) and (3.14):

$$\partial_t \left( \frac{\partial \mathcal{L}}{\partial \dot{X}} \dot{X} - \mathcal{L} \right) + \operatorname{div} \left( (DX)^{-1} \det DX \frac{\partial \mathcal{L}}{\partial d} \dot{X} \right) = 0, \tag{3.15}$$

**Additional state variable.** It might also happen that the physical modeling requires the introduction of an extra independent variable (sometimes called *order parameter*), say  $\alpha$  for the sake of clarity. The state of the system is characterized by the pair  $(X, \alpha)$ . This should be of course viewed in the Lagrangian setting which can depend on the variable  $\alpha$  and its, for example, first time and space derivatives  $\partial_t \alpha$  and  $\nabla \alpha$  (dispersive systems). and adapt Hamilton's principle by imposing that  $(X, \alpha)$  is critical. We skip the technical details and just note that the Euler-Lagrange equations in the Lagrangian coordinates in this case become

$$\left\{ \frac{\partial \mathcal{L}}{\partial \alpha} - \frac{\partial}{\partial t} \left( \frac{\partial \mathcal{L}}{\partial (\partial_t \alpha)} \right) - \operatorname{div} \left( \frac{\partial \mathcal{L}}{\partial (\nabla \alpha)} \right) = 0, \tag{3.16a}
\right.$$

$$\left. \left\{ \partial_t \left( \frac{\partial \mathcal{L}}{\partial \dot{X}} \right) + \operatorname{div} \left( (DX)^{-1} \det DX \frac{\partial \mathcal{L}}{\partial d} \right) = 0. \tag{3.16b}
\right. \right.$$

As before, the energy conservation equation can be derived as a consequence of the governing equations:

$$\partial_t \left( \frac{\partial \mathcal{L}}{\partial \dot{X}} \dot{X} + \frac{\partial \mathcal{L}}{\partial (\partial_t \alpha)} \partial_t \alpha - \mathcal{L} \right) + \operatorname{div} \left( (DX)^{-1} \det DX \frac{\partial \mathcal{L}}{\partial d} \dot{X} + \frac{\partial \mathcal{L}}{\partial (\nabla \alpha)} \partial_t \alpha \right) = 0. \tag{3.17}$$

Such an approach was successfully applied for the Eulerian description of wave propagation in fluids containing gas bubbles [Drui et al., 2019, Gavriluk and Saurel, 2002], in dispersive hydrodynamics [Favre and Gavriluk, 2017] and even in quantum mechanics [Dhaouadi et al., 2019]. For the sake of clarity, in the following we consider the most simple situation where the dynamics does not depend on the space and time derivatives of  $\alpha$  and Eq. (3.16a) reduces to

$$\frac{\partial \mathcal{L}}{\partial \alpha} = 0. \tag{3.18}$$

### 3.3 Mixtures as CDMs with different velocities

The multi-phase flow modeling is a vast and challenging topic. Multiple important applications necessitate the construction of physically reliable and mathematically well-posed models. Classical averaging approach [Drew and Passman, 2006], [Ishii and Hibiki, 2010], [Nigmatulin, 1990] usually produces an underdetermined system of equations. Different closure hypotheses, even if they are reasonable, can drastically change the mathematical structure of the governing equations. A simple idea could be to formulate the Lagrangian of a complex multi-phase system directly in terms of average variables. At least it allows us to keep the same mathematical structure of the governing equations. In a second step, one takes in consideration friction forces which, at a mathematical level, amounts to add dissipative terms that are compatible with the entropy inequality. Thus, the main issue is the formulation of the non-dissipative equations (Euler–Lagrange equations) defined solely by the Lagrangian of the system. We want to model the case of a continuous deformable medium that is composed of two types of particles. In the most general case, each component is animated by its own movement. We consider the two CDM  $(\Omega_0, X_1)$ ,  $(\Omega_0, X_2)$  with the property that

$$X_1(t, \bar{\Omega}_0) = X_2(t, \bar{\Omega}_0) \quad \forall t \geq 0. \quad (3.19)$$

We are saying that at  $t = 0$ , any position  $x \in \Omega_0$  is occupied by two particles of different types. As time evolves, the particles change positions but in such a way that at any  $t > 0$ , the global domain occupied by the first specie is the same as the domain occupied by the second specie. Hence, no pure phase is allowed and (3.19) can be seen as a “no-segregation property”. Note also that at time  $t$  a geometrical point  $y \in X_1(t, \bar{\Omega}_0) = X_2(t, \bar{\Omega}_0)$  is not necessarily occupied by particles that left the same position at the initial time (see Figure 3).

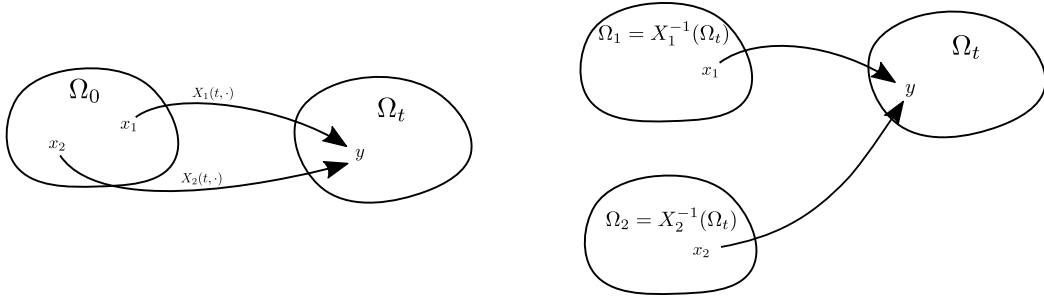


Figure 3: Two formal representations of the system. On the left, the initial configuration of the system  $\Omega_0$  and its image  $\Omega_t = X_1(t, \Omega_0) = X_2(t, \Omega_0)$ . On the right, the two reference spaces  $\Omega_1 = X_1^{-1}(t, \Omega_t)$ ,  $\Omega_2 = X_2^{-1}(t, \Omega_t)$ . One passes from one reference space  $\Omega_j$  to the other one  $\Omega_i$  through the application  $X_i^{-1} \circ X_j(t, \cdot)$ .

What remains for a complete description of the mixture is to say “how much volume at a point” does a particle of type  $a \in \{1, 2\}$  occupy. To that end, we introduce

$$\alpha_a^\ell(t, x) = \begin{cases} \text{the volume fraction at time } t \text{ of the particle of type } a \\ \text{measured at } X_a(t, x). \end{cases}$$

More precisely, for a given point  $(t, x)$ , we denote  $y = X_a(t, x)$  and consider a representative mixture volume containing  $y$ ,  $\omega = \omega_1 \cup \omega_2$  where  $\omega_a$  is the volume occupied by the phase  $a$ . We define the characteristic function of phase  $a$  as

$$\chi_a^e(t, z) = \begin{cases} 1 & \text{if } z \in \omega_a, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\alpha_a^e(t, y) := \frac{\int_{\omega} \chi_a^e(t, z) dz}{\text{Vol}(\omega)} = \frac{\text{Vol}(\omega_a)}{\text{Vol}(\omega)}.$$

In particular, with hypothesis (3.19) we impose

$$0 < \alpha_a^e(t, y) < 1 \quad \forall t \geq 0, y \in X_a(t, \Omega_0).$$

**Remark 3.3** *It is important to note that the definition of the volume fraction depends on the averaging process which is performed. Several averaging methods (e.g. ensemble averaging, statistical averaging, time averaging, etc.) may be found in the literature and we refer the interested reader to the books [Drew and Passman, 2006], [Ishii and Hibiki, 2010] and [Nigmatulin, 1990].*

The two volume fractions are linked through the relation

$$\alpha_1^\ell(t, x) + \alpha_2^\ell(t, X_2^{-1}(t, X_1(t, x))) = \alpha_1^\ell(t, X_1^{-1}(t, X_2(t, x))) + \alpha_2^\ell(t, x) = 1. \quad (3.20)$$

Indeed, the sum of the volume fractions is 1 at the geometrical point  $y = X_1(t, x)$ , position which is occupied at time  $t$  by the particle of the second specie that left  $X_2^{-1} \circ X_1(t, x)$  at time  $t_0 = 0$ . In order to ease the reading, when there is no possible confusion, we shall drop the  $\ell$  upper-script in the computations that follow.

We would like to emphasize the fact that the choice of a physically meaningful Lagrangian  $\mathcal{L}_{mix}$  is always a difficult problem. *A priori*, it cannot be reduced to a linear combination of the pure phases Lagrangians because of the interaction between phases. In particular, a new scalar variable  $|\dot{X}_1 - \dot{X}_2|$  should be added for the description of the virtual mass effects (when a moving particle of type 1 will displace surrounding fluid particles of type 2). The Lagrangian can also depend on topological characteristics of multiphase flows, as, for example, on the total area of contact between two fluids, *i.e.* the interfacial area (see the recent study [Cordes et al., 2019]). In the present notes, we shall stick to most simple physics and assume that the Lagrangian characterizing the dynamics of the mixture has the form

$$\mathcal{L}_{mix} = \mathcal{L}_1(\dot{X}_1, \det DX_1, \alpha_1, x) + \mathcal{L}_2(\dot{X}_2, \det DX_2, \alpha_2, x).$$

The associated Hamiltonian action between the times  $t_0$  and  $t_1$  is then

$$\begin{aligned} \mathcal{A}_{t_0, t_1} [X_1, X_2, \alpha_1] &= \mathcal{A}_{1; t_0, t_1} [X_1, \alpha_1] + \mathcal{A}_{2; t_0, t_1} [X_2, \alpha_2] \\ &= \int_{t_0}^{t_1} \int_{\Omega_0} \left\{ \mathcal{L}_1(\dot{X}_1, \det DX_1, \alpha_1, x) + \mathcal{L}_2(\dot{X}_2, \det DX_2, \alpha_2, x) \right\} dx dt. \end{aligned} \quad (3.21)$$

We postulate that the equations governing  $(X_1, X_2, \alpha_1)$  are obtained as the Euler-Lagrange equations associated to the action  $\mathcal{A}$ . These equations are obtained by imposing that the variations of  $\mathcal{A}$  with respect to  $X_1, X_2$  and  $\alpha_1$  are all 0. In the derivation of the equations it is important, even crucial, to remember the relation (3.20) which implies a cross-contribution of  $\mathcal{L}_2$  in  $\delta X_1$ , and conversely.

We begin with the variation of  $\mathcal{A}$  with respect to  $\alpha_1$ . Of course, the delicate part is to obtain the contribution coming from  $\mathcal{A}_2$ . Owing to the relation (3.20), we put  $\mathcal{A}_2$  under the form

$$\mathcal{A}_{2; t_0, t_1} = \int_{t_0}^{t_1} \int_{\Omega_0} \mathcal{L}_2(\dot{X}_2, \det DX_2, \alpha_2, x) dx dt$$

$$\begin{aligned}
&= \int_{t_0}^{t_1} \int_{\Omega_0} \left[ \mathcal{L}_2 \left( \dot{X}_2, \det DX_2, \alpha_2, x \right) \right]_{|X_2^{-1} \circ X_1(t, x')} \det D(X_2^{-1} \circ X_1)|_{(t, x')} dx' dt \\
&= \int_{t_0}^{t_1} \int_{\Omega_0} \mathcal{L}_2 \left( \dot{X}_2|_{X_2^{-1} \circ X_1(t, x')}, \det DX_2|_{X_2^{-1} \circ X_1(t, x')}, 1 - \alpha_1(t, x'), (X_2^{-1} \circ X_1)(t, x') \right) \\
&\quad \times \det D(X_2^{-1} \circ X_1)|_{(t, x')} dx' dt.
\end{aligned}$$

Thus we get that

$$\begin{aligned}
\delta_{\alpha_1} \left[ \mathcal{A}_{t_0, t_1}[X_1, X_2, \alpha_1] \right] &= \int_{t_0}^{t_1} \int_{\Omega_0} \left\{ \frac{\partial \mathcal{L}_1}{\partial \alpha_1} \left( \dot{X}_1, \det DX_1, \alpha_1, x \right) \right. \\
&\quad \left. - \left[ \frac{\partial \mathcal{L}_2}{\partial \alpha_2} \left( \dot{X}_2, \det DX_2, \alpha_2, x \right) \right]_{|X_2^{-1} \circ X_1} \det D(X_2^{-1} \circ X_1) \right\} dx dt,
\end{aligned}$$

and thus we obtain the first local equation

$$\frac{\partial \mathcal{L}_1}{\partial \alpha_1} \left( \dot{X}_1, \det DX_1, \alpha_1, x \right) = \left[ \frac{\partial \mathcal{L}_2}{\partial \alpha_2} \left( \dot{X}_2, \det DX_2, \alpha_2, x \right) \right]_{|X_2^{-1} \circ X_1} \det D(X_2^{-1} \circ X_1). \quad (3.22)$$

Next, we study the variation with respect to  $X_1$ . Again, the variation of  $\mathcal{A}_{1; t_0, t_1}$  is treated as in (3.13), we focus only on the contribution coming from  $\mathcal{A}_{2; t_0, t_1}$ :

$$\begin{aligned}
\delta_{\mathbb{X}_1} \mathcal{A}_2 &= \int_{t_0}^{t_1} \int_{\Omega_0} \delta_{\mathbb{X}_1} \mathcal{L}_2 \left( \dot{X}_2, \det DX_2, \alpha_2, x \right) dx dt \\
&= \int_{t_0}^{t_1} \int_{\Omega_0} \frac{\partial \mathcal{L}_2}{\partial \alpha_2} \left( \dot{X}_2, \det DX_2, \alpha_2, x \right) \delta_{\mathbb{X}_1} \alpha_2 dx dt.
\end{aligned}$$

But writing that

$$\alpha_2(t, x) = 1 - \alpha_1(t, X_1^{-1}(t, X_2(t, x))),$$

we see that

$$\delta_{\mathbb{X}_1} \alpha_2(t, x) = -D\alpha_1|_{(t, X_1^{-1} \circ X_2(x))} \delta_{\mathbb{X}_1} (X_1)^{-1}(t, X_2(t, x)).$$

We get then

$$\begin{aligned}
\delta_{\mathbb{X}_1} \alpha_2|_{(t, X_2^{-1} \circ X_1(t, x))} &= -D\alpha_1 \delta_{\mathbb{X}_1} [(X_1)^{-1}(t, X_1(t, x))] \\
&= D\alpha_1 D(X_1^{-1})|_{(t, X_1(t, x))} \delta_{\mathbb{X}_1}(t, x).
\end{aligned}$$

The passage to the last equation is obtained by differentiating the relation  $X_1^{-1} \circ X_1 = \text{Id}$  with respect to  $\mu$ , *i.e.*

$$\frac{\partial X_1^{-1}}{\partial \mu}(t, X_1(t, x)) + D(X_1^{-1})|_{(t, X_1)} \frac{\partial X_1}{\partial \mu}(t, x) = 0,$$

and for  $\mu = 0$  we get

$$\delta_{\mathbb{X}_1} (X_1)^{-1}(t, X_1(t, x)) = -D(X_1^{-1})|_{(t, X_1)} \delta_{\mathbb{X}_1}(t, x).$$

We may then write that

$$\begin{aligned}
&\delta_{\mathbb{X}_1} \mathcal{A}_{2; t_0, t_1} \\
&= \int_{t_0}^{t_1} \int_{\Omega_0} \frac{\partial \mathcal{L}_2}{\partial \alpha_2} \left( \dot{X}_2, \det DX_2, \alpha_2, x \right) \delta_{\mathbb{X}_1} \alpha_2 dx dt \\
&= \int_{t_0}^{t_1} \int_{\Omega_0} \left[ \frac{\partial \mathcal{L}_2}{\partial \alpha_2} \left( \dot{X}_2, \det DX_2, \alpha_2, x \right) \delta_{\mathbb{X}_1} \alpha_2 \right]_{|X_2^{-1} \circ X_1} \det D(X_2^{-1} \circ X_1) dx dt
\end{aligned}$$

$$= \int_{t_0}^{t_1} \int_{\Omega_0} \left\{ \left[ \frac{\partial \mathcal{L}_2}{\partial \alpha_2} \left( \dot{X}_2, \det DX_2, \alpha_2, x \right) \right]_{|X_2^{-1} \circ X_1} D\alpha_1 (DX_1)^{-1} \det D (X_2^{-1} \circ X_1) \right\} \delta \mathbb{X}_1 dx dt.$$

Consequently, we have

$$\begin{aligned} & \delta_{\mathbb{X}_1} \mathcal{A}_{t_0, t_1} \\ &= - \int_{t_0}^{t_1} \int_{\Omega_0} \left\{ \frac{\partial}{\partial t} \left( \frac{\partial \mathcal{L}_1}{\partial \dot{X}_1} \right) + \operatorname{div} \left( (DX_1)^{-1} \frac{\partial \mathcal{L}_1}{\partial d_1} \det DX_1 \right) \right\} \delta \mathbb{X}_1 dx dt \\ &+ \int_{t_0}^{t_1} \int_{\Omega_0} \left\{ \left[ \frac{\partial \mathcal{L}_2}{\partial \alpha_2} \right]_{|X_2^{-1} \circ X_1} D\alpha_1 (DX_1)^{-1} \det D (X_2^{-1} \circ X_1) \right\} \delta \mathbb{X}_1 dx dt, \end{aligned}$$

which yields the Euler-Lagrange equation associated to the variation in  $X_1$ :

$$\begin{aligned} & \frac{\partial}{\partial t} \left[ \frac{\partial \mathcal{L}_1}{\partial \dot{X}_1} \right] + \operatorname{div} \left( (DX_1)^{-1} \det DX_1 \frac{\partial \mathcal{L}_1}{\partial d_1} \right) \\ &= \left[ \frac{\partial \mathcal{L}_2}{\partial \alpha_2} \left( \dot{X}_2, \det DX_2, \alpha_2, x \right) \right]_{|X_2^{-1} \circ X_1} D\alpha_1 (DX_1)^{-1} \det D (X_2^{-1} \circ X_1). \end{aligned} \quad (3.23)$$

Obviously we obtain a similar equation when considering the variation with respect to  $X_2$ :

$$\begin{aligned} & \frac{\partial}{\partial t} \left[ \frac{\partial \mathcal{L}_2}{\partial \dot{X}_2} \right] + \operatorname{div} \left( (DX_2)^{-1} \det DX_2 \frac{\partial \mathcal{L}_2}{\partial d_2} \right) \\ &= \left[ \frac{\partial \mathcal{L}_1}{\partial \alpha_1} \left( \dot{X}_1, \det DX_1, \alpha_1, x \right) \right]_{|X_1^{-1} \circ X_2} D\alpha_2 (DX_2)^{-1} \det D (X_1^{-1} \circ X_2). \end{aligned} \quad (3.24)$$

Thus the Euler-Lagrange equations associated to the variation of the Hamiltonian action are given by (3.22), (3.23) and (3.24). As it will be seen in application, the Eulerian version of the above equations have a simpler form.

## 4 Recovering the Euler equations

Formally, a fluid is a CDM  $(\Omega_0, X)$  to which we associate two state functions  $(\rho^\ell, \eta^\ell)$  representing respectively the *density* and the *entropy* of the fluid. We denote

$$\rho_{|t=0}^\ell = \rho_0, \quad \eta_{|t=0}^\ell = \eta_0$$

the density and the entropy at the initial time  $t_0 = 0$ .

In the calculations that follow it may be sometimes convenient to work with the inverse of the density,

$$\tau^\ell := \frac{1}{\rho^\ell}, \quad (4.1)$$

which is called the *specific volume*.

Before derivating the equations describing the fluid motion, we need to precise two fundamental assumptions.

**Mass conservation** We assume that the physical system is closed, *i.e.* no matter is created or destroyed in the system. Then the mass  $M^e(t, X(t, \omega))$  at time  $t \geq 0$  of the material volume  $X(t, \omega)$  defined as

$$M^e(t, X(t, \omega)) := \int_{X(t, \omega)} \rho^e(t, y) dy = \int_{\omega} \rho^\ell(t, x) \det DX(t, x) dx,$$



remains constant in time. That is, for all  $\omega \subset \Omega_0$  and  $t \geq 0$

$$\int_{\omega} \rho^\ell(t, x) \det DX(t, x) dx = \int_{\omega} \rho^\ell(0, x) \det DX(0, x) dx = \int_{\omega} \rho_0(x) dx,$$

which amounts to ask that

$$\rho^\ell(t, x) \det DX(t, x) = \rho_0(x) \quad \forall t \geq 0, x \in \Omega_0. \quad (4.2)$$

Obviously the last identity, along with (4.1), yields

$$\tau^\ell(t, x) = \frac{\det DX(t, x)}{\rho_0(x)}.$$

**Transport of the entropy** We assume that the flow is reversible, *i.e.* the entropy is conserved along particle trajectories

$$\eta^\ell(t, x) = \eta_0(x) \quad \forall t \geq 0, x \in \Omega_0. \quad (4.3)$$

#### 4.1 The case of a compressible fluid

We define the *Lagrangian kinetic energy density* of the fluid as

$$T(\dot{X}, x) := \frac{1}{2} \rho_0(x) |\dot{X}|^2, \quad (4.4)$$

and the *Lagrangian potential energy density*

$$W(\det DX, x) := \rho_0(x) \varepsilon\left(\frac{\det DX}{\rho_0(x)}, \eta_0(x)\right) = \rho_0(x) \varepsilon(\tau^\ell, \eta^\ell). \quad (4.5)$$

where we call  $\varepsilon = \varepsilon(\tau, \eta)$  the (Lagrangian) internal energy per unit mass which related to the (Lagrangian) local pressure through the relation:

$$p(\tau, \eta) = -\partial_\tau \varepsilon(\tau, \eta). \quad (4.6)$$

We postulate that the Lagrangian that encodes the equations of movement of the fluid is defined as:

$$\begin{aligned} \mathcal{L}(\dot{X}, \det DX, x) &:= T(\dot{X}, x) - W(\det DX, x) \\ &= \frac{1}{2} \rho_0(x) |\dot{X}|^2 - \rho_0(x) \varepsilon\left(\frac{\det DX}{\rho_0(x)}, \eta_0(x)\right). \end{aligned}$$

We immediately see that

$$\frac{\partial \mathcal{L}}{\partial \dot{X}}(\dot{X}, \det DX, x) = \rho_0(x) \dot{X},$$

while, using (4.2), we get that

$$\frac{\partial \mathcal{L}}{\partial d}(\dot{X}, \det DX, x) = -\rho_0 \partial_\tau \varepsilon\left(\frac{\det DX}{\rho_0}, \eta_0\right) \frac{1}{\rho_0} = p(\tau^\ell, \eta^\ell).$$

Then, recalling that  $\dot{X}(t, x) = v^\ell(t, x)$ , the Euler-Lagrange equation (3.14) becomes

$$\rho_0 \partial_t v^\ell + \operatorname{div}((DX)^{-1} p(\tau^\ell, \eta^\ell) \det DX) = 0.$$

Multiplying the last equation by  $1/\det DX$  and using the lagrangian-eulerian passing formulae (3.2), (3.6) and (3.4) we get that

$$\rho^e (\partial_t v^e + v^e \cdot \nabla v^e) + \nabla p(\tau^e, \eta^e) = 0.$$

Let us summarize the above discussion. The Lagrangian equations for the evolution of a fluid are given by

$$\begin{cases} \rho^\ell \det DX = \rho_0, \\ \eta^\ell = \eta_0, \\ \rho_0 \partial_t v^\ell + \operatorname{div} ((DX)^{-1} p(\tau^\ell, \eta^\ell) \det DX) = 0. \end{cases}$$

In Eulerian coordinates, we recover the compressible Euler system

$$\begin{cases} \partial_t \rho^e + \operatorname{div}(\rho^e v^e) = 0, \\ \partial_t \eta^e + v^e \cdot \nabla \eta^e = 0, \\ \rho^e (\partial_t v^e + v^e \cdot \nabla v^e) + \nabla p(\tau^e, \eta^e) = 0. \end{cases} \quad (4.7)$$

## 4.2 Incompressible Euler equations

When the volume of any material element  $(X(t, \omega))_{t \geq 0}$  does not change in time, the flow is said to be *incompressible*. Remembering formula (3.7), the incompressibility condition reads

$$\det DX(t, x) = 1 \quad \forall t \geq 0, x \in \Omega_0, \quad (4.8)$$

from which we infer

$$\rho^\ell(t, x) = \rho_0(x) \quad \forall t \geq 0, x \in \Omega_0.$$

Since  $\det DX$  is constant, there is no energy coming from the deformation of the fluid and a naive choice for the Lagrangian would be:

$$\mathcal{L}(\dot{X}, \det DX, x) = T(\dot{X}, x) = \frac{1}{2} \rho_0(x) |\dot{X}|^2.$$

Nonetheless, recalling the discussion of Subsection 2.2, the constraint (4.8) leads us to consider the modified Lagrangian

$$\tilde{\mathcal{L}}(\dot{X}, \det DX, p^\ell) := \frac{1}{2} \rho_0(x) |\dot{X}|^2 + p^\ell (\det DX - 1). \quad (4.9)$$

The new variable  $p^\ell$  is called pressure of the fluid. However, it is important to note that the two pressures  $p^\ell$  and the ‘‘compressible pressure’’ (4.6) have different meanings. For an incompressible flow, the pressure is seen as a Lagrange multiplier associated to the constraint  $\det DX(t, x) = 1$ . Considering the variations with respect to  $\dot{X}$  and  $p^\ell$  of the modified Lagrangian, we have

$$\begin{cases} \frac{\partial \tilde{\mathcal{L}}}{\partial \dot{X}} = \rho_0(x) \dot{X}, \\ \frac{\partial \tilde{\mathcal{L}}}{\partial p^\ell} = p^\ell, \end{cases}$$

so that the Euler-Lagrange equations now read

$$\begin{cases} \rho^\ell = \rho_0, \\ \eta^\ell = \eta_0, \\ \rho_0 \partial_t v^\ell + \operatorname{div} ((DX)^{-1} p^\ell) = 0, \\ \det DX = 1. \end{cases}$$

Using (3.2), (3.6) and (3.4), we easily translate the first three equations in the Eulerian variables. In the Eulerian setting, we recover from the last equation the famous relation  $\operatorname{div} v^e = 0$  (we recall (3.8)). We thus have

$$\begin{cases} \partial_t \rho^e + u^e \cdot \nabla \rho^e = 0, \\ \partial_t \eta^e + u^e \cdot \nabla \eta^e = 0, \\ \rho^e (\partial_t v^e + v^e \cdot \nabla v^e) + \nabla p^e = 0, \\ \operatorname{div} v^e = 0. \end{cases}$$

## 5 Dynamics of two-phase flows

We have previously explained in Section 3.3 what we mean by a mixture of two species  $a \in \{1, 2\}$  occupying an initial configuration  $\Omega_0$  and that are animated by the two motions  $X_1, X_2$ . Under the “no-segregation assumption” (3.19), we saw that the system is also described by volume fractions of the two components which, as before, we call  $\alpha_1^\ell, \alpha_2^\ell$ . The relation linking them is

$$\alpha_1^\ell(t, x) + \alpha_2^\ell(t, X_2^{-1} \circ X_1(t, x)) = 1, \quad (5.1)$$

where, by  $X_2^{-1} \circ X_1(t, x)$ , we should understand  $X_2^{-1}(t, X_1(t, x))$ . Each component of the mixture is characterized by the state variables  $\rho_a$  and  $\eta_a$ . We define the apparent density  $\tilde{\rho}_a$  of the  $a$ -component which measures the density of the  $a$ -component in the mixture. It is different from the characteristic density, both being linked through the relation

$$\rho_a^\ell = \frac{\tilde{\rho}_a^\ell}{\alpha_a^\ell} \quad \text{for } a \in \{1, 2\}.$$

*We emphasize that the quantities  $(\tilde{\rho}_a^\ell, \eta_a^\ell)$  follow the particles at their own characteristic velocity so that  $\tilde{\rho}_1^\ell(t, x), \tilde{\rho}_2^\ell(t, x)$  do not represent in general the density in a same physical point !*

As in the single-phase case presented in the previous section, we assume that the mass of each phase is conserved, that is

$$\tilde{\rho}_a^\ell \det DX_a = \tilde{\rho}_{a,0} \quad \text{for } a \in \{1, 2\}, \quad (5.2)$$

where  $\tilde{\rho}_{1,0}, \tilde{\rho}_{2,0} : \Omega_0 \rightarrow [0, \infty)$  are the apparent densities at the initial time. Moreover, we assume that the entropies of the constituents remain constant along particle paths meaning that

$$\eta_a^\ell = \eta_{a,0}, \quad (5.3)$$

where  $\eta_{1,0}, \eta_{2,0} : \Omega_0 \rightarrow \mathbb{R}$  are the entropies at the initial time. We consider two functions modeling the internal energies of the components

$$\varepsilon_a = \varepsilon_a(\tau_a^\ell, \eta_a^\ell),$$

where  $\tau_a^\ell = 1/\rho_a^\ell$  and we denote by

$$p_a(\tau_a^\ell, \eta_a^\ell) = -\partial_{\tau_a^\ell} \varepsilon_a(\tau_a^\ell, \eta_a^\ell)$$

the corresponding pressures.

### 5.1 One velocity two-phase models

A first class of models for mixtures is obtained by assuming that the particles move with the same velocity, *i.e.*  $v_1 = v_2 = v$ , which amounts to making the following simplification:

$$X_1 = X_2 = X.$$

Therefore the following equations hold true

$$\begin{aligned} \alpha_1^\ell(t, x) + \alpha_2^\ell(t, x) &= 1 \quad \forall t \geq 0, x \in \Omega_0, \\ \tilde{\rho}_a^\ell \det DX &= \tilde{\rho}_{a,0}, \quad \eta_a^\ell = \eta_{a,0} \quad \text{for } a \in \{1, 2\}. \end{aligned}$$

We define the *Lagrangian kinetic energy density* (resp. *Lagrangian potential energy density*) of the mixture as the sum of the kinetic (resp. potential) energies of each component (of course this choice is questionable, see the discussion in Subsection 3.3):

$$T(\dot{X}, x) = \frac{1}{2} (\tilde{\rho}_{1,0}(x) + \tilde{\rho}_{2,0}(x)) |\dot{X}|^2,$$

$$W(\det DX, \alpha_1^\ell, x) = \tilde{\rho}_{1,0}(x) \varepsilon_1 \left( \frac{\alpha_1^\ell \det DX}{\tilde{\rho}_{1,0}(x)}, \eta_{1,0}(x) \right) + \tilde{\rho}_{2,0}(x) \varepsilon_2 \left( \frac{\alpha_2^\ell \det DX}{\tilde{\rho}_{2,0}(x)}, \eta_{2,0}(x) \right).$$

We postulate that the Lagrangian which encodes the dynamics of the mixture is defined as

$$\mathcal{L}(\dot{X}, \det DX, \alpha_1^\ell, x) = T(\dot{X}, x) - W(\det DX, \alpha_1^\ell, x).$$

The Euler-Lagrange equation associated to the variation of  $X$  is computed following the same steps as in the previous sections:

$$\frac{\partial \mathcal{L}}{\partial \dot{X}}(\dot{X}, \det DX, \alpha_1^\ell, \cdot) = (\tilde{\rho}_{1,0} + \tilde{\rho}_{2,0}) \dot{X} = (\tilde{\rho}_{1,0} + \tilde{\rho}_{2,0}) v^\ell,$$

and

$$\frac{\partial \mathcal{L}}{\partial d}(\dot{X}, \det DX, \alpha_1^\ell, \cdot) = \alpha_1^\ell p_1^\ell + \alpha_2^\ell p_2^\ell =: p^\ell,$$

where  $p^\ell$  is defined as the pressure of the mixture. Consequently, the Euler-Lagrange equation reads

$$(\tilde{\rho}_{1,0} + \tilde{\rho}_{2,0}) \partial_t v^\ell + \operatorname{div}((DX)^{-1} p^\ell \det DX) = 0.$$

The system is closed by considering the variation according to the volume fraction which leads to (3.18) (cf Section 3.2), *i.e.*

$$\frac{\partial \mathcal{L}}{\partial \alpha_1}(\dot{X}, \det DX, \alpha_1^\ell, x) = 0. \quad (5.4)$$

Since the kinetic energy is independent of  $\alpha_1$ , we just have to analyze the variation of the potential energy with respect to  $\alpha_1$ . We observe that

$$\frac{\partial W_a}{\partial \alpha_a} = \tilde{\rho}_{a,0} \partial_\tau \varepsilon_a \left( \frac{\alpha_a \det DX}{\tilde{\rho}_{a,0}}, \eta_{a,0} \right) \frac{\det DX}{\tilde{\rho}_{a,0}} \quad \text{for } a \in \{1, 2\}.$$

Hence, taking into account the constraint

$$\alpha_1^\ell + \alpha_2^\ell = 1,$$

Equation (5.4) yields

$$p_1^\ell \det DX = p_2^\ell \det DX.$$

Gathering the equations, we obtain that

$$\begin{cases} \eta_a^\ell = \eta_{a,0}, \\ \alpha_a^\ell \rho_a^\ell \det DX = \tilde{\rho}_{a,0}, \\ \rho_0 \partial_t v^\ell + \operatorname{div}((DX)^{-1} (p^\ell) \det DX) = 0, \\ p^\ell = p_1^\ell = p_2^\ell, \end{cases}$$

where have set

$$\rho_0 := \tilde{\rho}_{1,0} + \tilde{\rho}_{2,0}.$$

Using the relations (3.4), (3.3) and (3.6), we obtain the corresponding Eulerian equations which are

$$\begin{cases} \partial_t \eta_a^e + v^e \cdot \nabla \eta_a^e = 0, \\ \partial_t (\alpha_a^e \rho_a^e) + \operatorname{div}(\alpha_a^e \rho_a^e v^e) = 0, \\ \rho^e (\partial_t v^e + v^e \cdot \nabla v^e) + \nabla p^e = 0, \\ p^e = p_1^e = p_2^e, \end{cases} \quad (5.5)$$

where the mixture density is defined as

$$\rho^e := \alpha_1^e \rho_1^e + \alpha_2^e \rho_2^e.$$

The system (5.5) can obviously be generalized to any number of components. It is sometime called *Kapila's model* and was obtained in [Kapila et al., 2001].

From the mathematical point of view, one can show that the equations (5.5) are hyperbolic if

$$\frac{\partial p_a}{\partial \tau_a} < 0 \quad \text{for any } a \in \{1, 2\}. \quad (5.6)$$

The sound speed  $c_W$  in such a system is given by the *Wood formula* :

$$\frac{1}{\rho^e c_W^2} = \sum_a \frac{\alpha_a}{\rho_a^e c_a^2}, \quad (5.7)$$

where  $c_a$  is the sound velocity of  $a$ -th component, *i.e.*

$$c_a^2 = -\tau_a^2 \frac{\partial p_a}{\partial \tau_a}.$$

## 5.2 Two velocities, one pressure models

We assume that the local kinetic and potential energy densities are given by

$$\begin{cases} E_a(\dot{X}_a, x) = \frac{1}{2} \tilde{\rho}_{a,0}(x) |\dot{X}_a|^2, \\ W_a(\dot{X}_a, \det DX_a, \alpha_a^\ell, x) = \tilde{\rho}_{a,0} \varepsilon_a \left( \frac{\alpha_a^\ell \det DX_a}{\tilde{\rho}_{a,0}}, \eta_{a,0} \right). \end{cases}$$

As we discussed in Section 3.3, the Lagrangian encoding the dynamic of the mixture is given by

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_1 + \mathcal{L}_2 \\ &= (T_1 - W_1) + (T_2 - W_2) \end{aligned}$$

where, of course, it is to be considered that  $X_1, X_2$  and  $\alpha_1^\ell$  are independent. In order to obtain the Euler-Lagrange equations we just have to express the abstract equations (3.22), (3.23) and (3.24) obtained in Section 3.3.

We begin with

$$\frac{\partial \mathcal{L}_a}{\partial \alpha_a} = -\frac{\partial W_a}{\partial \alpha_a} = \tilde{\rho}_{a,0} p_a \left( \frac{\alpha_a^\ell \det DX_a}{\tilde{\rho}_{a,0}}, \eta_{a,0} \right) \frac{\det DX_a}{\tilde{\rho}_{a,0}} = p_a(\tau_a^\ell, \eta_a^\ell) \det DX_a.$$

Also, we have that

$$\begin{aligned} & \left[ \frac{\partial \mathcal{L}_2}{\partial \alpha_2}(\dot{X}_2, \det DX_2, \alpha_2^\ell, x) \right]_{|X_2^{-1} \circ X_1} \det D(X_2^{-1} \circ X_1) \\ &= p_2 \left( \frac{1 - \alpha_1^\ell}{\tilde{\rho}_2^\ell \circ X_2^{-1} \circ X_1}, \eta_2^\ell \right) \det DX_2(X_2^{-1} \circ X_1) \det D(X_2^{-1} \circ X_1) \\ &= p_2 \left( \frac{1 - \alpha_1^\ell}{\tilde{\rho}_2^\ell \circ X_2^{-1} \circ X_1}, \eta_2^\ell \circ X_2^{-1} \circ X_1 \right) \det DX_1. \end{aligned} \quad (5.8)$$

We see that the equation becomes

$$\begin{aligned} & p_1 \left( \frac{\alpha_1^\ell(t, x)}{\tilde{\rho}_1^\ell(t, x)}, \eta_1^\ell(t, x) \right) \det DX_1 \\ &= p_2 \left( \frac{1 - \alpha_1^\ell}{\tilde{\rho}_2^\ell \circ X_2^{-1} \circ X_1}, \eta_2^\ell(t, X_2^{-1} \circ X_1(t, x)) \right) \det DX_1 \end{aligned} \quad (5.9)$$

Obviously the last equation says that the Eulerian pressures are equal

$$p_1(\tau_1^e, \eta_1^e) = p_2(\tau_2^e, \eta_2^e). \quad (5.10)$$

Let us express the equation coming from the variation of  $X_1$ . Using (5.8) along with (5.9) we get that

$$\begin{aligned} & \left[ \frac{\partial \mathcal{L}_2}{\partial \alpha_2} \left( \dot{X}_2, \det DX_2, \alpha_2^\ell, x \right) \right]_{|X_2^{-1} \circ X_1} D\alpha_1^\ell (DX_1)^{-1} \det D(X_2^{-1} \circ X_1) \\ &= p_2 \left( \frac{1 - \alpha_1^\ell}{\tilde{\rho}_2^\ell \circ X_2^{-1} \circ X_1}, \eta_2^\ell \circ X_2^{-1} \circ X_1 \right) \det DX_1 D\alpha_1^\ell (DX_1)^{-1} \\ &= p_1^\ell \det DX_1 D\alpha_1^\ell (DX_1)^{-1} \end{aligned}$$

and thus the corresponding Euler-Lagrange equation is

$$\tilde{\rho}_{\alpha,0} \partial_t v_a^\ell + \operatorname{div} \left( (DX_a)^{-1} \alpha_a^\ell p_a^\ell \det DX_a \right) = \det DX_a p_1^\ell D\alpha_a^\ell (DX_a)^{-1} \quad (5.11)$$

or, in Eulerian variables

$$\alpha_a^e \rho_a^e \frac{Dv_a^e}{Dt} + \nabla (\alpha_a^e p_a^e) = p_a^e \nabla \alpha_a^e.$$

Gathering all the equations, we find that

$$\begin{cases} \eta_a^\ell = \eta_{a,0}, \\ \alpha_a^\ell \rho_a^\ell \det DX_a = \tilde{\rho}_{\alpha,0}, \\ \tilde{\rho}_{\alpha,0} \partial_t v_a^\ell + \operatorname{div} \left( (DX_a)^{-1} \alpha_a^\ell p_a^\ell \det DX_a \right) = \det DX_a p_1^\ell D\alpha_a^\ell (DX_a)^{-1}, \\ \alpha_1^\ell + \alpha_2^\ell|_{X_2^{-1} \circ X_1} = 1, \\ p_1^\ell = [p_2^\ell]_{|X_2^{-1} \circ X_1}. \end{cases}$$

Using the relations (3.4), (3.3) and (3.6) we obtain the corresponding Eulerian equations which are

$$\begin{cases} \partial_t \eta_a^e + v_a^e \cdot \nabla \eta_a^e = 0, \\ \partial_t (\alpha_a^e \rho_a^e) + \operatorname{div} (\alpha_a^e \rho_a^e v_a^e) = 0, \\ \partial_t (\alpha_a^e \rho_a^e v_a^e) + \operatorname{div} (\alpha_a^e \rho_a^e v_a^e \otimes v_a^e) + \nabla (\alpha_a^e p_a^e) = \alpha_a^e \nabla p_a^e, \\ \alpha_1^e + \alpha_2^e = 1, \\ p_1^e = p_2^e. \end{cases} \quad (5.12)$$

A simple analysis shows that, even if the equations admit the conservation of the total momentum and total energy, they cannot be rewritten as a conservative system of equations: the number of conservation laws admitted by the system is smaller than the number of unknowns. A non-classical definition of weak solutions is thus needed. Moreover, for small relative velocity  $w = v_1 - v_2$  the equations are not hyperbolic: complex characteristics appear. Thus the well-posedness of the Cauchy problem for (5.12) is questionable.

A possibility to evacuate the problem of complex characteristics is to use *relaxation methods*. Let us consider the system

$$\begin{cases} \alpha_a \rho_a^e \theta_a^e (\partial_t \eta_a^e + v_a^e \cdot \nabla \eta_a^e) = f_a, \\ \partial_t (\alpha_a^e \rho_a^e) + \operatorname{div} (\alpha_a^e \rho_a^e v_a^e) = 0, \\ \alpha_1^e \rho_1^e (\partial_t v_1^e + v_1^e \cdot \nabla v_1^e) + \nabla (\alpha_1^e p_1^e) = \alpha_1^e \nabla p_1^e + \lambda (v_2^e - v_1^e), \\ \alpha_2^e \rho_2^e (\partial_t v_2^e + v_2^e \cdot \nabla v_2^e) + \nabla (\alpha_2^e p_2^e) = \alpha_2^e \nabla p_2^e - \lambda (v_2^e - v_1^e), \\ \alpha_1^e + \alpha_2^e = 1, \\ \partial_t \alpha_1^e + v_1^e \cdot \nabla \alpha_1^e = \mu (p_1^e - p_2^e). \end{cases} \quad (5.13)$$

Here  $\theta_a$  are the phase temperatures :  $\theta_a = \partial_{\eta_a} \varepsilon_a$ ,  $p_I$  and  $v_I$  are the interface pressure and velocity, respectively,  $\lambda$  is the friction coefficient, and  $f_a$  are the entropy production terms. The parameter

$\mu$  is large, so we formally recover the “equal pressure” model as  $\mu \rightarrow \infty$  whereas we recover a “one-velocity” model in the limit  $\lambda \rightarrow \infty$ .

The total energy equation and the following entropy inequality

$$\partial_t \left( \sum_a \alpha_a^e \rho_a^e \eta_a^e \right) + \operatorname{div} \left( \sum_a \alpha_a^e \rho_a^e \eta_a^e v_a^e \right) = \sum_a \frac{f_a}{\theta_a} \geq 0 \quad (5.14)$$

allow us to find simple (phenomenological) expressions for  $f_a$  [Saurel et al., 2003, Gavriluk, 2011]. The system (5.13) is also called *Baer-Nunziato model* [Baer and Nunziato, 1986] (in the case  $v_I = v_1$ ,  $p_I = p_2$  or *vice versa*). It is weakly hyperbolic, *i.e.* all eigenvalues are real but one cannot guarantee a full set of the eigenvectors in the resonant case (eigenvectors can coincide on some hypersurfaces [Embed and Baer, 1992], [Andrianov and Warnecke, 2004]). The choice of interfacial variables  $v_I$  and  $p_I$  is not unique see [Hérard and Hurisse, 2005] and [Saurel et al., 2003].

Let us mention that the relaxation equation for the volume fraction (last equation of (5.13)) can be obtained from the barotropic Navier-Stokes equations by a homogenization procedure [Bresch and Huang, 2011, Bresch and Hillaire, 2019, Bresch and Hillaire, 2015, Hillaire, 2007, Bresch et al., 2020]. Loosely speaking, if the density of a fluid mixture wildly oscillates between two reference densities, the volume fractions  $\alpha_a^e$  can be interpreted as the probability to find a particle of fluid  $a$  at a given position and time instant. The coefficient  $\mu$  in such a homogenization procedure is the inverse of the mean viscosity of the two phases.

## Conclusion

The aim of this short course was the introduction into variational approach to the modeling of fluid mixtures. The only scalar function we needed to know is the full energy of the system (or, equivalently, the Lagrangian of the mechanical system). Due to the Noether theorem, this approach guarantees the fulfillment of the basic physical conservation laws, but, *a priori*, not well-posedness in the sense of Hadamard of the governing equations. The introduction of dissipative terms can provide the well-posedness. The simplest and most popular models of fluid mixtures (5.5) and (5.13) were derived.

The models obtained can also be used for the description of the interface between pure fluids. For example, in the problem of shock-interface interaction the interface between fluids becomes unstable (Meshkov-Richtmyer instability) and can be considered as a mixture of fluids [Karni, 1994, Saurel and Abgrall, 1999]. This method called also the method of diffuse interfaces, was recently extended to the problems involving solid-fluid mixtures [Ndanou et al., 2015] and to the description of interfaces with capillary effects [Schmidmayer et al., 2017].

The method can be extended to more complex situations involving mixtures of continua depending on higher order deformation gradients and the time derivatives of the deformation gradient, like bubbly fluids or the Green-Naghdi equations for dispersive shallow water flows (see [Gavriluk, 2011]).

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