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LORENTZIAN 3-MANIFOLDS AND DYNAMICAL SYSTEMS

CHARLES FRANCES

1. AN INVITATION TO LORENTZIAN DYNAMICS

The following article aims at presenting classical aspects of dynamical systems preserving a geometric structure, focusing on 3-dimensional Lorentzian dynamics. The reader won't find here any new result, but rather an expository approach of classical ones. Our main goal, in particular, is to introduce part of the techniques and arguments used in [Fraa] to obtain the classification of closed 3-dimensional Lorentzian manifolds admitting a non-compact isometry group. Doing so, we will present a self-contained, and somehow shortened proof of Zeghib's classification [Zeg96] of non-equicontinuous Lorentzian isometric flows in dimension 3.

1.1. From dynamical systems to geometry. A flow φ_X^t , generated by a vector field X , on a closed manifold M is said to be Anosov when it is non-singular, and there exists a φ_X^t -invariant splitting $TM = E^- \oplus \mathbb{R}X \oplus E^+$ such that vectors of E^- (resp. of E^+) are exponentially contracted (resp. dilated) under φ_X^t . Precisely, there exist positive constants c and λ such that

$$\|D\varphi_X^t(u^-)\| \leq ce^{-\lambda t}\|u^-\|$$

for every $u^- \in E^-$, $t \geq 0$, and

$$\|D\varphi_X^{-t}(u^+)\| \leq ce^{-\lambda t}\|u^+\|$$

for every $u^+ \in E^+$, $t \geq 0$. Here, the norms are taken with respect to any auxiliary Riemannian metric on M .

Examples of Anosov flows are provided by the geodesic flows of a closed, negatively curved, Riemannian manifolds (N, h) (the manifold M then corresponds to T^1N , the unit tangent bundle of N).

The very definition of Anosov flows already involves a (weak) geometric structure, namely the splitting $TM = E^- \oplus \mathbb{R}X \oplus E^+$. A beautiful aspect of the theory is that under certain circumstances, additional and more rigid invariant structures naturally appear. Assume, for simplicity, that our manifold M is 3-dimensional, so that E^- and E^+ are 1-dimensional. Assume moreover that the flow φ_X^t preserves a smooth volume form ω (this is for instance the case if φ_X^t is the geodesic flow of a smooth Riemannian surface). One can then define a metric g on M , given by the relations $g(X, X) = 1$, $g(u^-, u^+) = \omega(X, u^-, u^+)$ for every $(u^-, u^+) \in E^- \times E^+$, and

$\mathbb{R}X$ is g -orthogonal to $E^- \oplus E^+$. Observe that this metric is Lorentzian, namely has signature $(-, +, +)$, and is φ_X^t -invariant by construction. Our volume-preserving condition thus turned our Anosov flow into an isometric Lorentzian flow.

All this nice picture is however damaged by regularity issues. In full generality, the splitting $E^- \oplus \mathbb{R}X \oplus E^+$ is only Hölder continuous, and so is our Lorentzian metric g . This low regularity is a strong limitation (at least until now) to take advantage of this new geometrical data. But if one assumes that the distributions E^- and E^+ are smooth (as well as the field X), then all the tools of differential geometry can be used to understand the dynamical system (M, φ_X^t) .

It seems that Kanai [Kan88] was one of the first to apply this kind of ideas in the study of the geodesic flow of negatively curved surfaces. E. Ghys went further and obtained the following beautiful classification of 3-dimensional Anosov flows with smooth distributions E^-, E^+ .

Theorem 1.1. [Ghy87] *Let φ_X^t be a smooth, orientable, Anosov flow on a closed 3-dimensional manifold M . Then φ_X^t is either the suspension of an Anosov diffeomorphism on the 2-torus, or smoothly orbit equivalent to the right diagonal flow, acting on a compact quotient $\Gamma \backslash \widetilde{\mathrm{SL}}(2, \mathbb{R})$.*

Actually, Ghys did not use an invariant Lorentzian metric to achieve his classification, but rather another rigid geometric structure often called “contact Lagrangian”. This structure comes from the geometrization of second order *ODE*’s, and its study can be traced back to E. Cartan. Theorem 1.1 can also be obtained using Lorentzian geometry, as will be seen in this article.

In arbitrary dimension, when the distributions E^+, E^- are smooth, and $E^+ \oplus E^-$ defines a contact distribution, then an Anosov flow φ_X^t preserves a smooth pseudo-Riemannian (not Lorentzian in general) metric, what allowed Y. Benoist, P. Foulon and F. Labourie to extend the initial 3-dimensional classification to all contact Anosov flows on closed manifolds, with smooth stable and unstable distributions (see [BFL92],[BL93]). A particularly inspiring discussion of that topic can be found in [DG91, Section 2].

1.2. Some nice Lorentzian dynamical systems. We now illustrate what was said before on a few paradigmatic examples.

1.2.1. Suspensions of Anosov diffeomorphisms. We call \mathfrak{sol} the 3-dimensional Lie algebra generated by T, Y, Z , with bracket relations $[T, Y] = Y$, $[T, Z] = -Z$ and $[Y, Z] = 0$. By, SOL, we mean the associated connected, simply connected Lie group.

There is an identification of SOL as the following subgroup of affine transformations:

$$\text{SOL} \simeq \left\{ \left(\begin{pmatrix} e^t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-t} \end{pmatrix} + \begin{pmatrix} u \\ t \\ v \end{pmatrix} \mid (t, u, v) \in \mathbb{R}^3 \right\}.$$

It is easily seen that this group acts freely and transitively on \mathbb{R}^3 , preserving the flat Lorentz metric $dx_1 dx_3 + dx_2^2$. Let us identify SOL with \mathbb{R}^3 via the orbit map $g \mapsto g.0$. If h^t denotes the 1-parameter group generated by the element T , then the isometric flow $x \mapsto x + te_2$ corresponds to the right multiplication by h^t . To summarize, we have endowed SOL with a complete flat metric, invariant by the group $\text{SOL} \times \{h^t\}$.

If we now pick Γ a lattice in SOL, and look at the right action of h^t on $\Gamma \backslash \text{SOL}$, we get an isometric flow on the (flat) Lorentzian manifold $\Gamma \backslash \text{SOL}$. It is easily checked that $\Gamma \backslash \text{SOL}$ is diffeomorphic to a hyperbolic torus bundle \mathbb{T}_A^3 , with $A \in \text{SL}(2, \mathbb{Z})$ a hyperbolic matrix, and the right action of h^t is the suspension flow of the action of A on \mathbb{T}^2 .

1.2.2. Examples derived from geodesic and horocyclic flows on hyperbolic surfaces. The Lie group $\text{PSL}(2, \mathbb{R})$ admits a lot of interesting left-invariant Lorentzian metric. The most symmetric one is the *anti-de Sitter* metric g_{AdS} . It is obtained by left-translating the Killing form of the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$. The space $(\text{PSL}(2, \mathbb{R}), g_{AdS})$ is a complete Lorentz manifold with constant sectional curvature -1 , called anti-de Sitter space \mathbf{AdS}_3 . Because the Killing form is Ad-invariant, the metric g_{AdS} is invariant by left and right multiplications of $\text{PSL}(2, \mathbb{R})$ on itself. It follows that for any uniform lattice $\Gamma \subset \text{PSL}(2, \mathbb{R})$, the metric g_{AdS} induces a Lorentz metric \bar{g}_{AdS} on the quotient manifold $\Gamma \backslash \text{PSL}(2, \mathbb{R})$, with a non-compact isometry group coming from the right action of $\text{PSL}(2, \mathbb{R})$ on $\Gamma \backslash \text{PSL}(2, \mathbb{R})$.

Exponentiating the matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ (resp. $\begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}$), one gets a unipotent (resp. \mathbb{R} -split) flows $\{u^t\}$ (resp. $\{h^t\}$) in $\text{PSL}(2, \mathbb{R})$. The right action of h^t (resp. of u^t) on $\Gamma \backslash \text{PSL}(2, \mathbb{R})$ is an algebraic model for the horocyclic (resp. geodesic) flow on the unit tangent bundle of the hyperbolic surface $\Gamma \backslash \mathbb{H}^2$ (when Γ is torsionfree).

A remarkable feature is the possibility of deforming in a non-trivial way the previous examples. Consider a representation $\rho_h : \Gamma \rightarrow \{h^t\}$ (resp. $\rho_u : \Gamma \rightarrow \{u^t\}$), and the group $\Gamma_{\rho_h} \subset \text{PSL}(2, \mathbb{R}) \times \text{PSL}(2, \mathbb{R})$ (resp. Γ_{ρ_u}) defined by $\{(\gamma, \rho_h(\gamma)) \mid \gamma \in \Gamma\}$ (resp. $\{(\gamma, \rho_u(\gamma)) \mid \gamma \in \Gamma\}$). When the representations ρ_h and ρ_u are close enough to the trivial representation, the groups Γ_{ρ_h} and Γ_{ρ_u} act properly discontinuously on $\text{PSL}(2, \mathbb{R})$, preserving the metric g_{AdS} .

We get in this way (anti-de Sitter) Lorentz manifolds $\Gamma_{\rho_h} \backslash \text{PSL}(2, \mathbb{R})$ and $\Gamma_{\rho_u} \backslash \text{PSL}(2, \mathbb{R})$, with a (right) isometric action of $\{h^t\}$ and $\{u^t\}$ respectively. While the topological type of the closed manifolds obtained is the same as that of $\Gamma \backslash \text{PSL}(2, \mathbb{R})$, the new flow $\{h^t\}$ is no longer conjugated, as a flow, to the geodesic flow. This was noticed by Ghys in [Ghy87, Section 2]. However,

there still exists a smooth orbit equivalence between the deformed flow and the initial one [Ghy87, Prop 2.4].

The adjoint action of each of the groups $\{h^t\}$ and $\{u^t\}$, admits invariant Lorentz scalar products on $\mathfrak{sl}(2, \mathbb{R})$, which are not equal to a multiple of the Killing form. One can left-translate those scalar products and get metrics g_u and g_h on $\mathrm{PSL}(2, \mathbb{R})$ which are respectively $\mathrm{PSL}(2, \mathbb{R}) \times \{u^t\}$ and $\mathrm{PSL}(2, \mathbb{R}) \times \{h^t\}$ -invariant. Actually there are families of such metrics g_u and g_h which are not pairwise isometric. In the sequel, the metric g_{AdS} and metrics of the form g_u or g_h , will be referred to as *Lorentzian, non-Riemannian, left-invariant metrics on $\mathrm{PSL}(2, \mathbb{R})$* . Those are the only left-invariant metrics on $\mathrm{PSL}(2, \mathbb{R})$, the isometry group of which does not preserve a Riemannian metric. Such metrics induce Lorentzian metrics \bar{g}_h (resp. \bar{g}_u) on the quotients $\Gamma_{\rho_h} \backslash \mathrm{PSL}(2, \mathbb{R})$ (resp. $\Gamma_{\rho_u} \backslash \mathrm{PSL}(2, \mathbb{R})$) which are invariant by the right action of $\{h^t\}$ (resp. of $\{u^t\}$).

1.3. Classification results. The previous sections are a motivation for the general study of 3-dimensional Lorentzian manifolds (M^3, g) , admitting an isometric flow φ_X^t which is “dynamically interesting”, namely which does not have compact closure in $\mathrm{Iso}(M, g)$.

Theorem 1.2. [Zeg96, Theorems 1 and 2] *Let (M, g) be a smooth, closed 3-dimensional Lorentz manifold. Let φ_X^t be a flow of Lorentzian isometries which is not relatively compact in $\mathrm{Iso}(M, g)$, then up to finite cover, and after replacing eventually φ_X^t by $\varphi_X^{\alpha t}$ for $\alpha \in \mathbb{R}^*$, we are in exactly one of the following situations:*

- (1) *The manifold M is a torus bundle \mathbb{T}_A^3 , $A \in \mathrm{SL}(2, \mathbb{Z})$ hyperbolic, and g is a flat metric. The flow φ_X^t is the suspension of the Anosov diffeomorphism A .*
- (2) *The manifold (M, g) is isometric to a quotient $\Gamma_{\rho_h} \backslash \mathrm{PSL}(2, \mathbb{R})$ (resp. $\Gamma_{\rho_u} \backslash \mathrm{PSL}(2, \mathbb{R})$) endowed with a metric \bar{g}_{AdS} or \bar{g}_h (resp. \bar{g}_{AdS} or \bar{g}_u). The flow φ_X^t coincides with the right action of $\{h^t\}$ (resp. of $\{u^t\}$).*

By “up to finite cover”, we mean that we allow taking finite covers and quotients. This statement implies Theorem 1.1 (at least for Anosov flows preserving a smooth volume form, as it is the case for instance for contact flows). Let us observe however, that the proof given in [Zeg96] uses Ghys’s result. We will try in the following to provide an alternative proof (which will be independant of Theorem 1.1).

The condition that there exists in $\mathrm{Iso}(M, g)$ a 1-parameter group which is not relatively compact in $\mathrm{Iso}(M, g)$, amounts to the non-compactness of the identity component $\mathrm{Iso}^o(M, g)$. One may wonder if Theorem 1.2 concludes the classification of 3-dimensional closed Lorentzian manifolds having a non-compact isometry group.

The answer is negative as the simple following example shows. Let us consider \mathbb{R}^3 with the flat metric $-dx_1^2 + dx_2^2 + dx_3^2$. This metric is translation

invariant, hence induces a metric \bar{g}_{flat} on the 3-torus \mathbb{T}^3 (obtained as the quotient of \mathbb{R}^3 by the action of \mathbb{Z}^3 by translations). The isometry group $\text{Iso}(\mathbb{T}^3, \bar{g}_{flat})$ is $\text{O}(1, 2)_{\mathbb{Z}} \times \mathbb{T}^3$, hence non-compact (recall that $\text{O}(1, 2)_{\mathbb{Z}}$ is a lattice in $\text{O}(1, 2)$). We see on this example, that the non-compactness comes from the discrete part of the isometry group. Other 3-dimensional closed Lorentz manifolds display this property (see [Fraa, Section 2]), and some of them (as \mathbb{T}^3) have a topological type which is not represented by the manifolds of Theorem 1.2. A complete description of the Lorentz 3-manifolds (M, g) for which $\text{Iso}(M, g)$ is non-compact was recently given in [Fraa].

Theorem 1.3. [Fraa, Theorem A] *Let (M, g) be a smooth, closed 3-dimensional Lorentz manifold. Assume that (M, g) is orientable and time-orientable, and that $\text{Iso}(M, g)$ is non-compact. Then M is homeomorphic to one of the following spaces:*

- (1) *A quotient $\Gamma \backslash \widetilde{\text{PSL}}(2, \mathbb{R})$, where $\Gamma \subset \widetilde{\text{PSL}}(2, \mathbb{R})$ is any uniform lattice.*
- (2) *A 3-torus \mathbb{T}^3 , or a torus bundle \mathbb{T}^3_A , where $A \in \text{SL}(2, \mathbb{Z})$ can be any hyperbolic or parabolic element.*

Conversely, any smooth compact 3-manifold homeomorphic to one of the examples above can be endowed with a smooth Lorentz metric with a non-compact isometry group.

Our aim in this article is to present some ideas involved in the proof of Theorem 1.3, and apply them to give a self-contained proof of Theorem 1.2.

2. CURVATURE, KILLING FIELDS AND THE INTEGRABILITY THEOREM

We consider in this section a 3-dimensional Lorentz manifold (M, g) (actually, the material presented in this section fits into the much more general framework of Cartan geometries, but we stick to the Lorentz case for the sake of simplicity).

Let $\pi : \hat{M} \rightarrow M$ denote the bundle of orthonormal frames on \hat{M} . This is a principal $\text{O}(1, 2)$ -bundle over M , and it is classical (see [KN63][Chap. IV.2]) that the Levi-Civita connection associated to g can be interpreted as an Ehresmann connection α on \hat{M} , namely a $\text{O}(1, 2)$ -equivariant 1-form with values in the Lie algebra $\mathfrak{o}(1, 2)$. Let θ be the soldering form on \hat{M} , namely the \mathbb{R}^3 -valued 1-form on \hat{M} , which to every $\xi \in T_{\hat{x}}\hat{M}$ associates the coordinates of the vector $\pi_*(\xi) \in T_x M$ in the frame \hat{x} . The sum $\alpha + \theta$ is a 1-form $\omega : T\hat{M} \rightarrow \mathfrak{o}(1, 2) \times \mathbb{R}^3$ called the *canonical Cartan connection* associated to (M, g) .

Observe that for every $\hat{x} \in \hat{M}$, $\omega_{\hat{x}} : T_{\hat{x}}\hat{M} \rightarrow \mathfrak{o}(1, 2) \times \mathbb{R}^3$ is an isomorphism of vector spaces, and the form ω is $\text{O}(1, 2)$ -equivariant (where $\text{O}(1, 2)$ acts on $\mathfrak{o}(1, 2) \times \mathbb{R}^3$ via the adjoint action).

2.1. Killing fields and $\mathfrak{kill}^{\text{loc}}$ -orbits. Recall that a local Killing field on (M, g) is a vector field Z , defined on some open subset $U \subset M$, such that the Lie derivatives $L_Z g$ vanishes identically. In other words, the local flow φ_Z^t consists in local isometries for g .

Given an open set U , the Killing fields on U define a finite-dimensional Lie algebra $\mathfrak{kill}(U)$. Hence, given a point $x \in M$, and a decreasing sequence (U_n) of nested neighborhoods of x , the isomorphism type of the Lie algebras $\mathfrak{kill}(U_n)$ stabilizes for n large. This define a Lie algebra $\mathfrak{kill}^{\text{loc}}(x)$ that we will call (abusively) the Lie algebra of Local Killing fields around x . For 3-dimensional Lorentz manifolds, the dimension of $\mathfrak{kill}^{\text{loc}}(x)$ is at most 6, with equality when g has constant curvature in a neighborhood of x . For analytic structures, the analytic continuation property for Killing fields ensures that the algebraic type of $\mathfrak{kill}^{\text{loc}}(x)$ does not depend on x . In the smooth category, which is the one we consider here, the situation is very different. The dimension, as well as the algebraic type of $\mathfrak{kill}^{\text{loc}}(x)$ may not be constant.

Observe that because local isometries map orthonormal frames to orthonormal frames, any local isometry lifts to a local bundle isomorphism on \hat{M} , which preserves ω . In the same way, any local Killing field Z defined on U lifts to a vector field defined on $\hat{U} := \pi^{-1}(U)$ which satisfies $L_Z \omega = 0$. In particular Z commutes with the $O(1, 2)$ -action on \hat{M} . Conversely, any vector field defined on a connected open subset of \hat{M} , and such that $L_Z \omega = 0$, projects to a local Killing field on M . We will thus indistinctly speak about local Killing fields on M and on \hat{M} .

The $\mathfrak{kill}^{\text{loc}}$ -orbit of a point x , is the set of points that can be reached from x , by flowing along finitely many successive local Killing fields.

2.2. Curvature. We assume now that the structures considered are of class C^∞ . The notion of Riemannian curvature for g , as well as its higher order covariant derivatives have a counterpart in \hat{M} . Actually, we will see that it is really fruitful to work on the bundle \hat{M} instead of the manifold M itself. In all the sequel, we will denote by \mathfrak{g} the Lie algebra $\mathfrak{o}(1, 2) \times \mathbb{R}^3$.

The *curvature* of the Cartan connection ω is a 2-form K on \hat{M} , with values in \mathfrak{g} , defined as follows. If X and Y are two vector fields on \hat{M} , the curvature is given by the relation:

$$K(X, Y) = d\omega(X, Y) + [\omega(X), \omega(Y)].$$

Because at each point \hat{x} of \hat{M} , the Cartan connection ω establishes an isomorphism between $T_{\hat{x}}\hat{M}$ and \mathfrak{g} , it follows that any k -differential form on \hat{M} , with values in some vector space \mathcal{W} , can be seen as a map from \hat{M} to $\text{Hom}(\otimes^k \mathfrak{g}, \mathcal{W})$. This remark applies for the curvature form K itself, yielding a *curvature map* $\kappa : \hat{M} \rightarrow \mathcal{W}_0$, where the vector space \mathcal{W}_0 is a sub $O(1, 2)$ -module of $\text{Hom}(\wedge^2(\mathfrak{g}/\mathfrak{o}(1, 2)); \mathfrak{g})$ (the curvature is antisymmetric and vanishes when one argument is tangent to the fibers of \hat{M}).

2.2.1. *The curvature module.* We give here more algebraic informations on the curvature module \mathcal{W}_0 in our special case of a 3-dimensional Lorentz manifold.

Let us consider on \mathbb{R}^3 the Lorentzian form, with matrix in a basis e, h, f given by $J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

We will identify in all what follows the group $O(1, 2)$ with the subgroup of $GL(3, \mathbb{R})$ preserving the bilinear form determined by J . Its Lie algebra is denoted by $\mathfrak{o}(1, 2)$, and admits the following basis :

$$E = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, F = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}.$$

Notice the commutation relations $[H, E] = E$, $[H, F] = -F$ and $[E, F] = H$.

Under the natural identification of $\mathfrak{g}/\mathfrak{o}(1, 2)$ with \mathbb{R}^3 , our curvature module \mathcal{W}_0 is a 6-dimensional $\text{Hom}(\wedge^2(\mathbb{R}^3), \mathfrak{o}(1, 2))$ (this constraint on the dimension comes from Bianchi's identities). Choosing $e \wedge h, e \wedge f, h \wedge f$ as a basis for $\wedge^2(\mathbb{R}^3)$, and E, H, F as a basis for $\mathfrak{o}(1, 2)$, an element of $\text{Hom}(\wedge^2(\mathbb{R}^3), \mathfrak{o}(1, 2))$ is merely given by a 3×3 matrix. The action of $O(1, 2)$ on $\text{Hom}(\wedge^2(\mathbb{R}^3), \mathfrak{o}(1, 2))$ corresponds to the conjugation on matrices.

Scalar matrices are $O(1, 2)$ -invariant, and form a 1-dimensional irreducible submodule (corresponding to constant sectional curvature).

The other irreducible submodule of the curvature module is 5-dimensional, spanned by the matrices:

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

2.2.2. *Generalized curvature map.* We now differentiate the map κ , getting a map $D\kappa : T\hat{M} \rightarrow \mathcal{W}_0$. As we already explained it, the connection ω allows to identify $D\kappa$ with a map $\mathcal{D}\kappa : \hat{M} \rightarrow \mathcal{W}_1$, where $\mathcal{W}_1 = \text{Hom}(\mathfrak{g}, \mathcal{W}_0)$. We can then differentiate once again and see $D(\mathcal{D}\kappa)$ as a map $\mathcal{D}^2\kappa : \hat{M} \rightarrow \mathcal{W}_2 = \text{Hom}(\mathfrak{g}, \mathcal{W}_1)$. Repeating this procedure, we can define the r th-derivative of the curvature $\mathcal{D}^r\kappa : \hat{M} \rightarrow \text{Hom}(\mathfrak{g}, \mathcal{W}_r)$ (with \mathcal{W}_r defined inductively by $\mathcal{W}_r = \text{Hom}(\mathfrak{g}, \mathcal{W}_{r-1})$). The *generalized curvature map* of our Lorentz manifold (M, g) is the map $\kappa^{\mathfrak{g}} = (\kappa, \mathcal{D}\kappa, \mathcal{D}^2\kappa, \dots, \mathcal{D}^{\dim(\mathfrak{g})}\kappa)$. The $O(1, 2)$ -module $\text{Hom}(\mathfrak{g}, \mathcal{W}_{\dim(\mathfrak{g})})$ will be rather denoted $\mathcal{W}_{\kappa^{\mathfrak{g}}}$ in the following.

Remark 2.1. *Of course, we could have replaced $\dim(\mathfrak{g})$ by 6 in the expressions above, but we want to emphasize the role of the bound $\dim(\mathfrak{g})$ when working in the general framework of Cartan geometries.*

2.2.3. *The integrability theorem.* We are now coming to the most important result of this section. To understand its statement, we first make the trivial

remark that given a local Killing field Z defined on some open subset $\hat{U} \subset \hat{M}$, then $D\kappa^g(Z) = 0$. The question is about the converse of this property. Given a point $\hat{x} \in \hat{M}$, and $\xi \in T_{\hat{x}}\hat{M}$ such that $D\kappa^g(\xi) = 0$, can we ensure that ξ is the evaluation at \hat{x} of a local Killing field Z ?

The remarkable fact is that the answer is positive if \hat{x} stays in a nice dense open subset of \hat{M} the *integrability locus* of \hat{M} . This locus is defined as the subset $\hat{M}^{\text{int}} \subset \hat{M}$ where the rank of the map κ^g is locally constant. Observe that \hat{M}^{int} is a dense open subset in \hat{M} , which is $O(1, 2)$ -invariant. We will denote by M^{int} its projection on M . The integrability theorem can then be stated as follows:

Theorem 2.2 (Integrability theorem). *Let $\hat{M}^{\text{int}} \subset \hat{M}$ the integrability locus of the smooth Lorentz manifold (M, g) .*

- (1) *For every $\hat{x} \in \hat{M}^{\text{int}}$, and every $\xi \in \text{Ker}(D_{\hat{x}}\kappa^g)$, there exists a local Killing field Z around \hat{x} such that $Z(\hat{x}) = \xi$.*
- (2) *The connected components of the fibers of κ^g in \hat{M}^{int} coincide with the fil^{loc} -orbits in \hat{M}^{int} .*

Some comments about this result are in order. The deepest, and most difficult part, of the theorem is the first point. A first, and weaker version of this statement, can be found in [Nom60]. Actually, the integrability condition stated in [Nom60] required the vector ξ to be in the kernel of *all* successive derivatives of the curvature map κ , which makes the conclusions less interesting. It was M. Gromov's insight to see that Killing generators of a *finite*, sufficiently large order are locally integrable. He stated this integrability result, as well as its amazing consequences for the structure of I_s^{loc} -orbits in [Gro88][Section 1.6]. Quite recently, those results were recast in the framework of Cartan geometries by K. Melnick in the analytic case (see [Mel11]). The reference [Pec16] gives an alternative approach, allowing to deal with *smooth* Cartan geometries, leading to the statement of Theorem 2.2. A proof that the integrability property actually holds on the set where the rank of κ^g is locally constant (first point of the theorem) follows Pecastaing's ideas and can be found in Annex A of [Frab].

2.3. Enriched structures. We have given the statement of Theorem 2.2 in the case where (M, g) is a (3-dimensional) Lorentz manifold. Actually, this theorem holds for all pseudo-Riemannian manifolds, and also for a much larger class of geometric structures called Cartan geometries. We invite the reader wanting to learn more about this topic, to look at the very comprehensive books [Sha97], and [vS09]. In particular, [vS09, Chap. 4] provides an extensive discussion of examples.

The integrability theorem is actually available for an even greater class of structures, that we call *enriched structures*, namely a Cartan geometry together with additional tensorial data (see [Pec16, Section 4] for a detailed account). The example which will be relevant for us is that of a 3-dimensional Lorentz manifold (M, g) , together with a Killing field X . The

geometric structure considered is then the triple (M, g, X) . Given $x \in M$, we can consider local Killing fields for our enriched structure (M, g, X) , defined in a neighborhood of x . They are the Killing vector fields for g which *moreover commute with X* . The Lie algebra of such local fields will be denoted by $\mathfrak{z}_X(x)$. With the same definition as above, we will speak about the \mathfrak{z}_X -orbit of a given point.

Our Killing field X lifts to a vector field (still denoted X) on the bundle \hat{M} , and provides there a $O(1, 2)$ -equivariant map $\lambda_X : \hat{M} \rightarrow \mathfrak{g}$ defined by $\lambda_X(\hat{x}) = \omega(X(\hat{x}))$. Any local vector field Z in $\mathfrak{z}_X(x)$ will lift to a vector field (still denoted Z) in a neighborhood of $\hat{x} \in \pi^{-1}\{x\}$ so that $L_Z\omega = 0$ and $Z.\lambda_X = 0$ (or in other words $[Z, X] = 0$).

We can now put the curvature κ and the map λ_X together, and build a new generalized curvature map

$$\kappa_X^{\mathfrak{g}} := (\kappa, \mathcal{D}\kappa, \dots, \mathcal{D}^{\dim(\mathfrak{g})}\kappa, \lambda_X, \mathcal{D}\lambda_X, \dots, \mathcal{D}^{\dim(\mathfrak{g})}\lambda_X).$$

This defines an integrability locus $\hat{M}_X^{\text{int}} \subset \hat{M}$ comprising all point where the rank of $\kappa_X^{\mathfrak{g}}$ is locally constant. As before this set is $O(1, 2)$ -invariant and projects on a dense open subset $M_X^{\text{int}} \subset M$. One can then formulate an integrability theorem similar to Theorem 2.3 for the enriched structure (M, g, X) .

Theorem 2.3. [Pec16, Th 4.4] *Let (M, g, X) be the geometric structure comprising the Lorentz metric g and the Killing field X .*

- (1) *For every $\hat{x} \in \hat{M}_X^{\text{int}}$, and every $\xi \in \text{Ker}(D_{\hat{x}}\kappa_X^{\mathfrak{g}})$, there exists a local field Z around \hat{x} , which is the lift an element of $\mathfrak{z}_X(x)$, and such that $\omega(Z(\hat{x})) = \xi$.*
- (2) *The connected components of the fibers of $\kappa_X^{\mathfrak{g}}$ in \hat{M}^{int} coincide with the \mathfrak{z}_X -orbits in \hat{M}^{int} .*

2.4. Components of the integrability locus. The integrability locus M_X^{int} is a dense open subset, which can be written as a (possibly infinite) union of connected components. Above each such component \mathcal{M} , the generalized curvature map $\kappa_X^{\mathfrak{g}}$ has constant rank, hence the dimension of $\mathfrak{z}_X(x)$ is constant as x ranges over \mathcal{M} . The isomorphism type of $\mathfrak{z}_X(x)$ is thus constant too, so that we will write $\mathfrak{z}_X(\mathcal{M})$ for the Lie algebra isomorphic to $\mathfrak{z}_X(x)$ for all $x \in \mathcal{M}$. Observe that when \mathcal{M} is not simply connected, monodromy phenomena may appear, so that generally, $\mathfrak{z}_X(\mathcal{M})$ won't admit a realization as a subalgebra of vector fields on \mathcal{M} .

3. BASIC FACTS ABOUT LORENTZIAN ISOMETRIC FLOWS

Before tackling the proof of Theorem 1.2, we recall briefly some elementary facts about isometries of Lorentz manifolds. The first one is a rigidity property of Lorentz isometries.

Proposition 3.1. *Let (M, g) be a smooth connected Lorentz manifold.*

- (1) *If an isometry f of (M, g) fixes a point $x \in M$, and satisfies $D_x f = id$, then f is the identity map on M .*
- (2) *If (f_k) is a sequence of $\text{Iso}(M, g)$, and if $x \in M$ is such that $f_k(x)$ is bounded in M , and $D_x f_k$ is bounded, then (f_k) has compact closure in $\text{Iso}(M, g)$.*

Proof: The fact is classical, and we only outline its proof, which relies essentially on the fact that the exponential map locally linearizes any Lorentz isometry. More precisely, if $\xi \mapsto \exp(x, \xi)$ denotes the exponential map at $x \in M$, and $f \in \text{Iso}(M, g)$, the relation:

$$f(\exp(x, \xi)) = \exp(f(x), D_x f(\xi))$$

holds for every $\xi \in T_x M$ in a small neighborhood of 0. In particular, this relation shows that if $f(x) = x$ and $D_x f = id$, then f is the identity on a neighborhood of x . Actually, we also get that the set of points where $f(y) = y$ and $D_y f = id$ is open. Since it is clearly closed, we conclude that f is the identity map.

The second point is a refinement of this reasoning. Under the assumption that $f_k(x)$ and $D_x f_k$ are bounded, we may find an extraction such that $f_k(x)$ converges to x_∞ , and $D_x f_k$ converges to a linear, invertible, map $A_\infty : T_x M \rightarrow T_{x_\infty} M$. Then the relation $f(\exp(x, \xi)) = \exp(f(x), D_x f(\xi))$ ensures that on a small neighborhood U of x , f_k converges for the C^∞ topology to a local isometry $f_\infty : U \rightarrow V$. As before, one shows that the set of points in M , having a neighborhood on which f_k converges is open and closed in M . As a consequence, one may extract from every subsequence of (f_k) a converging sequence, showing that (f_k) is relatively compact in $\text{Iso}(M, g)$. \diamond

We now focus on flows of Lorentz isometries on closed 3-dimensional manifolds. The following result was proved in [Zeg96, Prop 4.1].

Lemma 3.2. *Let (M^3, g) be a smooth, closed, 3-dimensional Lorentz manifold. Let X be a Killing field on M generating a flow φ_X^t which is not relatively compact in $\text{Iso}(M, g)$. Then*

- (1) *The Lorentz norm $g(X, X)$ is constant on M , and nonnegative.*
- (2) *The field X has no singularities.*
- (3) *If X is lightlike, then any Killing field on M commuting with X is of the form λX , for some $\lambda \in \mathbb{R}$.*
- (4) *If X is spacelike, then any lightlike geodesic whose direction u is orthogonal to X is complete.*

Proof:

We introduce the (smooth) function $\psi = g(X, X)$, and we assume that it is not constant to zero (if not the first point is proved). We pick $a \in \mathbb{R} \setminus \{0\}$ a regular value of ψ . Let Σ be the level set $\psi^{-1}\{a\}$. It is a smooth submanifold of M . We pick $x \in \Sigma$. By compactness of Σ , we can choose a sequence (t_k) tending to infinity in \mathbb{R} , such that $y_k := \varphi_X^{t_k}(x)$ converges to y_∞ in Σ .

We choose U and V small open neighborhoods of x and y_∞ , as well as $\mathcal{R} = (E_1, E_2, E_3)$ a field of orthonormal frames on $U \cup V$. If we read $D_x \varphi_X^{t_k}$ in this frame field, we get a sequence A_k of matrices in $O(1, 2)$.

We first assume $a < 0$. Then the identity $D_x \varphi_X^{t_k}(X(x)) = X(y_k)$ yields a timelike vector u such that $A_k \cdot u$ converges in $\mathbb{R}^{1,2}$. It follows easily that (A_k) is bounded in $O(1, 2)$, and Proposition 3.1 implies that $\{\varphi_X^t\}$ is relatively compact in $\text{Iso}(M, g)$. This is in contradiction with our hypotheses.

If $a > 0$ we complete X into a local frame field (X, E_1, E_2) on $U \cup V$, with (E_1, E_2) spanning X^\perp and satisfying $g(E_1, E_1) = g(E_2, E_2) = 0$, $g(E_1, E_2) = 1$. Considering A_k as above we have $A_k(E_1) = \lambda_k E_1$ and $A_k(E_2) = \frac{1}{\lambda_k} E_2$, for a sequence $\lambda_k \rightarrow 0$. The invariance of ψ under φ_X^t shows that $\psi(\exp(x, sE_1)) = \exp(y_k, \lambda_k sE_1)$. It follows that $s \mapsto \exp(x, sE_1)$ is included in Σ , hence $E_1(x) \in T_x \Sigma$. Then $E_1(y_k) \in T_{y_k} \Sigma$ for all $k \in \mathbb{N}$, and finally $E_1(y_\infty) \in T_{y_\infty} \Sigma$.

One also has $\psi(\exp(x, s\lambda_k E_2)) = \exp(y_k, sE_2)$, what shows that $s \mapsto \exp(y_\infty, sE_2)$ is included in Σ . We reach the conclusion that the three independent vectors $E_1(y_\infty), E_2(y_\infty), X(y_\infty)$ belong to $T_{y_\infty} \Sigma$, contradicting the fact that Σ is a surface.

Observe also that in this last case $a > 0$, if U and V are shrunk so that all geodesics of direction $E^1(y)$ or $E^2(y)$ are defined on $(-\tau, \tau)$, $\tau > 0$, then the geodesics of direction $E^1(x)$ and $E^2(x)$ will be defined on $(-\lambda_k^{-1}\tau, \lambda_k^{-1}\tau)$ for every $k \in \mathbb{N}$. It follows that those geodesics are complete, proving the last point of the proposition.

We now show that X does not have any singularity. This of course obvious if $g(X, X)$ is constant and positive. Now if $g(X, X)$ is constant equal to 0. Let us assume that x_0 is a singularity of X , and let us look at the action of $D_{x_0} \varphi_X^t$ on the 1-sphere corresponding to lightlike directions at x_0 . If this action is nontrivial, then there exist some x on the local lightcone with vertex x_0 (denoted $C(x_0)$), such that $X(x)$ is tangent to $C(x_0)$ and transverse to the lightlike geodesic through x and x_0 . This forces $g(X, X) > 0$ at x , a contradiction.

If the action on the lightlike directions at x_0 is trivial, then $D_{x_0} \varphi_X^t = id$, and we get that X is identically 0 by Proposition 3.1. This is again a contradiction. Observe that we have proved the general fact that a nontrivial lightlike Killing field has no singularity.

It remains to prove the third point of the proposition. Let Y be a Killing field commuting with X , where X is Killing and lightlike. If Y is colinear to X at each $x \in M$, then there exists a real λ such that $Y - \lambda X$ has a singularity. Since $Y - \lambda X$ is Killing and lightlike, the remarks above imply that $Y - \lambda X = 0$, and we are done.

It remains to investigate the case where X and Y are linearly independent in a neighborhood of a point $x \in M$. Since φ_X^t preserves the Lorentz volume on M , which is finite by compactness of M , we may assume that x is a recurrent point for φ_X^t . In this case, there exists a local

frame field (E_1, E_2, E_3) around x , with $g(E_1, E_2) = g(E_3, E_3) = 1$ and $g(E_1, E_1) = g(E_2, E_2) = g(E_1, E_3) = g(E_2, E_3) = 0$, such that $D_x \varphi_X^t$ expressed in the frame (E_1, E_2, E_3) is the identity. The second point of Proposition 3.1 then implies that φ_X^t is relatively compact. This is a contradiction.

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4. A KILLING FIELD FACTORY

4.1. Recurrence produces Killing fields. We are now proving an essential fact, namely that a non-compact flow φ_X^t produces extra local Killing fields. In the following statement, we use the notation $\mathfrak{I}\mathfrak{s}_X(x)$ for the isotropy at x , namely the subalgebra of $\mathfrak{z}_X(x)$ comprising all local Killing fields in $\mathfrak{z}_X(x)$ vanishing at x .

Proposition 4.1. *Let (M^3, g) be a smooth, closed, 3-dimensional Lorentz manifold. Let X be a Killing field on M generating a flow φ_X^t which is not relatively compact in $\text{Iso}(M, g)$.*

- (1) *For every $x \in M_X^{\text{int}}$, the isotropy algebra $\mathfrak{I}\mathfrak{s}_X(x)$ is 1-dimensional, and generates a non-compact subgroup of $\text{O}(T_x M)$. This subgroup is hyperbolic (resp. parabolic) if X is spacelike (resp. X is lightlike).*
- (2) *On each component $\mathcal{M} \subset M_X^{\text{int}}$, the Lie algebra $\mathfrak{z}_X(\mathcal{M})$ has dimension 3 or 4. In the first case, the \mathfrak{z}_X -orbits on \mathcal{M} have dimension 2. In the second case, the component \mathcal{M} is a single 3-dimensional \mathfrak{z}_X -orbit.*

Proof: The metric g defines a Lorentz volume (attributing volume 1 to any orthonormal frame), which is finite on M and φ_X^t -invariant. Poincaré recurrence theorem ensures that the set of points which are recurrent for φ_X^t has full measure. Let x be such a recurrent point for φ_X^t and choose $\hat{x} \in \hat{M}$ in the fiber of x . The recurrence hypothesis means that there exist sequences (t_k) and (p_k) in \mathbb{R} and $\text{O}(1, 2)$ respectively, satisfying $|t_k| \rightarrow \infty$ and $\varphi_X^{t_k}(\hat{x}) \cdot p_k^{-1} \rightarrow \hat{x}$. Observe that (p_k) tends to infinity in $\text{O}(1, 2)$, because $\text{Iso}(M, g)$ acts properly on \hat{M} (Proposition 3.1). By equivariance of the generalized curvature map, we also have

$$p_k \cdot \kappa_X^g(\hat{x}) \rightarrow \kappa_X^g(\hat{x}).$$

The linear action of $\text{O}(1, 2)$ on $\mathcal{W}_{\kappa_X^g}$ is algebraic, hence its orbits are locally closed. As a consequence, there exists a sequence (ϵ_k) in $\text{O}(1, 2)$ with $\epsilon_k \rightarrow id$ and $\epsilon_k \cdot p_k \cdot \kappa_X^g(\hat{x}) = \kappa_X^g(\hat{x})$. We infer that the stabilizer $S(\hat{x})$ of $\kappa_X^g(\hat{x})$ in $\text{O}(1, 2)$ is a non-compact subgroup. This group is algebraic, hence its identity component is non-compact too, and the Lie algebra $\mathfrak{s}(\hat{x})$ is nontrivial. But Theorem 2.3 actually says that elements in $\mathfrak{s}(\hat{x})$ are images under ω of evaluations (at \hat{x}) of elements of $\mathfrak{I}\mathfrak{s}_X(x)$. In particular $\mathfrak{I}\mathfrak{s}_X(x)$ is not reduced to $\{0\}$. If $Y \in \mathfrak{I}\mathfrak{s}_X(x)$, then because $[X, Y] = 0$, the 1-parameter subgroup $e^{t\omega_{\hat{x}}(Y(\hat{x}))}$ fixes the vector $\omega_{\hat{x}}(X(\hat{x}))$. In particular, the horizontal part of $\omega_{\hat{x}}(X(\hat{x}))$, which is nonzero by point (2) of Lemma 3.2,

is fixed and thus $\mathfrak{Is}_X(x)$ is 1-dimensional. The flow $e^{t\omega_{\hat{x}}(Y(\hat{x}))}$ is hyperbolic if $X(x)$ is spacelike and parabolic if $X(x)$ is lightlike. The first point of the proposition follows by density of recurrent points in M_X^{int} .

To prove the second point, we start with $x \in M_X^{\text{int}}$, and consider a simply connected neighborhood $U \subset M_X^{\text{int}}$ of x . Then, every algebra $\mathfrak{Is}_X(y)$, $y \in U$, is realized as a Lie algebra of Killing fields defined on U . By the first part of our proof, there exists Y a nontrivial Killing field of \mathfrak{z}_X on U , such that $Y(x) = 0$. The zero locus of Y is a nowhere dense set in U . We can thus pick $y \in U$ satisfying $Y(y) \neq 0$, and apply again the first point of the proof at y . We get a second nontrivial Killing field Z defined on U , and vanishing at y . Let us consider a lightlike direction $u \in T_y M$ such that $g_y(u, Y(y)) \neq 0$, and u is transverse to the zero locus of Z . Let $t \mapsto \gamma_u(t)$ be the geodesic passing through y at $t = 0$ and such that $\dot{\gamma}_u(0) = u$. Then Clairaut's equation ensures that $g(\dot{\gamma}_u, Y)$ is constant on γ_u , hence does not vanish on γ_u by our choice of u .

Clairaut's equation also ensures that $g_{\gamma_u}(\dot{\gamma}_u, Z) = 0$, and $Z(\gamma_u(t)) \neq 0$ for small, nonzero, values of t . Hence, on some open subset of U , the \mathfrak{z}_X -orbits have dimension ≥ 2 , while for every point $z \in U$, the dimension of $\mathfrak{Is}_X(z)$ is ≥ 1 by the first part of the proof. It follows that the dimension of $\mathfrak{z}_X(\mathcal{M})$ is at least 3. Because the dimension of M is 3, the dimension of $\mathfrak{z}_X(\mathcal{M})$ can be 3 (in which case the \mathfrak{z}_X -orbits on \mathcal{M} are 2-dimensional) or 4 (in which case \mathcal{M} is a single 3-dimensional \mathfrak{z}_X -orbit).

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4.2. Hyperbolic and parabolic components. Proposition 4.1 shows that components of M_X^{int} split into two categories. The ones for which the isotropy algebra generates a 1-parameter hyperbolic group will be called *hyperbolic components*. Those for which the isotropy algebra generates a 1-parameter parabolic group will be called *parabolic components*. Hyperbolic and parabolic components can themselves be divided into two subcategories. The ones for which $\mathfrak{z}_X(\mathcal{M})$ is 4-dimensional, which are locally homogeneous, and the others for which $\mathfrak{z}_X(\mathcal{M})$ is 3-dimensional. We will call those later components non locally homogeneous. Observe that this is a property about the local action of \mathfrak{z}_X , and there is no obstruction for the metric g itself to be locally homogeneous on a non locally homogeneous component.

4.3. Existence of closed \mathfrak{z}_X -orbits. One key consequence of the previous results is that components \mathcal{M} for which the dimension of $\mathfrak{z}_X(\mathcal{M})$ is minimal, contain closed fill^{loc} -orbits.

Proposition 4.2. *Let (M^3, g) be a smooth, closed, 3-dimensional Lorentz manifold. Let X be a Killing field on M generating a flow φ_X^t which is not relatively compact in $\text{Iso}(M, g)$. Then every component $\mathcal{M} \subset M_X^{\text{int}}$ such that $\mathfrak{z}_X(\mathcal{M})$ has minimal dimension contains a fill^{loc} -orbit which is closed in M .*

Proof: The proof follows essentially from two remarks.

The first one is that the stabilizer of $\kappa_X^g(\hat{x})$ in $O(1, 2)$ is 1-dimensional for every $\hat{x} \in \hat{M}$. Indeed, Proposition 4.1 and the density of \hat{M}_X^{int} in \hat{M} shows that this stabilizer is always at least 1-dimensional. Since there are no 2-dimensional stabilizers in a finite dimensional linear representation of $O(1, 2)$, the only other possibility is that $\kappa_X^g(\hat{x})$ is stabilized by $O^o(1, 2)$. Because the map κ_X^g has the map λ_X as one of its factors, this would imply that $\lambda_X(\hat{x})$ is $O^o(1, 2)$ -invariant, hence equal to zero. Contradiction with Lemma 3.2.

Because for an algebraic action, orbits can accumulate only on orbits of smaller dimension, we infer that for every $\hat{x} \in \hat{M}$, the $O(1, 2)$ -orbit (as well as the $O^o(1, 2)$ -orbit) of $\kappa_X^g(\hat{x})$ is closed in $\kappa_X^g(\hat{M})$.

We then call $\hat{\Omega}$ the open subset of \hat{M} , where the rank of κ_X^g achieves its maximal value, that we call r_{max} . Theorem 2.3 ensures that $\hat{\Omega} \subset \hat{M}_X^{\text{int}}$. Let us denote by $\hat{\Lambda}$ the set of points in \hat{M} where the rank of κ_X^g is $\leq r_{max} - 1$. Sard's theorem says that the r_{max} -dimensional Hausdorff measure of the set $\kappa_X^g(\hat{\Lambda})$ is zero. It follows that there must be $\hat{x}_0 \in \hat{\Omega}$ such that $\Sigma := (\kappa_X^g)^{-1}(\kappa_X^g(\hat{x}_0))$ is included in $\hat{\Omega}$. Now Σ is a closed submanifold of \hat{M} . Theorem 2.3 says that the projection $\pi(\Sigma)$ is a submanifold of M_X^{int} the connected components of which are \mathfrak{z}_X -orbits. Our first remarks show that the saturation of Σ by the action of $O(1, 2)$ is also closed in \hat{M} , since it is the inverse image of a $O(1, 2)$ -orbit in $\mathcal{W}_{\kappa_X^g}$ and such orbits are closed in $\kappa_X^g(\hat{M})$. We infer that $\pi(\Sigma)$ is closed in M , and the proposition is proved.

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5. HEISENBERG STRUCTURES AND THE GEOMETRY OF PARABOLIC COMPONENTS

5.1. Heisenberg structures on surfaces. We consider the following realisation of the 3-dimensional Heisenberg group as a subgroup of affine transformations of the plane:

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} x \\ y \end{pmatrix}, \quad x, y, t \in \mathbb{R}$$

We will denote by Heis this group.

If Σ is a surface, we will call a $(\text{Heis}, \mathbb{R}^2)$ -structure on Σ the data of an atlas (U_i, φ_i) , taking values in \mathbb{R}^2 , so that transition maps $\varphi_j \circ \varphi_i^{-1}$ are restrictions of elements of Heis. Observe that the action of Heis on \mathbb{R}^2 preserves the (degenerate) metric dy^2 , and commutes with the 1-parameter group given by the transformations $(x, y) \mapsto (x + t, y)$, $t \in \mathbb{R}$. It follows that any $(\text{Heis}, \mathbb{R}^2)$ -structure \mathcal{S} on Σ defines accordingly a degenerate metric λ on Σ , as well as a flow ψ^t . We will say that λ (resp. ψ^t) is the standard metric (resp. the standard flow) associated to the structure \mathcal{S} . In the following, we will often use the terminology *Heisenberg structure* instead of $(\text{Heis}, \mathbb{R}^2)$ -structure.

Here is an equivalent point of view for Heisenberg structures. Let us consider the Lie algebra $\mathfrak{h} \subset \mathfrak{o}(1,2) \times \mathbb{R}^3$, spanned by the vectors e, h, E . The corresponding Lie subgroup $H \subset \mathrm{O}(1,2) \times \mathbb{R}^3$ is isomorphic to Heis, and the action of Heis on \mathbb{R}^2 is conjugated to that of H on the homogeneous space $H/\{e^{tE}\}$. We will retain in the sequel $H/\{e^{tE}\}$ as a model space for Heisenberg structures. Actually, we will also use the following dual point of view for Heisenberg structure. Let us consider ω_H the Maurer-Cartan form on the group H . Then a Heisenberg structure on a surface Σ is equivalent to the data of a $\{e^{tE}\}$ -principal bundle $\hat{\Sigma}$ over Σ , as well as a 1-form ω on $\hat{\Sigma}$ with values in \mathfrak{h} , such that the fiber bundle $(\hat{\Sigma}, \omega)$ is locally isomorphic to (H, ω_H) .

We end up this section with the following completeness result.

Proposition 5.1. *Let Σ be a closed surface, endowed with an Heisenberg structure \mathcal{S} . Then:*

- (1) *The surface Σ is a 2-torus, and the structure is complete, namely is the quotient of \mathbb{R}^2 by a discrete subgroup of Heis isomorphic to \mathbb{Z}^2 .*
- (2) *For such a structure, the standard flow ψ^t on Σ is periodic.*

Proof: An Heisenberg structure provides an orientation of Σ , and because the standard flow ψ^t does not have singularity, the Euler characteristic of Σ is zero. Hence Σ is a 2-torus.

The Heisenberg structure on Σ yields a developing map $\delta : \tilde{\Sigma} \rightarrow \mathbb{R}^2$, which is a smooth immersion. Here, $\tilde{\Sigma}$ stands for the universal cover of Σ . The map δ is ρ -equivariant for a representation $\rho : \pi_1(\Sigma) \rightarrow \mathrm{Heis}$. The standard flow ψ^t is complete on Σ . It follows that the image $\delta(\tilde{\Sigma})$ is saturated by lines of the form

$$D_u = \{(t, u) \mid t \in \mathbb{R}\}.$$

Let us call I the projection of $\delta(\tilde{\Sigma})$ on the second coordinate axis, namely

$$I = \{u \in \mathbb{R}, D_u \subset \delta(\tilde{\Sigma})\}.$$

Then I is invariant by the action of $\rho(\pi_1(\Sigma))$ on the set of lines of the form D_u . It turns out that this action is by translations, and it must also be cocompact since δ is ρ -equivariant and $\pi_1(\Sigma)$ acts cocompactly on $\tilde{\Sigma}$. This forces $I = \mathbb{R}$, and δ is onto.

To check that δ is injective, we consider W a vector field on Σ , transverse to the orbits of ψ^t , and satisfying $\lambda(W, W) = 1$, where λ stands for the standard metric on Σ defined by the Heisenberg structure. We lift W to $\tilde{\Sigma}$ to get a (complete) flow ϕ_W^t , whose orbits are transverse to those of ψ^t . Let us pick $x \in \tilde{\Sigma}$, and consider the curve $\gamma : s \mapsto \phi_W^s \cdot x$. The map δ is injective in restriction to γ . Indeed, if we write $\delta(\gamma(s)) = (u(s), v(s))$, we notice that $|v'(s)| = 1$, since $\lambda(W, W) = 1$. It follows also that $v(\mathbb{R}) = \mathbb{R}$, and we conclude that $\{\psi^t(\phi_W^s(x)) \mid (s, t) \in \mathbb{R}^2\}$ is mapped injectively onto \mathbb{R}^2 . One deduces easily that δ has the path-lifting property, and δ is a covering between $\tilde{\Sigma}$ and \mathbb{R}^2 , hence a diffeomorphism. We conclude that $\rho(\pi_1(\Sigma))$ is

a discrete subgroup of Heis isomorphic to \mathbb{Z}^2 , proving the first point of the proposition.

All discrete copies of \mathbb{Z}^2 in Heis intersect the center of Heis nontrivially. This shows that the standard flow ψ^t is periodic. \diamond

5.2. Existence of Heisenberg structures on \mathfrak{z}_X -orbits. We now study in more details the geometry of non locally homogeneous components. This section will be devoted to the proof of the next proposition.

Proposition 5.2. *Let \mathcal{M} be a parabolic component which is not locally homogeneous. Then:*

- (1) *The Lie algebra $\mathfrak{z}_X(\mathcal{M})$ is isomorphic to the 3-dimensional Heisenberg algebra \mathfrak{heis} .*
- (2) *Any \mathfrak{z}_X -orbit of \mathcal{M} is endowed with an invariant Heisenberg structure, for which the restriction of φ_x^t (resp. of the Lorentz product g) coincides with the standard flow (resp. the standard metric).*

Let us fix a point $x \in \mathcal{M}$. We work in the fiber bundle \hat{M} (and lift all local Killing fields there). Let Y be a generator of the local isotropy at x . Then we can find $\hat{x} \in \hat{M}$ in the fiber of x such that $\omega(Y(\hat{x})) = E$. Because X commutes with Y , we must have $\omega(X(\hat{x})) = e + \alpha E$. Finally, there is a third vector field in $\mathfrak{z}_X(x)$ such that $\omega(Z(\hat{x})) = h + \beta H + \gamma F$.

We call κ_0 the element of $\text{Hom}(\wedge^2(\mathbb{R}^3), \mathfrak{o}(1,2))$ corresponding to the identity matrix, namely κ_0 maps $e \wedge h$ to E , $e \wedge f$ to H and $h \wedge f$ to F . We also call κ_1 the element of $\text{Hom}(\wedge^2(\mathbb{R}^3), \mathfrak{o}(1,2))$ corresponding to the matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ (see Section 2.2.1).}$$

The two dimensional vector space spanned by κ_0 and κ_1 is the set of fixed points of the action of $\{e^{tE}\}_{t \in \mathbb{R}}$ on the curvature module. We infer that $\kappa(\hat{x})$ is of the form $\kappa = \sigma \kappa_0 + b \kappa_1$. In particular, the following identities hold at \hat{x} :

$$(1) \quad \kappa(e \wedge h) = \sigma E, \quad \kappa(e \wedge f) = \sigma H, \quad \kappa(h \wedge f) = bE + \sigma F.$$

Notice that $\sigma, b, \alpha, \beta, \gamma$ depend on x and \hat{x} , but since those points are fixed once for all in the following, there will be considered as constant.

Cartan's formula $L_X \omega = \iota_X d\omega + d(\iota_X \omega)$ shows that whenever U, V are two Killing fields on \hat{M} , the following relation holds:

$$(2) \quad \omega([U, V]) = K(U, V) - [\omega(U), \omega(V)]$$

We now write Equation (2) at \hat{x} , using identities (1), when U and V range over Z, X, Y . For instance, the first equation is (at \hat{x}):

$$\omega([X, Z]) = (\sigma + \alpha\beta)E + (\beta - \alpha)e + \gamma h - \gamma\alpha H.$$

Now $[X, Z] = 0$ hence $\gamma = 0$, $\beta = \alpha$ and $\sigma = -\alpha^2$.

We now write the second equation (still at \hat{x}):

$$\omega([Y, Z]) = -e + \alpha E = \omega(-X + 2\alpha Y).$$

Next, two Killing fields which coincide at \hat{x} must be equal (by freeness of the action of isometries on the orthonormal frames). We get the following relation at the level of the Lie algebra $\mathfrak{z}_X(x)$:

$$[Y, Z] = -X + 2\alpha Y.$$

We see in particular that at each $z \in \mathcal{M}$, if U is a local Killing field around z generating the isotropy at z , then $\text{ad}(U)$ is a nilpotent endomorphism of $\mathfrak{z}_X(z)$.

At x , $X(x)$ is lightlike and nonzero and $Z(x)$ is spacelike, orthogonal to $X(x)$. The orthogonal to $Z(x)$ at x is a Lorentzian plane spanned by $X(x)$ and another vector $w \in T_x M$. Let us call $t \mapsto \gamma(t)$ the geodesic through x satisfying $\dot{\gamma}(0) = w$. Clairault's equation ensures that the quantities $g(\dot{\gamma}(t), Z(\gamma(t)))$, $g(\dot{\gamma}(t), X(\gamma(t)))$ and $g(\dot{\gamma}(t), Y(\gamma(t)))$ do not depend on t . In particular, for $t > 0$, both $Y(\gamma(t))$ and $Z(\gamma(t))$ are orthogonal to $\dot{\gamma}(t)$ while $X(\gamma(t))$ is not. For $t > 0$ small enough, $Z(\gamma(t))$ is still spacelike, hence nonzero, and $\gamma(t)$ belongs to \mathcal{M} . In particular, the \mathfrak{z}_X -orbit at $\gamma(t)$ is not 3-dimensional, so that there must exist some real number λ_t for which $Y(\gamma(t)) = \lambda_t Z(\gamma(t))$. Observe finally that w is transverse to the curve where Y vanishes, hence for $t > 0$ small $\lambda_t \neq 0$. Using our bracket relation $[Y, Z] = -X + 2\alpha Y$, we compute

$$\text{Trace}(\text{ad}(Y - \lambda_t Z)) = 2\lambda_t \alpha.$$

Because we observed that $\text{ad}(Y - \lambda_t Z)$ must be nilpotent at $\gamma(t)$, we get $\alpha = 0$.

To summarize, we have shown that $\alpha = 0$ and $\sigma = \alpha^2 = 0$ (in particular $\kappa_{\hat{x}}(e \wedge h) = 0$). Moreover, at \hat{x} :

$$\omega(X) = e, \quad \omega(Y) = E, \quad \omega(Z) = h,$$

and the bracket relations for $\mathfrak{z}_X(\mathcal{M})$ are

$$[X, Y] = [X, Z] = 0, \quad [Z, Y] = X,$$

showing that $\mathfrak{z}_X(\mathcal{M})$ is isomorphic to \mathfrak{heis} .

Finally, let us consider $\hat{\Sigma}$ the \mathfrak{z}_X -orbit of \hat{x} . Calling, as in Section 5.1, \mathfrak{h} the Lie subalgebra spanned by e, h, E in $\mathfrak{o}(1, 2) \times \mathbb{R}^3$, the invariance of ω under local killing flows show that $\omega(T\hat{\Sigma}) \equiv \mathfrak{h}$. This makes $\hat{\Sigma}$ into a $\{e^{tE}\}$ -principal bundle over Σ . The restriction of ω to $T\hat{\Sigma}$ yields a Cartan connexion $\omega' : T\hat{\Sigma} \rightarrow \mathfrak{h}$. The curvature of this induced Cartan connection ω' is given by $\kappa(e \wedge h)$, which is identically 0 on $\hat{\Sigma}$ since we have $\sigma \equiv 0$. It thus follows that $(\hat{\Sigma}, \omega')$ is locally isomorphic, as $\{e^{tE}\}$ -principal bundle to (H, ω_H) fibering on $H/\{e^{tE}\}$, where ω_H stands for the Maurer-Cartan form on H . As explained in Section 5.1, this defines an Heisenberg structure on Σ . Because $\omega' = \omega|_{T\hat{\Sigma}}$, and because $\omega(X(\hat{y})) = e$ for every $\hat{y} \in \hat{\Sigma}$, it follows

that the restriction of g to Σ yields the standard metric of the Heisenberg structure, and φ_X^t coincides with the standard flow.

6. LOCAL HOMOGENEITY

The next step is to show that under our assumption that the flow φ_X^t is not relatively compact, our manifold (M, g) is locally homogeneous.

Proposition 6.1. *Let (M^3, g) be a smooth, closed, 3-dimensional Lorentz manifold. Let X be a Killing field on M generating a flow φ_X^t which is not relatively compact in $\text{Iso}(M, g)$. Then M consists in a single \mathfrak{z}_X -orbit, hence is locally homogeneous.*

Proposition 6.1 will be proved if we can show there are no closed 2-dimensional \mathfrak{z}_X -orbits. Indeed, Proposition 4.2, together with Proposition 4.1 then imply that all \mathfrak{z}_X -orbits in M_X^{int} have dimension 3, hence are open. Proposition 4.2 again shows that one of those \mathfrak{z}_X -orbits must be closed, hence is the entire manifold M . By those remarks, Proposition 6.1 will just be a consequence of Lemmas 6.2 and 6.3 below.

Lemma 6.2. *Let (M^3, g) be a smooth, closed, 3-dimensional Lorentz manifold. Let X be a Killing field on M generating a flow φ_X^t which is not relatively compact in $\text{Iso}(M, g)$. Then there are no non locally homogeneous hyperbolic components.*

Proof: By Proposition 4.1, it is sufficient to prove that for any hyperbolic component \mathcal{M} , the \mathfrak{z}_X -orbits are not 2-dimensional. Assume for a contradiction that such \mathcal{M} contains a point x having a 2-dimensional \mathfrak{z}_X -orbit, denoted Σ . Let us denote by The Lie algebra $\mathfrak{Is}_X(x)$ is generated by a local Killing field Y around x , vanishing at x , and such that the flow $\{D_x\phi_Y^t\} \subset \text{O}(T_xM)$ is a hyperbolic flow. Linearizing Y around x thanks to the exponential map, we see there are two distinct lightlike directions u and v in T_xM such that the two geodesics $\gamma_u : s \mapsto \exp(x, su)$ and $\gamma_v : s \mapsto \exp(x, sv)$ are left invariant by ϕ_X^t . In particular, for $s \neq 0$ close to 0, $\dot{\gamma}_u(s)$ and $\dot{\gamma}_v(s)$ are colinear to Y , hence tangent to the \mathfrak{z}_X -orbits at $\gamma_u(s)$ and $\gamma_v(s)$ respectively. By continuity, this property must still hold for $s = 0$. We infer that u and v are tangent to Σ at x , hence Σ has Lorentz signature. By local homogeneity of Σ , the Gaussian curvature of Σ is constant. The Lie algebra $\mathfrak{z}_X(\mathcal{M})$ should then be isomorphic to the Lie algebra $\mathfrak{so}l$ (case of curvature 0) or $\mathfrak{o}(1, 2)$ (case of nonzero constant curvature). This yields a contradiction since the center of both $\mathfrak{so}l$ and $\mathfrak{o}(1, 2)$ is trivial (we have implicitly used the fact that a local Killing field on \mathcal{M} which vanishes on Σ is trivial, what is easily proved using proposition 3.1). \diamond

Lemma 6.3. *Let (M^3, g) be a smooth, closed, 3-dimensional Lorentz manifold. Let X be a Killing field on M generating a flow φ_X^t which is not relatively compact in $\text{Iso}(M, g)$. Then there are no non locally homogeneous parabolic components.*

Proof:

If such a non locally homogeneous parabolic component \mathcal{M} existed, then Proposition 4.2 would provide a $\mathfrak{fill}^{\text{loc}}$ -orbit $\Sigma \subset \mathcal{M}$ which is a closed surface. Proposition 5.2 ensures that Σ is endowed with an Heisenberg structure, for which the standard flow ψ^t coincides with φ_X^t . Then, Proposition 5.1 shows that the restriction of φ_X^t to Σ is cyclic, namely there exists $T > 0$ such that φ_X^T restricts to identity on Σ . But one easily checks that a 1-parameter group in $\mathfrak{o}(1, 2)$ fixing pointwise an hyperplane must be trivial, so that point (1) of Proposition 3.1 ensures that φ_X^T is the identity on M . This contradicts the fact that $\{\varphi_X^t\}$ is not relatively compact in $\text{Iso}(M, g)$.

◇

7. LOCAL MODELS AND THE CLASSIFICATION THEOREM

7.1. Three homogeneous 3-dimensional Lorentzian manifolds.

7.1.1. *Lorentz-Heisenberg geometry.* We call \mathfrak{heis} the 3-dimensional Heisenberg Lie algebra, namely the Lie algebra generated by X, Y, Z , with relation $[Y, Z] = X$. Let Heis be the connected, simply connected, associated Lie group. The Lorentz scalar product defined by $\langle X, Y \rangle = 1$, $\langle Z, Z \rangle = 1$, and all other products are zero, can be left-translated on Heis to give an homogeneous Lorentz metric g_{LH} called *the Lorentz-Heisenberg metric* on Heis .

The 1-dimensional group of automorphism I_h of Heis , which acts on \mathfrak{heis} by the matrices $\begin{pmatrix} e^t & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $t \in \mathbb{R}$ (in the basis Y, Z, X) acts isometrically on (Heis, g_{LH}) . The group G_h generated by I_h and the left translations coincides with identity component of $\text{Iso}(\text{Heis}, g_{LH})$. Its Lie algebra \mathfrak{g}_h is generated by X, Y, Z , and a fourth element T satisfying $[T, Y] = Y$, $[T, Z] = -Z$ and $[T, X] = 0$. It is isomorphic to the semi-direct product $\mathbb{R} \rtimes_h \mathfrak{heis}$ (the subscript h means that the \mathbb{R} -factor integrates into a group of hyperbolic automorphisms of \mathfrak{heis}). We call \mathbf{X}_{LH} the pair (Heis, g_{LH}) , modelling the Lorentz-Heisenberg geometry. Notice that $\mathbf{X}_{LH} = G_h/I_h$ as an homogeneous space.

7.1.2. *The geometry \mathbf{X}_h .* We still consider the group G_h , and we call I_u the Lie subgroup generated by $Y + Z$. It is easily checked that the adjoint action of I_u on $\mathfrak{g}_h/\mathfrak{J}_u$ is unipotent and preserves a Lorentz scalar product. This defines a G_h -invariant Lorentz metric on G_h/I_u (or more accurately, a 2-parameter family of such metrics, since there is such a family of Lorentz scalar products preserved by $\text{Ad}(I_u)$). The corresponding Lorentzian homogeneous space G_h/I_u will be denoted by \mathbf{X}_h . The identity component of its isometry group is G_h .

7.1.3. *The geometry \mathbf{X}_e .* We now consider a different 4-dimensional Lie algebra \mathfrak{g}_e , generated by T, X, Y, Z , and defined in the following way. The elements X, Y, Z generate a Lie algebra isomorphic to \mathfrak{heis} , with bracket relation $[Y, Z] = X$. The element T normalizes this subalgebra, and the relations are $[T, Y] = -Z$, $[T, Z] = Y$, $[T, X] = 0$.

The algebra \mathfrak{g}_e is again a semi-direct product $\mathbb{R} \rtimes_e \mathfrak{heis}$ (subscript e indicates that the \mathbb{R} -factor induces a group of elliptic automorphisms on \mathfrak{heis}). Let I be the 1-dimensional group generated by $Y + Z$. As in the previous example, it is easily checked that $\text{Ad}(I)$ acts on $\mathfrak{g}_e/\mathfrak{J}$ by unipotent transformations, preserving a Lorentzian scalar product (actually, again a 2-parameters family of such scalar products). This yields a homogeneous Lorentzian manifold G_e/I , denoted by \mathbf{X}_e . The identity component of its isometry group is G_e .

7.1.4. *Natural foliations, and characteristic flow.* We discuss now some remarkable geometric features of the spaces \mathbf{X}_h and \mathbf{X}_e .

The groups G_h and G_e have a derived subgroup H isomorphic to Heis . The center of H is a 1-parameter subgroup $\{z^t\}$, which coincides with the center of G_h (resp. G_e). It follows that z^t induces a well defined flow on any manifold M locally modelled on \mathbf{X}_{LH} , \mathbf{X}_h or \mathbf{X}_e . This flow will be called the *characteristic flow* of the structure.

The foliation of G_h (resp. G_e) the leaves of which are given by the classes gH , induces a G_h -invariant (resp. G_e -invariant) codimension one foliation \mathcal{F} on \mathbf{X}_h (resp. \mathbf{X}_e). All leaves are lightlike and naturally endowed with an Heisenberg structure (see Section 5.1).

Let $\bar{\alpha}$ be the degenerate bilinear form on \mathfrak{g}_h (resp. \mathfrak{g}_e) such that X, Y, Z generate the Kernel of $\bar{\alpha}$, and $\bar{\alpha}(T, T) = 1$. Observe that the adjoint action of I_h (resp. of I) leaves $\bar{\alpha}$ invariant, resulting on a G_h (resp. G_e) invariant degenerate metric on \mathbf{X}_h (resp. \mathbf{X}_e). It follows that any manifold M locally modelled on \mathbf{X}_h or \mathbf{X}_e is naturally endowed with a metric α of signature $(+, 0, 0)$, such that the foliation \mathcal{F} integrates the Kernel of α . This yields a transverse Riemannian structure for the foliation \mathcal{F} .

7.2. Identifying the Lie algebra \mathfrak{z}_X and the homogeneous models.

We are now in position to describe all the possible local homogeneous models, for a Lorentzian manifold (M^3, g) with a non-relatively compact isometric flow.

Proposition 7.1. *Let (M^3, g) be a smooth, closed, 3-dimensional Lorentz manifold. Let X be a Killing field on M generating a flow φ_X^t which is not relatively compact in $\text{Iso}(M, g)$. Then either φ_X^t preserves a metric of constant curvature g_0 on M , or the geometry (M, g) is locally modelled on \mathbf{X}_{LH} , \mathbf{X}_h or \mathbf{X}_e , and φ_X^t coincides with the characteristic flow.*

The list of local models is pretty short, and will be shortened even more in the next section.

Obtaining those local models amounts to determine the possibilities for the algebra \mathfrak{z}_X , as well as for the isotropy algebras. It turns out that this can be obtained by very simple algebraic computations, similar to those made in the proof of Proposition 5.2 (we keep the notations of this proof in the following).

Let us start with the computations when the isotropy algebra is hyperbolic (namely M coincides with a hyperbolic component). We pick $x \in M$, $\hat{x} \in \hat{M}$ in the fiber of X , and T, X, Y, Z generating \mathfrak{z}_X (and Y generating the isotropy). We can choose T, Y, Z and \hat{x} so that the following relations hold at \hat{x} .

$$\begin{aligned}\omega(X) &= h + \alpha H, \omega(Y) = H, \\ \omega(Z) &= e + \beta E + \gamma F, \omega(T) = f + \lambda E + \mu F,\end{aligned}$$

for some real numbers $\alpha, \beta, \gamma, \lambda, \mu$.

The curvature $\kappa(\hat{x})$ is invariant under the isotropy, namely the flow $\{e^{tH}\}$, hence of the form $\kappa(\hat{x}) = \sigma\kappa_0 + b\kappa_2$ (see notation in Section 2.2.1).

In particular, at \hat{x} , $\kappa(e \wedge h) = (\sigma + b)E$, $\kappa(e \wedge f) = (\sigma - 2b)H$, and $\kappa(h \wedge f) = (\sigma + b)F$.

Recall the equation

$$(3) \quad \omega([U, V]) = K(U, V) - [\omega(U), \omega(V)]$$

available for every pair U, V of local Killing fields around \hat{x} . We first express that X is in the center of \mathfrak{z}_X (all equalities below are available at \hat{x}).

$$\begin{aligned}\omega([Z, X]) &= 0 = \kappa(e \wedge h) + [h + \alpha H, e + \beta E + \gamma F] \\ &= (\alpha - \beta)e + \gamma f + (\sigma + b + \alpha\beta)E + \alpha\gamma F\end{aligned}$$

which implies

$$\gamma = 0, \alpha = \beta, \sigma + b + \alpha^2 = 0.$$

We use these relations to compute

$$\begin{aligned}\omega([X, T]) &= 0 = \kappa(h \wedge f) + [f + \lambda E + \mu F, h + \alpha H] \\ &= \lambda e + (\alpha - \mu)f - \alpha\lambda E\end{aligned}$$

leading to the extra relations

$$\lambda = 0, \alpha = \mu.$$

From the simplified relations $\omega(X) = h + \alpha H$, $\omega(Z) = e + \alpha E$ and $\omega(T) = f + \alpha F$, one proceeds writing

$$\begin{aligned}\omega([Z, T]) &= \kappa(e \wedge f) + [f + \alpha F, e + \alpha E] = 2\alpha\omega(X) + (3\sigma - \alpha^2)\omega(Y), \\ \omega([Y, Z]) &= -[H, e + \alpha E] = -\omega(Z) \\ \omega([Y, T]) &= -[H, f + \alpha F] = \omega(T).\end{aligned}$$

Because two Killing fields which coincide at \hat{x} coincide, we obtain the bracket relations for \mathfrak{z}_X . The possibilities split into three cases.

- *Case 1: α and σ are zero.* This implies $b = 0$ because of the relation $\sigma + b + \alpha^2 = 0$, so that the geometry is flat. The only nontrivial bracket relations are $[Y, Z] = -Z$ and $[Y, T] = T$. The Lie algebra \mathfrak{z}_X is isomorphic to $\mathfrak{sol} \oplus \mathbb{R}$.
- *Case 2: $\alpha \neq 0$ and $\alpha^2 = 3\sigma$.* Then we put $X' = 2\alpha X$, and we have $[Z, T] = X'$. It follows that Z, T, X' generate a subalgebra isomorphic to \mathfrak{heis} , that acts locally freely on M , and \mathfrak{z}_X is isomorphic to \mathfrak{g}_h . Our manifold (M, g) is locally modelled on the Lorentz-Heisenberg geometry \mathbf{X}_{LH} . Let us make a further observation. Our computations led to the relations $\omega(Z) = e + \alpha E$ and $\omega(T) = f + \alpha F$. Because $[E, e] = 0 = [F, f]$, we get that the curves $t \mapsto e^{tY}$ and $t \mapsto e^{tZ}$ are (lightlike) geodesics for the Lorentz-Heisenberg metric.
- *Case 3: $\alpha \neq 0$ and $\alpha^2 \neq 3\sigma$.* Then we can define $Y' = 2\alpha X + (3\sigma - \alpha^2)Y$, and the bracket relations become $[Z, T] = Y'$, $[Y', Z] = -(3\sigma - \alpha^2)Z$, $[Y', T] = 3(3\sigma - \alpha^2)T$. We see that Y', Z, T generate a Lie algebra isomorphic to $\mathfrak{sl}(2, \mathbb{R})$, that acts locally freely on M . The Lie algebra \mathfrak{z}_X is isomorphic to $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathbb{R}$. We conclude that we have a φ_X^t -invariant (G, \mathbf{X}) -structure on M , where $\mathbf{X} = \widetilde{\text{PSL}}(2, \mathbb{R})$, and $G = \widetilde{\text{PSL}}(2, \mathbb{R}) \times \{\tilde{h}^t\}$. Here $\{\tilde{h}^t\}$ is the 1-dimensional subgroup of $\widetilde{\text{PSL}}(2, \mathbb{R})$ obtained by exponentiating $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Because the group $\widetilde{\text{PSL}}(2, \mathbb{R}) \times \{\tilde{h}^t\}$ preserves the bi-invariant metric on $\widetilde{\text{PSL}}(2, \mathbb{R})$ coming from the Killing form, it follows that φ_X^t preserves a Lorentz metric g_0 of constant curvature -1 on M .

Similar computations can be carried out to determine \mathfrak{z}_X in the case where the isotropy is parabolic. Actually, those computations were already detailed in [Frab, Section 4.3.3], where all possibilities for local geometries having a 4-dimensional Lie algebra of Killing fields, and a parabolic isotropy, were listed. We restrict here to Lie algebras having a nontrivial center (which are the only candidates for \mathfrak{z}_X). Again, if we rule out the possibility of (M, g) being of constant curvature, [Frab, Prop 4.3] yields three more possibilities for the Lie algebra \mathfrak{z}_X .

- *Case 4.* The Lie algebra \mathfrak{z}_X is isomorphic to \mathfrak{g}_h (and the isotropy is parabolic). The local model for (M, g) is then \mathbf{X}_h .
- *Case 5.* The Lie algebra \mathfrak{z}_X is isomorphic to \mathfrak{g}_e . The local model for (M, g) is then \mathbf{X}_e .
- *Case 6.* The Lie algebra \mathfrak{z}_X is isomorphic to $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathbb{R}$. This time the isotropy is parabolic, so that the local model for (M, g) is $\widetilde{\text{PSL}}(2, \mathbb{R})$, endowed with a Lorentz metric invariant under $\widetilde{\text{PSL}}(2, \mathbb{R}) \times \{\tilde{u}^t\}$. Here $\{\tilde{u}^t\}$ is the 1-dimensional subgroup of $\widetilde{\text{PSL}}(2, \mathbb{R})$ obtained by exponentiating $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Again, there is then a Lorentz metric g_0 of constant curvature -1 on M , which is invariant under φ_X^t .

7.3. Completeness results. The final step toward the classification theorem 1.2 is the global understanding of the manifold M . In this section, \mathbf{X} will denote indistinctly $\mathbf{X}_h, \mathbf{X}_e$ or \mathbf{X}_{LH} . Any Lorentzian manifold (M, g) locally isometric to \mathbf{X} is automatically endowed with a $(\text{Iso}(\mathbf{X}), \mathbf{X})$ -structure (see [Thu97] for more material on this notion). Such a (G, \mathbf{X}) -structure on M gives rise to a developing map $\delta : \tilde{M} \rightarrow \mathbf{X}$ and a holonomy morphism $\rho : \text{Iso}(\tilde{M}, \tilde{g}) \rightarrow \text{Iso}(\mathbf{X})$.

Considering a finite cover of (M, g) we may assume that (M, g) is endowed with a $(G_h, \mathbf{X}_h), (G_h, \mathbf{X}_{HL})$ or (G_e, \mathbf{X}_e) -structure.

Of crucial importance then, are completeness results for the (G, \mathbf{X}) -structures we are considering. For the geometries $\mathbf{X}_e, \mathbf{X}_h$ and \mathbf{X}_{LH} , this completeness is respectively proved in [DZ10, Prop 7.3] and [DZ10, Section 8.1]. Observe that the proof given in [DZ10], of the completeness of compact manifolds locally modelled on the Lorentz-Heisenberg geometry uses Theorem 1.2. To avoid any vicious circle, we give below an independent proof (under our standing assumption that φ_X^t is not relatively compact), which follows the arguments of [Ghy87, Thm. 5.7].

Proposition 7.2. *Any compact Lorentzian manifold locally modelled on $\mathbf{X}_{LH}, \mathbf{X}_h$ or \mathbf{X}_e , and for which the characteristic flow is not relatively compact is complete. Namely, the structure is the quotient of the model space by a discrete subgroup of isometries.*

Proof:

We first sketch the arguments for structures modelled on \mathbf{X}_h or \mathbf{X}_e . In Section 7.1.4, we explained that any M modelled on \mathbf{X}_h or \mathbf{X}_e is endowed with a lightlike, codimension 1 foliation \mathcal{F} , the leaves of which have a Heisenberg structure. The foliation \mathcal{F} integrates the distribution X^\perp . Replacing M by a double cover, we can pick W a global vector field on M such that $g(W, W) = 1$ and $g(W, X) = 0$. This vector field W is complete, and the proof of Proposition 5.1 works *verbatim* to show that the Heisenberg structure of each leaf of \mathcal{F} is complete. Taking again a double cover of M , we can also pick another vector field W' on M which is transverse to \mathcal{F} and satisfies $\alpha(W', W') = 1$ (here α is the degenerate metric of signature $(+, 0, 0)$ introduced in Section 7.1.4 and defining the transverse Riemannian structure of \mathcal{F}). Then, just as in the proof of Proposition 5.1, we get that if \tilde{F} is any lift of a leaf F to the universal cover \tilde{M} , and if ψ^t is the flow of the lift of W' to \tilde{M} , then $\cup_{t \in \mathbb{R}} \psi^t(\tilde{F})$ is mapped diffeomorphically on \mathbf{X}_e (resp. \mathbf{X}_h) by the developing map $\delta : \tilde{M} \rightarrow \mathbf{X}_e$ (resp. \mathbf{X}_h). One easily deduced that δ has the path-lifting property, hence is a diffeomorphism from \tilde{M} to \mathbf{X}_e (resp. \mathbf{X}_h).

Let us now consider the case of Lorentz-Heisenberg geometry. We have again a developing map $\delta : \tilde{M} \rightarrow \mathbf{X}_{LH}$. We recall the basis T, X, Y, Z introduced in Section 7.1.1. Our remark in the study of case 2 (Section 7.2) says that for every $g \in \text{Heis}$, the curves $t \mapsto e^{tY}$ and $t \mapsto e^{tZ}$ are lightlike geodesics for the Lorentz-Heisenberg metric. Those geodesics are

orthogonal to the flow generated by the center z^t . Our hypothesis that the characteristic flow φ_X^t is not relatively compact ensures that lightlike geodesics orthogonal to X in M are complete (Lemma 3.2). It follows that any piecewise smooth curve of Heis, obtained by successively flowing along the right multiplication under e^{tY} , e^{tZ} and $z^t = e^{tX}$, can be lifted to \tilde{M} through the map δ , with arbitrary initial condition. Let us fix $\epsilon > 0$ such that the map $f : (r, s, t) \mapsto \text{Heis}$ defined by $f(r, s, t) = e^{rY} e^{sZ} z^t$ yields a diffeomorphism from $(-\epsilon, \epsilon) \times (-\epsilon, \epsilon) \times (-\epsilon, \epsilon)$ to its image B_ϵ . Our previous remark about completeness ensures that whenever \tilde{x} is a point of \tilde{M} , such that $\delta(\tilde{x}) = g$, then gB_ϵ can be lifted to U containing \tilde{x} , such that $\delta : U \rightarrow gB_\epsilon$ is a diffeomorphism. One easily deduces that $\delta : \tilde{M} \rightarrow \mathbf{X}_{LH}$ has the path-lifting property, hence is a diffeomorphism.

◇

Corollary 7.3. (see [Zeg96, Section 14]) *For every compact Lorentzian manifold locally modelled on \mathbf{X}_{LH} , \mathbf{X}_h or \mathbf{X}_e , the characteristic flow is relatively compact.*

Proof: Our manifold (M, g) is endowed with a (G, \mathbf{X}) structure, where \mathbf{X} is \mathbf{X}_{LH} , \mathbf{X}_h or \mathbf{X}_e , and G is G_h or G_e . We denote by Γ the image of $\pi_1(M)$ by the holonomy morphism ρ . Because we have shown that the developing map δ is a diffeomorphism, Γ is a discrete subgroup of G . The flow φ_X^t is mapped by ρ to the center $\{z^t\}$ of G .

The algebraic structure is in all the three cases that of a semi-direct product $\mathbb{R} \ltimes \text{Heis}$. Moreover, G is always the identity component of an algebraic group, so that it makes sense to consider $\bar{\Gamma}$, the identity component of the Zariski closure of Γ in G .

- *Case 1:* $\bar{\Gamma} \cap \text{Heis} = \{1\}$. In this case $\bar{\Gamma}$ projects injectively on \mathbb{R} , hence is abelian. The only connected abelian subgroups of G have dimension 1 or 2. If the dimension of $\bar{\Gamma}$ is one, then $\bar{\Gamma}$ is the exponentiation of some $U \in \mathfrak{g}$. Observe that U is centralized by an index 2 subgroup of Γ . If U lies in the center of \mathfrak{g} , then φ_X^t is a periodic flow. If U is not in the center of \mathfrak{g} , then U gives rise to a Killing field on M which is not proportional to X and commutes with X . Lemma 3.2 says that φ_X^t is relatively compact. If the dimension of $\bar{\Gamma}$ is 2, we directly get that $\bar{\gamma}$ contains the center $\{z^t\}$, so that $\{z^t\}$ is relatively compact in $\bar{\Gamma}/\Gamma$. It follows again that φ_X^t is relatively compact.
- *Case 2:* $\bar{\Gamma} \cap \text{Heis} \neq \{1\}$. In this case $\Gamma' = \Gamma \cap \text{Heis}$ is a nontrivial, discrete, normal subgroup of Γ . Either Γ' is cyclic, in which case it is included in a 1-parameter group of Heis. If this group is the center, then φ_X^t is periodic. If this group is not the center, we again get a Killing field on M which commutes with X , without being proportional to X , so that φ_X^t is relatively compact by Lemma

3.2. Any non-cyclic discrete subgroup of Heis intersects the center nontrivially. hence, if Γ' is not cyclic, then φ_X^t is periodic.

◇

7.4. Structures with constant curvature and classification. All our work so far says that whenever (M, g) is a closed 3-dimensional Lorentz manifold, and φ_X^t is a non relatively compact isometric flow, then either g is flat, or φ_X^t preserves a metric g_0 with constant curvature -1 on M . We are thus reduced to study structures of constant curvature having a non relatively compact flow of isometries.

Thanks to the work of Y. Carrière [Car89] (curvature zero) and B. Klingler [Kli96] (arbitrary curvature), closed Lorentz manifolds of constant curvature are complete, namely are obtained as a quotient of the 1-connected, complete model space of constant curvature by a discrete subgroup Γ of isometries (it follows that the curvature has to be zero or negative). Those completeness results are deeper, and more difficult, than the ones presented above for geometries \mathbf{X}_e , \mathbf{X}_h , \mathbf{X}_{LH} , and we can't present them here. They will allow us to recover the classification of Theorem 1.2.

7.4.1. *Flat structures.* The model space in this case is Minkowski space $\mathbb{R}^{1,2}$, namely \mathbb{R}^3 endowed with the metric $dx_1 dx_3 + dx_2^2$. The flow φ_X^t has constant norm (Lemma 3.2), hence its orbits are geodesics. It follows that φ_X^t is mapped by the holonomy morphism to a flow z^t of translations, which may be spacelike or lightlike (the timelike case is ruled out by lemma 3.2). We thus may assume, after conjugating in $\text{Iso}(\mathbb{R}^{1,2})$, that z^t is the flow of lightlike translations $T_{e_1}^t$, or of spacelike translations $T_{e_2}^t$.

The following result states the completeness of closed flat Lorentz manifolds ([Car89]), as well as a Bieberbach type theorem proved in [FG83] (see also [GK84]):

Theorem 7.4 (Bieberbach's theorem for flat Lorentz manifolds). *Let (M, g) be a closed, 3-dimensional, flat Lorentz manifold. There exists a discrete subgroup $\Gamma \subset \text{Iso}(\mathbb{R}^{1,2})$ such that (M, g) is isometric to the quotient $\Gamma \backslash \mathbb{R}^{1,2}$. Moreover, there exists a connected 3-dimensional Lie group $G \subset \text{Iso}(\mathbb{R}^{1,2})$, which is isomorphic to \mathbb{R}^3 , Heis or SOL, and which acts simply transitively on $\mathbb{R}^{1,2}$, satisfying that $\Gamma_0 = G \cap \Gamma$ has finite index in Γ and is a uniform lattice in G .*

Considering a finite cover of (M, g) we may assume $\Gamma = \Gamma_0$ in the previous statement. The centralizer of $T_{e_1}^t$ in $\text{Iso}(\mathbb{R}^{1,2})$ is the following 4-dimensional Lie group

$$G_1 = \left\{ \left(\begin{pmatrix} 1 & s & -\frac{s^2}{2} \\ 0 & 1 & -s \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} t \\ u \\ v \end{pmatrix} \mid (s, t, u, v) \in \mathbb{R}^4 \right\}.$$

The centralizer of $T_{e_2}^t$ in $\text{Iso}(\mathbb{R}^{1,2})$ is the following 4-dimensional Lie group

$$G_2 = \left\{ \left(\begin{pmatrix} e^s & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-s} \end{pmatrix} + \begin{pmatrix} u \\ t \\ v \end{pmatrix} \mid (s, t, u, v) \in \mathbb{R}^4 \right\}.$$

There are no copies of SOL in the group G_1 , and all copies of \mathbb{R}^3 and Heis contain the translations along e_1 , namely the 1-parameter group $\{z^t\}$. We infer that any lattice Γ in such a group meets $\{z^t\}$ non-trivially, implying that φ_X^t is cyclic. We thus rule out this case, and focus on the case where φ_X^t is spacelike, and the centralizer of $\{z^t\}$ is G_2 .

There are no copies of Heis in G_2 . The only copy of \mathbb{R}^3 is the subgroup of translations, hence any lattice Γ in it will meet $\{z^t\}$ non-trivially. We again get a cyclic flow φ_X^t in this case, what contradicts our hypothesis of non-compactness. The only remaining case is when $G < G_2$ is isomorphic to SOL. It is thus a group of the form

$$G = \left\{ \left(\begin{pmatrix} e^s & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-s} \end{pmatrix} + \begin{pmatrix} u \\ \beta s \\ v \end{pmatrix} \mid (s, u, v) \in \mathbb{R}^3 \right\},$$

for some $\beta \in \mathbb{R}^*$.

The action of the flow $T_{e_2}^t$ on the quotient $\Gamma \backslash G$ is (after a possible reparametrization $T_{e_2}^{\alpha t}$, $\alpha \in \mathbb{R}^*$) the suspension of a hyperbolic toral automorphism. We are in the first case of Theorem 1.2.

7.4.2. Anti-de Sitter structures. We investigate now the case when φ_X^t preserves a Lorentz metric g_0 of curvature -1 on M . Klingler's completeness theorem [Kli96] ensures that (M, g_0) is the quotient of $(\widetilde{\text{PSL}}(2, \mathbb{R}), \tilde{g}_{AdS})$ (the universal cover of anti-de Sitter space, see Section 1.2.2) by a discrete group of isometries $\tilde{\Gamma}$. An important result, known as *finiteness of level*, says that $\tilde{\Gamma}$ must intersect the center of $\widetilde{\text{PSL}}(2, \mathbb{R})$ non trivially. This property was first stated in [KR85]. A detailed proof can be found in [Sal99, Theorem 3.3.2.3]. It follows that up to finite cover, (M, g) is actually isometric to a quotient of $(\text{PSL}(2, \mathbb{R}), g_{AdS})$ by a discrete subgroup $\Gamma \subset \text{PSL}(2, \mathbb{R}) \times \text{PSL}(2, \mathbb{R})$. The structure of the group Γ is well understood. Up to conjugacy, there exists Γ_0 a uniform lattice in $\text{PSL}(2, \mathbb{R})$, and a representation $\rho : \Gamma_0 \rightarrow \text{PSL}(2, \mathbb{R})$ such that

$$\Gamma = \Gamma_\rho = \{(\gamma, \rho(\gamma)) \in \text{PSL}(2, \mathbb{R}) \times \text{PSL}(2, \mathbb{R}) \mid \gamma \in \Gamma_0\}.$$

This was established in [KR85, Theorem 5.2] when Γ is torsion-free. For a group with torsion, the adapted proof can be found in [Tho14, Lemma 4.3.1]. Denoting by z^t the image of φ_X^t by the holonomy morphism, we see that z^t must centralize the group Γ . It is thus a flow of the form $\{id\} \times \{u^t\}$ (in which case Γ_ρ is of the form Γ_{ρ_u} , see Section 1.2.2), or of the form $\{id\} \times \{h^t\}$ (in which case Γ_ρ is of the form Γ_{ρ_h}).

The manifold (M, g_0) is isometric to a quotient $\Gamma_{\rho_u} \backslash \mathrm{PSL}(2, \mathbb{R})$ (resp. $\Gamma_{\rho_h} \backslash \mathrm{PSL}(2, \mathbb{R})$) endowed with a metric \bar{g}_{AdS} . The flow φ_X^t coincides with the right action of $\{h^t\}$ (resp. of $\{u^t\}$). Regarding the initial metric g , it must come from a $\mathrm{Ad}(u^t)$ -invariant (resp. $\mathrm{Ad}(h^t)$ -invariant) Lorentz scalar product on $\mathfrak{sl}(2, \mathbb{R})$. It is thus of the form \bar{g}_u or \bar{g}_{AdS} (resp. \bar{g}_h or \bar{g}_{AdS}). This completes the proof of Theorem 1.2.

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