High-gain observer for 3x3 linear heterodirectional hyperbolic systems

Constantinos Kitsos, Gildas Besancon, Christophe Prieur

High-gain observer for 3×3 linear heterodirectional hyperbolic systems

Constantinos Kitsos [a,b], Gildas Besançon [b], Christophe Prieur [b]

[a] *LAAS-CNRS, Univ. of Toulouse, CNRS, Toulouse, France.*

[b] *Univ. Grenoble Alpes, CNRS, Grenoble INP°, GIPSA-lab, Grenoble, France.*

° *Institute of Engineering*

Abstract

The problem of High-Gain Observer Design is addressed for a class of 3×3 inhomogeneous linear hyperbolic systems with possibly distinct characteristic velocities and considering distributed measurement of part of the state. Applying an infinite-dimensional state transformation, the system is mapped into a new set of partial differential equations, satisfying an appropriate form for observer design. The observer includes output correction terms and also spatial derivatives of the output, and ensures arbitrarily fast state estimation. The result is illustrated with a simulated example.

Key words: linear hyperbolic systems, high-gain observers, C^1-convergence

1 Introduction

High-gain observer design for nonlinear finite-dimensional systems has been considerably developed (see e.g. Khalil (2017) and references therein). In short, it relies on a single tuning coefficient, to be chosen large enough so as to ensure exponential - and arbitrarily fast - convergence. High-gain observers apply to a large class of cases corresponding to uniformly observable systems Gauthier and Bornard (1981). In the recent papers Kitsos et al (2018), Kitsos et al (2020), this approach was extended to infinite-dimensional systems, namely, quasi-linear hyperbolic systems made of $n \times n$ balance laws with one characteristic velocity and considering distributed measurement of a part of the state. In Kitsos et al (2019) the more general case of two distinct characteristic velocities was considered, but for 2×2 systems written in a triangular form.

Other observer designs for hyperbolic systems can be found in the literature, but mainly considering the full state vector on the boundaries as measurement, and without the high gain features. Amongst others, one can mention Di Meglio et al (2013), Hasan et al (2016), and Anfinsen et al (2016) for backstepping designs, Besançon et al (2006) for a discretization approach (early-lumping techniques), Castillo et al (2013) for direct infinite-dimension-based Lyapunov techniques (see also Besançon et al (2013)), or Nguyen et al (2016) for optimization methods. For semigroup and spectral-based methods, see for instance Curtain (1982), Demetriou (2004) and Schaum et al (2015). In Christophides and Daoutidis (1996), there appear distributed observers relying on operator theoretic and geometric techniques.

The present paper aims at providing a solution to a high-gain observer design problem (H-GODP) for a class of 3×3 inhomogeneous linear hyperbolic systems of balance laws with one measurement, written in a triangular form, as an extension of Kitsos et al (2020), Kitsos et al (2018) for the case of systems with distinct velocities. This case presents technical difficulties, even for linear systems. Similar difficulties appear in the notable work of Alabau-Boussouira et al (2017), as well as in previous works of the same authors, about the internal controllability of hyperbolic systems in cascade form with reduced number of control laws. In these works, stronger regularity of the solutions is required. The approach here on observer design with reduced number of observations reveals some duality with this framework. The present paper also extends the results of Kitsos et

⋆ The work of the third author has been partially supported by MIAI@Grenoble Alpes (ANR-19-P3IA-0003).

Email addresses: constantinos.kitsos@laas.fr (Constantinos Kitsos), gildas.besancon@grenoble-inp.fr (Gildas Besançon), christophe.prieur@grenoble-inp.fr (Christophe Prieur).

al (2019) up to 3×3 strictly hyperbolic systems. Moreover, it can be noticed that, even though the system is linear, the use of high gain is instrumental, as a result of the considered domain of the hyperbolic operator. The class of systems that we study, written in an appropriate triangular form, can be found in various situations, like plug flow chemical reactors, where, by measuring the jacket temperature, we would expect to estimate the concentrations of masses of chemicals, or in linearized hyperbolic Lotka-Volterra systems with three species in cascade form (see for instance Zhang et al (2018)). Stability and controllability for linear hyperbolic systems have been widely studied, see for instance Bastin and Coron (2016), Prieur and Winkin (2018), Espitia et al (2017) by employing Lyapunov and operator-theoretic strategies. In Coron and Nguyen (2019) and Auriol and di Meglio (2016), the problem of minimum-time control of such systems arises, suggesting that a faster observer than a boundary one, which overcomes the transport phenomena restrictions, would be desirable.

The main contribution here is the proof of solvability of the H-GODP for 3×3 strictly hyperbolic triangular systems via an indirect approach. An important idea of this paper is to perform an appropriate infinite - dimensional state transformation to overcome the lack of a commutative property, yet needed in the Lyapunov stability analysis. Note that constraints on the source term can be found in some studies of stability problems as in Bastin and Coron (2016), Coron and Bastin (2015) and in Prieur et al (2014) (Proposition 2.1), which allow a similar commutation, while this is not the case here. The transformation we use is related to the system's triangularity and maps the original hyperbolic system into a target system of partial differential equations (PDEs). This methodology requires stronger regularity for system's dynamics and additionally to the output correction terms, ouput's spatial derivatives up to order $q-1$ are injected in the high-gain observer dynamics, where q is an indicator of the number of different characteristic velocities. The convergence of the observer is then proven for the sup spatial norm.

The paper is organized as follows. The description of the H-GODP, the sufficient conditions and a solution to it are presented in Section 2, where Theorem 1 constitutes the main result. In Section 3 we prove Theorem 1 and in Section 4, we illustrate our methodology via a plug flow chemical reactor. Conclusions and perspectives are discussed in Section 5.

Notation: For a given w in $\mathbb{R}^n$, $|w|$ denotes its usual Euclidean norm. For a given matrix A in $\mathbb{R}^{n\times n}$, $A^\top$ denotes its transpose, $|A| := \sup\{|Aw|, |w| = 1\}$ is its induced norm and $\mathrm{Sym}(A) = \frac{A+A^\top}{2}$ stands for its symmetric part. By $\underline{\mathrm{eig}}(A)$ we denote the minimum eigenvalue of a symmetric matrix A. By I_n we denote the identity matrix of dimension n. By $\mathrm{sgn}(x)$ we denote the signum function $\mathrm{sgn}(x) = \frac{\mathrm{d}}{\mathrm{d}x}|x|$, when $x \neq 0$, with $\mathrm{sgn}(0) = 0$. For given $\xi : [0, +\infty) \times [0, L] \to \mathbb{R}^n$ and time $t \geq 0$ we use the notation $\xi(t)(x) := \xi(t, x)$ to refer to the profile at time t. For a real-valued function f, by f' we denote its first derivative. For a continuous (C^0) map $[0, L] \ni x \mapsto \xi(x) \in \mathbb{R}^n$ (or $[0, L] \ni x \mapsto A(x) \in \mathbb{R}^{n\times n}$) we adopt the notation $\|\xi\|_\infty := \max\{|\xi(x)|, x \in [0, L]\}$ ($\|A\|_\infty := \max\{|A(x)|, x \in [0, L]\}$ respectively). For a q - times continuously differentiable (C^q) map $[0, L] \ni x \mapsto \xi(x) \in \mathbb{R}^n$ we adopt the notation $\|\xi\|_q := \sum_{i=0}^{q} \|\partial_x^i \xi\|_\infty$.

2 Problem statement and main result

We are concerned with one-dimensional, first-order linear hyperbolic systems of balance laws, described by the following equations on a strip $\Sigma := [0, +\infty) \times [0, L]$

$$\xi_t(t, x) + \Lambda(x)\xi_x(t, x) = M(x)\xi(t, x), \tag{1a}$$

where $\xi = \begin{pmatrix} \xi_1 & \xi_2 & \xi_3 \end{pmatrix}^\top : \Sigma \to \mathbb{R}^3$.

Consider also a distributed measurement, provided in the output, of the form

$$y(t, x) = C\xi(t, x), \tag{1b}$$

where y is a mapping from Σ to $\mathbb{R}$ and the matrix of characteristic velocities

$$\Lambda(x) = \mathrm{diag}\{\lambda_1(x), \lambda_2(x), \lambda_3(x)\},$$

with $\lambda_1(x) > 0$ for all $x \in [0, L]$ (without loss of generality), contains $m \in \{1, 2, 3\}$ positive characteristic velocities $\lambda_i(x)$ and $3 - m$ negative ones, implying that these are of constant sign in $[0, L]$. We have the following algebraic structures for the involved matrices

$$M(x) = \begin{pmatrix} m_{1,1}(x) & m_{1,2}(x) & 0 \\ m_{2,1}(x) & m_{2,2}(x) & m_{2,3}(x) \\ m_{3,1}(x) & m_{3,2}(x) & m_{3,3}(x) \end{pmatrix},$$
$$C = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}.$$

To give appropriate boundary conditions, let us first define a permutation matrix denoted by $I_\pi \in \mathbb{R}^{3\times 3}$, which re-orders the two last components of ξ according to the signs of the related second and third characteristic velocities, namely, putting the m elements corresponding to positive velocities in ξ^+ and the $3 - m$ to negative velocities in ξ^-. More explicitly,

$$I_\pi \xi := \begin{pmatrix} \xi^+ \\ \xi^- \end{pmatrix}.$$

Boundary conditions are written in the form

$$\begin{pmatrix} \xi^+(t,0) \\ \xi^-(t,L) \end{pmatrix} = K \begin{pmatrix} \xi^+(t,L) \\ \xi^-(t,0) \end{pmatrix}, \tag{2a}$$

where K is a matrix of the form $K = \begin{pmatrix} K_{00} & K_{01} \\ K_{10} & K_{11} \end{pmatrix}$, with $K_{00} \in \mathbb{R}^{m\times m}, K_{01} \in \mathbb{R}^{m\times(3-m)}, K_{10} \in \mathbb{R}^{(3-m)\times m}$, $K_{11} \in \mathbb{R}^{(3-m)\times(3-m)}$.

We also consider initial condition

$$\xi(0,x) =: \xi^0(x), x \in [0,L]. \tag{2b}$$

Notice that the considered system provides an internal measurement of the first state only, while not any information on the two other states is given, except for the general relationship between incoming and outgoing information on the boundaries, given by a general law (2).

Remark 1 *We assumed, without loss of generality, that the first characteristic velocity λ_1 is positive and we perform all the calculations that follow for this particular case. Indeed, if λ_1 is negative, then applying the space variable transformation $x \mapsto L - x$, we get a hyperbolic system with opposite sign of all three velocities. Then, the observer design methodology remains unchanged for the new system. Therefore, in this paper, we may assume that the first characteristic velocity is positive.*

At this point, let us define a, roughly speaking, "index of strict hyperbolicity" as follows

$$q := \min\{i : \lambda_i \equiv \lambda_j, \forall j = i, \dots, 3\},$$

where we used the equivalence relation $\lambda_i \equiv \lambda_j \Leftrightarrow \lambda_i(x) = \lambda_j(x)$, for all x in $[0,L]$. By this definition, we have $q \in \{1,2,3\}$, and in case of a strictly hyperbolic system, we have $q = 3$. The case where $q = 1$ (a single velocity), as a particular case of the general formulation here, has been already addressed in Kitsos et al (2018), while the case where $q \neq 1$ is more complex and needs a different treatment. We further define

$$\bar{q} := \max(1, q-1).$$

We are now in a position to present the main assumptions.

Assumption 1 *Functions λ_i and $m_{i,j}$ belong to $C^{\bar{q}}([0,L];\mathbb{R})$. Initial condition ξ^0 in $C^{\bar{q}}([0,L];\mathbb{R}^3)$ satisfies compatibility conditions of order $\bar{q}$ (see (Bastin and Coron , 2016, Chapter 4.5.2) for definition of compatibility conditions of any order).*

The following assumption concerns the structure of the matrix K linking the incoming with the outgoing information on the boundaries.

Assumption 2 *If $\lambda_3(\cdot) > 0$, then K_{11} is invertible and if $\lambda_3(\cdot) < 0$, then K_{00} is invertible*

The previous requirement is trivially met if all velocities λ_1, λ_2, and λ_3 are positive. Also, a case where this assumption is trivially satisfied is when $K = I_3$. In that case, boundary conditions are x-periodic. Note that an x- periodicity assumption is made in Alabau-Boussouira et al (2017) for the somehow dual problem of controllability of underactuated systems.

The nature of the next assumption is revealed in Kitsos et al (2018), where a triangular form is introduced for the case of quasi-linear hyperbolic systems. This assumption allows us to obtain a target system, that we introduce later, and is also a sufficient condition for the observer design.

Assumption 3 *The following condition is satisfied for all x in $[0,L]$:*

$$m_{1,2}(x), m_{2,3}(x) \neq 0.$$

Given the previous assumption, system (1a) satisfies some triangular structure, which presents an analogy to the finite-dimensional case (see Khalil (2017)).

The following fact results from classical arguments borrowed from the theory of linear hyperbolic systems, combined with the manipulation of the extra regularity (see (Bastin and Coron , 2016, App. A, Chap. 4.5) and references therein).

Fact 1 *Under Assumption 1, there exists a unique global solution to system (1a)-(2) in $C^{\bar{q}}([0,+\infty)\times[0,L];\mathbb{R}^3)$.*

The problem that we address in this work is stated in the following definition.

Definition 1 *(H-GODP) The High-Gain Observer Design Problem is solvable for a system given by (1a)-(2) with output (1b), while output's spatial derivatives of order $q-1$ might also be available, if there exists a well-posed observer system, which estimates the state of (1a) with a convergence speed that can be arbitrarily tuned via a single parameter (high-gain constant).*

The observer design problem under consideration relies on a distributed measurement of a part of the state. The main feature of it is the arbitrarily fast convergence rate, similarly as in the finite dimensions and to achieve this property, distributed measurement on the whole domain is assumed. We note here that, although boundary observers with the full-state measurement are preferred for practical reasons, in the present formulation, distributed measurement of part of the state might be available in many cases of distributed parameter systems. For instance, some setups include thermal cameras for chemical reactors or alternative methods, see for instance Zogg et al (2004), Pradere et al (2009), providing the desired measurements. Also, approximations with distributed measurements within the domain would provide an approximated measurement on the whole domain. Furthermore, as indicated in the H-GODP definition, stronger regularity of the solutions to the initial systems is required for some classes of systems, coming from Assumption 1, since the observer dynamics may include higher-order spatial derivatives of the output. These spatial derivatives might be available, since their knowledge is causal, contrary to the time-derivatives which are strictly avoided. Approximations of such derivatives via kernel convolutions might be investigated more. In the following remark, we discuss the problem of solvability of the H-GODP, if instead of a distributed measurement, we had a boundary one.

Remark 2 *H-GODP is not solvable in case of boundary measurement, instead of internal measurement as in* (1b). *First, arbitrary convergence condition would not be fulfilled, since a boundary observer for hyperbolic systems would experience a limitation with respect to convergence speed. The rate of convergence is limited by a minimal observation time which depends on the size of the domain L and the characteristic velocities λ_i in that case (see Li (2008) for minimum time of observability due to transport phenomena, and Deutschmann et al (2016) for some comments on the convergence of boundary observers). Solvability of the H-GODP implies that the convergence rate can be arbitrary fast. Second, following a boundary observer design methodology as in Castillo et al (2013), in the presence of the most general form of boundary conditions, where a general law couples the incoming with the outgoing information on the boundaries, see* (2)*, a reduced number of observations would not be enough to lead to the observer convergence. In the present case, a dissipativity property on the boundaries that would lead to observer convergence cannot be achieved by just one measurement, namely the first state ξ_1. Furthermore, in Di Meglio et al (2013) observer design is achieved with reduced number of boundary observations (see also Coron et al (2013) for the control design for underactuated system). In these cases, the considered boundary conditions allow a backstepping transformation of the observer error equations into a system with dissipative boundary conditions which is stable, but this is not the case here, since the present boundary conditions cannot allow us to obtain this dissipativity (see Coron and Bastin (2015) about linking dissipativity of boundary conditions with stability).*

To proceed to the observer design, we introduce an indirect approach to deal with the generality of the considered hyperbolic operator. Although system (1a)-(1b)-(2) is written in an appropriate triangular form, as it was introduced in Kitsos et al (2018), Kitsos et al (2020), it seems that in presence of distinct characteristic velocities, we need to employ a different strategy, in order to find solutions for the H-GODP. The problem comes from the fact that the balance laws in (1a) do not allow the choice of a diagonal Lyapunov functional to be used in the stability analysis of the observer error equations. A non-diagonal Lyapunov functional does not permit an integration by parts when taking its time-derivative, since the Lyapunov matrix and the matrix of velocities do not commute. To address this problem, we perform a transformation including spatial derivations of the state up to order $q-2$, in order to write the system in an appropriate form for which a Lyapunov approach is feasible. Then, for the obtained target system, we design the high-gain observer and, finally, returning to the initial coordinates, solvability of H-GODP is guaranteed. Note that H-GODP for 2×2 quasilinear hyperbolic systems with two velocities has been already solved in Kitsos et al (2019). The latter is a simpler problem than the present one of the 3×3 case, since no state transformation is required. The increased difficulties with respect to the presence of distinct velocities appear in the somehow dual problems of internal controllability with reduced numbers of controls (see comments on algebraic solvability in Alabau-Boussouira et al (2017)) and the main technical difficulty of the present work is to tackle this problem.

Let us define a Banach space $\mathcal{X}$ by

$$\mathcal{X} := C^{\bar{q}}([0,L];\mathbb{R}) \times C^1([0,L];\mathbb{R}^2),$$

equipped with the norm $\|\xi\|_{\mathcal{X}} := \|\xi_1\|_{\bar{q}} + \|\xi_2\|_1 + \|\xi_3\|_1$. Assume that there exists a linear bounded and injective transformation $\mathcal{T} : (\mathcal{X}, \|\cdot\|_{\mathcal{X}}) \to (\mathcal{X}, \|\cdot\|_{\mathcal{X}})$, with bounded inverse, which maps system (1a)-(2) into a target system ζ, as follows

$$\begin{gathered}\zeta = \mathcal{T}\xi; \\ \text{with } \zeta_1 = \xi_1.\end{gathered} \tag{3}$$

The desired target system (T) of PDEs that we consider in this work satisfies the following equations on Σ, distinguishing two boundary cases, depending on the sign of $\lambda_1(\cdot)$

$$(\mathrm{T})\begin{cases}\zeta_t(t,x) + \lambda_3(x)\zeta_x(t,x) = \tilde{M}(x)\zeta(t,x) + \mathcal{M}\zeta_1(t)(x), \\ \begin{pmatrix}\zeta^+(t,0)\\ \zeta^-(t,L)\end{pmatrix} = K\begin{pmatrix}\zeta^+(t,L)\\ \zeta^-(t,0)\end{pmatrix} + \mathcal{K}_1\zeta_1(0) + \mathcal{K}_2\zeta_1(L), \\ y_\zeta(t,x) = y(t,x) = C\zeta(t,x),\end{cases} \tag{4}$$

with initial condition $\zeta(0,x) := \zeta^0(x) = \mathcal{T}\xi^0(x)$, where $\mathcal{M} : C^{q-1}([0,L];\mathbb{R}) \to C^0([0,L];\mathbb{R}^3), \mathcal{K}_1, \mathcal{K}_2 : C^{q-1}([0,L];\mathbb{R}) \to \mathbb{R}^n$ are linear differential operators acting on ζ_1, to be determined in the sequel, depending on the choice of $\mathcal{T}$, $\tilde{M}(x)$ is a matrix satisfying the same structure as $M(x)$ and y_ζ is target system's output, which remains equal to original system's output y. The existence of such a transformation $\mathcal{T}$ is shown in the following section.

The proposed high-gain observer for (4) satisfies the following equations on Σ

$$\hat{\zeta}_t(t,x) + \lambda_3(x)\hat{\zeta}_x(t,x) = \tilde{M}(x)\hat{\zeta}(t,x) - \Theta N(x)\left(y(t,x) - C\hat{\zeta}(t,x)\right) + \mathcal{M}y(t)(x), \tag{5a}$$

$$\begin{pmatrix}\hat{\zeta}^+(t,0)\\ \hat{\zeta}^-(t,L)\end{pmatrix} = K\begin{pmatrix}\hat{\zeta}^+(t,L)\\ \hat{\zeta}^-(t,0)\end{pmatrix} + \mathcal{K}_1 y(0) + \mathcal{K}_2 y(L), \tag{5b}$$

with initial condition $\hat{\zeta}^0(x) := \hat{\zeta}(0,x)$ (for a function $\hat{\zeta}^0$ in $\mathcal{X}$), where

$$\Theta := \operatorname{diag}\left\{\theta, \theta^2, \theta^3\right\},$$

with $\theta \geq 1$ the candidate high-gain constant of the observer, to be selected precisely later. In the above equations, we considered, also, a vector gain $N(\cdot)$ in $C^{\bar{q}}\left([0,L];\mathbb{R}^3\right)$, selected in a way, such that, for $P(\cdot)$ in $C^{\bar{q}}\left([0,L];\mathbb{R}^{3\times3}\right)$ symmetric and positive definite, a Lyapunov equation of the following form is satisfied

$$2\mathrm{Sym}\left(P(x)\left(M_1(x) + N(x)C\right)\right) = -Q(x) \tag{6}$$

for some symmetric and positive definite $Q(x)$ of class $C^{\bar{q}}$, where $M_1(x)$ is derived by $M(x)$ keeping only its 1 - diagonal, namely,

$$M_1(x) = \begin{pmatrix} 0 & m_{1,2}(x) & 0 \\ 0 & 0 & m_{2,3}(x) \\ 0 & 0 & 0 \end{pmatrix}.$$

Lyapunov equation (6) is solvable by a symmetric and positive definite $P(x)$ for choice of $N(x)$, such that $M_1(x)+N(x)C$ is Hurwitz for all x in $[0,L]$. The latter is feasible due to observability of the pair $(M_1(x), C)$ coming from Assumption 3. In addition, we note that the solution $P(\cdot)$ of (6), used in a Lyapunov functional that we introduce in the sequel, is never diagonal, meaning that it would not commute in general with the matrix of velocities $\Lambda(\cdot)$. This is a problem for Lyapunov-based stability analysis, which motivates for the proposed infinite-dimensional transformation.

We are now in a position to present our main result on the solvability of the H-GODP.

Theorem 1 *Consider system* (1a)-(2)*, defined on* Σ *with output* (1b) *and suppose that Assumptions 1 - 3 hold. Let also* P *in* $C^{\bar{q}}\left([0,L];\mathbb{R}^{3\times3}\right)$ *be symmetric and positive definite and let* N *in* $C^{\bar{q}}\left([0,L];\mathbb{R}^3\right)$*, both satisfying* (6) *for some* Q *symmetric and positive definite in* $C^{\bar{q}}\left([0,L];\mathbb{R}^{3\times3}\right)$*. Then, the H-GODP is solvable by* $\mathcal{T}^{-1}\hat{\zeta}$ *(where* $\hat{\zeta}$ *is the unique solution to* (5)*), for* $\theta \geq 1$ *as a high gain and initial condition* $\mathcal{T}^{-1}\hat{\zeta}^0(x)$*, with* $\hat{\zeta}^0$ *satisfying zero and one-order compatibility conditions. More precisely, for every* $\kappa > 0$*, there exists* $\theta_0 \geq 1$*, such that for every* $\theta \geq \theta_0$*, the following holds for all* $t \geq 0$*:*

$$\|\mathcal{T}^{-1}\hat{\zeta}(t,\cdot) - \xi(t,\cdot)\|_\infty \leq \ell e^{-\kappa t}\|\mathcal{T}^{-1}\hat{\zeta}^0(\cdot) - \xi^0(\cdot)\|_{\mathcal{X}}, \tag{7}$$

with $\ell > 0$ *a polynomial in* θ*.*

This observer convergence result is based on the existence of a transformation $\mathcal{T}$ as introduced in (3), that we show in the next section, along with the observer convergence of Theorem 1. We note also that in the study of internal controllability for underactuated systems, the phenomenon of loss of derivatives appears, as the regularity of the dynamics is stronger than the regularity of the control laws, whenever the velocities are distinct (see Theorem 3.1 in Alabau-Boussouira et al (2017)). In the present framework of the solutions to the H-GODP, the regularity of system's dynamics needs to be stronger than the regularity of the space in the norm of which the asymptotic convergence of the observer is exhibited (sup spatial norm).

3 Proof of Theorem 1

In this section, we prove Theorem 1. We first show the existence of the infinite-dimensional state transformation $\mathcal{T}$ and then we prove the observer convergence via Lyapunov analysis.

3.1 Transformation and target system

We show here the existence of $\mathcal{T}$ and provide the corresponding dynamics of the target system (T).

Let us choose $\mathcal{T}$ in (3) by

$$\mathcal{T} := I_3 + \tilde{\mathcal{T}}; \qquad \tilde{\mathcal{T}} := \begin{pmatrix} 0 & 0 & 0 \\ \tau(x)\partial_x & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{8}$$

where

$$\tau(x) := \frac{\lambda_2(x) - \lambda_3(x)}{m_{1,2}(x)}.$$

Obviously, this transformation is bounded, invertible, with bounded inverse from $\mathcal{X}$ to $\mathcal{X}$, independently of boundary conditions. Applying this transformation to system (1a)-(2), we obtain target system (T) with

$$\begin{aligned}
\tilde{M}(x) :=& M(x) + \tau(x)\begin{pmatrix} 0 & 0 & 0 \\ m'_{1,1}(x) & m'_{1,2}(x) & 0 \\ 0 & 0 & 0 \end{pmatrix},\\
\mathcal{M} :=& \begin{pmatrix} a_1(x)\partial_x \\ a_2(x)\partial_x + a_3(x)\partial_x^2 \\ a_4(x)\partial_x \end{pmatrix},\\
\mathcal{K}_1 :=& \frac{1+\mathrm{sgn}(\lambda_1(x))}{2} I_\pi \tilde{\mathcal{T}} C^\top - \frac{1-\mathrm{sgn}(\lambda_1(x))}{2} K I_\pi \tilde{\mathcal{T}} C^\top,\\
\mathcal{K}_2 :=& -\frac{1+\mathrm{sgn}(\lambda_1(x))}{2} K I_\pi \tilde{\mathcal{T}} C^\top + \frac{1-\mathrm{sgn}(\lambda_1(x))}{2} I_\pi \tilde{\mathcal{T}} C^\top,
\end{aligned}$$

where

$$\begin{aligned}
a_1(x) :=& \lambda_3(x) - \lambda_1(x) - m_{1,2}(x)\tau(x),\\
a_2(x) :=& \tau(x) m_{1,1}(x) - \tau(x)\lambda_1'(x) - \tau(x)(m_{1,2}(x)\tau(x))'\\
& - \tau(x) m_{2,2}(x) + \tau'(x)\lambda_2(x),\\
a_3(x) :=& \left(\lambda_2(x) - \lambda_1(x) - \tau(x) m_{1,2}(x)\right)\tau(x),\\
a_4(x) :=& -\tau(x) m_{3,2}(x).
\end{aligned}$$

By the above forms of mappings $\mathcal{M}$ and $\mathcal{K}_1, \mathcal{K}_2$, we see that the target system of PDEs (4) contains spatial derivatives of the first state ζ_1 up to order 2 in its dynamics and up to order 1 in its boundaries. Notice also that for the case of 3 positive velocities ($m = 3$) as a special case, we have $I_\pi = I_3$ and mappings $\mathcal{K}_1$ and $\mathcal{K}_2$ take the simpler forms

$$\mathcal{K}_1 := \begin{pmatrix} 0 \\ \tau(0)\partial_x \\ 0 \end{pmatrix}, \mathcal{K}_2 := -K_{00}\begin{pmatrix} 0 \\ \tau(L)\partial_x \\ 0 \end{pmatrix}.$$

3.2 Observer convergence

We are in a position to prove that the proposed observer is a high-gain observer for the target system, which is mapped from the original system via $\mathcal{T}$. Observer's exponential convergence is proven for appropriate spatial norm. Injectivity of $\mathcal{T}$ and boundedness of its inverse, then, guarantees that $\mathcal{T}^{-1}\hat{\zeta}$ approaches exponentially the state ξ of the original system and, thus, solves the H-GODP.

We start from a prerequisite lemma for the well-posedness of the observer, which is a direct consequense of Assumption 1 and linearity of the system. The proof relies on the method of characteristics and a combination of classical arguments from proofs found in (Bastin and Coron , 2016, Chap. 4.5, App. A), therefore, it is omitted here.

Lemma 1 *Under the regularity assumptions for the dynamics and for any y in $C^{\bar{q}}([0,+\infty)\times[0,L];\mathbb{R})$, the problem described by* (5) *on domain Σ with initial condition $\hat{\zeta}^0(x)$ satisfying compatibility conditions of order* 1 *admits a unique solution $\hat{\zeta}$ in $C^1\left([0,+\infty)\times[0,L];\mathbb{R}^3\right)$.*

Consider, now, observer (5) for target system (T). We define a scaled observer error by $\varepsilon := \Theta^{-1}\left(\hat{\zeta} - \zeta\right)$, for which we derive the following hyperbolic equations on Σ

$$\begin{aligned}
&\varepsilon_t(t,x) + \lambda_3(x)\varepsilon_x(t,x) = \theta\left(M_1(x) + N(x)C\right)\varepsilon(t,x)\\
&+ \Theta^{-1}(\tilde{M}(x) - M_1(x))\Theta\varepsilon(t,x), &(9a)\\
&\varepsilon(t,0) = \Theta^{-1}\bar{K}\Theta\varepsilon(t,L), \text{when } \lambda_3(\cdot) > 0, &(9b)\\
&\varepsilon(t,L) = \Theta^{-1}\bar{\bar{K}}\Theta\varepsilon(t,0), \text{when } \lambda_3(\cdot) < 0, &(9c)
\end{aligned}$$

where

$$\begin{aligned}
\bar{K} :=& I_\pi^\top \begin{pmatrix} K_{00} - K_{01}K_{11}^{-1}K_{10} & K_{01}K_{11}^{-1} \\ -K_{11}^{-1}K_{10} & K_{11}^{-1} \end{pmatrix} I_\pi,\\
\bar{\bar{K}} :=& I_\pi^\top \begin{pmatrix} K_{00}^{-1} & -K_{00}^{-1}K_{01} \\ K_{10}K_{00}^{-1} & K_{11} - K_{10}K_{00}^{-1}K_{01} \end{pmatrix} I_\pi.
\end{aligned}$$

To prove exponential stability of with respect to its origin of the solutions to observer error system, we adopt a Lyapunov-based approach inspired by methodologies presented in Bastin and Coron (2016). Similar p-functionals have appeared in Kitsos et al (2018), Kitsos et al (2020). The stability is proven for the 1- spatial norm. We define a Lyapunov functional $\mathcal{W}_p : C^1([0,L];\mathbb{R}^3) \to \mathbb{R}$ by

$$\mathcal{W}_p[\varepsilon] := \left(\int_0^L \pi(x) e^{p\mu_\theta x}\mathcal{G}_p[\varepsilon](x)\mathrm{d}x\right)^{1/p}; \tag{10a}$$

$$\mathcal{G}_p[\varepsilon](x) := \left(\varepsilon^\top(x)P(x)\varepsilon(x) + \varepsilon_t^\top(x)P(x)\varepsilon_t(x)\right)^p \tag{10b}$$

and p in $\mathbb{N}$, $P(\cdot)$ is of class C^1, symmetric and positive definite, satisfying (6), $\pi : [0,L] \to \mathbb{R}$ is a function given by

$$\pi(x) := (\bar{\pi} - 1)\frac{x}{L} + 1; \bar{\pi} := \frac{\max_{x\in[0,L]}\lambda_3(x)}{\min_{x\in[0,L]}\lambda_3(x)} \tag{11}$$

and μ_θ is a constant given by

$$\begin{aligned}
\mu_\theta :=& \frac{1+\mathrm{sgn}(\lambda_3(x))}{2L}\ln\left(\frac{|\bar{K}|^2\|P(\cdot)\|_\infty}{\min_{x\in[0,L]}\underline{\mathrm{eig}}(P(x))}\theta^4\right)\\
&+ \frac{1-\mathrm{sgn}(\lambda_3(x))}{2L}\ln\left(\frac{\min_{x\in[0,L]}\underline{\mathrm{eig}}(P(x))}{|\bar{\bar{K}}|^2\|P(\cdot)\|_\infty\theta^4}\right). \qquad (12)
\end{aligned}$$

Note here that, by its definition, π is bounded as follows

$$1 \le \pi(x) \le \bar{\pi}, \forall x \in [0,L]. \tag{13}$$

By invoking Lemma 1 and Fact 1, we are in a position to define $G_p, W_p : [0, +\infty) \to \mathbb{R}_+$ by

$$G_p(t) := \mathcal{G}_p[\varepsilon](t), W_p(t) := \mathcal{W}_p[\varepsilon](t), t \geq 0. \tag{14}$$

By temporarily assuming that ε has some extra regularity, i.e., it is C^2, we obtain the following hyperbolic equations for ε_t

$$\varepsilon_{tt}(t,x) + \lambda_3(x)\varepsilon_{xt}(t,x) = \theta\left(M_1(x) + N(x)C\right)\varepsilon_t(t,x)$$
$$+ \Theta^{-1}(\tilde{M}(x) - M_1(x))\Theta\varepsilon_t(t,x), \tag{15a}$$
$$\varepsilon_t(t,0) = \Theta^{-1}\bar{K}\Theta\varepsilon_t(t,L), \lambda_3(\cdot) > 0, \tag{15b}$$
$$\varepsilon_t(t,L) = \Theta^{-1}\bar{\bar{K}}\Theta\varepsilon_t(t,0), \lambda_3(\cdot) < 0. \tag{15c}$$

For the time-derivative $\dot{W}_p$ along the C^1 solutions of (15) we get the following

$$\dot{W}_p = \frac{1}{p}W_p^{1-p}\int_0^L p\pi(x)e^{p\mu_\theta x}G_{p-1}(x)$$
$$\times\sum_{i=0}^{1}\left(\partial_t^{i+1}\varepsilon^\top(x)P(x)\partial_t^i\varepsilon(x) + \partial_t^i\varepsilon^\top(x)P(x)\partial_t^{i+1}\varepsilon(x)\right)\mathrm{d}x$$
$$= W_p^{1-p}\left(\frac{1}{p}T_{1,p} + T_{2,p}\right), \tag{16}$$

where

$$T_{1,p} := -\int_0^L \lambda_3(x)\pi(x)e^{p\mu_\theta x}\Big[\partial_x G_p(x)$$
$$-pG_{p-1}(x)\sum_{i=0}^{1}\partial_t^i\varepsilon^\top(x)P_x(x)\partial_t^i\varepsilon(x)\Big]\mathrm{d}x,$$
$$T_{2,p} := \int_0^L (2\theta\pi(x)e^{p\mu_\theta x}G_{p-1}(x)$$
$$\times\sum_{i=0}^{1}\partial_t^i\varepsilon^\top(x)\mathrm{Sym}\left((P(x)(M_1(x) + N(x)C\right)\partial_t^i\varepsilon(x)$$
$$+2\pi(x)e^{p\mu_\theta x}G_{p-1}(x)\sum_{i=0}^{1}\partial_t^i\varepsilon^\top(x)$$
$$\times\mathrm{Sym}\left(P(x)\Theta^{-1}(\tilde{M}(x) - M_1(x))\Theta\right)\partial_t^i\varepsilon(x)\Big)\mathrm{d}x.$$

Using an integration by parts in term $T_{1,p}$ and utilizing (13) and other trivial bounds, we get

$$T_{1,p} \leq -\lambda_3(L)\pi(L)e^{p\mu_\theta L}G_p(L) + \lambda_3(0)\pi(0)G_p(0)$$
$$+\left(\omega_1 + p\omega_2 + p|\mu_\theta|\|\lambda_3(\cdot)\|_\infty\right)W_p^p,$$

where $\omega_1 := \frac{\bar{\pi}-1}{L}\|\lambda_3(\cdot)\|_\infty + \|\lambda_3'(\cdot)\|_\infty$, $\omega_2 := \frac{\|\lambda_3(\cdot)\|_\infty\|P_x(\cdot)\|_\infty}{\min_{x\in[0,L]}\underline{\mathrm{eig}}(P(x))}$. Substituting (15b), for each of the cases $\lambda_3(\cdot) > 0$ and $\lambda_3(\cdot) < 0$, the above yields

$$T_{1,p} \leq \Big[-\min_{x\in[0,L]}\lambda_3(x)\pi(L)\left(\min_{x\in[0,L]}\underline{\mathrm{eig}}(P(x))\right)^p e^{p\mu_\theta L}$$
$$+\pi(0)\max_{x\in[0,L]}\lambda_3(x)\left(\theta^2|\bar{K}|\right)^{2p}\|P(\cdot)\|_\infty^p\Big]$$
$$\times\left(\sum_{i=0}^{1}|\partial_t^i\varepsilon(L)|^2\right)^p + (\omega_1 + p\omega_2$$
$$+p|\mu_\theta|\|\lambda_3(\cdot)\|_\infty)W_p^p, \text{when } \lambda_3(x) > 0,$$
$$T_{1,p} \leq \Big[-\min_{x\in[0,L]}\lambda_3(x)\pi(L)\|P(\cdot)\|_\infty^p\left(\theta^2|\bar{\bar{K}}|\right)^{2p}$$
$$\times e^{p\mu_\theta L} + \pi(0)\max_{x\in[0,L]}\lambda_3(x)$$
$$\left(\min_{x\in[0,L]}\underline{\mathrm{eig}}(P(x))\right)^p\Big]\left(\sum_{i=0}^{1}|\partial_t^i\varepsilon(0)|^2\right)^p + (\omega_1$$
$$+p\omega_2 + p|\mu_\theta|\|\lambda_3(\cdot)\|_\infty)W_p^p, \text{ when } \lambda_3(x) < 0,$$

which, by virtue of (11), (12), is bounded as follows

$$T_{1,p} \leq \left(\omega_1 + p\omega_2 + p|\mu_\theta|\|\lambda_3(\cdot)\|_\infty\right)W_p^p. \tag{17}$$

Next, for $\theta \geq 1$ and invoking (6), we obtain for $T_{2,p}$

$$T_{2,p} \leq \left(-\theta\omega_3 + \omega_4\right)W_p^p, \tag{18}$$

where $\omega_3 := \frac{\min_{x\in[0,L]}\underline{\mathrm{eig}}(P(x))\min_{x\in[0,L]}\underline{\mathrm{eig}}(Q(x))}{\|P(\cdot)\|_\infty}$, $\omega_4 := 2\frac{\|P(\cdot)\|_\infty\|\tilde{M}(\cdot)-M_1(\cdot)\|_\infty}{\min_{x\in[0,L]}\underline{\mathrm{eig}}(P(x))}$. By (16), in conjunction with (17), (18) we get

$$\dot{W}_p \leq \left(-\theta\omega_3 + |\mu_\theta|\|\lambda_3(\cdot)\|_\infty + \omega_1 + \omega_2 + \omega_4\right)W_p. \tag{19}$$

We obtained the above estimate of $\dot{W}_p$ for ε of class C^2, By invoking density arguments, the results remain valid with ε only of class C^1 (see Coron and Bastin (2015) for further details). Now, one can select the high gain θ, such that $\theta \geq \theta_0$, where $\theta_0 \geq 1$ is such that

$$-\theta\omega_3 + |\mu_\theta|\|\lambda_3(\cdot)\|_\infty + \omega_1 + \omega_2 + \omega_4 \leq -2\kappa_\theta, \forall\theta \geq \theta_0,$$

for some $\kappa_\theta > 0$. One can easily check that for any $\kappa_\theta > 0$, there always exists a $\theta_0 \geq 1$, dependent on the involved constants, such that the previous inequality is satisfied. Subsequently, (19) yields

$$W_p(t) \leq e^{-2\kappa_\theta t}W_p(0). \tag{20}$$

Taking also into account (13), we get the following property holding for continuous $\varepsilon, \varepsilon_t$

$$\lim_{p\to\infty} W_p = \lim_{p\to\infty} \|e^{\mu_\theta \cdot}\pi(\cdot)^{\frac{1}{p}} G_p^{1/p}(\cdot)\|_{L^p(0,L)} \tag{21}$$

$$= \sum_{i=0}^{1} \|e^{\mu_\theta \cdot}\partial_t^i \varepsilon^\top(\cdot) P(\cdot)\partial_t^i \varepsilon(\cdot)\|_\infty. \tag{22}$$

By (20), in conjunction with (21), we derive

$$\sum_{i=0}^{1} \|e^{\mu_\theta \cdot}\partial_t^i \varepsilon^\top(\cdot) P(\cdot)\partial_t^i \varepsilon(\cdot)\|_\infty \leq e^{-2\kappa_\theta t} \times \sum_{i=0}^{1} \|e^{\mu_\theta \cdot}\partial_t^i \varepsilon_0^\top(\cdot) P(\cdot)\partial_t^i \varepsilon_0(\cdot)\|_\infty, \tag{23}$$

where $\varepsilon_0(x) := \varepsilon(0, x)$. Now, from error dynamics (15) in conjunction with continuous differentiability of the dynamics, we can perform simple differentiations, so as to calculate constants $\rho_{\theta,i}, \sigma_{\theta,i}$, depending polynomially on θ, such that

$$\sum_{i=0}^{1} \rho_{\theta,i}\|\partial_x^i \varepsilon(\cdot)\|_\infty \leq \sum_{i=0}^{1} \|\partial_t^i \varepsilon(\cdot)\|_\infty \leq \sum_{i=0}^{1} \sigma_{\theta,i}\|\partial_x^i \varepsilon(\cdot)\|_\infty.$$

Combining the above estimates with the following inequality

$$e^{\frac{\mu_\theta - |\mu_\theta|}{2}L} \min_{x\in[0,L]} \underline{\mathrm{eig}}(P(x)) \left(\sum_{i=0}^{1} \|\partial_t^i \varepsilon(\cdot)\|_\infty\right)^2 \leq \sum_{i=0}^{1} \|e^{\mu_\theta \cdot}\partial_t^i \varepsilon^\top(\cdot) P(\cdot)\partial_t^i \varepsilon(\cdot)\|_\infty \leq e^{\frac{\mu_\theta + |\mu_\theta|}{2}L} \|P(\cdot)\|_\infty \left(\sum_{i=0}^{1} \|\partial_t^i \varepsilon(\cdot)\|_\infty\right)^2,$$

we obtain an exponential stability result as follows

$$\|\hat{\zeta}(t,\cdot) - \zeta(t,\cdot)\|_1 \leq \bar{\ell}_\theta e^{-\kappa_\theta t}\|\hat{\zeta}^0 - \zeta^0\|_1, \tag{24}$$

where $\bar{\ell}_\theta$ is a polynomial in θ (as in high-gain observers in finite dimensions). To return to the original coordinates, we notice that $\mathcal{T}$ is bounded from $\mathcal{X}$ to $\mathcal{X}$, $\mathcal{X}$ is continuously embedded in $C^1([0, L]; \mathbb{R}^3)$. Also the extension of $\mathcal{T}^{-1}$ on $C^0[0, L]; \mathbb{R}^3)$ is bounded in $C^0[0, L]; \mathbb{R}^3)$ and $C^1([0, L]; \mathbb{R}^3)$ is continuously embedded in $C^0([0, L]; \mathbb{R}^3)$. Thereby, by (24), we can calculate constant ℓ, polynomial again in θ, such that (7) is satisfied with $\kappa = \kappa_\theta$.

The proof of Theorem 1 is complete.

Remark 3 *Notice that although the considered system is linear, the high-gain technique is of special interest to dominate "extra terms" in the derivative of the Lyapunov function, similarly to the nonlinear terms in finite-dimensional high-gain observers. In the present case, indeed, there appears a term μ_θ in the Lyapunov derivative (see (17)), coming from the boundary conditions, and having an effect similar to nonlinearities in finite-dimensional systems. In an abstract sense, passing from the finite dimensions to the infinite dimensions, the hyperbolic (differential) operator imposes extra difficulties, even for linear systems.*

4 Simulation

In this section, we apply the high-gain observer design to a system of an exothermic plug flow chemical reactor. Control and observer designs for chemical reactors have been widely investigated (see for instance Boskovic and Krstic (2002) and Christophides and Daoutidis (1996)).

Here we consider a linearized model, where system's states ξ_1, ξ_2, ξ_3 represent the deviation with respect to their steady values, i.e., $\xi_1 = T_c - T_c^*, \xi_2 = T_r - T_r^*, \xi_3 = C_A - C_A^*$, where T_c is the coolant temperature, T_r is the reactor temperature and C_A is the concentration of the chemical components (see (Bastin and Coron , 2016, 5.1.1) for more details). The hyperbolic dynamics satisfy (1a) with

$$\Lambda = \begin{pmatrix} V_c & 0 & 0 \\ 0 & V_r & 0 \\ 0 & 0 & V_r + \epsilon \end{pmatrix}, M(x) = \begin{pmatrix} k_0 & -k_0 & 0 \\ -k_0 & k_0 + k_1\phi_1(x) & k_1\phi_0(x) \\ 0 & -\phi_1(x) & -\phi_0(x) \end{pmatrix}$$

for positive $V_c, V_r, \epsilon, k_0, k_1$, and boundary conditions

$$\xi_1(t, 0) = 2\xi_1(t, L), \xi_2(t, 0) = \xi_2(t, L), \xi_3(t, 0) = \xi_3(t, L)$$

(chosen to be fictitious in order to get unstable trajectories for the sake of illustration), while $\phi_0(x) = (a + b)\exp\left(-\frac{E}{RT_r^*(x)}\right)$,
$\phi_1(x) = \left(C_A^*(x) - \frac{b}{a+b}C_A^{\text{in}}\right) \frac{E}{R(T_r^*(x))^2}\phi_0(x)$, for constants $a, b, E, R, C_A^{\text{in}}$. The steady states satisfy the following differential equations over $[0, L]$:

$$\begin{aligned} V_r \frac{\mathrm{d}}{\mathrm{d}x} T_r^* &= -k_0(T_c^* - T_r^*) + k_1 r(T_r^*, C_A^*), \\ V_r \frac{\mathrm{d}}{\mathrm{d}x} C_A^* &= -r(T_r^*, C_A^*), \\ V_c \frac{\mathrm{d}}{\mathrm{d}x} T_c^* &= k_0(T_c^* - T_r^*), \end{aligned}$$

with $r(T_r, C_A) = \left((a + b)C_A - bC_A^{\text{in}}\right)\exp\left(-\frac{E}{RT_r}\right)$ the reaction rate and boundary conditions $T_r^*(0) = T_r^{\text{in}}, C_A^*(0) = C_A^{\text{in}}, T_c^*(0) = T_c^{\text{in}}$. For simulation, numerical values are as follows: $T_r^{\text{in}} = 340\text{K}, C_A^{\text{in}} =$

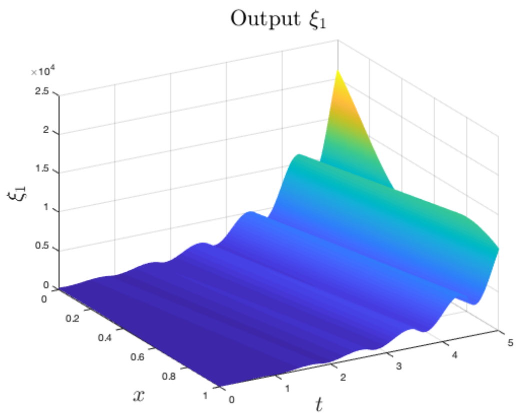

Fig. 1. Time and space evolution of system's output

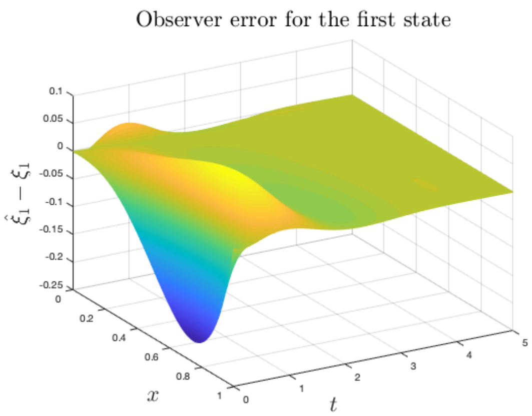

Fig. 2. First observer error for $\theta = 4$

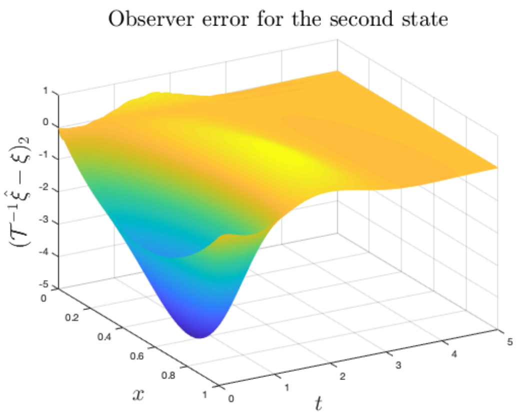

Fig. 3. Second observer error for $\theta = 4$

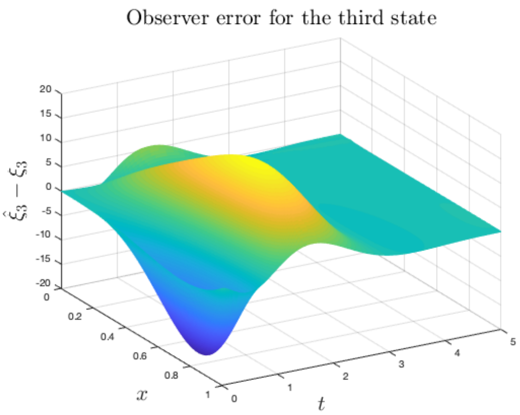

Fig. 4. Third observer error for $\theta = 4$

$0.02\text{mol} \cdot \text{L}^{-1}, T_c^{\text{in}} = 293\text{K}$. The length of the reactor is $L = 1\text{m}$, the reactive fluid velocity in the reactor is $V_r = 0.025\text{m} \cdot \text{s}^{-1}$, the coolant velocity in the jacket is $V_c = 1.13\text{m} \cdot \text{s}^{-1}$, the activation energy is $E = 11250\text{cal} \cdot \text{mol}^{-1}$, $a = 0.56\text{s}^{-1}$ and $b = 0.12\text{s}^{-1}$ are rate constants, and $R = 1.986\text{cal} \cdot \text{mol}^{-1} \cdot \text{K}^{-1}$ is the Boltzmann constant. We also add an artificial constant $\epsilon = 0.005\text{m} \cdot \text{s}^{-1}$ in the third characteristic velocity to make system strictly hyperbolic and deal with the problem in its full generality (with $\epsilon \neq 0$, we have $q = 3$ instead of $q = 2$, for $\epsilon = 0$). Assume that measured output is the cooling temperature, i.e., $y(t, x) = \xi_1(t, x)$. We now follow the steps of the H-GODP described in the previous sections. In particular, we apply the transformation (8), introduced in Subsection 3.1, with $\tau = \epsilon/k_0$. We choose some initial conditions, such that Assumption 1 is satisfied with $\bar{q} = 2$, and now system meets all sufficient conditions for solvability of the H-GODP. We apply Theorem 1, with $\theta = 4$ and $N(x) = \begin{pmatrix} -1 & 5 & 10 \end{pmatrix}^\top$. As expected, the convergence of the inversely transformed observer state to the unknown state ξ is guaranteed by Theorem 1. In Fig. 1 we represent the output ξ_1. In figures 2 - 4 we see the observation errors for each of the original states ξ_1, ξ_2 and ξ_3, after choosing arbitrary observer initial conditions satisfying appropriate compatibility conditions. In Fig. 5 we represent the time evolution of the sup spatial norm of the second observer error for different values of high-gain constant θ. For $\theta = 1$, the observer converges very slowly, while for increasing values of θ above 4, the convergence becomes faster, but with a trade off with respect to transients (and overshoot) to be handled in practice (as expected from (7)). In Fig. 6 we see the second observer error when we consider a smaller gain ($\theta = 0.1$), which is divergent, as we expected.

5 Conclusion

A high-gain observer for a class of 3×3 linear hyperbolic systems in triangular form with possibly distinct characteristic velocities has been presented, considering distributed measurement of a part of the state. This result constitutes an extension of the high-gain observer design for finite-dimensional systems to a class of hyperbolic systems and, also, an extension of previous works to the case of systems with distinct velocities. To that end, the hyperbolic system is first mapped into a target system

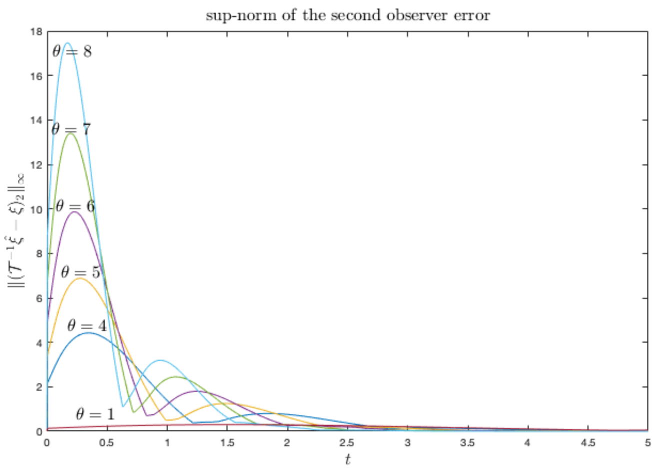

Fig. 5. sup spatial norm of the second observer error for various high-gain constants

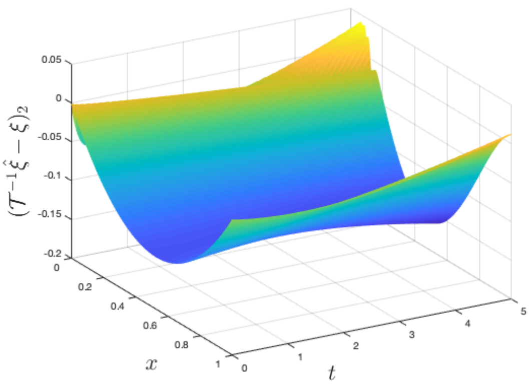

Fig. 6. Second observer error for $\theta = 0.1$

of PDEs and an observer for this system is designed, utilizing output correction terms and injection of output spatial derivatives. The extension of this methodology to wider classes of infinite-dimensional systems will be subject to our future work.

References

F. Alabau-Boussouira, J.-M. Coron, and G. Olive, Internal controllability of first order quasilinear hyperbolic systems with a reduced number of controls, *SIAM J. Control and Optimization*, vol. 55, no. 1, pp. 300–323, 2017.

H. Anfinsen, M. Diagne, O. M. Aamo, and M. Krstic, An adaptive observer design for n + 1 coupled linear hyperbolic PDEs based on swapping, *IEEE Transactions on Automatic Control*, vol. 61, no. 12, pp. 3979–3990, 2016.

J. Auriol and F. Di Meglio, Minimum time control of heterodirectional linear coupled hyperbolic PDEs, *Automatica,* vol. 71, pp. 300–307, 2016.

G. Bastin and J.-M. Coron, Stability and boundary stabilization of 1-D hyperbolic systems, *Progress in Nonlinear Differential Equations and Their Applications. Springer International Publishing*, 2016.

G. Besançon, D. Georges, and Z. Benayache, Sur la commande en dimension finie d'une classe de systèmes non linéaires de dimension infinie, *Conf. Internationale Francophone d'Automatique*, Bordeaux, France, 2006.

G. Besançon, B. Pham, and D. Georges, Robust state estimation for a class of convection-diffusion-reaction systems, *IFAC Workshop on Control of Systems Governed by Partial Differential Equations*, Paris, France, 2013.

D. Boskovic and M. Krstic, Backstepping control of chemical tubular reactors, *Computers and Chemical Engineering*, vol. 26, pp. 1077–1085, 2002.

F. Castillo, E. Witrant, C. Prieur, and L. Dugard, Boundary observers for linear and quasi-linear hyperbolic systems with application to flow control, *Automatica*, vol. 49, no. 11, pp. 3180–3188, 2013.

P. D. Christofides and P. Daoutidis, Feedback control of hyperbolic pde systems, *AIChE Journal*, vol. 42, no. 11, pp. 3063–3086, 1996.

J.-M. Coron and G. Bastin, Dissipative boundary conditions for one-dimensional quasi-linear hyperbolic systems: Lyapunov stability for the $C1$-norm, *SIAM J. Control Optim.*, vol. 53, no. 3, pp. 1464–1483, 2015.

J.-M. Coron, R. Vazquez, M. Krstić, and G. Bastin, Local exponential stabilization of a 2x2 quasilinear hyperbolic system using backstepping, *SIAM J. Control Optim.*, vol. 51, no. 3, pp. 2005—2035, 2013.

J.-M. Coron and H. M. Nguyen, Optimal time for the controllability of linear hyperbolic systems in one dimensional space, *SIAM Journal on Control and Optimization*, vol. 57, no. 2, pp. 1127–1156, 2019.

R. F. Curtain, Finite-dimensional compensator design for parabolic distributed systems with point sensors and boundary input, *IEEE Transactions on Automatic Control*, vol. 27, no. 1, pp. 98–104, 1982.

M. Demetriou, Second order observers for second order distributed parameter systems, *Syst. Control Lett.*, vol. 51, pp. 225—234, 2004.

A. Deutschmann, L. Jadachowski, and A. Kugi, Backstepping-based boundary observer for a class of time-varying linear hyperbolic PIDEs, *Automatica*, vol. 68, pp. 369–377, 2016.

F. Di Meglio, R. Vazquez, and M. Krstic, Stabilization of a system of $n+1$ coupled first-order hyperbolic linear PDEs with a single boundary input, *IEEE Transactions on Automatic Control*, vol. 58, pp. 3097–3111, 2013.

N. Espitia, A. Tanwani, and S. Tarbouriech, Stabilization of boundary controlled hyperbolic PDEs via Lyapunov-based event triggered sampling and quantization, *IEEE Conference on Decision and Control*, pp. 3112–3117, Melbourne, Australia, 2017.

J. P. Gauthier and G. Bornard, Observability for any $u(t)$ of a class of nonlinear systems, *IEEE Transactions on Automatic Control*, vol. 26, no. 4, pp. 922–926, 1981.

A. Hasan, O. M. Aamo, and M. Krstic, Boundary observer design for hyperbolic PDE-ODE cascade systems, *Automatica*, vol. 68, pp. 75–86, 2016

H. K. Khalil, High-gain observers in nonlinear feedback control, Advances in Design and Control, SIAM, 2017.

C. Kitsos, G. Besançon, and C. Prieur, High-gain observer design for a class of hyperbolic systems of balance laws, *IEEE Conference on Decision and Control*, Miami, USA, pp. 1361–1366, 2018.

C. Kitsos, G. Besançon, and C. Prieur, A high-gain observer for a class of 2x2 hyperbolic systems with C1 exponential convergence, *IFAC Workshop on Control of Systems Governed by Partial Differential Equations*, Oaxaca, Mexico, pp. 174–179, 2019.

C. Kitsos, G. Besançon, and C. Prieur, High-gain observer design for a class of integro-differential quasilinear hyperbolic systems - Application to an epidemic model, conditionally accepted in *IEEE Transactions on Automatic Control*, 2020.

T-T. Li, Exact boundary observability for quasilinear hyperbolic systems, *ESAIM: Control, Optimisation and Calculus of Variations*, vol. 14, no. 4, pp. 759–766, 2008.

V. Nguyen, D. Georges, and G. Besançon, State and parameter estimation in 1-d hyperbolic PDEs based on an adjoint method, *Automatica*, vol. 67, pp. 185–191, 2016.

C. Pradere, M. Joanicot, J. C. Batsale, J. Toutain,and C. Gourdon, Processing of temperature fields in chemical microreactors with infrared temperature fields. *QIRT J.*, vol. 3, pp. 117–135, 2009.

C. Prieur, A. Girard, and E. Witrant, Stability of switched linear hyperbolic systems by Lyapunov techniques, *IEEE Transactions on Automatic Control*, vol. 59, no. 8, pp. 2196–2202, 2014.

C. Prieur and J. Winkin, Boundary feedback control of linear hyperbolic systems: application to the Saint-Venant–Exner equations, *Automatica*, vol. 88, pp. 44–51, 2018.

A. Schaum, J. A. Moreno, J. Alvarez, and T. Meuer, A simple observer scheme for a class of 1-D semi-linear parabolic distributed parameter systems, *European Control Conf.*, Linz, Austria, pp. 49–54, 2015.

L. Zhang, C. Prieur, and J. Qiao, Local Exponential Stabilization of Semi-Linear Hyperbolic Systems by Means of a Boundary Feedback Control, *IEEE Control Systems Letters*, vol. 2(1), pp. 55–60, 2018.

A. Zogg, F. Stoessel, U. Fischer, and K. Hungerbuhler, Isothermal reaction calorimetry as a tool for kinetic analysis, *Thermochimica Acta*, vol. 419, pp. 1—17, 2004.