

Existence in BV^s for 2×2 strictly hyperbolic systems of conservation laws

Stéphane Junca

Université Côte d'Azur, CNRS, Inria, LJAD, Nice, France

NEPDE, Shanghai Jiao-Tong, online, December 2, 2020

Joint work with: Christian BOURDARIAS, Anupam CHOUDHURY, Shyam GHOSHAL, Billel GUELMAME, Boris HASPOT, Animesh JANA

$$\partial_t U + \partial_x F(U) = 0, \quad U(0, x) = U_0(x)$$

The theory of global weak entropy solutions of hyperbolic conservation laws is usually in BV or in L^∞

- BV

- + compactness in L^1_{loc}
- + $u \in BV$ has left and right traces: shock structure
- + existence for small initial data: $TVU_0 \ll 1$, Glimm 1969
- Not all solutions are in BV , K.S. Cheng 1983

- L^∞

- no compactness
- no traces
- + compensated compactness, Tartar 1979
- + smoothing effect, Lax & Oleinik 1957

$$BV \subset \dots ? \dots \subset L^\infty$$

1 Fractional BV spaces BV^s , $0 < s < 1$

- BV -like properties
- A short story on BV^s and BV_ϕ

2 Scalar 1D conservation laws

- Propagation of BV^s regularity
- Optimal 1D smoothing effect

3 2×2 Hyperbolic Systems

- Two genuinely nonlinear fields (GNL)
- Two linearly degenerate fields (LD)
- One GNL & one LD
 - Existence in $BV^{1/3}$
 - Blow-up out of $BV^{1/3}$

$$BV \subset BV^s \subset L^\infty$$

$$0 < s < 1, \quad p = 1/s > 1$$

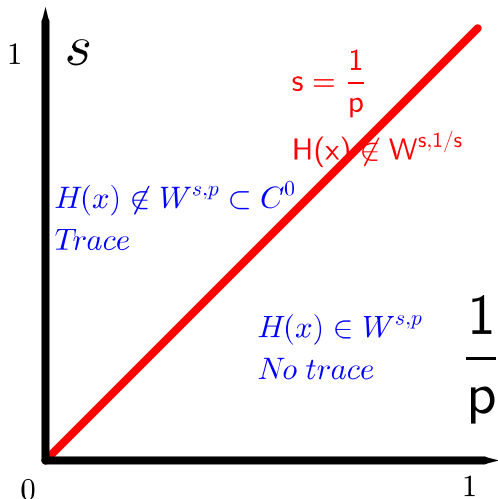
$$TV^s u = \sup_{\dots < x_j < x_{j+1} < \dots} \sum_i |u(x_{i+1}) - u(x_i)|^p$$

$$BV^s = \{u, TV^s u < \infty\}$$

- $BV = BV^1$
- $L^\infty = BV^0$ (convention)
- **Compactness** in L^1_{loc} : Helly's Theorem, also for generalized BV spaces Φ - BV , Musielak, Orlicz 1959
- **Traces**: left and right \rightarrow shock structure ! Musielak, Orlicz 1959
- Sobolev regularity: $\forall \varepsilon > 0, BV^s \subset W^{s-\varepsilon, 1/s}$, $BV^s \neq W^{s, 1/s}$ without traces, Bourdarias, Gisclon, J. 2014

BV^s is not a Sobolev space $W^{s,p}$

$$H(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{else} \end{cases}$$



A short story on BV^s and Φ - BV

functions with bounded " p -variation", $p = 1/s$

- BV : Jordan, graph length, convergence of $\sum c_n e^{inx}$ 1881
- 2-variation ($BV^{1/2}$), Wiener 1924, $\sum c_n e^{inx}$
- p -variation (BV^s), Love & Young 1936, $\int fdg$
- Φ -variation, (Φ - BV), Musielak & Orlicz, 1959
- (BV^s), M. Bruneau, Lect. notes in Math., 332 p., 1974
- Probabilities: On rough paths, K. Friz, M. Hairer, 251 p., 2014

$$\partial_t u + \partial_x f(u) = 0, \quad u(0, x) = u_0(x) \in L^\infty \Rightarrow u \in BV? \text{ No}$$

- Conservation laws and compensated compactness 1979: Tartar, interpolation between L^∞ and BV ?
- TOD schemes (Φ - BV), Perthame, Westdickenberg, Num. Math. 2005.
- 2×2 systems, δ -shock, ($BV^{1/2}$), M. Sever, 65 p., JHDE 2010
- BV^s , scalar, Bourdarias, Gisclon, J., JHDE 2014
- $BV^{1/3}$, 2×2 , **Chemistry**, Bourdarias, Gisclon, J., Peng, CMS, 2016
- Φ - BV , strictly convex C^1 flux: Castelli, J., JMAA 2017
- Non convex $f \in C^3$, $u \in \Phi$ - BV or BV^s , Marconi, 69 p., JHDE 2018
- Kolmogorov compactness, Ancona, Glass, Nguyen, AIHP 2019

- Lipschitz convex flux, $\Phi-BV$, Guelmame, J., Clamond, CMS 2019
- BV^s for weakly coupled systems, J., Lombard, SIAP 2020
- **Optimal BV^s estimates for all time**, Ghoshal, Guelmame, Jana, J., NoDEA 2020
- **Triangular systems with a transport equation & $BV^{1/3}$** , Bourdarias, Choudhury, Guelmame, J., prep. 2020
- **Existence in $BV^{1/3}$ for 2×2 systems with a genuinely nonlinear field and a linear degenerate one**, Haspot, J., prep. 2020
- **Discontinuous flux & $BV^{1/2}$** , Ghoshal, J., Parmar, prep. 2021

Theorem (BV^s decay, Bourdarias-Gisclon-J. JHDE 2014)

$f \in Lip(\mathbb{R}, \mathbb{R})$

$$\begin{aligned} \partial_t u + \partial_x f(u) &= 0, & u(x, 0) &= u_0(x), \\ TV^s u(t, \cdot)(\mathbb{R}) &\leq TV^s u_0(\cdot)(\mathbb{R}) \end{aligned}$$

$TV^s D$ schemes:

- Glimm scheme
- Godunov scheme
- Wave Front Tracking Algorithm

Theorem (Optimal smoothing effect,
Bourdarias-Gisclon-J., 2014, Castelli-Jabin-J., 2018)

Degenerate C^1 convex flux: $u_0 \in L^\infty(\mathbb{R})$, $p > 1$

$$0 < \inf_{\mathbb{R} \times \mathbb{R}} \frac{|f'(u) - f'(v)|}{|u - v|^p} \implies u(t > 0, \cdot) \in BV_{loc}^s(\mathbb{R}_x), \quad s = \frac{1}{p}$$

- BV -smoothing effect for $p = 1$, $f \in C^2$, $f'' > 0$, Lax & Oleinik, 1957
Burgers, $f(u) = u^2 \rightarrow BV$
- Examples: power law $f(u) = |u|^{1+p}$, $f(u) = u^4 \rightarrow BV^{1/3}$,
- Optimality, Castelli-J., 2014, 2017, Ghoshal-Guelmame-Jana-J., 2020
- $C^{\geq 3}$ nonconvex flux, $f(u) = u^3 \rightarrow BV^{1/2}$, Marconi, 2018

2×2 strictly Hyperbolic Systems

$$\partial_t U + \partial_x F(U) = 0,$$

$U = (u, v)$, $A = DF(U)$, $Ar_i = \lambda_i r_i$, $\lambda = \lambda_1 > \mu = \lambda_2$

$N \times N$ systems, **existence with small BV initial data**, Glimm 1969

2×2 systems, existence of **Riemann invariants**

$U \rightarrow W = (w_1, w_2)$, $w = w_1$, $z = w_2$, for smooth solutions,

$$\partial_t w + \lambda(w, z) \partial_x w = 0, \quad \partial_t z + \mu(w, z) \partial_x z = 0$$

- small data, U near a constant state $\bar{U} \in \Omega$
- a genuinely nonlinear field: $\frac{\partial \lambda_i}{\partial w_j} = \nabla \lambda_i \cdot r_i \neq 0$ on Ω
- a linearly degenerate field: $\frac{\partial \lambda_i}{\partial w_j} = \nabla \lambda_i \cdot r_i = 0$ on Ω

Two genuinely nonlinear fields

Existence in L^∞

- Large data,
by **compensated compactness** or averaging lemmas,
for Euler equations, ...
R. DiPerna 1983, G.-Q Chen 1985, ..., Lions-Perthame-Tadmor 1994
- Small data,
by Glimm scheme with a **BV smoothing effect**
Glimm-Lax, Mem. AMS 1970, Bianchini-Colombo-Monti, JDE 2010

a too simple Temple example: $\partial_t u_i + \partial_x u_i^2 / 2 = 0, i = 1, 2$

Two linearly degenerate fields

This is a special Temple system:

- two Riemann invariants
- only contact discontinuities

Existence in L^∞ , G.-Q. Chen 92, D. Serre 92, Y.-J Peng 98

Propagation of BV^s regularity with a Wave Front Tracking Algorithm

Linear case, Riemann problem

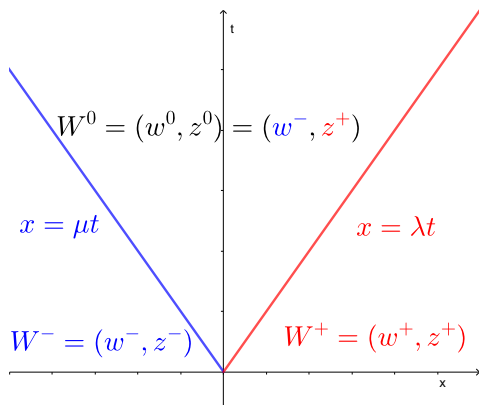


Figure: Riemann problem for a linear system

Linear case: interaction of waves

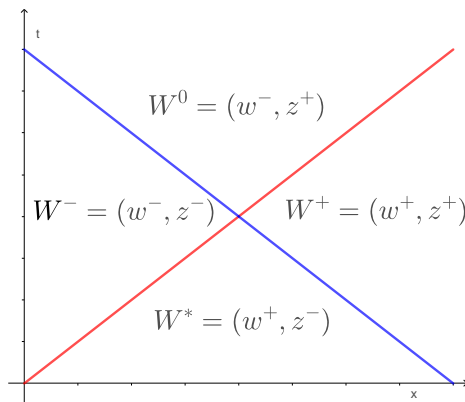


Figure: Same oscillations for w and z before and after the interaction

2 LD, general case

- The constant linear case: λ_j constant

$$w_j(t, x) = w_j(0, x - \lambda_j t)$$

- Nonlinear case:

$$\lambda = \lambda(z), \quad \mu = \mu(w)$$

similar pictures for the interactions.

$$TV^s w = TV^s w_0,$$

$$TV^\sigma z = TV^\sigma z_0$$

$s > 0, \sigma > 0 \Rightarrow$ compactness

Nonlinear case: interaction of waves

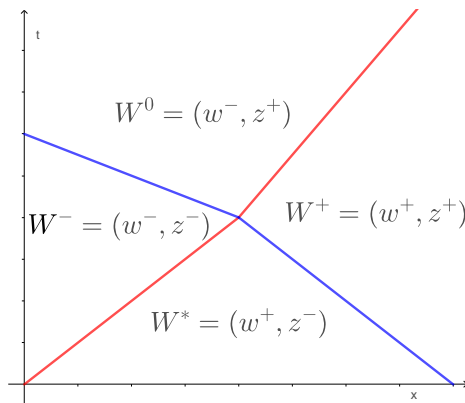


Figure: Different speeds of wave after the interaction

λ genuinely nonlinear & μ linearly degenerate

Theorem (Existence in $BV^{1/3}$, Haspot, J. 2020)

λ is GNL & μ is LD,

If $w_0 \in BV^s$, $s \geq 1/3$ and $z_0 \in L^\infty$

Then there exists a global weak entropy solution $(u, v) \leftrightarrow (w, z)$,

$$w \in L^\infty([0, +\infty[, BV^s(\mathbb{R}, \mathbb{R})) \cap W^{s, \infty}([0, +\infty[, L_{loc}^{1/s}(\mathbb{R}, \mathbb{R}))$$

$$z \in L^\infty([0, +\infty[\times \mathbb{R}, \mathbb{R})$$

for smooth solutions:

$$\partial_t w + \lambda(w, z) \partial_x w = 0,$$

$$\partial_w \lambda \neq 0,$$

$$\partial_t z + \mu(w) \partial_x z = 0$$

A triangular system

A subclass of 2×2 systems with 1 GNL & 1 LD
to prove existence in $BV^{1/3}$ and the optimality of $s = 1/3$.

$$\partial_t u + \partial_x f(u) = 0$$

$$\partial_t v + \partial_x (a(u)v) = 0$$

eigenvalues: $\lambda = f'(u) > \mu = a(u)$
 λ GNL means $\pm f'' > 0$

Riemann invariants: $w = u$, $z = v \exp(A(u))$, $A'(u) = \frac{a'(u)}{a(u) - f'(u)}$

for smooth solutions:

$$\partial_t u + f'(u) \partial_x u = 0,$$

$$\partial_t z + a(u) \partial_x z = 0.$$

2 equations: natural but bad idea

Solving first the scalar conservation law

$$\partial_t u + \partial_x f(u) = 0$$

then a linear transport equation with a discontinuous transport speed:

$$\partial_t v + \partial_x (a(u)v) = 0$$

$a(u)$ discontinuous yields to delta-shock: measure solutions, no uniqueness of solutions

Poupaud-Rascle 1997, Bouchut-James 1998

A whole system: good idea

Solving the whole system

$$\partial_t u + \partial_x f(u) = 0$$

$$\partial_t v + \partial_x (a(u)v) = 0$$

For instance a shock occurs only on the first family: $[u] = u^+ - u^-$,

$$s[u] = [f(u)] \quad \Rightarrow \quad s = \frac{[f(u)]}{[u]} = \frac{f(u^+) - f(u^-)}{u^+ - u^-}$$

fixes the speed of the discontinuity and then the intermediate state between v^- and v^+ is given by solving the equation

$$s[v] = [a(u)v] \quad \Rightarrow \quad v^0 = \frac{s - a(u^+)}{s - a(u^-)} v^+, \quad z^0 = v^0 e^{A(u^-)}$$

Riemann problem for the triangular system

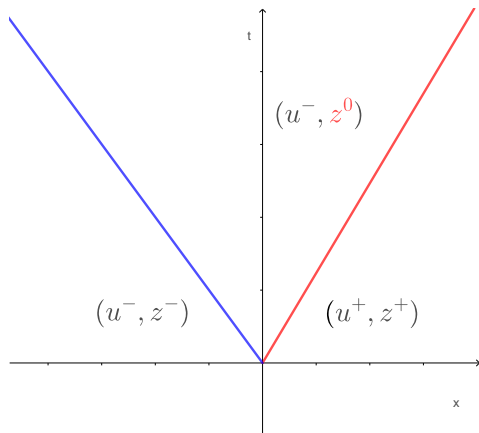


Figure: A contact discontinuity and a shock

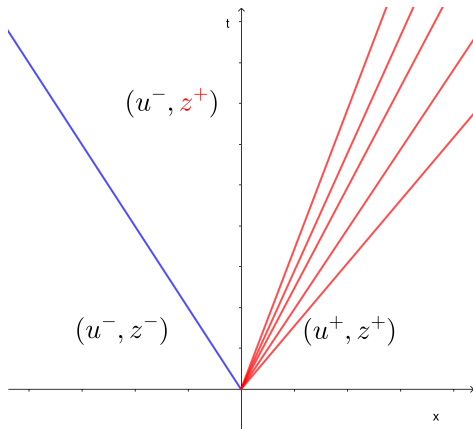


Figure: A contact discontinuity and a rarefaction

Lax wave curves

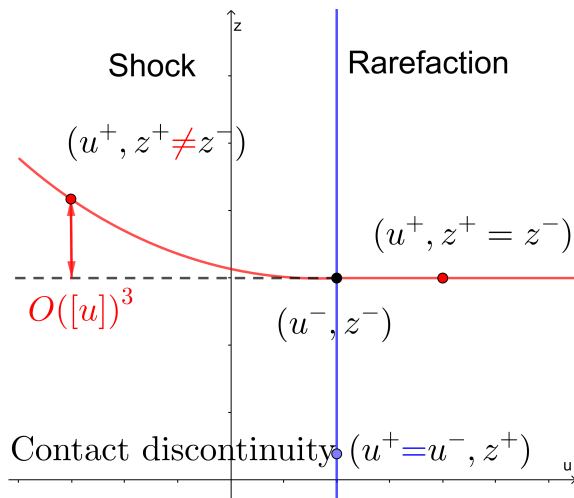


Figure: $[z] = O([u])^3$ along the Lax Shock curve

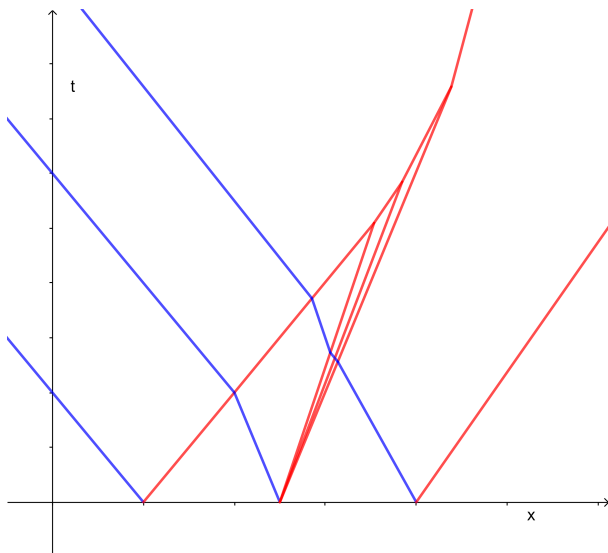
Interaction of waves for the triangular system

- The “master” and “slave” families
 - 1 First or “master” family: u , Rarefaction (R) or Shock (S) independent of the second family
 BV^S decay as a scalar conservation law: $TV^S u \leq TV^S u_0$
 - 2 Second or “slave” family: z , contact discontinuity (CD)
- Wave interactions between:
 - 1 first and second families: R-CD \rightarrow CD-R, S-CD \rightarrow CD-S
 - 2 first family: R-S (or S-R) \rightarrow R-CD or S-CDThe variation of u is constant (1) or decreases (2).

It is like a Temple system for u but not for z .

The variation of z can increase by $O([u]^3)$

Wave Front Tracking (WFT)



L^∞ estimates for z along the second characteristics

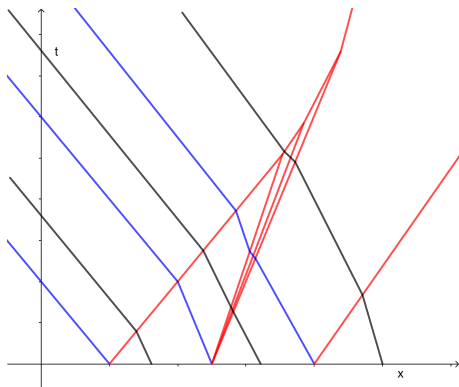


Figure: Black characteristics $\Gamma: \frac{dx}{dt} = a(u(t, x)), x(0) = x_0$,
 $\sup_{\Gamma} |z| \leq \sup |z_0| + O(TV^{1/3}u_{\Gamma}) \leq \sup |z_0| + O(TV^{1/3}u_0)$

Blow-up out of $BV^{1/3}$

A triangular system

$$\begin{aligned}\partial_t u + \partial_x(u^2/2) &= 0 \\ \partial_t v + \partial_x((u-3)v) &= 0\end{aligned}$$

Uniform strict hyperbolicity: If $\sup |u_0| < 1$ then

$$1 > \lambda = u > -1 \quad > \quad -2 > \quad \mu = u - 3 > -4$$

Riemann Invariants: $w = u$, $z = v \exp(-u/3)$

u_0 piecewise constant with compact support,
 $z_0 \equiv 1$, i.e, $v_0 = \exp(u_0/3)$

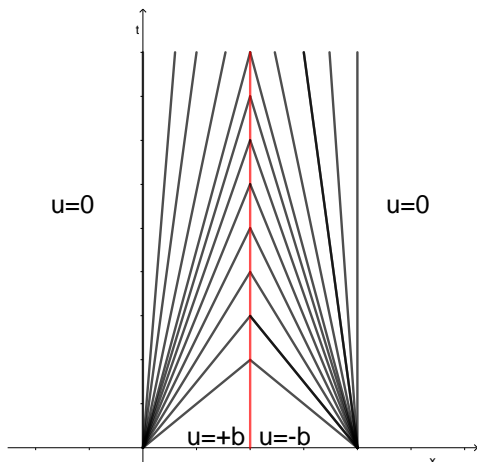


Figure: Building block for $u(t, x)$, Adimurthi, Ghoshal, Gowda, 2014

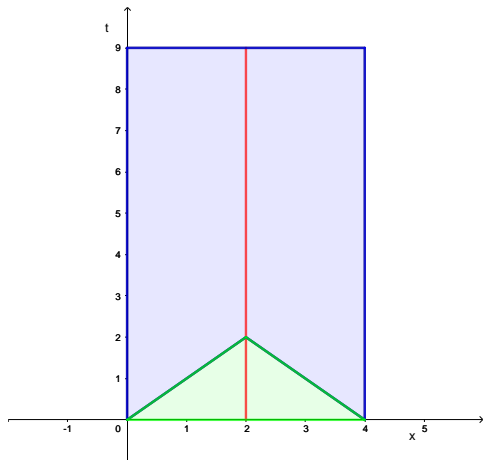


Figure: Schematic building block,
 blue: rarefaction, white: $u = 0$, green: $u = \pm 1$

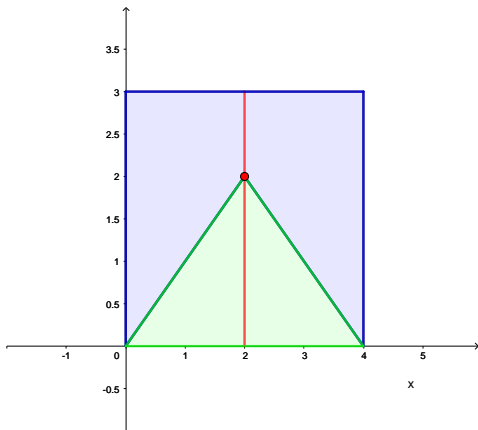


Figure: Zoom on the green part where $u = \pm 1$

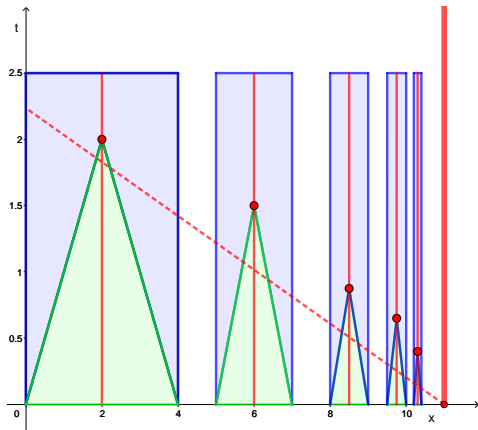


Figure: Exact solution u

For any $t > 0$, $u(t, \cdot) \in BV_x$, Lax-Oleinik smoothing effect 1957.
The blow-up cannot start at a positive time.

$u_0 \notin BV^{1/3}$ thus, on the dotted line, $u(\cdot, \cdot) \notin BV^{1/3}$

Along a 2-characteristic z is constant except through a shock.

Through a shock, z increases from the left to the right, as $O([u]^3)$,
Lax curves C^2 and piecewise C^3 regularity, 1957.

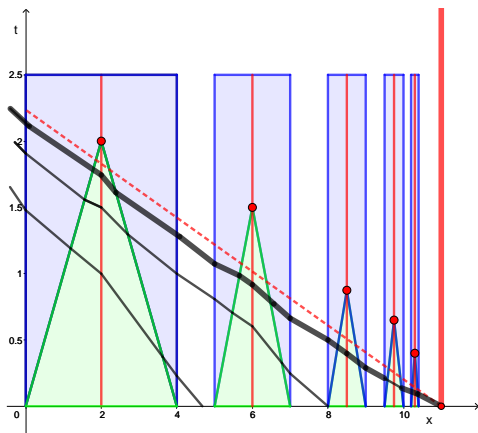


Figure: z along a black characteristic Γ has the size of $TV^{1/3}u$ on Γ .
 $z = +\infty$ on the thick black characteristic

Conclusion: $BV \subset BV^s \subset L^\infty$

Existence of the solution and regularity of the Riemann invariants $x \mapsto (w, z)(t, x)$ for $t > 0$ depending on the number of GNL fields and the regularity of the bounded initial data, $(w_0, z_0) \in L^\infty \times L^\infty$

GNL	LD	(w_0, z_0)	$(w, z)(t, \cdot)$
2	0	$L^\infty \times L^\infty$	$BV \times BV$
0	2	$BV^{s>0} \times BV^{\sigma>0}$	$BV^s \times BV^\sigma$
1	1	$BV^{s \geq 1/3} \times L^\infty$	$BV^s \times L^\infty$
1	1	$w_0 \notin BV^{1/3}$	$z(t, \cdot) \notin L^\infty$

small BV^s data, so, big BV data

1 GNL & 1 LD , a critical regularity $s = 1/3$ for the GNL component

Thank you
for
your attention !

$$BV \subset \Phi\text{-}BV \subset L^\infty$$

$$\Phi\text{-}TV[u] = \sup_{\dots < x_j < x_{j+1} < \dots} \sum_i \Phi(|u(x_{i+1}) - u(x_i)|)$$

$$\Phi\text{-}BV = \{u, \Phi\text{-}TV[u] < \infty\} \subset L^\infty(\mathbb{R}, \mathbb{R})$$

- convex function $\Phi : \Phi(0) = 0, \Phi(u) > 0, u > 0$
- $BV^s := p\text{-}BV = \Phi\text{-}BV, \Phi(|u|) = |u|^p, 0 < s < 1 < p = s^{-1}$
- **Compactness** in L^1_{loc} : Helly's Theorem,
Traces: left and right \rightarrow shock structure !
 Musielak, Orlicz 1959
- Sobolev regularity: $\forall \varepsilon > 0, BV^s \subset W^{s-\varepsilon, 1/s}$,
 $BV^s \not\subset W^{s, 1/s}$ without traces, Bourdarias, Gisclon, J. 2014