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DUALS OF QUANTUM SEMIGROUPS WITH INVOLUTION

YULIA KUZNETSOVA

ABSTRACT. We define a category QSI of quantum semigroups with involution which carries a corepresentation-based duality map $M \mapsto \widehat{M}$. Objects in QSI are von Neumann algebras with comultiplication and coinvolution, we do not suppose the existence of a Haar weight or of a distinguished spatial realisation. In the case of a locally compact quantum group \mathbb{G} , the duality $\widehat{}$ in QSI recovers the universal duality of Kustermans: $\widehat{L^\infty(\mathbb{G})} = C_0^u(\widehat{\mathbb{G}})^{**} = C_0^u(\mathbb{G})^{**}$, and $\widehat{L^\infty(\widehat{\mathbb{G}})} = C_0^u(\mathbb{G})^{**} = C_0^u(\widehat{\mathbb{G}})^{**}$. Other various examples are given.

1. INTRODUCTION

In the theory of locally compact groups, a crucial role is played by the Haar measure; in the theory of locally compact quantum groups, its place is taken by a pair of left and right Haar weights. It is known that in the classical case, the existence of a Haar measure is a theorem, whereas in the quantum case this is a part of axioms.

The dual of a locally compact quantum group in the setting of Kustermans and Vaes [8] is defined via an explicit construction involving the Haar weight; in the setting of Woronowicz [17], the spatial realisation of the multiplicative unitary gives readily the corresponding pair of quantum group algebras in duality.

This should be compared with the classical Pontryagin duality: we consider first the group \widehat{G} of continuous characters of G , which is meaningful for any topological group, and then prove that if G is locally compact abelian, then so is \widehat{G} , and $\widehat{\widehat{G}}$ is isomorphic to G .

It is known that in the quantum setting, one can define a dual in a measure-independent way, but arriving then to a different (bigger) object called the *universal dual*. This was done by Kirchberg [6] in the particular case of Kac algebras, extended to locally compact quantum groups (LCQG) by Kustermans [7], and later realized by Sołtan and Woronowicz [14] in the setting of multiplicative unitaries.

The problem is however that this approach does not give any existence theorem except for the case of LCQG or multiplicative unitaries respectively. In general, there is no guarantee that the dual of the universal dual exists, and thus we cannot speak of a Pontryagin-like isomorphism between an algebra and its bidual.

The present paper introduces a category QSI of *quantum semigroups with involution* which solves this problem. For every $M \in QSI$, we define its dual object $\widehat{M} \in QSI$, so that it extends the universal duality of Kustermans: if $L^\infty(\mathbb{G})$ is a von Neumann algebraic LCQG and $\widehat{}$ denotes the dual in QSI , then $\widehat{L^\infty(\mathbb{G})} = C_0^u(\widehat{\mathbb{G}})^{**} = C_0^u(\mathbb{G})^{**}$, and $\widehat{L^\infty(\widehat{\mathbb{G}})} = C_0^u(\mathbb{G})^{**} = C_0^u(\widehat{\mathbb{G}})^{**}$. Moreover, the construction produces first the C^* -algebras $C_0^u(\widehat{\mathbb{G}})$ and $C_0^u(\mathbb{G})$ and only then we pass to their universal enveloping von Neumann algebras (identified with the linear biduals).

An object in QSI is a von Neumann algebra M with comultiplication $\Delta : M \rightarrow M \otimes M$ and a densely defined coinvolution $S : D(S) \subset M \rightarrow M$, which satisfy certain natural axioms (see Section 2). This class includes all LCQG, but a more general example would be the algebra $C(P)$ of continuous functions on a compact semitopological semigroup with involution P (Example 2.3).

The structure on M induces on the predual M_* a structure of a Banach algebra, with an involution defined on a subalgebra M_{**} . By passing to a quotient of M , we can guarantee that M_{**} is dense

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in M_* . We consider representations of M_* which are involutive on M_{**} , and, as usual, call such a representation unitary if it is generated by a unitary $U \in M \otimes B(H)$ (Section 4).

The central point is to consider the ideal $M_{**}^0 \subset M_*$ defined as the common kernel of all *irreducible non-unitary* representations. It appears that this ideal contains all necessary information on the dual algebra.

We set $\widehat{M} = C^*(M_{**}^0)^{**}$. By definition (except for degenerate cases when we set $\widehat{M} = \{0\}$), irreducible representations of this algebra correspond bijectively to the unitary irreducible representations of M_* ; by disintegration, this bijection extends also to reducible representations, see Section 4.

Every dual algebra carries a structure of a quantum semigroup with involution. Moreover, it is a Hopf algebra in a certain sense (Subsection 5.2), what suggest to consider it as a quantum *group* rather than a semigroup.

If $M = \widehat{N}$ is a dual of some $N \in \mathcal{QSZ}$ and is either commutative or cocommutative, then it is isomorphic to $C_0(G)^{**}$ or $C^*(G)^{**}$ respectively for a classical locally compact group G . Moreover, in all examples known (Sections 7 and 8) the dual algebra \widehat{M} coincides with a dual of a LCQG, so in particular \widehat{M} is isomorphic to the third dual of M . This allows to conjecture that $\widehat{M} \simeq \widehat{\widehat{M}}$ in general.

This article is a development of [10], where the coinvolution of M (termed antipode there) was supposed to be bounded.

2. DEFINITIONS AND NOTATIONS

Our main objects are von Neumann bialgebras, which we allow to be just zero spaces. Below, \otimes denotes the von Neumann algebraic tensor product, and \mathbb{I} the identity operator.

Definition 2.1. A *von Neumann bialgebra* M is a von Neumann algebra endowed with a comultiplication $\Delta : M \rightarrow M \otimes M$ which is a normal coassociative $*$ -homomorphism: $\Delta^* = (* \otimes *)\Delta$ and $(\mathbb{I} \otimes \Delta)\Delta = (\Delta \otimes \mathbb{I})\Delta$.

Note that we do not require Δ to be unital.

It is known that in the assumptions above the predual M_* of M is a completely contractive Banach algebra.

In addition to this structure, we postulate the existence of a *coinvolution* S on M . As it will be seen, the requirements on S are weaker than on an antipode, and this notion corresponds rather to an *involution* of a semigroup than to a group inverse. For this reason the author prefers to use the term coinvolution and not antipode as it was done in [10].

Definition 2.2. Let M be a von Neumann bialgebra. A linear map $S : D(S) \subset M \rightarrow M$ is called a *proper coinvolution* if it satisfies the following conditions:

- (1) $D(S)$ is σ -weakly dense in M ;
- (2) $D(S)$ is closed under multiplication and $S : D(S) \rightarrow M$ is an anti-homomorphism;
- (3) $(*S)(D(S)) \subset D(S)$ and $(*S)^2 = \mathbb{I}_{D(S)}$;
- (4) if $\mu, \nu \in M_*$ are such that $\mu \circ S$ and $\nu \circ S$ extend to normal functionals on M , then for all $x \in D(S)$ holds $\Delta(x)(\nu \circ S \otimes \mu \circ S) = \Delta S(x)(\mu \otimes \nu)$.

A von Neumann bialgebra equipped with a proper coinvolution is called a *quantum semigroup with involution*.

The last formula is a replacement of the identity $\theta(S \otimes S)\Delta = \Delta S$ which may have no sense a priori. The antipode of a locally compact quantum group satisfies these conditions, see [8, Lemma 5.25]. Note that we do not require S to be closeable.

Example 2.3. Let P be a compact semitopological semigroup with involution. Recall that an involution on a semigroup is a map $*$: $P \rightarrow P$ such that $(x^*)^* = x$ and $(xy)^* = y^*x^*$ for all $x, y \in P$. Let $SC(P \times P)$ be the space of separately continuous functions on $P \times P$, then we get a natural map $\check{\Delta} : C(P) \rightarrow SC(P \times P)$, $\check{\Delta}(f)(s, t) = f(st)$. Set $M = C(P)^{**}$, with the usual structure of a von Neumann algebra. Since $SC(P \times P)$ is canonically imbedded into $M \otimes M$, $\check{\Delta}$ can be viewed as a map from $C(P)$ to $M \otimes M$, and it is known [2] that it is a unital $*$ -homomorphism and as such can be extended by normality to M , so that the extension Δ satisfies $(\Delta \otimes \text{id})\check{\Delta} = (\text{id} \otimes \Delta)\check{\Delta}$. Altogether, this

implies that Δ is a comultiplication on M . Set $D(S) = C(P)$ and $(Sf)(t) = f(t^*)$ for $f \in C(P)$, $t \in P$. It is easily seen that S satisfied conditions (1)–(4), so that $M = C(P)^{**}$ is a quantum semigroup with involution.

Definition 2.4. For every $\mu \in M_*$, define $\bar{\mu} \in M_*$ by $\bar{\mu}(a) = \overline{\mu(a^*)}$, $a \in M$. Let M_{**} be the subspace of all $\mu \in M_*$ such that $\bar{\mu} \circ S$ extends to a bounded normal functional on M . We will denote by μ^* this extension, so that $\mu^*(x) = \bar{\mu}(Sx)$ for $x \in D(S)$.

Note that in M_* as a predual space, M_{**} is $\sigma(M_*, M)$ -dense iff it is norm dense.

Proposition 2.5. *If M_{**} is dense in M_* then S is closeable.*

Proof. Suppose that M_{**} is dense in M_* , that is, separates the points of M . Towards a contradiction, suppose also that S is not closeable. Then there exist $a_n \rightarrow a$, $b_n \rightarrow a$ such that $\alpha = \lim Sa_n \neq \lim Sb_n = \beta$. Then for every $\mu \in M_{**}$ we have: $\bar{\mu}(\alpha) = \lim \bar{\mu}(Sa_n) = \lim \mu^*(a_n) = \mu^*(a) = \lim \mu^*(b_n) = \lim \bar{\mu}(Sb_n) = \bar{\mu}(\beta)$. Thus $\mu(\alpha^* - \beta^*) = 0$ for all $\mu \in M_{**}$ while $\alpha - \beta \neq 0$, and we arrive at a contradiction. \square

Proposition 2.6. *On M_* , $\mu \mapsto \bar{\mu}$ is a homomorphism. If $S^2 = \mathbb{I}$, it is involutive.*

Proof.

$$(\bar{\mu} \cdot \bar{\nu})(x) = (\bar{\mu} \otimes \bar{\nu})(\Delta(x)) = \overline{(\mu \otimes \nu)(\Delta(x)^*)} = \overline{(\mu \otimes \nu)(\Delta(x^*))} = \overline{(\mu \cdot \nu)(x^*)} = \overline{(\mu \cdot \nu)}(x).$$

\square

Proposition 2.7. *M_{**} is a subalgebra in M_* , and $(\mu\nu)^* = \nu^*\mu^*$ for all $\mu, \nu \in M_{**}$. With the norm $\|\mu\|_* = \max(\|\mu\|, \|\mu^*\|)$, M_{**} is a Banach $*$ -algebra.*

Proof. For $\mu, \nu \in M_{**}$ and $x \in D(S)$, we have with 2.2(4):

$$\overline{\mu\nu} \circ S(x) = (\bar{\mu} \cdot \bar{\nu})(Sx) = (\bar{\mu} \otimes \bar{\nu})(\Delta(Sx)) = (\bar{\nu} \circ S \otimes \bar{\mu} \circ S)(\Delta(x)) = (\nu^*\mu^*)(x).$$

The first statement follows.

It is immediate that $\|\mu\nu\|_* \leq \|\mu\|_*\|\nu\|_*$. It remains to prove that $(M_{**}, \|\cdot\|_*)$ is complete. If $(\mu_n) \subset M_{**}$ is a Cauchy sequence with respect to $\|\cdot\|_*$, then in particular $\mu_n \rightarrow \mu$, $\mu_n^* \rightarrow \nu$ with respect to $\|\cdot\|$ for some $\mu, \nu \in M_*$. For every $a \in D(S)$, we have $\bar{\mu}(Sa) = \overline{\mu((Sa)^*)} = \lim \mu_n((Sa)^*) = \lim \mu_n^*(a) = \nu(a)$, thus $\mu \in M_{**}$ and $\mu^* = \nu$, moreover $\mu_n \rightarrow \mu$ in $(M_{**}, \|\cdot\|_*)$. \square

Recall that on M_* , there is a two-sided action of M . For $\mu \in M_*$, $a \in M$ one sets $(\mu.a)(b) = \mu(ab)$ and $(a.\mu)(b) = \mu(ba)$, $b \in M$. As in the case of a locally compact group [7, Lemma 2.1], M_{**} is stable under multiplication by $D(S)$:

Lemma 2.8. *M_{**} is a two-sided module over $D(S)$.*

Proof. Let μ be in M_{**} , $a \in D(S)$. Then for $b \in D(S)$ we have

$$\begin{aligned} \overline{(a.\mu)}(Sb) &= \overline{(a.\mu)((Sb)^*)} = \overline{\mu((Sb)^*a)} = \overline{\mu((Sb)^*(S(Sa)^*)^*)} \\ &= \overline{\mu((S(b(Sa)^*)^*)^*)} = \mu^*(b(Sa)^*) = ((Sa)^*.\mu^*)(b). \end{aligned}$$

This means that $\overline{(a.\mu)} \circ S$ extends to the bounded functional $(Sa)^*.\mu^* = (a.\mu)^*$ on M , so $a.\mu \in M_{**}$. The proof for the right action is similar. \square

Definition 2.9. Let M_{**} be dense in M_* . Set

$$D(\tilde{S}) = \{a \in M : \exists C > 0 \text{ such that } |\mu^*(a)| \leq C\|\mu\| \text{ for all } \mu \in M_{**}\}.$$

For every $a \in D(\tilde{S})$, define $\tilde{S}(a) \in M$ by $\mu(\tilde{S}(a)) = \overline{\mu^*(a)}$.

Proposition 2.10. *Let M_{**} be dense in M_* . Then \tilde{S} is a closed operator.*

Proof. If $a_n \rightarrow a$, $\tilde{S}(a_n) \rightarrow b$ in M , then for every $\mu \in M_{**}$ $\mu^*(a_n) \rightarrow \mu^*(a)$; $\mu^*(a_n) = \overline{\mu(\tilde{S}(a_n))} \rightarrow \overline{\mu(b)}$ as $n \rightarrow \infty$. It follows that $\mu^*(a) = \overline{\mu(b)}$ whence $a \in D(\tilde{S})$ and $\tilde{S}(a) = b$. \square

Remark 2.11. Let M_{**} be dense in M_* . Then $D(S) \subset D(\tilde{S})$ and $\tilde{S}(a) = *S(a)$ for $a \in D(S)$. In particular, $\tilde{S}D(S) \subset D(S)$.

Proposition 2.12. *Let M_{**} be dense in M_* . Then $\tilde{S} : D(\tilde{S}) \rightarrow M$ is $\sigma(M, M_{**})$ continuous.*

Proof. Follows from the equality $|\mu(\tilde{S}(a))| = |\mu^*(a)|$ for all $\mu \in M_{**}$, $a \in M$. \square

Proposition 2.13. *If M_{**} is dense in M_* then:*

- (i) $D(\tilde{S})$ is closed under multiplication and $\tilde{S} : D(\tilde{S}) \rightarrow M_*$ is a homomorphism;
- (ii) M_{**} is a two-sided module over $D(\tilde{S})$;
- (iii) $\tilde{S}(D(\tilde{S})) \subset D(\tilde{S})$ and $\tilde{S}^2 = \mathbb{I}_{D(\tilde{S})}$.

Proof. (i). First note that $\tilde{S} = *S$ is a homomorphism on $D(S)$.

Next we prove that $ab \in D(\tilde{S})$ and $\tilde{S}(ab) = \tilde{S}(a)\tilde{S}(b)$ for $a \in D(\tilde{S})$, $b \in D(S)$. By assumption 2.2.(1), there exist $a_i \in D(S)$ such that $a_i \rightarrow a$ σ -weakly. Then $a_i b \rightarrow ab$, and by Proposition 2.12, $\tilde{S}(a_i) \rightarrow \tilde{S}(a)$, both σ -weakly. For every $\mu \in M_{**}$

$$\begin{aligned} \overline{\mu^*(ab)} &= \lim \overline{\mu^*(a_i b)} = \lim \mu(\tilde{S}(a_i b)) = \lim \mu(\tilde{S}(a_i)\tilde{S}(b)) \\ &= \lim \tilde{S}(b) \cdot \mu(\tilde{S}(a_i)) = (\tilde{S}(b) \cdot \mu)(\tilde{S}(a)) = \mu(\tilde{S}(a)\tilde{S}(b)), \end{aligned}$$

what proves that $ab \in D(\tilde{S})$ and $\tilde{S}(ab) = \tilde{S}(a)\tilde{S}(b)$.

In the same way we prove the equality for $a \in D(S)$, $b \in D(\tilde{S})$.

We can now prove (ii): for $a \in D(\tilde{S})$, $\mu \in M_{**}$, $x \in D(S)$,

$$\begin{aligned} (\mu \cdot \tilde{S}(a))^- (Sx) &= \overline{(\mu \cdot \tilde{S}(a))((Sx)^*)} = \overline{\mu(\tilde{S}(a)(Sx)^*)} \\ &= \overline{\mu(\tilde{S}(a)\tilde{S}(x))} = \overline{\mu(\tilde{S}(ax))} = \mu^*(ax) = (\mu^* \cdot a)(x). \end{aligned}$$

This implies that $\mu \cdot \tilde{S}(a) \in M_{**}$ and $(\mu \cdot \tilde{S}(a))^* = \mu^* \cdot a$. Similarly one shows that $\tilde{S}(a) \cdot \mu \in M_{**}$.

If now $a, b \in D(\tilde{S})$, then for $\mu \in M_{**}$

$$\mu^*(ab) = (\mu^* \cdot a)(b) = (\mu \cdot \tilde{S}(a))^*(b) = \overline{(\mu \cdot \tilde{S}(a))(\tilde{S}(b))} = \overline{\mu(\tilde{S}(a)\tilde{S}(b))};$$

this shows that $ab \in D(\tilde{S})$ and $\tilde{S}(ab) = \tilde{S}(a)\tilde{S}(b)$.

For (iii), suppose that $a \in D(\tilde{S})$. For $\mu \in M_{**}$,

$$\mu^*(\tilde{S}(a)) = \overline{\mu^{**}(a)} = \overline{\mu(a)},$$

which implies that $\tilde{S}(a) \in D(\tilde{S})$ and $\tilde{S}(\tilde{S}(a)) = a$. \square

Proposition 2.14. *Let M_{**} be dense in M_* . Then $\bar{S} = *\tilde{S}$, defined on $D(\tilde{S})$, satisfies the axioms (1)-(4) of a proper coinvolution.*

Proof. Clearly $\bar{S} = S$ on $D(S)$. The axiom (1) holds obviously; (2) and (3) follow immediately from 2.13. For (4), suppose that $\mu, \nu \in M_*$ are such that $\mu \circ \bar{S}$ and $\nu \circ \bar{S}$ extend to normal functionals on M . Then $\bar{\mu}, \bar{\nu} \in M_{**}$, and $\mu \circ \bar{S} = (\bar{\mu})^*$, $\nu \circ \bar{S} = (\bar{\nu})^*$. For $x \in D(\bar{S}) = D(\tilde{S})$,

$$\begin{aligned} (\nu \circ \bar{S} \otimes \mu \circ \bar{S})(\Delta(x)) &= ((\bar{\nu})^* \otimes (\bar{\mu})^*)(\Delta(x)) = ((\bar{\nu})^*(\bar{\mu})^*)(x) = ((\bar{\mu}\bar{\nu})^*)(x) \\ &= \overline{(\bar{\mu}\bar{\nu})(\tilde{S}(x))} = (\mu\nu)(\tilde{S}(x)^*) = (\mu \otimes \nu)(\Delta(\tilde{S}(x))). \end{aligned}$$

\square

Remark 2.15. Let M_{**} be dense in M_* . The set $\{\mu \in M_* : \bar{\mu} \circ \bar{S} \text{ extends to a normal functional on } M\}$ is equal to M_{**} , and the application of Definition 2.9 to \bar{S} leads to the same operator \tilde{S} , now with domain $D(\tilde{S}) = D(\bar{S})$. Thus, $\bar{S} = \tilde{S}$.

Proposition 2.16. *Let M_{**} be dense in M_* . Then S can be extended to a closed proper coinvolution, with the same space M_{**} . From now on, we assume that S is closed and $D(S) = D(\tilde{S})$.*

Proposition 2.17. M_{**}^\perp is a (σ -weakly closed) ideal in M .

Proof. Denote $L = M_{**}^\perp$. Obviously it is closed. By Lemma 2.8, if $a \in D(S)$, $b \in L$, $\mu \in M_{**}$, then $\mu(ab) = (a \cdot \mu)(b) = 0$, so $D(S)L \subset L$, and we have, with $[\cdot]$ meaning σ -weak closure in M :

$$ML = [D(S)]L \subset [D(S)L] \subset [L] = L,$$

and similarly for the other inclusion. \square

Notations 2.18. Proposition 2.17 implies that $M_{**}^\perp = (1 - \zeta)M$ for a central projection ζ , which will be denoted also by ζ_M to indicate the algebra M . By general relations in Banach spaces, $[M_{**}] = (M_{**}^\perp)^\perp$, where $[\cdot]$ means the norm closure in M_* and $(\cdot)^\perp$ the annihilator in M_* of a subset of M . Denote $M_r = M/M_{**}^\perp$ and let $Q : M \rightarrow M_r$ be the quotient map.

The isomorphism $Z : M_r \rightarrow \zeta M$ is a right inverse for Q , and its preadjoint map $Z_* : M_* \rightarrow (M_r)_*$ is left inverse to $Q_* : (M_r)_* \rightarrow M_*$.

Proposition 2.19. *With the quotient structure, M_r is a von Neumann bialgebra, with the predual isomorphic to $[M_{**}]$ (the norm closure of M_{**} in M_*).*

Proof. In the notations of Remark 2.18, $Q(a) = Q(\zeta a)$ for all $a \in M$. We define the comultiplication on M/M_{**}^\perp by $\tilde{\Delta}(Qa) = (Q \otimes Q)(\Delta(a))$, $a \in M$. To show that it is well defined, suppose that $Qa = 0$, that is $a \in M_{**}^\perp$. Then for $\mu, \nu \in M_*$

$$(\mu \otimes \nu)((\zeta \otimes \zeta)\Delta(a)) = (Z_*\mu \otimes Z_*\nu)(\Delta(a)) = (Z_*\mu \cdot Z_*\nu)(a) = 0,$$

since $Z_*\mu \cdot Z_*\nu \in M_{**}$. It follows that

$$(Q \otimes Q)(\Delta(a)) = (Q \otimes Q)(\zeta \otimes \zeta)(\Delta(a)) = 0.$$

It is immediate to verify that $\tilde{\Delta}$ is a comultiplication indeed. Note that by definition, Q is a coalgebra morphism. \square

Proposition 2.20. *On M_r , the formula $S_r(Qa) = Q(Sa)$, $a \in D(S)$, defines a proper coinvolution with the domain $D(S_r) = Q(D(S))$.*

Proof. Check first that S_r is well-defined. If $Qa = 0$ for $a \in D(S)$, then $a \in M_{**}^\perp$. Let us show first that $\tilde{S}(a) \in M_{**}^\perp$. For $\mu \in M_{**}$, $\mu(\tilde{S}(a)) = \overline{\mu^*(a)} = 0$ since $\mu^* \in M_{**}$. But $S(a) = (S(a))^* \in M_{**}^\perp$ since the latter is a self-adjoint ideal.

The property 2.2(1) follows from the fact that Q is surjective and σ -weak continuous; 2.2(2,3) from it being a $*$ -homomorphism. For 2.2(4), take $x \in D(S)$ and $\mu, \nu \in [M_{**}]$. If $\mu \circ S_r$ and $\nu \circ S_r$ extend to normal functionals on M_r then $\mu \circ S_r \circ Q = \mu \circ Q \circ S$ and $\nu \circ Q \circ S$ extend to normal functionals on M , and

$$\begin{aligned} \tilde{\Delta}(Qx)(\nu \circ S_r \otimes \mu \circ S_r) &= (S_r Q \otimes S_r Q)\Delta(x)(\nu \otimes \mu) = (QS \otimes QS)\Delta(x)(\nu \otimes \mu) \\ &= \Delta(x)(\nu \circ Q \circ S \otimes \mu \circ Q \circ S) = \Delta S(x)(\mu \circ Q \otimes \nu \circ Q) \\ &= \tilde{\Delta}QS(x)(\mu \otimes \nu) = \tilde{\Delta}S_r(Qx)(\mu \otimes \nu). \end{aligned}$$

\square

Corollary 2.21. *M_r with the quotient structure is a quantum semigroup with involution, with the coinvolution S_r ; moreover, $(M_r)_{**}$ is dense in $(M_r)_*$, and S_r is closeable.*

Proposition 2.22. *The restriction of $Z_* : M_* \rightarrow (M_r)_*$ to M_{**} is a $*$ -homomorphism.*

Proof. First note that $\mu(x) = \mu(\zeta x)$ for every $\mu \in M_{**}$, $x \in M$. For $\mu, \nu \in M_{**}$, $x \in M$ we have:

$$\begin{aligned} Z_*(\mu\nu)(Qx) &= (\mu\nu)(ZQx) = (\mu\nu)(\zeta \cdot x) = (\mu\nu)(x) = (\mu \otimes \nu)(\Delta(x)) = (\mu \otimes \nu)((\zeta \otimes \zeta)\Delta(x)) \\ &= (\mu \otimes \nu)((ZQ \otimes ZQ)(\Delta(x))) = (Z_*\mu \otimes Z_*\nu)(\tilde{\Delta}(Qx)) = (Z_*\mu Z_*\nu)(Qx), \end{aligned}$$

which implies $Z_*(\mu\nu) = Z_*(\mu)Z_*(\nu)$.

Again, for $\mu \in M_{**}$, $x \in D(S)$

$$\begin{aligned} Z_*(\mu^*)(Qx) &= \mu^*(ZQx) = \mu^*(\zeta \cdot x) = \mu^*(x) = \overline{\mu((Sx)^*)} \\ &= \overline{\mu(\zeta \cdot (Sx)^*)} = \overline{\mu(ZQ(*Sx))} = \overline{Z_*\mu((QSx)^*)} = \overline{Z_*\mu(*\tilde{S}(Qx))} = (Z_*\mu)^*(Qx), \end{aligned}$$

whence $Z_*(\mu^*) = (Z_*(\mu))^*$. \square

On M_* , Z_* might not be a homomorphism.

3. REPRESENTATIONS VS. THEIR COEFFICIENTS

Theorem 3.1 below is an analogue of [4, 1.4.2] and of [7, Proposition 2.5]. Though the proofs in this section are very close to the cited publications, our situation is somewhat more general and we prefer to include the proofs.

Theorem 3.1. *Suppose M_{**} is dense in M_* . Let I be a set, and let $x_{\alpha\beta}: \alpha, \beta \in I$ be elements in $D(S) \subset M$ which satisfy the relations $S(x_{\alpha\beta}) = x_{\beta\alpha}^*$ and*

$$\Delta(x_{\alpha\beta}) = \sum_{\gamma} x_{\alpha\gamma} \otimes x_{\gamma\beta}, \quad (1)$$

the series converging absolutely σ -weakly, for all $\alpha, \beta \in I$. Then there exists a $*$ -representation π of M_{**} on $\ell^2(I)$ such that

$$\mu(x_{\alpha\beta}) = \langle \pi(\mu)e_{\beta}, e_{\alpha} \rangle \text{ for all } \mu \in M_{**} \text{ and all } \alpha, \beta \in I. \quad (2)$$

Recall [12, 11.3] that a positive functional φ on a Banach $*$ -algebra \mathcal{A} is called representable if there exists a $*$ -representation $T: \mathcal{A} \rightarrow B(H)$ and a (topologically) cyclic vector $\xi \in H$ such that $\varphi(a) = \langle T(a)\xi, \xi \rangle$ for all $a \in \mathcal{A}$. A positive functional is representable if and only if there is a constant C such that $\varphi(a)^2 \leq C\varphi(a^*a)$, $a \in \mathcal{A}$. In this case $\|\varphi\| = \|\xi\|^2 \leq C$. (See [12, 11.3.4].) Below we consider the algebra M_{**} with its norm $\|\cdot\|_*$ which makes it a Banach $*$ -algebra.

Lemma 3.2. *In the assumptions of Theorem 3.1, let $\xi \in \ell^2(I)$ be finitely supported, and let $\omega_{\xi} = \sum_{\alpha, \beta \in I} \xi_{\alpha} \bar{\xi}_{\beta} x_{\alpha\beta}$. Then ω_{ξ} is a positive representable functional on M_{**} and $\|\omega_{\xi}\|_{M_{**}} \leq \|\xi\|_{\ell^2(I)}^2$.*

Proof. For $\mu \in M_{**}$,

$$\omega_{\xi}(\mu^* \mu) = \Delta(\omega_{\xi})(\mu^* \otimes \mu) = \sum_{\alpha, \beta} \xi_{\alpha} \bar{\xi}_{\beta} \left(\sum_{\gamma} x_{\alpha\gamma} \otimes x_{\gamma\beta} \right) (\mu^* \otimes \mu) = \sum_{\gamma} \left(\sum_{\alpha, \beta} \xi_{\alpha} \bar{\xi}_{\beta} x_{\alpha\gamma}(\mu^*) x_{\gamma\beta}(\mu) \right),$$

where we change the summation order by absolute convergence. Since $x_{\alpha\gamma}(\mu^*) = (Sx_{\alpha\gamma})(\bar{\mu}) = (x_{\gamma\alpha}^*)(\bar{\mu}) = x_{\gamma\alpha}(\mu)$,

$$\omega_{\xi}(\mu^* \mu) = \sum_{\gamma} \left(\sum_{\alpha, \beta} \xi_{\alpha} \bar{\xi}_{\beta} \overline{x_{\gamma\alpha}(\mu)} x_{\gamma\beta}(\mu) \right) = \sum_{\gamma} \left| \sum_{\alpha} \bar{\xi}_{\alpha} x_{\gamma\alpha}(\mu) \right|^2 \geq 0,$$

and we see that ω is positive. Next,

$$|\omega_{\xi}(\mu)|^2 = \left| \sum_{\alpha} \xi_{\alpha} \sum_{\beta} \bar{\xi}_{\beta} x_{\alpha\beta}(\mu) \right|^2 \leq \|\xi\|_{\ell^2(I)}^2 \sum_{\alpha} \left| \sum_{\beta} \bar{\xi}_{\beta} x_{\alpha\beta}(\mu) \right|^2 = \|\xi\|_{\ell^2(I)}^2 \omega_{\xi}(\mu^* \mu).$$

This shows that ω_{ξ} is representable $\|\omega_{\xi}\|_{M_{**}} \leq \|\xi\|_{\ell^2(I)}^2$. \square

Proof of the Theorem. Let $(e_{\alpha})_{\alpha \in I}$ be the canonical base in $\ell^2(I)$. For $\mu \in M_{**}$ and finitely supported $\xi = \sum_{\alpha} \xi_{\alpha} e_{\alpha} \in \ell^2(I)$, set $\pi(\mu)\xi = \sum_{\alpha, \beta} \xi_{\alpha} x_{\beta\alpha}(\mu) e_{\beta}$. By the calculation in the Lemma above,

$$\left\| \sum_{\alpha, \beta} \xi_{\alpha} x_{\beta\alpha}(\mu) e_{\beta} \right\|^2 = \sum_{\beta} \left| \sum_{\alpha} \xi_{\alpha} x_{\beta\alpha}(\mu) \right|^2 = \omega_{\xi}(\mu^* \mu) \leq \|\omega_{\xi}\| \|\mu^*\|_* \|\mu\|_* = \|\xi\|^2 \|\mu\|_*^2.$$

This shows that $\pi(\mu)$ extends to a bounded operator on $\ell^2(I)$ of norm at most $\|\mu\|_*$. By definition, $x_{\alpha\beta}(\mu) = \langle \pi(\mu)e_{\beta}, e_{\alpha} \rangle$.

For $\mu, \nu \in M_{**}$,

$$\begin{aligned} \langle \pi(\mu\nu)e_{\alpha}, e_{\beta} \rangle &= x_{\beta\alpha}(\mu\nu) = \Delta(x_{\beta\alpha})(\mu\nu) = \sum_{\gamma} x_{\beta\gamma}(\mu) x_{\gamma\alpha}(\nu) = \sum_{\gamma} \langle \pi(\mu)e_{\gamma}, e_{\beta} \rangle \langle \pi(\nu)e_{\alpha}, e_{\gamma} \rangle \\ &= \langle \pi(\nu)e_{\alpha}, \pi(\mu)^* e_{\beta} \rangle = \langle \pi(\mu)\pi(\nu)e_{\alpha}, e_{\beta} \rangle. \end{aligned}$$

This proves that π is multiplicative. Next,

$$\langle \pi(\mu^*)e_{\alpha}, e_{\beta} \rangle = x_{\beta\alpha}(\mu^*) = \bar{\mu}(Sx_{\beta\alpha}) = \bar{\mu}(x_{\alpha\beta}^*) = \overline{\mu(x_{\alpha\beta})} = \overline{\langle \pi(\mu)e_{\beta}, e_{\alpha} \rangle} = \langle e_{\alpha}, \pi(\mu)e_{\beta} \rangle,$$

so that π is involutive on M_{**} . \square

Corollary 3.3. *Suppose M_{**} is dense in M_* . Let I be a set, and let $x_{\alpha\beta}: \alpha, \beta \in I$ be elements in $D(S) \subset M$ which satisfy the relations $S(x_{\alpha\beta}) = x_{\beta\alpha}^*$ and*

$$\Delta(x_{\alpha\beta}) = \sum_{\gamma} x_{\gamma\beta} \otimes x_{\alpha\gamma}, \quad (3)$$

then there exists a $*$ -antirepresentation π of M_{**} on $\ell^2(I)$ with the condition (2).

Proof. Set $z_{\alpha\beta} = x_{\beta\alpha}$, then $(z_{\alpha\beta})_{\alpha, \beta \in I}$ satisfy the assumptions of Theorem 3.1. For the $*$ -representation ρ of M_{**} on $\ell^2(I)$ given by the Theorem,

$$\mu(x_{\alpha\beta}) = \mu(z_{\beta\alpha}) = \langle \rho(\mu)e_{\alpha}, e_{\beta} \rangle$$

for all $\mu \in M_{**}$ and all $\alpha, \beta \in I$. Consider now the $\overline{\ell^2(I)}$ with the conjugate-linear structure and the product $\langle x, y \rangle_- = \langle y, x \rangle$. For every μ , $\rho(\mu)^*$ is a linear operator on $\overline{\ell^2(I)}$, which we can denote $\pi(\mu)$. Moreover,

$$\langle \pi(\mu)e_{\beta}, e_{\alpha} \rangle_- = \langle \rho(\mu)e_{\alpha}, e_{\beta} \rangle = \mu(x_{\alpha\beta}),$$

so that π satisfies (2). Finally, π is obviously anti-multiplicative and involutive. \square

Proposition 3.4. *Let π be a $*$ -representation of M_* , and let $(x_{\alpha\beta})_{\alpha, \beta \in I}$ be its coefficients in a basis $(e_{\alpha})_{\alpha \in I}$. Then the series (1) converges absolutely σ -weakly for all $\alpha, \beta \in I$.*

Proof. This is a direct verification: for $\mu, \nu \in M_*$,

$$\begin{aligned} \left(\sum_{\gamma} x_{\alpha\gamma} \otimes x_{\gamma\beta} \right) (\mu \otimes \nu) &= \sum_{\gamma} x_{\alpha\gamma}(\mu) x_{\gamma\beta}(\nu) = \sum_{\gamma} \langle \pi(\mu)e_{\gamma}, e_{\alpha} \rangle \langle \pi(\nu)e_{\beta}, e_{\gamma} \rangle \\ &= \langle \pi(\nu)e_{\beta}, \pi(\mu)^* e_{\alpha} \rangle = \langle \pi(\mu)\pi(\nu)e_{\beta}, e_{\alpha} \rangle = \langle \pi(\mu\nu)e_{\beta}, e_{\alpha} \rangle = \Delta(x_{\alpha\beta})(\mu \otimes \nu). \end{aligned}$$

The series converges absolutely, since this is a decomposition of a scalar product. \square

4. ABSOLUTELY CONTINUOUS IDEAL

Every $*$ -representation of M_{**} is bounded in the norm $\|\cdot\|_*$, but not necessarily in the norm of M_* . In [7], J. Kustermans proves that if M is a locally compact quantum group then one implies the other. But in general we have no reason to expect this. Thus, we restrict the class of representations to those which have a “generator” (defined below) in $B(H) \otimes M$, what implies in particular that they are completely bounded as representations of M_* (but maybe not involutive, and maybe the image of M_* is not self-adjoint).

There are some complications arising from the fact that M_{**} might not be dense in M_* . Recall that this never occurs if M is a locally compact quantum group [7, Section 3].

By Corollary 2.21, we can always pass to M_r and have $(M_r)_{**}$ dense in $(M_r)_*$. In the sequel, we define the dual von Neumann algebra \widehat{M} first in the case when M_{**} is dense, and set then $\widehat{M} := \widehat{M}_r$. Next, we show that \widehat{M} has a structure of a quantum semigroup with involution.

4.1. The case of M_{} dense in M_* .** Let M be as above a quantum semigroup with involution, and suppose that $M_{**} \subset M_*$ is dense. Let π be an involutive representation of M_{**} on a Hilbert space. It is automatically continuous with respect to $\|\cdot\|_*$, but not necessarily with respect to $\|\cdot\|$. But in the case it is, π extends by continuity to M_* , and this extension is a homomorphism (with a possibly non-selfadjoint range). Representations of M_* which are involutive on M_{**} will be called below just $*$ -representations of M_* .

M_* can be considered as an operator space with the structure induced by duality to M . Recall the following fact.

Proposition 4.1. *Let π be a representation of M_* on a Hilbert space K . Then π is completely bounded if and only if there exists $U \in B(K) \otimes M \simeq (B(K)_* \widehat{\otimes}_{\text{op}} M_*)^*$ such that $U(\omega, \mu) = \omega(\pi(\mu))$ for every $\mu \in M_*$, $\omega \in N(K)$. In this case $\|U\| = \|\pi\|_{cb}$.*

Notations 4.2. Let π be a representation of M_* on a Hilbert space K with a basis (f_{α}) . Denote by $\pi_{\alpha\beta} \in M$ the linear functional on M_* defined by $\pi_{\alpha\beta}(\mu) = \langle \pi(\mu)f_{\beta}, f_{\alpha} \rangle$, $\mu \in M_*$.

In the next definition, the equivalence of conditions (1) and (2) has been proved in [10, Theorem 3.5]. The proof makes no use of involution on M_* or of the coinvolution (coinvolution) on M , so it remains valid in the present case. In fact, it is not even necessary to suppose that π is a representation.

Definition 4.3. Let $\pi : M_* \rightarrow B(K)$ be a completely bounded $*$ -representation of M_* on a Hilbert space K . Then π is called *unitary* if one of the following equivalent conditions holds:

- (1) Exists a unitary $U \in B(K) \bar{\otimes} M$ called a *generator* of π such that

$$U(\omega, \mu) = \omega(\pi(\mu)) \quad (4)$$

for every $\mu \in M_*$, $\omega \in B(K)_*$;

- (2) π is non-degenerate and in some basis of K ,

$$\sum_{\gamma} \pi_{\gamma\alpha}^* \cdot \pi_{\gamma\beta} = \sum_{\gamma} \pi_{\alpha\gamma} \cdot \pi_{\beta\gamma}^* = \begin{cases} 1, & \alpha = \beta \\ 0, & \alpha \neq \beta \end{cases} \quad (5)$$

for every α, β , the series converging absolutely in the M_* -weak topology of M .

From the Theorem 3.5 of [10] it follows that this definition does not depend, in fact, on the choice of a basis.

Moreover, one can verify that the following is true:

Proposition 4.4. *Let the elements $\pi_{\alpha\beta} \in M$, $\alpha, \beta \in A$, be such that (5) holds. Then there is a Hilbert space K with a basis $(e_{\alpha})_{\alpha \in A}$, a unitary operator $U \in B(K) \otimes M$ and a linear completely contractive map $\pi : M_* \rightarrow B(K)$ such that $\pi_{\alpha\beta}(\mu) = \langle \pi(\mu)e_{\beta}, e_{\alpha} \rangle = U(\omega_{\alpha\beta}, \mu)$ for all $\mu \in M_*$ and $\alpha, \beta \in A$.*

For the proof, one realizes M on a Hilbert space H and defines U as an operator on $K \otimes H$ by

$$\langle U(e_{\alpha} \otimes x), (e_{\beta} \otimes y) \rangle = \pi_{\beta\alpha}(\mu_{x,y})$$

for $x, y \in H$, $\alpha, \beta \in A$. It follows then from (5) that U is unitary, what implies the existence of π with required properties.

Similarly to [10], we give the following definitions:

Definition 4.5. We call *non-unitary* the $*$ -representations of M_{**} , completely bounded or not, which are not unitary. Let $M_{**}^{\times} \subset M_{**}$ be the (intersection of M_{**} with the) common kernel of all *irreducible* non-unitary representations. If there are none, let $M_{**}^{\times} = M_{**}$. This is a two-sided $*$ -ideal in M_{**} , which is called the *absolutely continuous ideal* of M_* . With the structure inherited from M_{**} , M_{**}^{\times} is a Banach $*$ -algebra.

By definition, M_{**}^{\times} is contained in M_{**} and might not be an ideal in M_* . However, if M_{**} is dense in M_* , one sees that the norm closure $[M_{**}^{\times}]$ is an ideal in M_* .

Note that the direct sum of a unitary and non-unitary representation is non-unitary; this is the reason to consider only irreducible representations in the definition above.

The main property of M_{**}^{\times} is that every representation of M_* which is irreducible on M_{**}^{\times} must be unitary. For the duality construction we need more: that unitary representations do not vanish on it. To guarantee this, we exclude all degenerate cases by the following definition:

Definition 4.6. Let I^0 be the weakly closed ideal in M generated by $(M_{**}^{\times})^{\perp}$, that is by the annihilator of M_{**}^{\times} . Set $M_{**}^0 = M_{**}^{\times}$ if $I^0 \neq M$ and $M_{**}^0 = \{0\}$ otherwise.

Remark 4.7. The following can be said also of any $*$ -ideal of M_{**} . Every non-degenerate representation of M_{**}^0 extends uniquely to M_{**} [12, 11.1.12]. It is easy to show that M_{**} is mapped into the weak closure of the image of M_{**}^0 , i.e. to the von Neumann algebra generated by M_{**}^0 . Conversely, if $\varphi_1, \varphi_2 : M_{**} \rightarrow N$ are two $*$ -homomorphisms to a von Neumann algebra N which agree on M_{**}^0 and are such that $\varphi_i(M_{**})$ is contained in the weak closure of $\varphi_i(M_{**}^0)$ for $i = 1, 2$, then $\varphi_1 = \varphi_2$.

Proposition 4.8. *If $M_{**}^0 \neq \{0\}$ then every unitary representation of M_* is non-degenerate on M_{**}^0 .*

Proof. Let $\pi : M \rightarrow B(H)$ be a unitary representation of M_* . First, it is nonzero on M_{**}^0 : otherwise we would have $\pi_{\alpha\beta} \in I^0$ for all its coefficients, and by (5) this would imply $1 \in I^0$, what is not true by assumption.

Next suppose that π is degenerate on M_{**}^0 . Let $L \subset H$ be the null subspace of $\pi(M_{**}^0)$. Then $\pi|_L$ is also unitary (with generator $(P_L \otimes 1)U$ where P_L is the projection onto L and U is the generator of π) and vanishes on M_{**}^0 , what is impossible. \square

Proposition 4.9. *Every $*$ -representation of M_{**}^0 extends to a completely contractive representation of M_* .*

Proof. Every irreducible representation of M_{**}^0 is by definition unitary and thus extends to a completely contractive representation of M_* . In general, the norm of a $*$ -representation $\pi : M_{**}^0 \rightarrow B(H)$ is majorated by the supremum over all irreducible representations; thus $\|\pi(x)\| \leq \|x\|_{M_*}$ for every $x \in M_{**}^0$, and it follows that π extends to a contractive representation of M_* by continuity.

To show that π is completely bounded, we need the disintegration theory in its a priori non-separable form [5]. Let A be the unital C^* -algebra generated by $\pi(M_{**}^0)$ and $\mathbb{1}_H$. There exists a measure ν on the state space Ω of A such that the identity representation of A on H is isomorphic to the direct integral $\int_{\Omega} \rho_{\varphi} d\nu(\varphi)$ of irreducible representations ρ_{φ} of A on respective spaces H_{φ} ; note that the direct integral is understood in the sense of W. Wils [16], a definition suitable for non-separable disintegration.

Every $\rho_{\varphi} \circ \pi$ is either irreducible on M_{**}^0 or vanishes on it. This allows to verify explicitly that the matrix norm of $\pi_n : M_n(M_{**}^0) \rightarrow M_n(B(\int_{\Omega} H_{\varphi}))$ is majorated by 1, what proves that π extends to a completely contractive representation of M_* . \square

The following is a generalization of Theorem 5.5 of [10]. The proof is changed in significant details to treat possible inequality of M_* and M_{**} .

Theorem 4.10. *Every bounded $*$ -representation π of M_* which is nondegenerate on M_{**}^0 is unitary.*

Proof. By Proposition 4.9 (extension is the same as π by non-degeneracy), π is in fact completely contractive. By Proposition 4.1, there exists $U \in B(H) \bar{\otimes} M$ such that $U(\omega, \mu) = \omega(\pi(\mu))$, and all we need is to prove that U is unitary.

Let M be realized on a Hilbert space K . We will need several times the following representation. Fix $x \in K$, $\xi \in H$ and bases $(e_{\alpha}) \subset K$, $(f_{\beta}) \subset H$. Then

$$\langle U(\xi \otimes x), f_{\beta} \otimes e_{\alpha} \rangle = \langle \pi(\mu_{xe_{\alpha}}) \xi, f_{\beta} \rangle,$$

so that

$$\begin{aligned} U(\xi \otimes x) &= \sum_{\alpha, \beta} \langle U(\xi \otimes x), f_{\beta} \otimes e_{\alpha} \rangle f_{\beta} \otimes e_{\alpha} = \sum_{\alpha, \beta} \langle \pi(\mu_{xe_{\alpha}}) \xi, f_{\beta} \rangle f_{\beta} \otimes e_{\alpha} \\ &= \sum_{\alpha} \left(\sum_{\beta} \langle \pi(\mu_{xe_{\alpha}}) \xi, f_{\beta} \rangle f_{\beta} \right) \otimes e_{\alpha} = \sum_{\alpha} \pi(\mu_{xe_{\alpha}}) \xi \otimes e_{\alpha} \end{aligned} \quad (6)$$

(convergence is everywhere in the Hilbert space norm). Similarly,

$$\begin{aligned} U^*(\xi \otimes x) &= \sum_{\alpha, \beta} \langle \xi \otimes x, U(f_{\beta} \otimes e_{\alpha}) \rangle f_{\beta} \otimes e_{\alpha} = \sum_{\alpha, \beta} \langle \pi(\mu_{e_{\alpha}x}) f_{\beta}, \xi \rangle^- f_{\beta} \otimes e_{\alpha} \\ &= \sum_{\alpha} \left(\sum_{\beta} \langle \pi(\mu_{e_{\alpha}x})^* \xi, f_{\beta} \rangle f_{\beta} \right) \otimes e_{\alpha} = \sum_{\alpha} \pi(\mu_{e_{\alpha}x})^* \xi \otimes e_{\alpha} \end{aligned} \quad (7)$$

We can suppose that M is realized in its standard form. Then every $\mu \in M_*$ is equal to μ_{xy} for some $x, y \in K$, so that we can identify M_* with $K \otimes \bar{K}$.

For subspaces $E, F \subset K$, let $M_{E,F} \subset M_*$ denote the closed subalgebra generated by μ_{xy} with $x \in E$, $y \in F$. Denote also $M_{E,F}^0 = M_{E,F} \cap M_{**}^0$. These subalgebras are not supposed to be self-adjoint. By $M_{E,F}^{\times}$, $M_{E,F}^{0 \times}$ we denote the $\|\cdot\|_*$ -closed $*$ -subalgebras generated by $M_{E,F} \cap M_{**}$ and $M_{E,F}^0$ respectively. Note that by definition $M_{E,F}^0 \subset M_{E,F}^{0 \times}$ but it might be $M_{E,F} \not\subset M_{E,F}^{\times}$.

Lemma 4.11. *A closed subspace $L \otimes E \subset H \otimes K$ is U -invariant if and only if $\pi(M_{E,K})L \subset L$ and $\pi(M_{E,E^{\perp}})L = \{0\}$. It is U^* -invariant if and only if $\pi(M_{K,E})^*L \subset L$ and $\pi(M_{E^{\perp},E})^*L = \{0\}$.*

The proof is identical to the proof of Lemma 5.6 of [10].

Lemma 4.12. *For every separable subspace $V = G \otimes F \subset H \otimes K$ there exist closed separable subspaces $E \subset K$, $L \subset H$ such that $V \cup UV \cup U^*V \subset L \otimes E$ and $M_{F,F} \subset \overline{M_{**} \cap M_{E,E}}$.*

Proof. It follows from the assumptions that F and G are separable. Let $X \subset M_{**}$ be countable and such that $M_{F,F} \subset \overline{X}$. By the isomorphism $M_* \simeq K \otimes \bar{K}$, we can write $X = \{\mu_{x_n y_n} : n \in \mathbb{N}\}$. Set $F_1 = \overline{\text{lin}\{x_n, y_n : n \in \mathbb{N}\}}$, then $X \subset M_{F_1, F_1}$.

Since V is separable, so is $V_2 = \overline{G \otimes F_1 + V + UV + U^*V}$. Pick a sequence (v_n) dense in V and orthonormal bases $(e_\alpha) \subset K$, $(f_\beta) \subset H$. Every v_n is contained in $\overline{\text{lin}\{f_\beta \otimes e_\alpha : \alpha \in A_n, \beta \in B_n\}}$ with countable A_n, B_n . Then for $E = \overline{\text{lin}\{e_\alpha : \alpha \in \cup A_n\}}$ and $L = \overline{\text{lin}\{f_\beta : \beta \in \cup B_n\}}$ we have $V \cup UV \cup U^*V \subset L \otimes E$, and $X \subset M_{E,E}$ so that E, L are as required. \square

Lemma 4.13. *Every $v \in K \otimes H$ can be embedded into a U, U^* -invariant separable subspace $L \otimes E$ such that L is essential for $\pi(M_{E,K}^0)|_L$ and $M_{E,E} \subset \overline{M_{**} \cap M_{E,E}}$.*

Proof. Construct separable subspaces E_k, L_k by induction as follows. Let $L_1 \otimes E_1$ be any separable subspace containing v . Suppose now that E_{k-1}, L_{k-1} are constructed for some $k \geq 2$. Since H is essential for M_{**}^0 , there are sequences $\mu_n^{(k)} \in M_{**}^0, \xi_n^{(k)} \in H$ such that $\overline{\text{lin}\{\pi(\mu_n^{(k)})\xi_n^{(k)}\}}$ is dense in L_{k-1} . Since M is in the standard form, every $\mu_n^{(k)}$ can be represented as $\mu_n^{(k)} = \mu_{x_n^{(k)}, y_n^{(k)}}$ with $x_n^{(k)}, y_n^{(k)} \in K$.

Set $E'_k = E_{k-1} + \overline{\text{lin}\{x_n^{(k)} : n \in \mathbb{N}\}}$, $L'_k = L_{k-1} + \overline{\text{lin}\{\xi_n^{(k)} : n \in \mathbb{N}\}}$. Then $\mu_n^{(k)} \in M_{E'_k, K}^0$ and $\xi_n^{(k)} \in L'_k$ for all n . By Lemma 4.12 there are separable subspaces E_k, L_k such that

$$L'_k \otimes E'_k \cup U(L'_k \otimes E'_k) \cup U^*(L'_k \otimes E'_k) \subset L_k \otimes E_k$$

and $M_{E'_k, E'_k} \subset \overline{M_{**} \cap M_{E_k, E_k}}$.

Set $E = \overline{\cup E_k}$ and $L = \overline{\cup L_k}$, then $L \otimes E = \overline{\cup (L_k \otimes E_k)}$ since E_k, L_k are increasing. We have $U(L_k \otimes E_k) \subset L_{k+1} \otimes E_{k+1}$ and $U^*(L_k \otimes E_k) \subset L_{k+1} \otimes E_{k+1}$ for all k , what implies the U, U^* -invariance of $L \otimes E$. Moreover, by construction the set $\{\pi(\mu\xi : \mu \in M_{E,K}^0, \xi \in L)\}$ is dense in L . Finally, $M_{E_k, E_k} \subset \overline{M_{**} \cap M_{E_{k+1}, E_{k+1}}} \subset \overline{M_{**} \cap M_{E,E}}$ for every k , so $M_{E,E} = \overline{\cup M_{E_k, E_k}} \subset \overline{M_{**} \cap M_{E,E}}$. \square

Proof of the theorem. Take any $v \in H \otimes K$. Let $L \otimes E \subset H \otimes K$ be U, U^* -invariant and separable, such that $v \in L \otimes E$, $M_{E,E} \subset \overline{M_{**} \cap M_{E,E}}$ and L is essential for $M_{E,K}^0$. It follows that $\pi(M_{E,K})L \subset L$; $\pi(M_{E,E^\perp})L = \{0\}$; $\pi(M_{K,E})^*L \subset L$ and as a consequence $\pi(M_{E,E}^\times)L \subset L$.

Fix an orthonormal base $(e_\alpha)_{\alpha \in A}$ in K such that the (countable) subset $(e_\alpha)_{\alpha \in A_1}$ is a base for E . Note that $(L \otimes E)^\perp$ is also U, U^* -invariant. For $x \in E^\perp$, $\xi \in L$ one has $U^*(\xi \otimes x) \in (L \otimes E)^\perp$; from (7) it follows that $\pi(\mu_{e_\alpha x})^* \xi \in L^\perp$ if $e_\alpha \in E$. It follows that $\pi(M_{E,E^\perp})^*L \subset L^\perp$. For $x \in K$, $\xi \in L^\perp$ we have again $U^*(\xi \otimes x) \in (L \otimes E)^\perp$; and from (7) it follows that $\pi(\mu_{e_\alpha x})^* \xi \in L^\perp$ if $e_\alpha \in E$. It follows that $\pi(M_{E,K})^*L^\perp \subset L^\perp$.

Let $C^*(X)$ denote the closed $*$ -algebra generated by X in its relevant space of operators, and let $r_L : B(H) \rightarrow B(L)$ be the reduction onto L . The reasoning above shows that $r_L : C^*(\pi(M_{E,K})) \rightarrow B(L)$ is a $*$ -representation which vanishes on $\pi(M_{E,E^\perp})$. Denote $\rho = r_L \circ \pi$; we have $\rho(M_{E,E^\perp}) = 0$.

Recall that $M_{E,E}^\times$ is closed in the $\|\cdot\|_*$ -norm; the assumption $M_{E,E} \subset \overline{M_{**} \cap M_{E,E}}$ implies moreover that in the norm of M_* we have $M_{E,E} \subset \overline{M_{E,E}^\times}$, so that $\rho(M_{E,K}) = \rho(M_{E,E}) \subset \overline{\rho(M_{E,E}^\times)} = C^*(\rho(M_{E,E}^\times))$. From the other side, obviously $C^*(\rho(M_{E,E}^\times)) \subset C^*(\rho(M_{E,E}))$, so in fact $C^*(\rho(M_{E,E}^\times)) = C^*(\rho(M_{E,K}))$.

Set $\mathcal{A} = C^*(\rho(M_{E,K}^0))$. Since $M_{E,E}$ is separable, so are $\rho(M_{E,E}) = \rho(M_{E,K}) \supset \rho(M_{E,K}^0)$ and \mathcal{A} . The identity representation of \mathcal{A} in $B(L)$ is decomposed into a direct integral of irreducible representations [3, 8.5.2]: there exist a set P equipped with a probability measure β ; an integrable field of Hilbert spaces $\Gamma \subset \{(H_p)_{p \in P}\}$; a field of representations $\sigma_p : \mathcal{A} \rightarrow B(H_p)$, $p \in P$, where every σ_p is irreducible; and an isometric isomorphism $V : L \rightarrow \Gamma = \int_P^\oplus H_p d\beta(p)$ such that: if $\xi \in L$ and $V(\xi) = \int_P \xi_p d\beta(p)$, then for every $a \in \mathcal{A}$ we have $V(a\xi) = \int_P \sigma_p(a)\xi_p d\beta(p)$.

For every $\mu \in M_{E,K}^{0 \times}$ and $\xi \in L$, denoting $V\xi = \int \xi_p$, we have

$$V(\rho(\mu)\xi) = \int_P \rho_p(\mu)\xi_p d\beta(p) \quad (8)$$

with irreducible representations $\rho_p = \sigma_p \circ \rho$ of $M_{E,K}^{0 \times}$.

$M_{E,K}^{0\times}$ is an ideal in $M_{E,K}^\times$ as well as $\rho(M_{E,K}^{0\times})$ in $\rho(M_{E,K}^\times)$, so we can extend σ_p uniquely and irreducibly to $\rho(M_{E,K}^\times)$, still denoting this extension by σ_p . By continuity, σ_p extends to $C^*(\rho(M_{E,K}^\times)) = C^*(\rho(M_{E,K}))$, and this allows to extend ρ_p to $M_{E,K}$. These considerations show also that (8) is valid in fact for all $\mu \in M_{E,K}$.

For every p , $\sigma_p \circ r_L$ is lifted [3, 2.10] from $C^*(\rho(M_{E,K}))$ to an irreducible representation $\tilde{\sigma}_p$ on $C^*(\pi(M_*))$, probably on a bigger space $\tilde{H}_p \supset H_p$. Set $\tilde{\rho}_p = \tilde{\sigma}_p \circ \pi$.

Since π is completely bounded, every $\tilde{\rho}_p$ is completely bounded too. For $\mu \in M_{E,K}$, $\tilde{\rho}_p(\mu)|_{H_p} = \sigma_p \circ \rho(\mu) = \rho_p(\mu)$. In particular, H_p is invariant under $\tilde{\rho}_p(M_{E,K})$, and $\tilde{\rho}_p(M_{E,E^\perp})|_{H_p} = 0$ (since $\rho(M_{E,E^\perp}) = 0$). Being irreducible and nonzero on M_{**}^0 , $\tilde{\rho}_p$ is unitary, with a unitary generator $U_p \in B(\tilde{H}_p) \otimes M$. By the reasoning above, we can apply Lemma 4.11 and conclude that $H_p \otimes E$ is U_p -invariant.

The rest of the proof is identical to that of [10, Theorem 5.5]. $\int (H_p \otimes E)_{p \in P}$ is also a field of Hilbert spaces, isomorphic to $L \otimes E$ under the isomorphism $\tilde{V} = V \otimes \text{id}$. We have then for $x \in E$, $\xi \in L$ that $\mu_{xe_\alpha} \in M_{E,K}$ and so $\pi(\mu_{xe_\alpha})\xi = \rho(\mu_{xe_\alpha})\xi$. Denote $V\xi = \int \xi_p$; we have

$$\begin{aligned} \tilde{V}(U(\xi \otimes x)) &= \tilde{V}\left(\sum_{\alpha \in A_1} \pi(\mu_{xe_\alpha})\xi \otimes e_\alpha\right) \\ &= \sum_{\alpha \in A_1} V(\rho(\mu_{xe_\alpha})\xi) \otimes e_\alpha = \sum_{\alpha \in A_1} \int (\rho_p(\mu_{xe_\alpha})\xi_p) \otimes e_\alpha \\ &= \sum_{\alpha \in A_1} \int (\rho_p(\mu_{xe_\alpha})\xi_p) \otimes e_\alpha. \end{aligned} \tag{9}$$

The last series converges in the Hilbert norm of $\int (H_p \otimes E)$.

From the other hand, for every p we have a formula similar to (6): if $x \in E$, $\xi_p \in H_p$, then

$$U_p(\xi_p \otimes x) = \sum_{\alpha \in A_1} \rho_p(\mu_{xe_\alpha})\xi_p \otimes e_\alpha,$$

so the series in (9) converges pointwise to $\int U_p(\xi_p \otimes x)$. Both imply convergence in measure in the following sense: denote $\varphi_{\alpha p} = \rho_p(\mu_{xe_\alpha})\xi_p \otimes e_\alpha$, then for every $\varepsilon > 0$

$$\beta\{p : \|(\tilde{V}(U(\xi \otimes x)))_p - \sum_{\alpha \in B} \int \varphi_{\alpha p}\| \geq \varepsilon\} \rightarrow 0,$$

$$\beta\{p : \|U_p(\xi_p \otimes x) - \sum_{\alpha \in B} \int \varphi_{\alpha p}\| \geq \varepsilon\} \rightarrow 0$$

as finite set of indices $B \subset A_1$ increases (the reasoning for real-valued functions applies verbatim). It follows that $(\tilde{V}(U(\xi \otimes x)))_p = U_p(\xi_p \otimes x)$ almost everywhere, that is

$$\tilde{V}(U(\xi \otimes x)) = \int U_p(\xi_p \otimes x) = \left(\int U_p\right)(\tilde{V}(\xi \otimes x)).$$

It follows that (U_p) , or strictly speaking $(U_p|_{H_p \otimes E})$, is a measurable field of operators on $\int H_p \otimes E$, and $\tilde{V}U = \int U_p \tilde{V}$. Since $\int U_p$ is unitary, so is U on $L \otimes E$.

As the initial vector v was arbitrary, we get that U is unitary on $H \otimes K$, what proves the theorem. \square

Definition 4.14. Denote by $C^*(M_{**}^0)$ the C^* -enveloping algebra of M_{**}^0 , that is the completion of M_{**}^0 with respect to the maximal C^* -seminorm on it. Denote by \widehat{M} the enveloping von Neumann algebra of $C^*(M_{**}^0)$ and by $\Phi : M_{**}^0 \rightarrow \widehat{M}$ the canonical map into it. As every non-degenerate representation, Φ extends uniquely to M_{**} (see remark 4.7), and by continuity to M_* . In the sequel, we always consider Φ as a map from M_* to \widehat{M} . Note that a priori Φ might not be injective.

By Proposition 4.9, Φ is completely contractive, and by Theorem 4.10 it is unitary, unless $\widehat{M} = \{0\}$. Denote also by $\widehat{\Phi}$ the preadjoint map $\Phi_* : \widehat{M}_* \rightarrow M$.

Remark 4.15. This definition coincides with the one given in [10] in the case of a bounded coinvolution S . In this case, in particular, M_{**} is always dense in M_* .

4.2. The case of M_{**} not dense in M_* .

Definition 4.16. In general, set $\widehat{M} = \widehat{M}_r$, $\Phi_M = \Phi_{M_r} \circ Z_*$, where $Z_* : M_* \rightarrow M_{r*}$ is the map defined in Remark 2.18. By Proposition 2.22, Φ_M is a $*$ -homomorphism on M_{**} . However, if M_{**} is not dense in M_* , then Φ_M might not be multiplicative on M_* . Still, Φ_M has a generator $V_M = (\mathbb{I} \otimes Z)(V_{M_r}) \in \widehat{M} \otimes M$, but it is not unitary, since V_M is contained in the ideal $\widehat{M} \otimes (\zeta M)$ (see notations in Definition 2.18).

Proposition 4.17. *Let N be a von Neumann algebra and $\pi : M_* \rightarrow N$ a $*$ -homomorphism. Then there is a unique $*$ -homomorphism $\tilde{\pi} : \widehat{M} \rightarrow N$ such that $\tilde{\pi} \circ \Phi = \pi$ on $[M_{**}]$ and $\pi(M_*)$ is contained in the weak closure of $\pi(M_{**}^0)$ in N . If M_{**} is dense in M_* , then the equality holds on M_* .*

Proof. From M_{**}^0 , π is lifted by the universality property to $\tilde{\pi} : \widehat{M} \rightarrow N$, so that $\tilde{\pi} \circ \Phi = \pi$ (on M_{**}^0). Now π and $\tilde{\pi} \circ \Phi$ are two extensions of π to M_{**} with the property that the image of M_{**} is contained in the weak closure of $\pi(M_{**}^0)$. As for any $*$ -ideal in M_{**} , this implies that $\pi = \tilde{\pi} \circ \Phi$ on M_{**} . Finally, by density this equality holds also on $[M_{**}]$, and on the whole of M_* is M_{**} is dense in it. \square

5. THE DUAL HOPF-VON NEUMANN ALGEBRA

In this section we still suppose that M_{**} is dense in M_* . General case will follow for granted since it is reduced to the case of M_{**} dense.

In order to define a comultiplication on \widehat{M} , we define first the Kronecker product $\Phi \times \Phi$ of Φ by itself, via its coordinates. Recall the procedure of how it is done. If $\widehat{M} \subset B(K)$ and $(e_\alpha)_{\alpha \in A}$ is a base of K , let $\Phi_{\alpha\beta}$ be the coefficients of Φ . Then $\Phi_{\alpha,\beta} \cdot \Phi_{\alpha',\beta'}$ with $\alpha, \beta, \alpha', \beta' \in A$ satisfy the equalities (5), so they generate a unitary $V \in \widehat{M} \otimes \widehat{M} \otimes M$ such that $V(\omega_{\alpha\beta}, \omega_{\alpha'\beta'}, \mu) = (\Phi_{\alpha,\beta} \cdot \Phi_{\alpha',\beta'}) (\mu)$ for all $\mu \in M_*$. The corresponding map $\Phi \times \Phi : M_* \rightarrow B(K \otimes K)$ has clearly its range in $\widehat{M} \otimes \widehat{M}$ and is multiplicative on M_* , since $\Delta(\Phi_{\alpha,\beta} \cdot \Phi_{\alpha',\beta'}) = \sum_{\gamma,\gamma'} \Phi_{\alpha,\gamma} \Phi_{\alpha',\gamma'} \otimes \Phi_{\gamma\beta} \Phi_{\gamma'\beta'}$ (and this follows from the multiplicativity of Φ).

Proposition 5.1. *Let π be a representation of M_* in a Hilbert space H . Assume that S is extended according to Corollary 2.16. Then π is involutive if and only if for every $\omega \in B(H)_*$, $\pi_*(\omega) \in D(S)$ and $S(\pi_*(\omega)) = \pi_*(\bar{\omega})^*$.*

Proof. Let π be involutive. For $\mu \in M_{**}$,

$$\pi_{\alpha\beta}(\mu^*) = \langle \pi(\mu^*)e_\beta, e_\alpha \rangle = \overline{\langle \pi(\mu)e_\alpha, e_\beta \rangle} = \overline{\pi_{\beta\alpha}(\mu)}.$$

By definition 2.9, this implies $\pi_{\alpha\beta} \in D(R)$ and $R(\pi_{\alpha\beta}) = \pi_{\beta\alpha}$, so that $S(\pi_*(\omega_{\alpha\beta})) = \pi_*(\bar{\omega}_{\alpha\beta})^*$. By the weak continuity of R , the statement follows for every ω .

Conversely, $R(\pi_{\alpha\beta}) = \pi_{\beta\alpha}$ for all α, β implies by the same calculation that $\pi(\mu^*) = \pi(\mu)^*$ for every $\mu \in M_{**}$. \square

Proposition 5.2. $\Phi \times \Phi$ is involutive on M_{**} .

Proof. Suppose first that M_{**} is dense in M_* . For $\mu \in M_{**}$ and $\omega, v \in \widehat{M}_*$, $(\Phi \times \Phi)_*(\omega \otimes v) = \widehat{\Phi}(\omega)\widehat{\Phi}(v) \in D(S)$ since $\widehat{\Phi}(\widehat{M}_*) \subset D(S)$, and

$$S((\Phi \times \Phi)_*(\omega \otimes v)) = S\widehat{\Phi}(v)S\widehat{\Phi}(\omega) = \widehat{\Phi}(\bar{v})^*\widehat{\Phi}(\bar{\omega})^* = ((\Phi \times \Phi)_*(\bar{\omega} \otimes \bar{v}))^*.$$

By Proposition 5.1, $\Phi \times \Phi$ is involutive.

We have $(\Phi \times \Phi)_{\alpha,\alpha',\beta,\beta'} = \Phi_{\alpha,\beta} \cdot \Phi_{\alpha',\beta'} \in D(R)$ for all $\alpha, \alpha', \beta, \beta'$, moreover $R((\Phi \times \Phi)_{\alpha,\alpha',\beta,\beta'}) = R(\Phi_{\alpha,\beta})R(\Phi_{\alpha',\beta'}) = \Phi_{\alpha',\beta'} \cdot \Phi_{\alpha,\beta} = (\Phi \times \Phi)_{\alpha',\alpha,\beta',\beta}$. This proves the statement. \square

Proposition 5.3. $\widehat{M}_{**} \supset \widehat{\Phi}^{-1}(\widehat{\Phi}(\widehat{M}_*) \cap \widehat{\Phi}(\widehat{M}_*)^*)$.

By universality, $\Phi \times \Phi$ lifts to a $*$ -homomorphism $\widehat{\Delta} : \widehat{M} \rightarrow \widehat{M} \otimes \widehat{M}$, so that $\widehat{\Delta}\Phi = \Phi \times \Phi$ on M_{**}^0 . Since both are non-degenerate, their extension to M_{**} is unique, thus the equality holds on M_{**} ; by continuity, it holds on M_* as well.

Proposition 5.4. $\widehat{\Delta}$ is a comultiplication on \widehat{M} .

Proof. $\widehat{\Delta}$ is involutive by construction. The fact that it is coassociative and unital is proved exactly as in Proposition 6.6 [10]. \square

Comultiplication on \widehat{M} turns \widehat{M}_* , as usual, into a Banach *-algebra.

Proposition 5.5. $\widehat{\Phi}$ is multiplicative.

Proof. For $v, \omega \in \widehat{M}_*$, $\mu \in M_*$ by definition

$$(v \cdot \omega)(\Phi(\mu)) = (v \otimes \omega)(\widehat{\Delta}\Phi(\mu)) = (v \otimes \omega)((\Phi \times \Phi)(\mu)).$$

At the same time, $(v \cdot \omega)(\Phi(\mu)) = \widehat{\Phi}(v \cdot \omega)(\mu)$ and $(v \otimes \omega)((\Phi \times \Phi)(\mu)) = (\widehat{\Phi}(v)\widehat{\Phi}(\omega))(\mu)$. This implies that $\widehat{\Phi}(v \cdot \omega) = \widehat{\Phi}(v)\widehat{\Phi}(\omega)$. \square

Definition 5.6. Let $V \in \widehat{M} \otimes M$ be the unitary generator of Φ . On $D(\widehat{S}) = \Phi(M_*)$, define a map \widehat{S} by $(\widehat{S}\Phi\mu, \omega) = V^*(\omega, \mu)$, $\mu \in M_*$, $\omega \in \widehat{M}_*$.

Proposition 5.7. If $\mu \in M_*$ is such that $\mu \circ S$ extends to a normal functional on M , then $\widehat{S}\widehat{\Phi}(\mu)(\omega) = (\mu \circ S)(\widehat{\Phi}(\omega))$ for all $\omega \in \widehat{M}_*$.

Proof. For any $\mu \in M_*$ and $\omega \in \widehat{M}_*$, we have:

$$\widehat{S}\widehat{\Phi}(\mu)(\omega) = V^*(\omega, \mu) = \overline{V(\bar{\omega}, \bar{\mu})} = \bar{\omega}(\Phi(\bar{\mu})) = \omega(\Phi(\bar{\mu})^*),$$

so that

$$\widehat{S}\widehat{\Phi}(\mu) = \Phi(\bar{\mu})^*. \quad (10)$$

The condition on μ is equivalent to the fact that $\bar{\mu} \in M_{**}$; if it holds, we can continue the calculation above as

$$\widehat{S}\widehat{\Phi}(\mu)(\omega) = \bar{\mu}^*(\widehat{\Phi}(\omega)) = (\mu \circ S)(\widehat{\Phi}(\omega)),$$

which proves the statement. \square

Proposition 5.8. If M_{**} is dense in M_* , then $\widehat{\Phi}(\widehat{M}_*) \subset D(S)$ and $S\widehat{\Phi}(\omega) = \widehat{\Phi}(\bar{\omega})^*$.

Proof. Follows immediately from Proposition 5.1. \square

Suppose that $M \subset B(K)$ and denote by θ the flip map on $K \otimes K$. Let H be another Hilbert space. Recall the leg numbering notation: for $V \in B(H) \otimes M$, we denote $V_{12} = V \otimes \mathbb{I}$ and $V_{13} = (\mathbb{I} \otimes \theta)(V \otimes \mathbb{I})(\mathbb{I} \otimes \theta)$. Exactly as in [4, 1.5.1], one can prove:

Proposition 5.9. Let $\pi : M_* \rightarrow B(H)$ be a completely bounded linear map, and let $V \in B(H) \otimes M$ be such that $V(\omega, \mu) = \omega(\pi(\mu))$ for all $\omega \in B(H)_*$, $\mu \in M_*$. Then

(1) π is multiplicative if and only if

$$(\mathbb{I} \otimes \Delta)(V) = V_{12}V_{13}; \quad (11)$$

(2) π is anti-multiplicative if and only if

$$(\mathbb{I} \otimes \Delta)(V) = V_{13}V_{12}. \quad (12)$$

Proposition 5.10. \widehat{S} is a proper coinvolution on \widehat{M} .

Proof. (1) $D(\widehat{S}) = \Phi(M_*)$ is by definition dense in \widehat{M} .

(2) $D(\widehat{S})$ is closed under multiplication since Φ is a homomorphism. Let V be the generator of Φ . Since Φ is a homomorphism, by Proposition 5.9

$$(\mathbb{I} \otimes \Delta)(V) = V_{12}V_{13}.$$

For V^* we have, since Δ is involutive and $\theta^* = \theta$:

$$(\mathbb{I} \otimes \Delta)(V^*) = (V^*)_{13}(V^*)_{12}.$$

By definition, V^* is the generator of $\widehat{S}\Phi$, thus by the same Proposition $\widehat{S}\Phi$ is anti-multiplicative. It follows that \widehat{S} is anti-multiplicative on its domain.

(3) A direct calculation shows that

$$V^*(\bar{\omega}, \mu) = \overline{V(\omega, \bar{\mu})}$$

for all $\mu \in M_*$, $\omega \in \widehat{M}_*$.

Next,

$$\omega(\Phi(\bar{\mu})^*) = \overline{\omega(\Phi(\bar{\mu}))} = \overline{V(\bar{\omega}, \bar{\mu})} = V^*(\omega, \mu) = \omega(\widehat{S}\Phi(\mu)) \quad (13)$$

It follows that $*\widehat{S}\Phi\mu = \Phi(\bar{\mu})$ for every $\mu \in M_*$. Now we get immediately $*\widehat{S}(*\widehat{S}\Phi\mu) = *\widehat{S}(\Phi\bar{\mu}) = \Phi(\bar{\mu}) = \Phi(\mu)$ and as a consequence $(*\widehat{S})^2 = \mathbb{I}$ as required.

(4) Let $u, v \in \widehat{M}_*$ be such that $u \circ \widehat{S}$ and $v \circ \widehat{S}$ extend to normal functionals on \widehat{M} . We need to show that for every $\mu \in M_*$,

$$\widehat{\Delta}(\Phi(\mu))(v \circ \widehat{S} \otimes u \circ \widehat{S}) = \widehat{\Delta}\widehat{S}\Phi(\mu)(u \otimes v). \quad (14)$$

The left hand side equals to

$$\widehat{\Delta}(\Phi\mu)(v \circ \widehat{S} \otimes u \circ \widehat{S}) = (\Phi \times \Phi)(\mu)(v \circ \widehat{S} \otimes u \circ \widehat{S}) = \mu(\widehat{\Phi}(v \circ \widehat{S}) \cdot \widehat{\Phi}(u \circ \widehat{S})). \quad (15)$$

One checks that for any $\nu \in M_*$

$$\nu(\widehat{\Phi}(u \circ \widehat{S})) = \widehat{S}\Phi(\nu)(u) = V^*(u, \nu) = \overline{V(\bar{u}, \bar{\nu})} = \overline{\bar{\nu}(\widehat{\Phi}(\bar{u}))} = \nu(\widehat{\Phi}(\bar{u})^*).$$

It follows that $\widehat{\Phi}(u \circ \widehat{S}) = \widehat{\Phi}(\bar{u})^*$, and similarly $\widehat{\Phi}(v \circ \widehat{S}) = \widehat{\Phi}(\bar{v})^*$. So we have in (15) (using $\overline{u \otimes v} = \bar{u} \otimes \bar{v}$):

$$\begin{aligned} \widehat{\Delta}(\Phi\mu)(v \circ \widehat{S} \otimes u \circ \widehat{S}) &= \mu(\widehat{\Phi}(\bar{v})^* \cdot \widehat{\Phi}(\bar{u})^*) = \mu((\widehat{\Phi}(\bar{u}) \cdot \widehat{\Phi}(\bar{v}))^*) = \overline{\bar{\mu}(\widehat{\Phi}(\bar{u}) \cdot \widehat{\Phi}(\bar{v}))} \\ &= \overline{(\Phi \times \Phi)(\bar{\mu})(\bar{u} \otimes \bar{v})} = \overline{\widehat{\Delta}(\Phi\bar{\mu})(\bar{u} \otimes \bar{v})} = \widehat{\Delta}(\Phi\bar{\mu})^*(u \otimes v) \\ &= \widehat{\Delta}(\Phi(\bar{\mu})^*)(u \otimes v) = \widehat{\Delta}\widehat{S}\Phi(\mu)(u \otimes v). \end{aligned}$$

□

We have obtained, in particular, that $S\widehat{\Phi}(\omega) = \widehat{\Phi}(\bar{\omega})^*$ for all $\omega \in \widehat{M}_*$.

We have proved that \widehat{M} is a quantum semigroup with involution. With respect to the involution on \widehat{M}_* defined as in Definition 2.4, that is $\omega^*(\Phi(\mu)) = \bar{\omega} \circ \widehat{S}(\Phi(\mu))$ for $\omega \in \widehat{M}_*$, $\mu \in M_*$, the map $\widehat{\Phi}$ is involutive, as (13) shows.

Proposition 5.11. *For $\omega \in \widehat{M}_*$, we have $\omega \in \widehat{M}_{**}$ if and only if $\widehat{\Phi}(\omega)^* \in \widehat{\Phi}(\widehat{M})$. In this case, $\omega^* = \widehat{\Phi}^{-1}(\widehat{\Phi}(\omega)^*)$.*

Proof. Take $\omega \in \widehat{M}_{**}$, $\mu \in M_{**}$. Then:

$$\omega(\widehat{S}\Phi\mu) = \overline{\omega((\widehat{S}\Phi\mu)^*)} = \overline{\omega(\Phi(\bar{\mu}))} = \overline{V(\omega, \bar{\mu})} = \overline{\widehat{V}(\bar{\mu}, \omega)} = \overline{\widehat{\Phi}(\omega)(\bar{\mu})} = \widehat{\Phi}(\omega)^*(\mu).$$

If ω^* is well defined, then the value above is also equal to $\omega^*(\Phi\mu) = \widehat{\Phi}(\omega^*)(\mu)$, what proves the proposition. □

The reasoning above is summarized in

Proposition 5.12. *Let M be a quantum semigroup with involution, such that M_{**} is dense in M_* . Then \widehat{M} is a quantum semigroup with involution.*

5.1. **The case of M_{**} not dense in M_* .** For general M , set $\widehat{M} = \widehat{M}_r$. Proposition 5.12 implies immediately

Theorem 5.13. *Let M be a quantum semigroup with involution. Then \widehat{M} is a quantum semigroup with involution.*

Note however that Φ_M might not be a homomorphism if M_{**} is not dense in M_* .

Proposition 5.14. *If $\widehat{M} \neq \{0\}$ then $\widehat{\Phi} : \widehat{M}_* \rightarrow ZM$ is unitary.*

Proof. Since $Z : M_r \rightarrow M$ is a *-homomorphism, $\widehat{\Phi}_M = Z_M \widehat{\Phi}_{M_r}$ is a *-homomorphism even if $M \neq M_r$. If $V_r \in \widehat{M} \otimes M_r$ is the generator of Φ_{M_r} (which is unitary), then $\widehat{V} = (Z_M \otimes 1)(\theta(V_r)) \in (Z_M M) \otimes \widehat{M}$, with the flip θ , is the generator of $\widehat{\Phi}_M$. □

Proposition 5.15. *$\widehat{\Phi}$ is injective.*

Proof. If $\widehat{\Phi}(\omega_1) = \widehat{\Phi}(\omega_2)$ for some $\omega_1, \omega_2 \in \widehat{M}_*$, then $\omega_1(\Phi(\mu)) = \omega_2(\Phi(\mu))$ for every $\mu \in M_*$, so that $\omega_1 = \omega_2$ on $\Phi(M_*)$. By weak density of $\Phi(M_*)$ in \widehat{M} , it follows that $\omega_1 = \omega_2$. \square

5.2. The axiom of the antipode. In the theory of Hopf algebras, the antipode S of a Hopf algebra \mathcal{M} satisfies the axiom

$$\mathfrak{m}(\mathbb{I} \otimes S)\Delta = \mathfrak{m}(S \otimes \mathbb{I})\Delta = \varepsilon 1, \quad (16)$$

where $\mathfrak{m} : \mathcal{M} \otimes \mathcal{M} \rightarrow \mathcal{M}$ is the multiplication and $\varepsilon : \mathcal{M} \rightarrow \mathbb{C}$ the counit.

Suppose that $\{0\} \neq M = \widehat{N}$ and set $\mathcal{M} = \Phi_N(N_*)$. On \mathcal{M} , a counit is well defined (and satisfies the usual axioms of a counit): $\varepsilon(\Phi(\nu)) = \nu(1_N)$, $\nu \in N_*$. Suppose that $N \subset B(H)$ is in its standard form, then every $\nu \in N_*$ has form $\nu = \nu_{xy} : a \mapsto \langle ax, y \rangle$, $a \in N$, $x, y \in H$. In this case, $\varepsilon(\nu_{xy}) = \langle x, y \rangle$. For every $\nu \in N_*$, $\Phi(\nu)$ is a coefficient of the unitary representation $\widehat{\Phi}_N : M_* \rightarrow N$. If we fix a basis (e_α) in H , then in particular $\Phi(\nu_{e_\alpha, e_\beta}) = \widehat{\Phi}_{\beta\alpha}$ in the notations 4.2 for every α, β .

Since Φ_N is a unitary representation of N_* , the formula (5) implies (by decomposing x and y in the basis) that

$$\sum_{\gamma} \Phi(\nu_{y, e_\gamma})^* \Phi(\nu_{x, e_\gamma}) = \langle x, y \rangle 1_M = \varepsilon(\nu_{xy}) 1_M. \quad (17)$$

At the same time,

$$\Delta(\Phi(\nu_{xy})) = \sum_{\gamma} \Phi(\nu_{e_\gamma, y}) \otimes \Phi(\nu_{x, e_\gamma}).$$

by Proposition 3.4. Together with $S_M \Phi(\nu_{e_\gamma, y}) = \Phi(\nu_{y, e_\gamma})^*$ which is valid by (10), the equality (17) takes form

$$\mathfrak{m}(S \otimes \mathbb{I})\Delta(a) = \varepsilon(a) 1_M$$

for $a = \nu_{xy}$. Similarly, one arrives at the other equality in (16). We see that the axiom of the antipode is satisfied for all $a \in \mathcal{M}$, in the sense described above. This suggests that every dual $M = \widehat{N}$ is a quantum *group* and not just a semigroup.

6. MORPHISMS AND SECOND DUALS

In this section we do not suppose that M_{**} is dense in M_* .

Definition 6.1. Let \mathcal{QST} be the category of quantum semigroup with involution. A morphism in \mathcal{QST} is a normal $*$ -homomorphism $\varphi : M \rightarrow N$ such that: $\Delta_N \circ \varphi = (\varphi \otimes \varphi)\Delta_M$, $\varphi(D(S_M)) \subset D(S_N)$ and $S_N \circ \varphi = \varphi \circ S_M$. Note that we do not require that $\varphi(1) = 1$.

Proposition 6.2. *If $\varphi : M \rightarrow N$ is a morphism in \mathcal{QST} , then $\varphi_* : N_* \rightarrow M_*$ is a $*$ -homomorphism. In particular, $\varphi_*(N_{**}) \subset M_{**}$.*

Proof. By definition, φ is ultraweakly continuous, so it has a pre-adjoint $\varphi_* : N_* \rightarrow M_*$. Since φ is a coalgebra morphism, it is standard to show that φ_* is a homomorphism. It remains to prove that φ_* is involutive.

For any $\nu \in N_*$, $\overline{\varphi_*(\nu)} = \varphi_*(\bar{\nu})$, since for any $x \in M$

$$\overline{\varphi_*(\nu)}(x) = \overline{\varphi_*(\nu)(x^*)} = \overline{\nu(\varphi(x^*))} = \overline{\nu(\varphi(x)^*)} = \bar{\nu}(\varphi(x)) = \varphi_*(\bar{\nu})(x).$$

If moreover $\nu \in N_{**}$, then for any $x \in D(S_M)$

$$\overline{\varphi_*(\bar{\nu})} \circ S_M(x) = \varphi_*(\bar{\nu})(S_M(x)) = \bar{\nu}(\varphi(S_M(x))) = \bar{\nu}(S_N \varphi(x)) = \nu^*(\varphi(x)) = \varphi_*(\nu^*)(x).$$

This calculation shows that $\varphi_*(\nu)^*$ is well defined and equals $\varphi_*(\nu^*)$. \square

Below we use the notations of Proposition 2.19.

Proposition 6.3. *If $\varphi : M \rightarrow N$ is a morphism in \mathcal{QST} , then there exists a morphism $\varphi_r : M_r \rightarrow N_r$ such that $\varphi_r Q_M = Q_N \varphi$.*

Proof. Recall that $\ker Q_M = (M_{**})^\perp$, $\ker Q_N = (N_{**})^\perp$. By Proposition 6.2, $\varphi_*(N_{**}) \subset M_{**}$. It follows immediately that $\varphi(\ker Q_M) \subset \ker Q_N$, so that φ_r which satisfies the equality $\varphi_r Q_M = Q_N \varphi$ is well defined. It is immediate to check that it is normal.

The fact that φ_r is a coalgebra morphism follows from the calculation below:

$$\begin{aligned} \Delta_{N_r} \varphi_r Q_M &= \Delta_{N_r} Q_N \varphi = (Q_N \otimes Q_N) \Delta_N \varphi = (Q_N \otimes Q_N) (\varphi \otimes \varphi) \Delta_M \\ &= (\varphi_r Q_M \otimes \varphi_r Q_M) \Delta_M = (\varphi_r \otimes \varphi_r) \Delta_{M_r} Q_M. \end{aligned}$$

For the coinvolution domains we have:

$$\varphi_r(D(S_{M_r})) = \varphi_r(Q_M(D(S_M))) = Q_N \varphi(D(S_M)) \subset Q_N D(S_N) = D(S_{N_r}),$$

and moreover

$$\varphi_r S_{M_r} Q_M = \varphi_r Q_M S_M = Q_N \varphi S_M = Q_N S_N \varphi = S_{N_r} Q_N \varphi = S_{N_r} \varphi_r Q_M. \quad \square$$

Proposition 6.4. *Let $\varphi : M \rightarrow N$ be a morphism in \mathcal{QSL} . Then there is a dual morphism $\widehat{\varphi} : \widehat{N} \rightarrow \widehat{M}$ such that $\widehat{\varphi} \circ \Phi_N = \Phi_M \circ \varphi_*$ on N_{**}^0 . If $\widehat{N} \neq \{0\}$, then the equality holds on N_{**} .*

Moreover, $\widehat{\varphi}_r = \widehat{\varphi}$.

Proof. The statement is trivial if $\widehat{N} = \{0\}$ or $\widehat{M} = \{0\}$ so we can assume that $N_{**}^0 \neq \{0\}$ and $\widehat{M} \neq \{0\}$.

Consider $\Phi_M \circ \varphi_* : N_{**}^0 \rightarrow \widehat{M}$. Since Φ_M, φ_* are *-homomorphisms on M_{**} and N_{**} respectively and $\varphi_*(N_{**}^0) \subset M_{**}$, their composition is a *-homomorphism on N_{**}^0 , so it is lifted to a normal *-homomorphism $\widehat{\varphi} : \widehat{N} \rightarrow \widehat{M}$ such that $\widehat{\varphi} \Phi_N = \Phi_M \varphi_*$ (on N_{**}^0). Both maps are in fact defined on N_* ; by Proposition 4.17, the equality holds on N_{**} , and if N_{**} is dense in N_* , on N_* .

Now we can prove that $\widehat{\varphi} = \widehat{\varphi}_r$. By density, it is sufficient to prove that $\widehat{\varphi} \Phi_{N_r} = \widehat{\varphi}_r \Phi_{N_r}$ on $(N_r)_{**}$, what is done as follows (recall Notations 2.18 and Definition 4.16):

$$\begin{aligned} \widehat{\varphi} \Phi_{N_r} &= \widehat{\varphi} \Phi_{N_r} Z_{N_*} Q_{N_*} = \widehat{\varphi} \Phi_N Q_{N_*} = \Phi_M \varphi_* Q_{N_*} \\ &= \Phi_M Q_{M_*} \varphi_{r*} = \Phi_{M_r} Z_{M_*} Q_{M_*} \varphi_{r*} = \Phi_{M_r} \varphi_{r*} = \widehat{\varphi}_r \Phi_{N_r}. \end{aligned}$$

From now on we can suppose that $\varphi = \varphi_r$, that is, M_{**} is dense in M_* and N_{**} is dense in N_* .

Next we prove that $\widehat{\varphi}$ is a coalgebra morphism: $\Delta_{\widehat{M}} \widehat{\varphi} = (\widehat{\varphi} \otimes \widehat{\varphi}) \Delta_{\widehat{N}}$. Since $\Delta_{\widehat{M}}$ is ultraweakly continuous, this equality is enough to check on $\Phi_N(N_{**}^0)$. Moreover, to check an equality in $\widehat{M} \otimes \widehat{M}$ where $\widehat{\Delta}_M$ takes its values, it is enough to consider evaluations on $x \otimes y$, with $x, y \in \widehat{M}_*$. We have, with any $\nu \in N_{**}^0$:

$$\begin{aligned} \Delta_{\widehat{M}} \widehat{\varphi}(\Phi_N(\nu))(x \otimes y) &= \widehat{\varphi}(\Phi_N(\nu))(xy) = \Phi_M(\varphi_*(\nu))(xy) = \varphi_*(\nu)(\widehat{\Phi}_M(x) \widehat{\Phi}_M(y)) \\ &= \nu(\varphi(\widehat{\Phi}_M(x)) \varphi(\widehat{\Phi}_M(y))). \end{aligned} \quad (18)$$

By definition, $\Delta_{\widehat{N}} \Phi_N = \Phi_N \times \Phi_N$, so from the other side:

$$(\widehat{\varphi} \otimes \widehat{\varphi}) \Delta_{\widehat{N}}(\Phi_N(\nu))(x \otimes y) = (\Phi_N \times \Phi_N)(\nu)(\widehat{\varphi}_* x \otimes \widehat{\varphi}_* y) = \nu(\widehat{\Phi}_N(\widehat{\varphi}_* x) \cdot \widehat{\Phi}_N(\widehat{\varphi}_* y)).$$

Recalling that $\varphi \widehat{\Phi}_M = \widehat{\Phi}_N \widehat{\varphi}_*$, we arrive at the required equality.

It remains to check the equality $S_{\widehat{M}} \widehat{\varphi} = \widehat{\varphi} S_{\widehat{N}}$. By definition $D(S_{\widehat{N}}) = \Phi_N(N_*)$; for $\xi = \Phi_N(\nu)$, $\nu \in N_*$, we have $\widehat{\varphi}(\xi) = \widehat{\varphi}(\Phi_N(\nu)) = \Phi_M(\varphi_*(\nu)) \in \Phi_M(M_*) = D(S_{\widehat{M}})$. Then

$$S_{\widehat{M}} \widehat{\varphi}(\xi) = S_{\widehat{M}} \Phi_M(\varphi_*(\nu)),$$

and by Definition 5.6, for any $\omega \in \widehat{M}$:

$$\begin{aligned} \omega(S_{\widehat{M}} \Phi_M(\varphi_*(\nu))) &= V_M^*(\omega, \varphi_*(\nu)) = \overline{V_M(\bar{\omega}, \varphi_*(\nu))} = \overline{V_M(\bar{\omega}, \varphi_*(\bar{\nu}))} \\ &= \overline{\omega(\Phi_M(\varphi_*(\bar{\nu})))} = \overline{\omega(\widehat{\varphi}(\Phi_N(\bar{\nu})))} = \overline{\widehat{\varphi}_*(\bar{\omega})(\Phi_N(\bar{\nu}))} \\ &= \overline{V_N(\widehat{\varphi}_*(\bar{\omega}), \bar{\nu})} = \overline{V_N(\widehat{\varphi}_*(\omega), \bar{\nu})} = V_N^*(\widehat{\varphi}_*(\omega), \nu) \\ &= \widehat{\varphi}_*(\omega)(S_{\widehat{N}} \Phi_N(\nu)) = \omega(\widehat{\varphi} S_{\widehat{N}}(\Phi_N(\nu))) = \omega(\widehat{\varphi} S_{\widehat{N}}(\xi)). \end{aligned}$$

It follows that $S_{\widehat{M}} \widehat{\varphi}(\xi) = \widehat{\varphi} S_{\widehat{N}}(\xi)$ as required. \square

Remark 6.5. If we have two QSL -morphisms $\varphi : M \rightarrow N$ and $\psi : N \rightarrow L$, then on L_{**}^0 we have:

$$\begin{aligned}\widehat{\psi \circ \varphi} \circ \Phi_L &= \Phi_M \circ (\psi \circ \varphi)_* = \Phi_M \circ \varphi_* \circ \psi_*, \\ \widehat{\varphi} \circ \widehat{\psi} \circ \Phi_L &= \widehat{\varphi} \circ \Phi_N \circ \psi_*.\end{aligned}$$

It might happen that $\psi_*(L_{**}^0)$ is not a subset of N_{**}^0 so we cannot continue the last line as $\Phi_M \circ \varphi_* \circ \psi_*$, in general. If $\widehat{N} \neq \{0\}$, then we can use the fact that $\widehat{\varphi} \circ \Phi_N = \Phi_M \circ \varphi_*$ on N_{**} and conclude that both displayed lines are equal, what implies $\widehat{\psi \circ \varphi} = \widehat{\varphi} \circ \widehat{\psi}$. If $\widehat{N} = \{0\}$, then necessarily $\widehat{\varphi} \circ \widehat{\psi} = 0$ but it might happen that $\widehat{\psi \circ \varphi} \neq 0$.

Proposition 6.6. *Let \mathbb{I}_M be the identity morphism of M , then $\widehat{\mathbb{I}}_M = \mathbb{I}_{\widehat{M}}$.*

Proof. One checks trivially that $(\mathbb{I}_M)_* = \mathbb{I}_{M^*}$. Next, by definition $\widehat{\mathbb{I}}_M \Phi_M = \Phi_M(\mathbb{I}_M)_*$, and then $\widehat{\mathbb{I}}_M \Phi_M = \Phi_M$ on M_{**}^0 . Since $\Phi_M(M_{**}^0)$ is dense in \widehat{M} , it follows that $\widehat{\mathbb{I}}_M = \mathbb{I}_{\widehat{M}}$. \square

Proposition 6.7. *For every M , there is a normal $*$ -homomorphism $D_M : \widehat{\widehat{M}} \rightarrow M$ such that $D_M \circ \Phi_{\widehat{M}}(x) = \widehat{\Phi}_M(x)$ for all $x \in \widehat{M}_{**}^0$. If $\widehat{M} \neq \{0\}$, then this equality holds actually on \widehat{M}_{**} . If M_{**} is dense in M_* , then D_M is a coalgebra morphism and thus a morphism in QSL .*

Proof. If $\widehat{M} = \{0\}$, the statement holds for $D_M = 0$ which is a morphism; below we suppose that $\widehat{M} \neq \{0\}$.

Denote $N = \widehat{M}$. The canonical map $\widehat{\Phi}_M : N_* \rightarrow M$ is a $*$ -homomorphism. In particular, $\widehat{\Phi}_M|_{N_*^0}$ is a $*$ -homomorphism, so it is extended uniquely to a normal homomorphism of its von Neumann envelope: $D_M : \widehat{N} \rightarrow M$. By definition, D_M satisfies the equality in the statement for $x \in N_{**}^0 = \widehat{M}_{**}^0$. If $\widehat{M} \neq \{0\}$, then by Propositions 5.14, 4.8 and 4.17 this equality holds also on \widehat{M}_{**} .

Similarly, there exists a unique $D_{M_r} : \widehat{\widehat{M}} \rightarrow M_r$ such that $D_{M_r} \circ \Phi_{\widehat{M}} = \widehat{\Phi}_{M_r}$ on \widehat{M}_{**} . Recalling that $\widehat{\Phi}_M = Z_M \widehat{\Phi}_{M_r}$, we infer that $D_M = Z_M D_{M_r}$.

To prove the last statement, we can suppose that M_{**} is dense in M_* and $Z_M = 1$. In this case $D_M \circ \Phi_{\widehat{M}} = \widehat{\Phi}_M$ on \widehat{M}_* , and the preadjoint map $(\widehat{\Phi}_M)_* : M_* \rightarrow \widehat{M}$, which is by definition Φ_M , equals also to $(\Phi_{\widehat{M}})_* \circ (D_M)_* = \widehat{\Phi}_{\widehat{M}} \circ (D_M)_*$. Also, Φ_M is a homomorphism.

To prove that D_M is a morphism of coinvolutive Hopf-von Neumann algebras, one should check the equality

$$\Delta_M D_M(x)(\mu \otimes \nu) = (D_M \otimes D_M) \Delta_{\widehat{N}}(x)(\mu \otimes \nu)$$

for every $x \in \widehat{N}$ and $\mu, \nu \in M_*$. By density, it is sufficient to consider $x = \Phi_N(y)$ with $y \in N_{**}^0$. Then we have:

$$\begin{aligned}\Delta_M D_M \Phi_N(y)(\mu \otimes \nu) &= \Delta_M \widehat{\Phi}_M(y)(\mu \otimes \nu) = \widehat{\Phi}_M(y)(\mu\nu) = y(\Phi_M(\mu\nu)) \\ &= y(\Phi_M(\mu)\Phi_M(\nu)) = y(\widehat{\Phi}_N((D_M)_*(\mu)) \cdot \widehat{\Phi}_N((D_M)_*(\nu))) \\ &= \Phi_N(y)((D_M)_*(\mu) \cdot (D_M)_*(\nu)) = \Delta_{\widehat{N}}(x)((D_M)_*(\mu) \otimes (D_M)_*(\nu)) \\ &= (D_M \otimes D_M)(\Delta_{\widehat{N}}(x))(\mu \otimes \nu).\end{aligned}$$

To verify that D_M agrees with the coinvolutions, let $y \in \widehat{M}_*$. Then $D_M \Phi_{\widehat{M}}(y) = \widehat{\Phi}_M(y) \in D(S_M)$ by Proposition 5.8, and $S_M(\widehat{\Phi}_M(y)) = \widehat{\Phi}_M(\bar{y})^*$ by formula (10). From the other side,

$$D_M S_{\widehat{M}} \Phi_{\widehat{M}}(y) = D_M(\Phi_{\widehat{M}}(\bar{y})^*) = (D_M \Phi_{\widehat{M}}(\bar{y}))^* = (\widehat{\Phi}_M(\bar{y}))^* = S_M(\widehat{\Phi}_M(y)) = S_M D_M \Phi_{\widehat{M}}(y).$$

This proves that $S_M D_M = D_M S_{\widehat{M}}$ on $\Phi_{\widehat{M}}(\widehat{M}_*)$. If $S_{\widehat{M}}$ is extended by Proposition 2.16, then by continuity (Proposition 2.12) we still have equality on the whole of $D(S_{\widehat{M}})$. \square

As it was shown already in the case of a bounded S in [10], the map D_M need not be neither injective nor surjective. For the algebra M from the example 5.10 of [10], $\widehat{\widehat{M}} = 0$, so $D_M = 0$. For a second example, take $M = L_\infty(G)$. Then $\widehat{\widehat{M}} = C_0(G)^{**}$, and D_M is a quotient map but is not injective.

If M is a dual of another algebra, then D_M is right invertible:

Proposition 6.8. *If $M = \widehat{N}$ for some N and $\widehat{M} \neq \{0\}$, then there is a QSL -morphism $E_M : M \rightarrow \widehat{M}$, such that $D_M \circ E_M = \text{id}_M$.*

Proof. Since $M = \widehat{N} = \widehat{N}_r$, we can assume that $N = N_r$. By Proposition 6.7 applied to N , since $\widehat{M} \neq \{0\}$, there is a QSL -morphism $D_N : \widehat{N} \rightarrow N$ such that $D_N \circ \Phi_{\widehat{N}}(x) = \widehat{\Phi}_N(x)$ for all $x \in \widehat{N}_{**} = M_{**}$. By Proposition 6.4, there exists a dual morphism $E_M = \widehat{D}_N : \widehat{N} = M \rightarrow (\widehat{N})^\wedge = \widehat{M}$.

By assumption $\widehat{M} = \widehat{N} \neq 0$, and in this case it was proved in Proposition 6.7 that $\Phi_N = \widehat{\Phi}_M \circ (D_N)_*$ on N_* . Moreover, since $\widehat{M} \neq \{0\}$, we have $D_M \circ \Phi_{\widehat{M}} = \widehat{\Phi}_M$ on \widehat{M}_{**} .

By Proposition 6.4, since $M \neq \{0\}$, we have $E_M \circ \Phi_N = \Phi_{\widehat{M}} \circ (D_N)_*$ on N_{**} . Then on N_{**} ,

$$D_M \circ (E_M \circ \Phi_N) = (D_M \circ \Phi_{\widehat{M}}) \circ (D_N)_* = \widehat{\Phi}_M \circ (D_N)_* = \Phi_N;$$

note that $(D_N)_*(N_{**}) \subset \widehat{M}_{**}$ by Proposition 6.2. Thus, $D_M \circ E_M = \text{id}_M$ on $\Phi_N(N_{**})$. Since this latter is weakly dense in M , this equality holds everywhere. \square

Corollary 6.9. *If $M = \widehat{N}$ for some N and $\widehat{M} \neq \{0\}$, then M_{**} is dense in M_* .*

Proof. In Proposition 6.7 it was proved that $D_M = Z_M D_{M_r}$. By Proposition 6.8, D_M is surjective. This implies that $\zeta_M = 1_M$ and M_{**} is dense in M_* . \square

Proposition 6.10. *If $M = \widehat{N}$ for some N and $\widehat{M}^{(4)} \neq \{0\}$ then E_M is unital.*

Proof. It is clear that $E_M(1) = p = p^2 = p^*$. Suppose that $p \neq 1$. Then $E_M(M)$ is contained in the proper weakly closed $*$ -subalgebra $I := p\widehat{M}p$. Consider $\pi = \Phi_M \circ E_{M_*} : \widehat{M}_* \rightarrow \widehat{M}$. Since M_{**} is dense in M_* , Φ_M is a $*$ -homomorphism; by Proposition 6.2, π is a $*$ -representation. For $\omega \in \widehat{M}_*$, $\omega(\pi(\mu)) = \Phi_M \circ E_{M_*}(\mu)(\omega) = \mu(E_M \circ \widehat{\Phi}_M(\omega))$, so $\pi_*(\omega) = E_M \circ \widehat{\Phi}_M(\omega)$ and the space of coefficients of π is contained in the subalgebra I . Then the equality (5) cannot hold, so π is not unitary. By Corollary 6.9, \widehat{M}_{**} is dense in \widehat{M}_* , so we can apply Proposition 4.10; it follows that π is degenerate on \widehat{M}_*^0 , or equivalently $\pi(\widehat{M}_*)$ is not contained in the weak closure of $\pi(\widehat{M}_{**}^0)$. In particular, $\pi(\widehat{M}_{**}^0)$ is not weakly dense in \widehat{M} .

Consider now $\widehat{E}_M : \widehat{M} \rightarrow \widehat{M}$. Since $\widehat{M} \neq \{0\}$, by Proposition 6.4 $\widehat{E}_M \Phi_{\widehat{M}} = \Phi_M E_{M_*} = \pi$ on \widehat{M}_{**} . and since \widehat{E}_M is weakly continuous, $\widehat{E}_M(\widehat{M})$ is contained in the closure of $\widehat{E}_M(\Phi_{\widehat{M}}(\widehat{M}_{**}^0)) = \pi(\widehat{M}_{**}^0)$, so \widehat{E}_M is not surjective. From the other side, since $\widehat{M} \neq \{0\}$, we have by Remark 6.5 equality $\widehat{E}_M \widehat{D}_M = (D_M E_M)^\wedge = \widehat{\mathbb{1}}_M = \mathbb{I}_{\widehat{M}}$ so \widehat{E}_M must be surjective. This contradiction proves that E_M is in fact unital. \square

Proposition 6.11. *$(D_M)_*(M_*)$ is a two-sided module over \widehat{M} .*

Proof. We write further D, E for D_M, E_M respectively. First, $D_*(M_*) = (\ker D)_\perp$ in \widehat{M}_* : For $\omega \in \widehat{M}_*$ if $\omega = D_*(\mu) \in D_*(M_*)$ then $\omega(x) = \mu(Dx) = 0$ for $x \in \ker D$; if $\omega \in (\ker D)_\perp$ then set $\mu = E_*(\omega)$; for $x \in \widehat{M}$, as $x - EDx \in \ker D$, we have $\omega(x) = \omega(EDx) = \mu(Dx)$, so that $\omega = D_*(\mu) \in D_*(M_*)$.

Now $(x.\omega)(y) = \omega(yx) = 0$ and $(\omega.x)(y) = \omega(xy) = 0$ for all $\omega \in D_*(M_*)$, $x \in \widehat{M}$, $y \in \ker D$, what proves the proposition. \square

7. LOCALLY COMPACT QUANTUM GROUPS

Every von Neumann algebraic quantum group is a quantum semigroup with involution in our definition (see for example [9]). Let us compare our construction with the universal dual of Kustermans [7].

The Kustermans' universal quantum group A_u is defined for a C^* -algebraic locally compact quantum group A . If M is a von Neumann algebraic locally compact quantum group, then [9] it contains

a canonically defined C^* -subalgebra A which is a C^* -algebraic locally compact quantum group with inherited structure.

The space $L^1(A)$ used as a starting point in [7] is isomorphic to M_* [8, p.913]. The subspace $L_*^1(A)$ which carries the involution is exactly our M_{**} , and is dense in M_* [7, p.303]. By [7, Corollary 4.3], for every non-degenerate $*$ -representation $\pi : L_*^1(A) \rightarrow B(H)$ there exists a unitary $U \in B(H) \otimes M(A)$ such that $\pi(\mu) = (\mathbb{I} \otimes \mu)(U)$ for all $\mu \in L_*^1(A)$. Since $M(A) \subset A'' = M$, it follows that $U \in B(H) \otimes M$ and π extends to a unitary representation of M_* .

This implies that $M_{**} = M_*^\times = M_{**}^0 = L_*^1(A)$, and $\widehat{A}_u = C^*(L_*^1(A)) = C^*(M_{**}^0)$. Our dual algebra $\widehat{M} = \widehat{A}_u^{**}$ is the enveloping von Neumann algebra of the universal dual \widehat{A}_u .

Theorem 7.1. *Let A be a C^* -algebraic locally compact quantum group and $M = A''$ its associated von Neumann algebraic locally compact quantum group. Then:*

- (1) $\widehat{M} = \widehat{A}_u^{**}$;
- (2) $\widehat{\widehat{M}} = A_u^{**}$.

In particular, $\widehat{\widehat{\widehat{M}}} = \widehat{M}$.

Proof. (1) follows from the discussion above.

(2) Since $\widehat{\widehat{M}} = C^*(\widehat{M}_{**}^0)^{**}$ and $A_u = C^*(L_1^*(\widehat{A}))$, the question is to show that $C^*(\widehat{M}_{**}^0) = C^*(L_1^*(\widehat{A}))$. The epimorphism $\widehat{\pi} : \widehat{A}_u \rightarrow \widehat{A}$ [7, 2.15] generates an epimorphism $\widehat{\pi}^{**} : \widehat{A}_u^{**} \rightarrow \widehat{A}^{**}$ and an imbedding $\widehat{\pi}^* : \widehat{A}^* \rightarrow \widehat{A}_u^* = \widehat{M}_*$.

The equality $(i \otimes \widehat{\Delta}_u)(\widehat{V}) = \widehat{V}_{13}\widehat{V}_{12}$ [7, 3.5] implies that λ_u^* is anti-multiplicative. Let us prove now that $\widehat{\pi}^*(\widehat{A}^*)$ is an ideal in \widehat{A}_u^* . It is sufficient to show that for any $\alpha \in \widehat{A}^*$ and any representation $\rho : \widehat{A}_u \rightarrow B(H)$ and every $\omega \in B(H)_*$ $\widehat{\pi}^*(\alpha)\rho^*(\omega) \in \widehat{\pi}^*(\widehat{A}^*)$. For $\mu \in L_1^*(A)$:

$$(\widehat{\pi}^*(\alpha)\rho^*(\omega))(\lambda_u(a)) = \lambda_u^*(\widehat{\pi}^*(\alpha)\rho^*(\omega))(a) = (\lambda_u^*\rho^*(\omega)\lambda_u^*\widehat{\pi}^*(\alpha))(a) = (\lambda_u^*\rho^*(\omega)\lambda^*(\alpha))(a) = \lambda^*(\xi)(a),$$

since $\lambda^*(\widehat{A}^*)$ is an ideal in A .

Every non-degenerate $*$ -representation of $L_1^*(\widehat{A})$ on a Hilbert space H is extended to a unitary representation of $L_1(\widehat{A})$ with a generator $U \in B(H) \otimes M(\widehat{A})$ [7, Corollary 4.3]. By [7, Proposition 3.13], there is unique $V_u \in B(H) \otimes M(\widehat{A}_u)$ such that $(\widehat{\pi}^{**} \otimes \mathbb{I})(V_u) = U$ and the corresponding map of $L_1(\widehat{A}_u)$ is a $*$ -representation. Inclusion $M(\widehat{A}_u) \subset \widehat{A}_u^{**} = \widehat{M}$ implies that $V \in B(H) \otimes \widehat{M}$, and $(\mathbb{I} \otimes \widehat{\pi}^{**})(V_u) = U$. Thus, every non-degenerate $*$ -representation of $\widehat{\pi}^*(L_1^*(\widehat{A}))$ is extended to a unitary representation of \widehat{M}_* , and since such the extension is unique, it follows that every non-unitary representation vanishes on $L_1^*(\widehat{A})$ and $L_1^*(\widehat{A}) \subset \widehat{M}_{**}^\times$.

From the other side, if $\rho : \widehat{M}_* \rightarrow B(H)$ vanishes on $L_1^*(\widehat{A})$ then its coefficients are contained in $L_1^*(\widehat{A})^\perp = L_1(\widehat{A})^\perp \subset \widehat{M}$. Let us show that $L_1(\widehat{A})^\perp$ (recall that we identify $L_1(\widehat{A})$ with $\widehat{\pi}^*(L_1(\widehat{A}))$) is a proper weakly closed ideal, then it will follow that ρ is non-unitary.

The epimorphism $\widehat{\pi} : \widehat{A}_u \rightarrow \widehat{A}$ is extended, by the universality property, to a $*$ -homomorphism $\tilde{\pi} : \widehat{M} = \widehat{A}_u^{**} \rightarrow \widehat{A}''$. Then, since $L_1(\widehat{A}) = (\widehat{A}'')_*$, one has $(\tilde{\pi}^*(L_1(\widehat{A})))^\perp = \ker \tilde{\pi}$. Since $\mu(\tilde{\pi}(x)) = \mu(\widehat{\pi}(x))$ for $x \in \widehat{A}_u$ and $\mu \in L^1(\widehat{A})$, by weak density of \widehat{A}_u in \widehat{M} it follows that $\tilde{\pi}^*(\mu) = \widehat{\pi}^*(\mu)$, so that $\widehat{\pi}^*(L_1(\widehat{A}))^\perp = \ker \tilde{\pi}$. This proves (2). \square

Theorems 7.4 and 7.6 of [10] are valid for our case also. The proofs are identical, with replacement of M_*^0 by M_{**}^0 , and adding the remark of Proposition 5.1. We will not repeat the proofs and just state the results:

Theorem 7.2. *Let $M \neq \{0\}$ be commutative and $M \simeq \widehat{N}$ for some N . Then there is a locally compact group G such that $M \simeq C_0(G)^{**}$.*

Corollary 7.3. *If $\{0\} \neq M \simeq \widehat{\widehat{M}}$ and M is commutative, then $M \simeq C_0(G)^{**}$ for a locally compact group G .*

Theorem 7.4. *Let M be cocommutative (i.e. M_* is commutative) and $M \simeq \widehat{N}$ for some N . Then, if $\widehat{M} \neq \{0\}$, there is a locally compact group G such that $M \simeq W^*(G)$.*

8. EXAMPLES

Outside the class of locally compact quantum groups, some examples of algebras and their duals are given in [10, Examples 5.10–5.12, 8.8]. The present construction extends the one of [10], so the same examples are valid for it. They give several algebras M , commutative or not, such that \widehat{M} is $\{0\}$ or \mathbb{C} . In [10, Example 5.11], the structure of $L^\infty(\mathbb{R}^2)$ is changed in such a way that its dual becomes isomorphic to $L^\infty(\mathbb{R})$. This demonstrates the idea that our duality cuts out the “non-unitary” part of a given algebra, and leaves the “unitary” one.

Below are presented some more examples. In every one constructed up to now, the dual \widehat{M} coincides with the dual of some locally compact quantum group.

Example 8.1. Let B be the quantum semigroup $C(\widetilde{S}_N^+)$ defined by Banica and Skalski [1]. Recall that it is defined starting with a “submagic” $N \times N$ matrix $u = (u_{ij})$ with entries in a unital C^* -algebra A . Being ‘submagic’ means that $u_{ij} = u_{ij}^* = u_{ij}^2$ for every i, j , and $u_{ij}u_{ik} = u_{ji}u_{ki} = 0$ for every i if $j \neq k$. By definition, B is the universal unital C^* -algebra with the relations above. The authors show that B admits a comultiplication (a unital coassociative $*$ -homomorphism) defined by the formula $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$, and a “sub-coinvolution” (everywhere defined $*$ -antihomomorphism) defined by $S(u_{ij}) = u_{ji}$.

Let M be the enveloping von Neumann algebra of B , then Δ and S extend obviously to normal maps on M . It is immediate to verify that S is a proper coinvolution on M . Since S is bounded, $M_{**} = M_*$. The elements (u_{ij}) are coefficients of a $*$ -representation of M_* (by Theorem 3.1 for example). One shows easily that it is irreducible.

However, u is clearly non-unitary, thus every u_{ij} belongs to the annihilator of M_{**}^\times (see Definition 4.5). Since (u_{ij}) generate B , by Definition 4.6 $M_{**}^0 = \{0\}$ and $\widehat{M} = \{0\}$.

Example 8.2. X. Li [11] defines a reduced C^* -algebra $C_r^*(P)$ of a discrete left cancellative semigroup P , using its regular representation on $\ell^2(P)$. The algebra $C_r^*(P)$ is generated by the translation operators T_p , $p \in P$, and their adjoints $T_p^* \in B(\ell^2(P))$. As a linear space, $C_r^*(P)$ is generated by $E_X L_g$, where X is an ideal in P , E_X is the operator of multiplication by the characteristic function of X , and $g = p_1^{\pm 1} \dots p_n^{\pm 1}$ with $p_j \in P$.

Set $M = C_r^*(P)^{**}$. Li does not define a coinvolution on his algebra. In order that it fits into our assumptions, set $S(T_p) = T_p^*$, and accordingly $S(T_p^*) = T_p$, $p \in P$, and extend it as a linear anti-homomorphism onto the algebra generated by these elements. For every $f \in \ell^2(P)$ and every $g_j \in G$, $X_j \subset P$, $j = 1, \dots, n$ the function $\sum \lambda_j E_{X_j} L_{g_j} f$ is the complex conjugate of $\sum \lambda_j E_{X_j} L_{g_j} \bar{f}$, and as $\|f\| = \|\bar{f}\|$, this implies that S is isometric (and bounded) on $C_r^*(P)$.

By definition $\Delta(T_p) = T_p \otimes T_p$, so that every T_p is a character of M_* . With the coinvolution above, it is involutive. It is unitary as an element of M if and only if p is invertible in P .

By construction, $\Delta(E_X) = E_X \otimes E_X$ for an ideal X in P , so that it is also a character, which is clearly unitary only if $X = \emptyset$ or $X = P$. It follows that M_{**}^\times is the linear dual space of $\text{lin}\{L_g : g \in P \cap P^{-1}\}$. As $H = P \cap P^{-1}$ is a group, M_{**}^\times is isomorphic to $\ell^1(H)^*$. It is readily seen that in fact, $M_{**}^0 = M_{**}^\times$. We conclude that $\widehat{M} = C^*(H)^{**}$.

8.1. Weakly almost periodic compactifications. Let G be a locally compact group, and let P be its weakly periodic compactification. It is known (see Example 2.3) that P is a compact semitopological semigroup, and $M = C(P)^{**}$ is a quantum semigroup with involution.

In P , there exists the minimal ideal J (so that $PJ = JP = J$ and J does not contain any other ideals), which has the form $J = eP$ for a central idempotent e and is isomorphic (and homeomorphic) to the Bohr compactification bG of G [13, Theorem III.1.9].

Let H be a locally compact group and let $\varphi_H : H \rightarrow P$ be a homomorphism which is also a homeomorphism of H onto $\varphi_H(H)$. We are interested in two cases: $H = G$, with the canonical imbedding into P , and $H = bG$. In this setting, let $p_H : C(P) \rightarrow C_b(H)$, $f \mapsto f \circ \varphi_H$, be the “restriction” map, and let $\tau_H : M(H) \rightarrow M(P)$ be the “extension” map: $\tau_H(\mu)(f) = \int p_H(f) d\mu$ for $\mu \in M(H)$, $f \in C(P)$. Its dual $\tau_H^* : C(P)^{**} \rightarrow C_0(H)^{**}$ is ultraweakly continuous and extends p , so it is a $*$ -homomorphism.

Proposition 8.3. *If $\pi : M(P) \rightarrow B(L)$ is a unitary representation, then $\pi \circ \tau_H$ is a unitary representation of $M(H)$.*

Proof. Fix a basis (e_α) in L . For every α, β we have $\pi_{\alpha\beta} \circ \tau \in C_0(H)^{**}$, and, by continuity,

$$\sum_{\gamma} (\pi_{\gamma\alpha} \circ \tau)^* \cdot \pi_{\gamma\beta} \circ \tau = \tau_H^* \left(\sum_{\gamma} \pi_{\gamma\alpha}^* \cdot \pi_{\gamma\beta} \right) = \begin{cases} 1, & \alpha = \beta \\ 0, & \alpha \neq \beta \end{cases}$$

as well as $\sum_{\gamma} \pi_{\alpha\gamma} \circ \tau \cdot (\pi_{\beta\gamma} \circ \tau)^*$. This proves that $\pi \circ \tau_H$ is unitary. \square

Proposition 8.4. *Every irreducible unitary representation π of $M(P)$ is finite-dimensional.*

Proof. As usual, let δ_t , $t \in P$, be the probability measure concentrated at t , and set $\bar{\pi}(t) = \pi(\delta_t)$. The equations (5) imply that $\bar{\pi}(t)$ is a unitary operator for every $t \in P$. It follows that $\bar{\pi} : P \rightarrow B(L)$ is a representation, which is unitary when restricted to $\varphi_H(H)$, and the Proposition above implies [10, Theorem 4.4] that $\bar{\pi} \circ \varphi_H$ is (strongly) continuous, and $\pi \circ \tau_H(\mu) = \int_H \bar{\pi} \circ \varphi_H(t) d\mu(t)$ for every $\mu \in M(H)$. It follows also that $\bar{\pi} \circ \varphi_H$ and $\pi \circ \tau_H$ have the same invariant subspaces.

For the idempotent e , $\bar{\pi}(e)$ is unitary and idempotent, so it is the identity operator. For every $\mu \in M(P)$, $\delta_e * \mu$ is in the image of the map τ_{bG} defined above. Indeed, $\delta_e * \mu = \tau_{bG}(\nu)$, where $\nu \in M(bG)$ is defined as $\nu(f) = \int_P f(\varphi_{bG}^{-1}(e\varphi_{bG}(t))) d\mu(t)$, $f \in C(bG)$. As $\pi(\mu) = \pi(\delta_e * \mu)$, we see that any τ_{bG} -invariant subspace is also π -invariant.

Suppose now that π is irreducible. Then, by discussion above, so is $\bar{\pi} \circ \varphi_{bG}$, and being continuous, it is finite-dimensional. \square

Conversely, if σ is an irreducible continuous representation of bG , then $\pi(\mu) = \int_P \sigma(\varphi_{bG}^{-1}(et)) d\mu(t)$ defines a representation of $M(P)$, and one easily verifies that the conditions (5) hold, so that π is unitary.

As shows the reasoning above, $I = \tau_{bG}(M(bG))$ is an ideal in $M(P)$. It follows that every irreducible representation π either vanishes on I or is irreducible on it. If π is non-unitary and $\pi|_I$ is irreducible, then $\pi \circ \tau_{bG}$ is non-unitary (otherwise π would be unitary, as the unique extension from I), so that $\pi \circ \tau_{bG}$ vanishes on $L^1(bG)$. We see that in any case π vanishes on $\tau_{bG}(L^1(bG))$, thus the latter is contained in M_{**}^\times . The annihilator of $\tau_{bG}(L^1(bG))$ in $C(P)^{**}$ is an ultraweakly closed (proper) ideal, what can be proved in a virtually the same way as in [10, Proposition 4.3]. This implies that $(M_{**}^\times)^\perp \neq C(P)^{**}$ and as a consequence $M_{**}^0 = M_{**}^\times$, according to Definition 4.6.

For a representation σ of $M(bG)$, set $\pi_\sigma(\mu) = \sigma(\tau_{bG}^{-1}(\mu * \delta_e))$, $\mu \in M(P)$ (it is clear that τ_{bG} is injective). Proposition 8.4 above implies that for $\mu \in M_{**}^0$,

$$\begin{aligned} & \sup\{\|\pi(\mu)\| : \pi \text{ is a } * \text{-representation of } M_{**}^0\} \\ &= \sup\{\|\pi_\sigma(\mu)\| : \sigma \text{ is a unitary } * \text{-representation of } M(bG)\}, \end{aligned}$$

so that $C^*(M_{**}^0)$ is isomorphic to $C^*(M(bG)^0)$. As we know, this equals to $C^*(bG)$, so finally $\widehat{M} = C^*(bG)^{**}$.

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