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A Study of Maximal and Minimal Ideals of n-Refined Neutrosophic Rings

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Abstract:
If R is a ring, then \( R_n(I) \) is called a refined neutrosophic ring. Every AH-subset of \( R_n(I) \) has the form \( P = \sum_{i=0}^{n} \sum_{j=0}^{i} a_{ij} = \{ a_0 + a_1 + \cdots + a_n : a_i \in P_i \} \), where \( P_i \) are subsets of the classical ring \( R \). The objective of this paper is to determine the necessary and sufficient conditions on \( P_i \) which make \( P \) an ideal of \( R_n(I) \). Also, this work introduces a full description of the algebraic structure and form for AH-maximal and minimal ideals in \( R_n(I) \).

Keywords:
n-Refined Neutrosophic Ring.
n-Refined AH-Ideal.
Maximal Ideal.
Minimal Ideal.

1. Introduction

Neutrosophy is a new kind of generalized logic proposed by Smarandache [12]. It becomes a useful tool in many areas of science such as number theory [16, 20], solving equations [18, 21], and medical studies [11, 15]. Also, there are many applications of neutrosophic structures in statistics [14], optimization [8], and decision making [7]. On the other hand, neutrosophic algebra began in [4], Smarandache and Kandasamy defined concepts such as neutrosophic groups and neutrosophic rings. These notions were handled widely by Agboola et al. in [6, 10], where homomorphisms and AH-substructures were studied [3, 13, 17].

Recently, there is an arising interest by the generalizations of neutrosophic algebraic structures. Authors proposed n-refined neutrosophic groups [9], rings [1], modules [2, 22], and spaces [5, 19].
If $R$ is a classical ring, then the corresponding refined neutrosophic ring is defined as follows:

$$R_n(I) = \{a_0 + a_1 I + \cdots + a_n I_n : a_i \in R\}.$$  

Addition and multiplication on $R_n(I)$ are defined as:

$$\sum_{i=0}^{n} x_i I_i + \sum_{i=0}^{n} y_i I_i = \sum_{i=0}^{n} (x_i + y_i) I_i,$$

$$\sum_{i=0}^{n} x_i I_i \times \sum_{j=0}^{n} y_j I_j = \sum_{i,j=0}^{n} (x_i \times y_j) I_i I_j.$$  

Where $\times$ is the multiplication defined on the ring $R$ and $I_i I_j = I_{\min(i,j)}$.

Every AH-subset of $R_n(I)$ has the form $P = \sum_{i=0}^{n} P_i I_i = \{a_0 + a_1 I + \cdots + a_n I_n : a_i \in P_i\}$. There is an important question arises here. This question can be asked as follows:

What are the necessary and sufficient conditions on the subsets $P_i$ which make $P$ be an ideal of $R_n(I)$? On the other hand, can we determine the structure of all AH-maximal and minimal ideals in the n-refined neutrosophic ring $R_n(I)$?

Through this paper, we try to answer the previous questions in the case of n-refined neutrosophic rings with unity. All rings through this paper are considered with unity.

2. Preliminaries

**Definition 1.** [1]. Let $(R, +, \times)$ be a ring and $I_k: 1 \leq k \leq n$ be $n$ indeterminacies. We define $R_n(I) = \{a_0 + a_1 I + \cdots + a_n I_n : a_i \in R\}$ to be n-refined neutrosophic ring. If $n=2$ we get a ring which is isomorphic to 2-refined neutrosophic ring $R(I_1, I_2)$.

Addition and multiplication on $R_n(I)$ are defined as:

$$\sum_{i=0}^{n} x_i I_i + \sum_{i=0}^{n} y_i I_i = \sum_{i=0}^{n} (x_i + y_i) I_i,$$

$$\sum_{i=0}^{n} x_i I_i \times \sum_{j=0}^{n} y_j I_j = \sum_{i,j=0}^{n} (x_i \times y_j) I_i I_j.$$  

Where $\times$ is the multiplication defined on the ring $R$.

It is easy to see that $R_n(I)$ is a ring in the classical concept and contains a proper ring $R$.

**Definition 2.** [1]. Let $R_n(I)$ be an n-refined neutrosophic ring, it is said to be commutative if $xy = yx$ for each $x, y \in R_n(I)$, if there is $I \in R_n(I)$ such $1. x = x. 1 = x$, then it is called an n-refined neutrosophic ring with unity.

**Theorem 1.** [1]. Let $R_n(I)$ be an n-refined neutrosophic ring. Then (a) $R$ is commutative if and only if $R_n(I)$ is commutative, (b) $R$ has unity if and only if $R_n(I)$ has unity, and (c) $R_n(I) = \sum_{i=0}^{n} R I_i = \{\sum_{i=0}^{n} x_i I_i : x_i \in R\}$.

**Definition 3.** [1]. (a) Let $R_n(I)$ be an n-refined neutrosophic ring and $P = \sum_{i=0}^{n} P_i I_i = \{a_0 + a_1 I + \cdots + a_n I_n : a_i \in P_i\}$ where $P_i$ is a subset of $R$, we define $P$ to be an AH-subring if $P_i$ is a subring of $R$ for all $i$.
subring is defined by the condition \( P_i = P_j \) for all \( i, j \). (b) \( P \) is an AH-ideal if \( P_i \) is an two sides ideal of \( R \) for all \( i \), the AHS-ideal is defined by the condition \( P_i = P_j \) for all \( i, j \). (c) The AH-ideal \( P \) is said to be null if \( P_i = R \) or \( P_i = \{0\} \) for all \( i \).

**Definition 4.** [1]. Let \( R_n(I) \) be an \( n \)-refined neutrosophic ring and \( P = \sum_{i=0}^{n} P_iI_i \) be an AH-ideal, we define AH-factor \( R(I)/P = \sum_{i=0}^{n}(R/P_i)I_i = \sum_{i=0}^{n}(x_i + P_i)I_i; x_i \in R \).

**Theorem 2.** [1]. Let \( R_n(I) \) be an \( n \)-refined neutrosophic ring and \( P = \sum_{i=0}^{n} P_iI_i \) be an AH-ideal: \( R_n(I)/P \) is a ring with the following two binary operations:

\[
\sum_{i=0}^{n}(x_i + P_i)I_i + \sum_{i=0}^{n}(y_i + P_i)I_i = \sum_{i=0}^{n}(x_i + y_i + P_i)I_i,
\]

\[
\sum_{i=0}^{n}(x_i + P_i)I_i \times \sum_{i=0}^{n}(y_i + P_i)I_i = \sum_{i=0}^{n}(x_i \times y_i + P_i)I_i.
\]

**Definition 5.** [1]. (a) Let \( R_n(I), T_n(I) \) be two \( n \)-refined neutrosophic rings respectively, and \( f_R; R \to T \) be a ring homomorphism. We define \( n \)-refined neutrosophic AHS-homomorphism as \( f: R_n(I) \to T_n(I); f(\sum_{i=0}^{n} x_iI_i) = \sum_{i=0}^{n} f(x_i)I_i \), (b) \( f \) is an \( n \)-refined neutrosophic AHS-isomorphism if it is a bijective \( n \)-refined neutrosophic AHS-homomorphism, and (c) AH-Ker \( f = \sum_{i=0}^{n} Ker(f)I_i = \sum_{i=0}^{n} x_iI_i; x_i \in Ker f_R \).

**3. Main Discussion**

**Theorem 3.** Let \( R_n(I) = \{a_0 + a_1I + \cdots + a_nI_n; a_i \in R\} \) be any \( n \)-refined neutrosophic ring with unity \( 1 \). Let \( P = \sum_{i=0}^{n} P_iI_i = \{a_0 + a_1I + \cdots + a_nI_n; a_i \in P_i \} \) be any AH-subset of \( R_n(I) \), where \( P_i \) are subsets of \( R \). Then \( P \) is an ideal of \( R_n(I) \) if and only if \( (a) P_i \) are classical ideals of \( R \) for all \( I \) and \( (b) P_0 \leq P_k \leq P_{k-1} \) for all \( 0 < k \leq n \).

**Proof.** First of all, we assume that \( (a), (b) \) are true. We should prove that \( P \) is an ideal. Since \( P_i \) are classical ideals of \( R \), then they are subgroups of \( (R, +) \), hence \( P \) is a subgroup of \( (R_n(I), +) \). Let \( r = r_0 + r_1I_1 + \cdots + r_nI_n \) be any element of \( R_n(I), x = x_0 + x_1I_1 + \cdots + x_nI_n \) be an arbitrary element of \( P \), where \( x_i \in P_i \). We have For \( n = 0 \), the statement \( r \cdot x \in P \) is true clearly. We assume that \( P \) is true for \( n = k \), we must prove it for \( k + 1 \).

\[
r \cdot x = (r_0 + r_1I_1 + \cdots + r_kI_k + r_{k+1}I_{k+1})(x_0 + x_1I_1 + \cdots + x_kI_k + x_{k+1}I_{k+1}) =
\]

\[
(r_0 + r_1I_1 + \cdots + r_kI_k)(x_0 + x_1I_1 + \cdots x_kI_k) + r_{k+1}I_{k+1}(x_0 + \cdots + x_kI_k + 1) + (r_0 + \cdots r_kI_k)x_{k+1}I_{k+1}.
\]

We see remark

\[
(r_0 + r_1I_1 + \cdots + r_kI_k)(x_0 + x_1I_1 + \cdots x_kI_k) \in P_0 + P_1I_1 + \cdots + P_kI_k (\text{by induction hypothesis}).
\]

On the other hand, we have

\[
r_{k+1}I_{k+1}(x_0 + \cdots + x_{k+1}I_{k+1}) = (r_{k+1}x_0 + r_{k+1}x_{k+1})I_{k+1} + r_{k+1}x_1I_1 + \cdots + r_{k+1}x_kI_k.
\]
Since all $P_i$ are ideals and $P_0 \leq P_{k+1}$, we have \( r_{k+1}x_i \in P_i \) and \( r_{k+1}x_0 + r_{k+1}x_{k+1} \in P_{k+1} \), hence \( r_{k+1}l_{k+1}(x_0 + \cdots + x_{k+1}l_{k+1}) \in P \). Also, \( (r_0 + \cdots + r_k l_k)x_{k+1}l_{k+1} = r_0x_{k+1}l_{k+1} + r_1x_{k+1}l_1 + \cdots + r_kx_{k+1}l_k \).

Under the assumption of theorem, we have \( r_0x_{k+1} \in P_{k+1} \) and \( r_1x_{k+1} \in P_{k+1} \leq P_i \).

For all \( 1 \leq i \leq k \). Thus \( P \) is an ideal.

For the converse, we assume that \( P \) is an ideal of \( R_n(I) \). We should prove (a) and (b).

It is easy to check that if \( P = P_0 + \cdots + P_nI_n \) is a subgroup of \((R_n(I),+)\), then every \( P_i \) is a subgroup of \((R,+)\). Now we show that (b) is true.

For every \( 1 \leq i \leq n \), we have an element \( l_i \), that is because \( R \) is a ring with unity, hence. Let \( x_0 \) be any element of \( p_0 \), we have \( x_0 \in P \), and \( x_0l_i \in P \).

Thus \( x_0 \in P_i \), which means that \( P_0 \leq P_i \) for all \( 1 \leq i \leq n \).

Also, for every \( x_i \in P_i \), we have \( x_i l_i \in P \), thus \( x_i l_i l_{i-1} = x_i l_{i-1} \in P \), so that \( x_i \in P_{i-1} \), which means that \( P_i \leq P_{i-1} \) and (b) holds.

**Example 1.** Let \( Z \) be the ring of integers, \( Z_3(I) = \{a + bi_1 + ci_2 + di_3; a, b, c, d \in Z\} \) be the corresponding 3-refined neutrosophic ring, we have:

\[
P = \langle 16 \rangle + \langle 2 \rangle + \langle 1 \rangle_4 + \langle 9 \rangle_8 = \{16x + 2y_1 + 4z_2 + 8t_3; x, y, z, t \in Z\}
\]

is an ideal of \( Z_3(I) \), that is because, \( \langle 16 \rangle \leq \langle 2 \rangle \leq \langle 9 \rangle \leq \langle 2 \rangle > \).

Now, we are able to describe all AH-maximal and minimal ideals in \( R_n(I) \).

**Theorem 4.** Let \( R_n(I) = \{a_0 + a_1 l + \cdots + a_n l_n; a_i \in R\} \) be any n-refined neutrosophic ring with unity 1.

Let \( P = \sum_{n=0}^{\infty} P_i l_i \) be any ideal of \( R_n(I) \). Then (a) non trivial AH-maximal ideals in \( R_n(I) \) have the form \( P_0 + RI_1 + \cdots + RI_n \), where \( P_0 \) is maximal in \( R \) and (b) non trivial AH-minimal ideals in \( R_n(I) \) have the form \( P_1 l_1 \), where \( P_1 \) is minimal in \( R \).

**Proof.** (a) assume that \( P \) is an AH-maximal ideal on the refined neutrosophic ring \( R_n(I) \), hence for every ideal \( M = (M_0 + M_1 l_1 + \cdots + M_n l_n) \) with property \( P \leq M \leq R_n(I) \), we have \( M = P \) or \( M = R_n(I) \). This implies that \( M_i = R \) or \( M_i = P_i \), which means that \( P_0 \) is maximal in \( R \). On the other hand, we have \( P_0 \leq P_k \leq P_{k-1} \). For all \( 0 < k \leq n \), thus \( P_i \in \{P_0, R\} \) for all \( 1 \leq i \leq n \). Now suppose that there is at least \( j \) such that \( P_j = P_0 \), we get that \( P_0 + \cdots + P_j l_j + \cdots + R l_n \leq P_0 + R l_1 + \cdots + R l_j + \cdots + R l_n \), hence \( P \) is not maximal. This means that \( P_0 + R l_1 + \cdots + R l_n \), where \( P_0 \) is maximal in \( R \) is the unique form of AH-maximal ideals.

For the converse, we suppose that \( P_0 \) is maximal in \( R \) and \( P_i = R \). For all \( 1 \leq i \leq n \). Consider \( M = (M_0 + M_1 l_1 + \cdots + M_n l_n) \) as an arbitrary ideal of \( R_n(I) \) with AH-structure. If \( P \leq M \leq R_n(I) \), then \( P_i \leq M_i \leq R \) and, this means that \( P_0 = M_0 \) or \( M_0 = R \), that is because \( P_0 \) is maximal.

According to **Theorem 3**, we have \( M_0 \leq M_i \leq M_{i-1} \). Now if \( M_0 = R \), we get \( M_i = R \), thus \( M = R_n(I) \).
If $M_0 = P_0$, we get $M = P$. This implies that $P$ is maximal.

(b) It is clear that if $P_j$ is minimal in $R$, then $P_j I_1$ is minimal in $R_n(I)$. For the converse, we assume that $P = P_0 + P_1 I_1 + \cdots + P_n I_n$ is minimal in $R_n(I)$, consider an arbitrary ideal with AH-structure $M = (M_0 + M_1 I_1 + \cdots + M_n I_n)$ of $R_n(I)$ with the property $M \leq P$, we have: $M = \{0\}$ or $M = P$ which means that $M_1 = P_1$ or $M_1 = \{0\}$. Hence $P_1$ is minimal.

According to Theorem 3, we have $M_0 \leq M_k \leq M_{k-1}$ for all $k$. Now, suppose that there is at least $\neq 1$ such that $P_j \neq \{0\}$, we get $P_j I_j \leq P_0 + P_1 I_1 + \cdots + P_n I_n$. Thus $P$ is not minimal, which is a contradiction with respect to assumption. Hence any non trivial minimal ideal has the form $P_1 I_1$, where $P_1$ is minimal in $R$.

Example 2. Let $R = Z$ be the ring of integers, $Z_n(I) = \{a_0 + a_1 I_1 + \cdots + a_n I_n; a_i \in Z\}$ be the corresponding n-refined neutrosophic ring, we have

(a) the ideal $P = \langle 2 \rangle + Z I_1 + \cdots + Z I_n$ is AH-maximal, that is because $\langle 2 \rangle$ is maximal in $R$ and (b) there is no AH-minimal ideals in $Z_n(I)$, that is because $R$ has no minimal ideals.

Example 3. Let $R = Z_{12}$ be the ring of integers modulo 12, $Z_{12n}(I)$ be the corresponding n-refined neutrosophic ring, we have

(a) the ideal $P = \langle 6 \rangle I_1 = \{0, 6 I_1\}$ is AH-minimal, that is because $\langle 6 \rangle$ is minimal in $R$.

(b) the ideal $Q = \langle 2 \rangle + Z_{12} I_1 + \cdots + Z_{12} I_n$ is maximal, that is because $\langle 2 \rangle$ is maximal in $R$.

Now, we show that Theorem 4 is not available if the ring $R$ has no unity, we construct the following example.

Example 4. Consider $2Z_2(I) = \{(2a + 2b I_1 + 2c I_2); a, b, c \in Z\}$ the 2-refined neutrosophic ring of even integers, let $P = (2Z + 4Z I_1 + 4Z I_2) = \{(2a + 4b I_1 + 4c I_2); a, b, c \in Z\}$ be an AH-subset of it. First of all, we show that $P$ is an ideal of $2Z_2(I)$. It is easy to see that $(P, +)$ is a subgroup. Let $x = (2m + 4n I_1 + 4t I_2)$ be any element of $P$, $r = (2a + 2b I_1 + 2c I_2)$ be any element of $2Z_2(I)$, we have $rx = (4am + [8an + 4bm + 8bt + 8ct + 4cm] + I_2[8at + 8ct + 4cm]) \in P$. Thus $P$ is an ideal and the inclusion’s condition is not available, that is because $2Z$ is not contained in $4Z$.

4. Conclusion

In this article, we have found a necessary and sufficient condition for any subset to be an ideal of any n-refined neutrosophic ring with unity. On the other hand, we have characterized the form of maximal and minimal ideals in this class of neutrosophic rings. As a future research direction, we aim to study Köthe’s Conjecture on n-refined neutrosophic rings about the structure of nil ideals and the maximality/minimality conditions if $R$ has no unity.
Open Problems

According to our work, we find two interesting open problems.

- Describe the algebraic structure of the group of units of any n-refined neutrosophic ring.
- What are the conditions of AH-maximal and minimal ideals if R has no unity?

References


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