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# A Variant of Wagner’s Theorem Based on Combinatorial Hypermaps

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## Abstract

Wagner’s theorem states that a graph is planar (i.e., it can be embedded in the real plane without crossing edges) iff it contains neither  $K_5$  nor  $K_{3,3}$  as a minor. We provide a combinatorial representation of embeddings in the plane that abstracts from topological properties of plane embeddings (e.g., angles or distances), representing only the combinatorial properties (e.g., arities of faces or the clockwise order of the outgoing edges of a vertex). The representation employs combinatorial hypermaps as used by Gonther in the proof of the four-color theorem. We then give a formal proof that for every simple graph containing neither  $K_5$  nor  $K_{3,3}$  as a minor, there exists such a combinatorial plane embedding. Together with the formal proof of the four-color theorem, we obtain a formal proof that all graphs without  $K_5$  and  $K_{3,3}$  minors are four-colorable. The development is carried out in Coq, building on the mathematical components library, the formal proof of the four-color theorem, and a general-purpose graph library developed previously.

**2012 ACM Subject Classification** Mathematics of computing → Graphs and surfaces; Theory of computation → Logic and verification

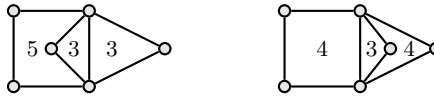
**Keywords and phrases** Coq, MathComp, Graph-Theory, Hypermaps, Planarity

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## 1 Introduction

Despite the importance of graph theory in mathematics and computer science, formalizations of graph theory results, as opposed to verified graph algorithms, remain few and spread between different systems. This includes early works in HOL4 [2, 3] and Mizar [12], as well as some landmark results such as the formalization of the four-color theorem [10] in Coq or the formal proof of the Kepler conjecture [11] in HOL Light and Isabelle. Unfortunately, none of these has led to the development of to a widely-used general-purpose graph theory library. Since we started to develop such a general-purpose library in 2017 [6, 7, 8], there has been some renewed interest in the formalization of graph theory [13, 14]. In [8], one of the main results is a formal proof that the graphs of treewidth at most two are precisely those that do not include  $K_4$ , the complete graph with four vertices, as a minor. Other classes of graphs can also be described in terms of excluded minors, and this paper is concerned with the characterization of planar graphs as those that contain neither  $K_5$  nor  $K_{3,3}$  (cf. Figure 1) as a minor. This is known as Wagner’s theorem.

The textbook definition (e.g. in [5]) of a graph being planar is that there exists a drawing (or embedding) in the real plane without crossing edges. However, much of the information provided by such a drawing (e.g., the precise location of vertices or the angles at which an edge leaves a vertex) are irrelevant for most proofs about planar graphs as they can be changed almost at will by shifting or deforming the drawing. A more abstract alternative would be to take the characterization in terms of excluded minors as the definition of planarity. However, this would not provide any geometric information at all. In particular, a graph can have multiple embeddings that differ in their combinatorial properties. For instance, consider the following two drawings of the same graph:



45

46 On the left, the (inner) faces have arities 5, 3, and 3, while the arities on the right are 4, 3, and 4.  
 47 Some proofs about planar graphs crucially rely on these kinds of combinatorial properties of  
 48 a given plane embedding. For instance, this is the case for the proof of the four-color theorem  
 49 (FCT), and the formal proof of the FCT in Coq [9, 10] represents drawings of graphs using  
 50 a structure called combinatorial hypermaps [4, 16]. This representation is quite far away  
 51 from the ordinary representations of graphs as a collection of vertices and edges, instead  
 52 representing vertices and edges as permutations on more primitive objects called “darts”.

53 In this paper, we use combinatorial hypermaps to represent embeddings of simple graphs,  
 54 and then give a formal and constructive proof that every simple graph containing neither  
 55  $K_5$  nor  $K_{3,3}$  as a minor can be represented by a planar hypermap.<sup>1</sup> This corresponds to  
 56 one direction of Wagner’s theorem, the direction that’s mathematically more interesting. In  
 57 particular, we bridge the gap between the hypermap representation of graphs used in [9, 10]  
 58 and the more standard representation of simple graphs as a finite type of vertices with an  
 59 edge relation. The latter representation is used pervasively in the graph theory library we  
 60 developed previously [8] and on which we base the parts of the argument that deal with  
 61 structural properties like minors and separators. As it comes to hypermaps, we build on the  
 62 formalization used in the proof of the four color theorem [9, 10]. Thus, as a corollary of this  
 63 work, we obtain a formal proof of a “structural” four-color theorem, i.e., a proof that every  
 64 graph not containing the aforementioned minors is four-colorable. This theorem does not  
 65 mention hypermaps in its statement. Hence, the question whether planar hypermaps are  
 66 a faithful representation of plane embeddings is secondary. What is important is that this  
 67 representation allows for machine-checked proofs of interesting properties.

68 The formal development underpinning this paper has been developed as a branch of  
 69 the `coq-community/graph-theory` library and the plan is to integrate the new proofs into  
 70 the main development. The development is available at: <https://coq-community.org/graph-theory/wagner/>  
 71

## 72 2 Graph Theory Preliminaries

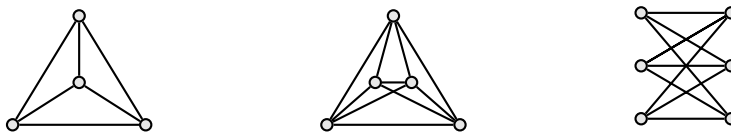
73 In this section we review some standard notions from graph theory that are used in the proof  
 74 of Wagner’s theorem. We mostly use the conventions and terminology from previous work [8].

75 A (*simple*) *graph* is a pair  $(G, -)$  where  $G$  is a finite type of objects called *vertices* and “ $-$ ”  
 76 is an irreflexive and symmetric relation on  $G$ . We use single capital letters  $F, G, \dots$  to denote  
 77 graphs as well as their underlying type of vertices. That is, we write  $x, y : G$  to denote that  
 78  $x$  and  $y$  are vertices of  $G$ . We also write  $x - y$  to say that  $x$  and  $y$  are linked by an edge and  
 79  $N(x) := \{y \mid x - y\}$  for the *open neighborhood* of  $x$ . If  $x, y : G$ , we write  $G + xy$  for  $G$  with  
 80 an additional  $xy$ -edge. For a set of vertices  $V$ , we write  $G[V]$  for the subgraph induced by  $V$ ,  
 81  $G - V := G[\bar{V}]$  for the subgraph induced by the complement of  $V$ , and  $G - x := G[\{\bar{x}\}]$  for  
 82 the graph that results from deleting the vertex  $x$  (and any incident edges) from  $G$ .<sup>2</sup>

83 We write  $|G|$  for the size of  $G$ , i.e. the number of vertices of  $G$ . We write  $G/xy$  for the  
 84 graph that results from merging the vertices  $x$  and  $y$  in  $G$ , which is implemented by removing

<sup>1</sup> For technical reasons, we also exclude graphs with isolated vertices (cf. Remark 21).

<sup>2</sup> Technically, the vertices of  $G[V]$  are dependent pairs of vertices  $x : G$  and proofs  $x \in V$ , but we will ignore this in the mathematical presentation (cf. [8]).



■ **Figure 1**  $K_4$  (left),  $K_5$  (middle), and  $K_{3,3}$  (right)

85 the vertex  $y$  and attaching its neighbors to  $x$ . We write  $K_n$  for the complete graph with  $n$   
 86 vertices and  $K_{3,3}$  for the complete bipartite graph with two times three vertices (cf. Figure 1).

87 A *path* (in some graph  $G$ ) is a nonempty sequence of vertices with subsequent vertices  
 88 linked by the edge relation, and an  $xy$ -path is a path starting at  $x$  and ending at  $y$ . A *cycle*  
 89 is an  $xy$ -path for some  $x, y : G$  such that  $x-y$ . A set of vertices  $A$  is *connected*, if any two  
 90 vertices in  $A$  are connected by a path contained in  $A$ . Two sets of vertices  $A$  and  $B$  are  
 91 *neighboring*, if there exist vertices  $x \in A$  and  $y \in B$  such that  $x-y$ .

92 A set of vertices  $S$  *separates*  $x$  and  $y$ , if  $x, y \notin S$  and every  $xy$ -path contains a vertex  
 93 from  $S$ . A set that separates any two vertices, i.e. whose removal would disconnect the graph,  
 94 is called a (*vertex*) *separator*. In particular,  $\emptyset$  is a separator iff  $G$  has multiple disconnected  
 95 components. A graph  $G$  is  $k$ -connected if  $k < |G|$  and every separator has size at least  $k$ .  
 96 In particular,  $K_{k+1}$  is  $k$ -connected, since there are no separators in a complete graph. A  
 97 *separation* of  $G$  is a pair  $(V_1, V_2)$  of sets of vertices such that  $V_1 \cup V_2$  covers  $G$  and there is  
 98 no edge from  $\overline{V_1}$  to  $\overline{V_2}$ . A separation  $(V_1, V_2)$  is *proper*, if both  $\overline{V_1}$  and  $\overline{V_2}$  are nonempty.

99 ► **Fact 1.** *Let  $G$  be a simple graph. Every separator  $S$  of  $G$  can be extended into a proper*  
 100 *separation  $(V_1, V_2)$  of  $G$  such that  $S = V_1 \cap V_2$ .*

101 We are interested in the characterization of planar graphs through excluded minors.  
 102 Intuitively, a minor of a graph is a graph that can be obtained from the original graph  
 103 through a series of edge deletions, vertex deletions, and edge contractions. Following our  
 104 previous work [8], we define the minor relation using functions we call minor maps:

105 ► **Definition 2.** *Let  $G$  and  $H$  be simple graphs. A function  $\phi : H \rightarrow 2^G$  is called a minor*  
 106 *map if:*

107 **M1.**  $\phi(x)$  is nonempty and connected for all  $x : H$ ,

108 **M2.**  $\phi(x) \cap \phi(y) = \emptyset$  whenever  $x \neq y$  for all  $x, y : H$ .

109 **M3.**  $\phi(x)$  neighbors  $\phi(y)$  for all  $x, y : H$  such that  $x-y$ .

110  $H$  is a minor of  $G$ , written  $H \prec G$  if there exists a minor map  $\phi : H \rightarrow 2^G$ .

111 If  $\phi : H \rightarrow 2^G$  is a minor map, then  $\phi(x)$  is the set of vertices being collapsed to  $x$  (by  
 112 contracting all the edges in  $\phi(x)$ ) when exhibiting  $H$  as a minor of  $G$ .

113 ► **Fact 3.**  $\prec$  is transitive.

114 ► **Definition 4.** A graph  $G$  is called  $H$ -free, if  $H$  is not a minor of  $G$ .

115 Note that if  $G$  is  $H$ -free, then, by transitivity, so is every minor of  $G$ . Also note that if  $x-y$ ,  
 116 then  $G/xy$  corresponds to an edge contraction. Hence, we have the following lemma.

117 ► **Lemma 5.** *If  $x-y$ , then  $G/xy \prec G$*

118 It is easy to see that  $G[V] \prec G$ , for any set  $V$  of vertices of  $G$ , and thus  $G[V]$  is  $H$ -free  
 119 whenever  $G$  is. However, when  $V$  is one of the two sides of a separation arising from a  
 120 separator  $\{x, y\}$ , we can even add an  $x-y$  edge, as shown below.

121 ► **Lemma 6.** *Let  $(V_1, V_2)$  be a proper separation of  $G$  with  $V_1 \cap V_2 = \{x, y\}$  with  $x \neq y$  and*  
 122  *$\{x, y\}$  a smallest separator. Then every minor of  $(G + xy)[V_1]$  is also a minor of  $G$ .*

123 **Proof.** If the  $xy$ -edge is used to justify  $H \prec (G + xy)[V_1]$  for some  $H$ , the  $xy$ -edge can always  
 124 be replaced by a path through  $V_2 \setminus V_1$ , which is not otherwise needed to establish  $H \prec G$ . ◀

### 125 3 Wagner's Theorem

126 Before we turn to the formal proof of Wagner's theorem using combinatorial hypermaps, we  
 127 first sketch the proof relying on an informal notion of plane embedding (i.e., drawings of  
 128 the graph without crossing edges), leaving the technical details of the modeling to Section 5.

129 The proof of Wagner's theorem consists of two parts. The main induction deals with the  
 130 case for 3-connected graphs. This is then extended to the general case though a number of  
 131 comparatively straightforward combinations of plane embeddings for subgraphs. Below, we  
 132 sketch the two arguments, including forward references to two types of lemmas: those that  
 133 are interesting from a mathematical point of view (marked with “★”) and those that depend  
 134 on the modeling of plane embeddings using hypermaps (marked with “\*\*”). The proofs are  
 135 inspired by those in [1, 5].

136 ► **Proposition 7.** *Let  $G$  be 3-connected,  $K_5$ -free, and  $K_{3,3}$ -free. Then  $G$  can be embedded in*  
 137 *the plane.*

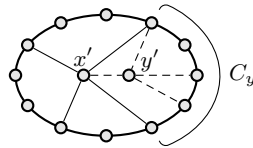
138 **Proof sketch.** The proof proceeds by induction on  $|G|$ .

- 139 1. Since  $G$  is 3-connected, we have  $4 \leq |G|$ . If  $|G| = 4$ , then is  $K_4$ , which can easily be  
 140 embedded in the plane (Figure 1, Proposition 22\*). Hence, we can assume  $5 \leq |G|$ .
- 141 2. Thus, we obtain  $x, y : G$  such that  $x-y$  and  $G/xy$  is again 3-connected (Theorem 11\*).
- 142 3. Since  $|G/xy| < |G|$ , we obtain a plane embedding for  $G/xy$  by induction (Lemma 5). Let  
 143  $v_{xy}$  be the vertex resulting from the contraction of the  $xy$ -edge and set

$$144 \quad H := G/xy - v_{xy}$$

145 Let  $X$  (resp.  $Y$ ) be the set of vertices in  $H$  that are neighbors of  $x$  (resp.  $y$ ) in  $G$ .

- 146 4. Since  $G/xy$  is 3-connected and since all vertices in  $X \cup Y$  are neighbors of  $v_{xy}$ , removing  $v_{xy}$   
 147 and all incident edges form the plane embedding of  $G/xy$  yields a plane embedding  $\hat{H}$   
 148 of  $H$  with a face whose boundary contains all vertices from  $X$  and  $Y$  (Lemma 28\*\*).
- 149 5. Since  $G/xy$  is 3-connected, we have that  $H$  is 2-connected. Hence, the face of  $\hat{H}$  whose  
 150 boundary contains  $X$  and  $Y$  is bounded by a (duplicate-free) cycle  $C$  (Theorem 25\*\*).
- 151 6. Splitting  $C$  at the elements of  $X$  yields a number of segments where every segment  
 152 overlaps with each of its two neighboring segments in exactly one element of  $X$  (unless  
 153 there are only two segments). Since  $K_5 \not\prec G$  and  $K_{3,3} \not\prec G$ , all elements of  $Y$  must be  
 154 contained in one of the segments of  $C$ ; call this segment  $C_y$  (Lemma 12\*).
- 155 7. Adding a vertex  $x'$  to  $\hat{H}$  inside  $C$  and making it adjacent to all vertices in  $X$  yields a  
 156 graph with an embedding that has a face containing  $x'$  and  $C_y$ . Thus, we can place a  
 157 vertex  $y'$  within this face and add edges to  $x'$  and all vertices in  $Y$  as shown below:



158

159 This yields a plane embedding of  $G$ . ◀

160 It remains to take care of the cases where  $G$  is not 3-connected.

161 ► **Theorem 8.** *Let  $G$  be  $K_5$ -free, and  $K_{3,3}$ -free. Then  $G$  can be embedded in the real plane.*

162 **Proof.** By induction on  $|G|$ . By Propositions 7 and 22, we can assume that  $5 \leq |G|$  and  
 163 that  $G$  has a minimal separator  $S$  with  $|S| \leq 2$ . We obtain a proper separation  $(V_1, V_2)$  with  
 164  $V_1 \cap V_2 = S$ . If  $S = \{x, y\}$ , we set  $H := G + xy$  and have that neither  $H[V_1]$  nor  $H[V_2]$   
 165 contains  $K_5$  or  $K_{3,3}$  as a minor (Lemma 6), allowing us to obtain plane embeddings of  $H[V_1]$   
 166 and  $H[V_2]$  by induction. Due to the added  $xy$ -edge, both embeddings must have a face with  
 167  $x$  and  $y$  adjacent on the boundary of some face. Without loss of generality, we can assume  
 168 that this is the (unbounded) outer face. By stretching and scaling, we can “glue” together  
 169 the two embeddings along these outer edges, obtaining a plane embedding of  $H$  (Lemma 30\*).  
 170 Removing the  $xy$  edge (or keeping it if it was present in  $G$ ), provides a plane embedding of  $G$ .  
 171 The cases for  $S = \emptyset$  and  $S = \{x\}$  are similar, but do not require the use of a “marker” edge. ◀

172 Note that the proof of Theorem 8 makes reference to intuitive operations such as stretching  
 173 and scaling. In particular, the fact that one can turn an arbitrary face into the outer face is  
 174 usually argued using a stereographic projection to the sphere and back to the plane [1]. All  
 175 of these will be no-ops for our representation of plane embeddings using hypermaps.

## 176 4 The Combinatorial Part

177 This section is concerned with the purely combinatorial part of the proof of Proposition 7,  
 178 justifying steps (2) and (6). The former amounts to locating an edge in a 3-connected graph  
 179 such that contracting this edge yields a smaller 3-connected graph. The latter is about  
 180 justifying (using the names from the proof of Proposition 7) that in the cycle  $C$  all the  
 181 neighbors of  $y$  are contained in a segment spanned by two successive neighbors of  $x$ . This  
 182 is the part of the proof where assumptions of  $K_5$ -freeness and  $K_{3,3}$ -freeness are used. Both  
 183 arguments are completely combinatorial, in the sense that neither argument makes any  
 184 reference to plane embeddings.

185 For step (2), the argument is based on minimal separators, and we repeatedly use the  
 186 following property:

187 ► **Proposition 9.** *If  $S$  is a minimal separator of  $G$ , then  $S$  neighbors every maximal component  
 188 of  $G - S$ .*

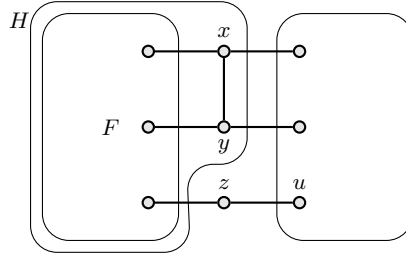
189 Recall that  $G/xy$  is implemented by removing  $y$  and updating the edge relation accordingly.

190 ► **Lemma 10.** *Let  $G$  be 3-connected with  $5 \leq |G|$ , and let  $x, y : G$  such that  $x-y$  and  
 191  $G/xy$  is not 3-connected. Then there exists some  $z : G$  such that  $\{x, y, z\}$  is a separator.*

192 **Proof.** Since  $G$  is 3-connected, we have that  $G/xy$  is 2-connected. Moreover,  $G/xy$  is not  
 193 3-connected by assumption. Hence,  $G/xy$  has a minimal separator  $S$  with  $|S| = 2$ . We have  
 194 that  $x \in S$ , because otherwise  $S$  would be a 2-separator of  $G$ . Thus,  $S = \{x, z\}$  for some  $z$ ,  
 195 and  $\{x, y, z\}$  is a separator of  $G$ . ◀

196 ► **Theorem 11.** *If  $G$  is 3-connected and  $5 \leq |G|$ , then there exists an  $xy$ -edge such that  
 197  $G/xy$  is 3-connected.*

198 **Proof.** Assume the theorem does not hold, i.e., assume that  $G/xy$  is not 3-connected for all  
 199  $x, y : G$  such that  $x-y$ . We obtain a contradiction as follows:



■ **Figure 2** Objects from the proof of Theorem 11 (cf. [1, Theorem 9.10])

200 By Lemma 10, every  $xy$ -edge can be extended to a separator  $\{x, y, z\}$ . Choose  $x, y, z$ ,  
 201 and  $F$  such that  $x-y, \{x, y, z\}$  is a separator,  $F$  is connected and disjoint from  $\{x, y, z\}$ , and  
 202 with  $|F|$  maximal for all possible choices of  $x, y, z$  and  $F$ . Now set  $H := F \cup \{x, y\}$ . Since  $G$   
 203 is 3-connected,  $\{x, y, z\}$  is indeed a minimal separator of  $G$ . Thus,  $x, y$ , and  $z$  are pairwise  
 204 distinct and by Proposition 9 there exists some vertex  $u \notin H$  such that  $z-u$  (cf. Figure 2).  
 205 Let  $v$  such that  $\{z, u, v\}$  is a separator (Lemma 10). Now it suffices to show that  $H \setminus \{v\}$   
 206 is connected, because this yields a component larger than  $F$ , contradicting the choice of  $F$ .  
 207 If  $v \notin H$  this is trivial and if  $v \in \{x, y\}$ , this follows since  $\{x, y, z\}$  is a minimal separator.  
 208 (Proposition 9 ensures that both  $x$  and  $y$  have neighbors in  $F$ .) Hence, we can assume  $v \in F$ .  
 209 Now if  $H \setminus \{v\}$  was disconnected, then there would be some vertex  $w$  such that every  $xw$ -path  
 210 in  $H$  passes through  $v$ . However, since  $F$  is maximal and therefore has no outgoing edges  
 211 other than those to  $x, y$ , and  $z$ , this would entail that  $\{v, z\}$  is a separator (separating  $x$   
 212 from  $w$ ), contradicting the assumption that  $G$  is 3-connected. ◀

213 We remark that, just like all the other results presented in this paper, the proof of  
 214 Theorem 11 does not require any classical axioms. The conclusion of the theorem involves  
 215 only decidable predicates and quantifiers over finite domains (i.e., the vertices of  $G$ ), and  
 216 these behave classically. Similarly, there are only finitely many choices for  $x, y, z$ , and  $F$ , so  
 217 we can easily obtain a combination where  $|F|$  is maximal among all possible choices.

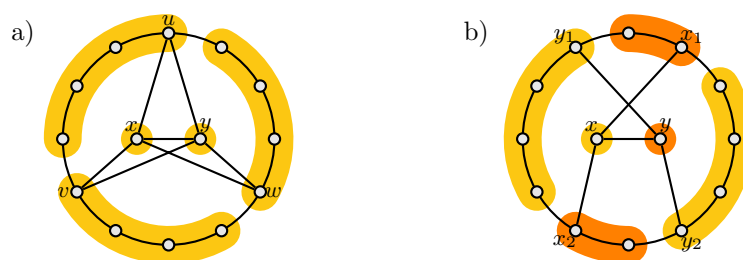
218 In order to formally state the lemma justifying step (6) of Proposition 7, we need to  
 219 introduce some operations on duplicate-free lists viewed as cycles. Let  $T$  be some type and  
 220 let  $C$  be a duplicate free list over  $T$ . For  $x \in C$ , we write  $\text{next } C x$  for the element following  
 221  $x$  in  $C$  or the first element of  $C$  if  $x$  is at the very end. For  $x, y \in C$  with  $x \neq y$ , we write  
 222  $\text{arc } C x y$  for the part of  $C$  (seen as a cycle) that starts at  $x$  and ends right before  $y$ . In  
 223 particular, the results of  $\text{next } C x$  and  $\text{arc } C x y$  are invariant under cyclic shifts of  $C$ .

224 ► **Lemma 12.** *Let  $G$  be a simple,  $K_5$ -free, and  $K_{3,3}$ -free graph, let  $x, y : G$  such that  $x-y$   
 225 and let  $C$  be a duplicate-free cycle in  $G$  containing neither  $x$  nor  $y$ . Let  $X$  be the sub-  
 226 sequence of  $C$  containing  $N(x)$  and let  $Y$  be the sub-sequence of  $C$  containing  $N(y)$ . If  $X$   
 227 and  $Y$  each contain at least two vertices, then there exists some vertex  $z \in X$  such that  
 228  $Y \subseteq \text{arc } C z (\text{next } X z) \cup \{\text{next } X z\}$ .*

229 **Proof.** We first show that there are at most two vertices in  $X \cap Y$ . Assume, for the sake of  
 230 contradiction, three distinct vertices  $u, v, w \in X \cap Y$ . W.l.o.g., we can assume that  $[u, v, w]$   
 231 is a sub-cycle of  $C$ . Hence, we obtain  $K_5$  as a minor of  $G$  by collapsing by mapping the  
 232 vertices of  $K_5$  to the sets  $\{x\}, \{y\}, \text{arc } C u v, \text{arc } C v w$ , and  $\text{arc } C w u$  as shown in Figure 3(a),  
 233 contradicting the assumption that  $G$  is  $K_5$ -free.

234 Next, we show that there cannot be a sub-cycle  $[x_1, y_1, x_2, y_2]$  of  $C$  such that  $\{x_1, x_2\} \subseteq X$   
 235 and  $\{y_1, y_2\} \subseteq Y$ . If such a sub-cycle were to exist, we could exhibit  $K_{3,3}$  as a minor of





■ **Figure 3** Obtaining  $K_5$  (left) and  $K_{3,3}$  (right) as minors in Lemma 12

236  $G$  by mapping the three pairwise-independent left-hand-side vertices to  $\{x\}$ ,  $\text{arc } C y_1 x_2$ ,  
 237 and  $\text{arc } C y_2 x_1$  and the three right hand side vertices to  $\{y\}$ ,  $\text{arc } C x_1 y_1$ , and  $\text{arc } C x_2 y_2$ ,  
 238 contradicting  $K_{3,3}$ -freeness of  $G$  (cf. Figure 3(b)).

239 Now, assume that the theorem does not hold, i.e., assume that for every  $x' \in X$ , there  
 240 exists some  $y' \in Y$  such that  $y' \notin \text{arc } C x'(\text{next } X x') \cup \{\text{next } X x'\}$ . We consider two cases:

- 241 ■ If  $Y \subseteq X$ , we have that  $Y = [y_1, y_2]$  for two distinct vertices  $y_1$  and  $y_2$ . Now  $\text{arc } C y_1 y_2$   
 242 must contain some vertex  $x_2 \in X \setminus \{y_1, y_2\}$ , for otherwise  $\text{next } X y_1 = y_2$  and both  $y_1$  and  
 243  $y_2$  are contained in  $\text{arc } C y_1 y_2 \cup \{y_2\}$ . By symmetry, we also have that  $\text{arc } C y_2 y_1$  must  
 244 contain some  $x_1 \in X \setminus \{y_1, y_2\}$ . However, then  $[x_1, y_1, x_2, y_2]$  is an alternating subcycle,  
 245 whose existence we excluded above. Contradiction.
- 246 ■ Otherwise, there exists some  $y_1 \in Y \setminus X$ . Let  $x_1$  such that  $y_1 \in \text{arc } C x_1(\text{next } X x_1)$  and set  
 247  $x_2 := \text{next } X x_1$ . By assumption, there must be some  $y_2 \in Y$  such that  $y_2 \notin \text{arc } C x_1 x_2 \cup x_2$ .  
 248 Hence,  $[x_1, y_1, x_2, y_2]$  is again an excluded alternating subcycle. Contradiction. ◀

249 Lemma 12 can be considered to be the combinatorial core argument underlying Wagner's  
 250 theorem. It is the place where absence of certain substructures (i.e., the minors  $K_5$  and  $K_{3,3}$ )  
 251 is turned into a positive statement that allows reversing the contraction of the  $xy$ -edge. We  
 252 remark that while the arc construction was already present in mathcomp, splitting a cycle  
 253 along a subcycle required a plethora of additional lemmas about arcs and cycles.

## 254 5 Hypermaps as plane embeddings

255 In this section we describe how we model plane embeddings of simple graphs using combina-  
 256 torial hypermaps. We first briefly review hypermaps and their most important properties  
 257 and then describe how we use hypermaps to model plane embeddings.

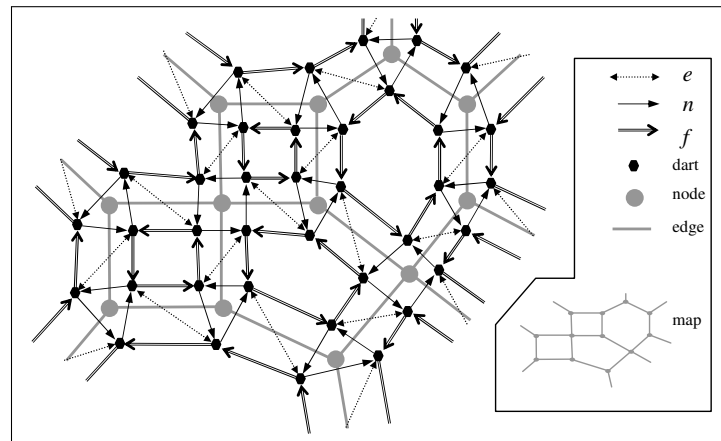
### 258 5.1 Combinatorial Hypermaps

259 Our presentation of hypermaps follows that of [9], because the formal development under-  
 260 pinning this paper is based on the formal proof of the four-color theorem presented there.  
 261 Consequently, none of the results in this section are new.

262 ► **Definition 13.** A (combinatorial) hypermap is a tuple  $\langle D, e, n, f \rangle$  where  $D$  is a finite type,  
 263 and  $e, n, f : D \rightarrow D$  such that  $n \circ f \circ e \equiv \text{id}_D$ . The elements of  $D$  are referred to as darts.

264 The condition  $n \circ f \circ e \equiv \text{id}_D$  ensures that the functions  $e$ ,  $n$ , and  $f$  are bijective (i.e.  
 265 permutations on  $D$ ). In particular, any two of the permutations determine the third. Each  
 266 of the permutations partitions the type  $D$  into a number of cycles and these cycles are used





■ **Figure 4** A hypermap. (Reprinted with permission from [9], ©2005 Georges Gonthier)

267 to represent the edges, nodes<sup>3</sup>, and faces of graphs. That is, a hypermap  $\langle D, e, n, f \rangle$  can be  
 268 seen as describing a graph embedded on a surface (not necessarily the plane) as follows (cf.  
 269 Figure 4):

- 270 ■ every  $n$ -cycle represents a node of the graph, listing incident edges in counterclockwise  
 271 order.
- 272 ■ every  $e$ -cycle represents an edge of the graph, linking the nodes (i.e.,  $n$ -cycles) it intersects.
- 273 ■ every  $f$ -cycle represents a face, listing in counterclockwise order one dart from every node  
 274 on the boundary of the face.

275 Even though one of the three permutations is technically redundant, keeping it makes  
 276 the definition completely symmetric and facilitates symmetry reasoning. In particular, if  
 277  $\langle D, e, n, f \rangle$  is a hypermap, then so are  $\langle D, f, e, n \rangle$  and  $\langle D, n, f, e \rangle$ . As we do for graphs, we  
 278 will usually use the same letter for a hypermap and its underlying type of darts.

279 ► **Definition 14.** Let  $\langle D, e, n, f \rangle$  be a hypermap.

- 280 ■  $D$  is called plain if every  $e$ -cycle has size 2.
- 281 ■  $D$  is called loopless if  $x$  and  $e(x)$  belong to different  $n$ -cycles for all  $x : D$ .
- 282 ■  $D$  is called simple if two  $n$ -cycles are linked by at most one  $e$ -cycle.

283 Plain hypermaps correspond to graphs where every edge is adjacent to two vertices, i.e.  
 284 graphs without hyperedges. As we will make precise later, plain loopless simple hypermaps  
 285 correspond to simple graphs, i.e., graphs without self loops and with at most one edge  
 286 between two vertices. The (partial) hypermap in Figure 4 satisfies all three properties, as  
 287 will most of the hypermaps we will be dealing with.

288 We fix a hypermap  $\langle D, e, n, f \rangle$  for the rest of the section. Moreover, we will use the same  
 289 letter  $D$  for the hypermap as a whole as well as the underlying type of darts.

290 The number of “holes” that would be needed in a surface in order to embed a given  
 291 hypermap in it can be computed using the Euler characteristic.

292 ► **Definition 15 (Genus).** The genus of  $D$  is  $((2C + |D|) - (E + N + F))/2$  where  $C$  is  
 293 the number of connected components of  $e \cup n \cup f$  (interpreting the functions as functional

<sup>3</sup> In line with the terminology of [9, 10], we say “node” when referring to an  $n$ -cycle of a hypermap. In line with [8], we continue to use “vertex” when referring to vertices of simple graphs.

relations) and  $E$ ,  $N$ , and  $F$  are the number of cycles of  $e$ ,  $n$ , and  $f$  respectively. A map of genus 0, i.e., a map satisfying the equation  $E + N + F = 2C + |D|$  is called planar.

The following general properties of hypermaps are established in [9].

► **Proposition 16.**  $E + N + F \leq 2C + |D|$ .

► **Proposition 17.**  $(2C + |D|) - (E + N + F)$  is even.

Proposition 16 implies that the (natural number) subtraction in Definition 15 is never truncating and Proposition 17 implies that the division in the genus formula is always an integer division without remainder.

For our use of hypermaps as representations of embeddings in the plane, we will need to modify hypermaps and prove that these modifications preserve planarity. Directly proving that an operation such as adding an edge across a face preserves the genus of the hypermap can be cumbersome. It is often simpler to express the operation in terms of more atomic planarity-preserving operations. The most important of these operations are the *Walkup* [15, 17] operations.

► **Definition 18.** For  $x : D$ ,  $\text{WalkupE}x$  is the hypermap where  $x$  has been removed by skipping over  $x$  in the  $n$  and  $f$  permutations and adapting  $e$  as necessary. Similarly,  $\text{WalkupN}x$  (resp.  $\text{WalkupF}x$ ) are the hypermaps where  $n$  (resp.  $f$ ) is the permutation being adapted after suppressing  $x$  from the other two.

As shown in [9], the *Walkup* operations never increase the genus of a hypermap and, in particular, always preserve planarity. In addition, the *Walkup* operations can be shown to preserve the genus in many circumstances, allowing us to prove preservation of planarity for operations that extend the hypermap by expressing them as inverse *Walkup* operations. Thus, the characterization of planarity in terms of Euler’s formula combined with expressing operations as combinations of *Walkup* operations provides for an easy means of proving that various operations on hypermaps preserve planarity.

In addition to showing that certain operations preserve planarity, we also need to establish some properties of planar hypermaps in general. For instance, we need to show that in every two-connected plane graph, all faces are bounded by (duplicate free) cycles (step (5)). For the topological model of plane graphs, this property is established using the Jordan curve theorem (JCT), which states that every closed simple curve divides the plain into an “inside” and an “outside”. Since hypermaps make no reference to the real plane, we could not use this theorem, even if it was available in Coq. However, the essence of the application of the JCT to plane graphs is captured by the following theorem on hypermaps:

► **Theorem 19** (Jordan curve theorem for hypermaps [9, 10]). Let  $\langle D, e, n, f \rangle$  be a hypermap. Then  $D$  is planar iff if there do not exist distinct darts  $x, y$  and a duplicate-free  $(n^{-1} \cup f)$ -path from  $x$  to  $n(y)$  visiting  $y$  before  $n(x)$  (with  $y = n(x)$  being allowed).

Note that when talking about hypermaps, an  $(n^{-1} \cup f)$ -path is a path in the relation  $(n^{-1} \cup f)$ . This is to be contrasted with the notion of an  $xy$ -path in a simple graph, where we mention the endpoints and leave the relation implicit. Paths in the relation  $(n^{-1} \cup f)$  are called *contour paths*, because they go around the outside of a group of faces (cf. Figure 4). Thus, a contour cycle in a planar map corresponds to a closed curve. The Jordan curve theorem for hypermaps establishes that in a planar hypermap there cannot be a contour path starting at the inside of a contour cycle and finishing on the outside without otherwise intersecting the cycle. In the theorem above, the contour cycle and the contour path are spliced together in order to obtain a simpler statement (cf. [9, 10]).

## 339 5.2 Combinatorial Embeddings

340 We now make precise what it means for a hypermap to represent an embedding of a graph  
 341 on some surface. To this end, we first introduce some additional notation. For relations  
 342  $r : D \rightarrow D \rightarrow \mathbb{B}$  over a finite type  $D$  (e.g., the darts of a hypermap) we write  $r^*$  for the  
 343 reflexive transitive closure of  $r$  and  $r^*(x)$  for the set  $\{y \mid r^* x y\}$ . In particular, we write  $f^*$   
 344 for the transitive closure of a function  $f : D \rightarrow D$  seen as the relation  $\lambda x y. f x = y$ . Note that  
 345  $f^*$  is symmetric if  $f$  is injective, as is the case for the permutations comprising hypermaps.

346 For a hypermap  $\langle D, e, n, f \rangle$ , we call two darts  $x$  and  $y$  *adjacent*, written  $\text{adjn } x y$ , if their  
 347 respective  $n$ -cycles are linked by an  $e$ -cycle (i.e., if there exists some dart  $z$  such that  $n^* x z$   
 348 and  $n^* y (e z)$ ).

349 ► **Definition 20.** *Let  $G$  be a simple graph and let  $\langle D, e, n, f \rangle$  be a plain hypermap. We call a*  
 350 *function  $g : D \rightarrow G$  a (combinatorial) embedding of  $G$  if it satisfies the following properties:*

- 351 1.  *$g$  is surjective*
- 352 2.  *$n^* x y$  iff  $g(x) = g(y)$ .*
- 353 3.  *$\text{adjn } x y$  iff  $g(x) - g(y)$ .*

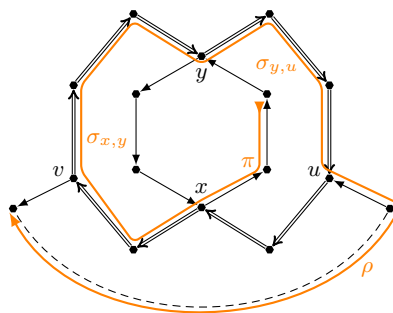
354 *An embedding where  $D$  is planar, is called a plane embedding, and an embedding where  $D$  is*  
 355 *simple is called a simple embedding. A graph together with a plane embedding is called a*  
 356 *plane graph.*

357 Note that, even though we refer to  $g$  as an embedding of a graph, the function maps darts of  
 358 the hypermap to vertices of the graph. This makes it easier to state the required properties.  
 359 Surjectivity of  $g$  ensures that  $D$  represents the whole graph. Condition (2) ensures that the  
 360 node cycles of  $D$  are in one-to-one correspondence to the vertices of  $G$ , and condition (3)  
 361 ensures that adjacent node cycles correspond to adjacent vertices of  $G$ . Note that we do *not*  
 362 require that the hypermap underlying an embedding is simple, i.e., we permit multiple parallel  
 363 edges. This reduces the number of conditions to check when constructing plane embeddings.  
 364 Parallel edges can always be removed, obtaining a simple embedding where needed.

365 ► **Remark 21.** Definition 20 abstracts not only from properties that can be changed by  
 366 continuously deforming the plane, it also do not single out a face as the “outer” face or  
 367 specify the relationships between the embeddings of disconnected components of a graph, i.e.,  
 368 we do not embed one component in a particular face of the embedding of another component.  
 369 Consequently, Definition 20 corresponds more to embedding every component of the graph  
 370 on its own sphere rather than embedding all components together in the plane. Moreover, the  
 371 degenerate case of a component consisting of a single isolated vertex cannot be represented  
 372 by hypermaps, because every dart of an  $n$ -cycle must also be part of an  $e$ -cycle. This is not  
 373 really an issue: isolated vertices are components without internal structure, and there would  
 374 be nothing to learn about such vertices from a combinatorial embedding.

375 With Definition 20 in place, we can now justify step (1) of the proof of Proposition 7,  
 376 i.e., obtain a plane embedding for  $K_4$ . The graph  $K_4$  has 6 edges, so we take the 12-element  
 377 type  $I_{12} := \Sigma n : \mathbb{N}. n < 12$  as the type of darts and provide the three permutations as well  
 378 as a mapping from  $I_{12}$  to the vertices of  $K_4$ . Since both  $K_4$  and its embedding are concrete  
 379 objects, we can use the depth-first search algorithm from `mathcomp` to compute the genus of  
 380 the map and check the correctness of the embedding. This requires brute-forcing various  
 381 quantifiers, which causes no problems due to the small size of their domain (i.e. 4 or 12).  
 382 Thus, we obtain:

383 ► **Proposition 22.** *There exists a plane embedding for  $K_4$ .*



■ **Figure 5** Moebious path from the proof of Lemma 26

384 We also show that  $K_{3,3}$  does not have a plane embedding. While this result does not  
 385 contribute to the main result of this paper, it serves as an example of how Definition 20 and  
 386 some of the properties described in Section 5.1 fit together.

387 ► **Proposition 23.** *There exists no plane embedding for  $K_{3,3}$ .*

388 **Proof.** Assume there was an embedding  $g : D \rightarrow K_{3,3}$  with  $D$  of genus 0. Without loss of  
 389 generality, we can assume that  $D$  is simple. Thus, we have  $N = 6$ ,  $E = 9$ ,  $|D| = 2 * E = 18$ ,  
 390 and  $C = 1$ . By the definition of genus, it suffices to show  $(5 - F)/2 > 0$  to obtain a  
 391 contradiction. Since every vertex of  $K_{3,3}$  has at least two neighbors and since  $D$  is simple,  
 392 every face-cycle must use at least 3 darts. Moreover,  $K_{3,3}$  has no odd-length cycles, so every  
 393 face-cycle of  $D$  must indeed use at last 4 darts. Thus  $F \leq 4$ , since  $|D| = 18$ . Finally,  $F \neq 4$   
 394 since the division in the genus formula is always without remainder (Proposition 17). ◀

395 We now come to the main result of this section, namely that the faces of 2-connected  
 396 plane graphs are bounded by irredundant cycles. In order to state his property precisely, we  
 397 define a notion of face for simple graphs relative to an embedding.

398 ► **Definition 24.** *If  $g : \langle D, e, n, f \rangle \rightarrow G$  is an embedding, a face of  $G$  under  $g$  is a cycle in  $G$   
 399 that can be obtained as the image of an  $f$ -cycle of  $D$  of under  $g$ .*

400 The theorem we want to prove is the following.

401 ► **Theorem 25.** *Let  $g$  be a plane embedding of a 2-connected graph  $G$ . Then all the faces  
 402 under  $g$  are duplicate-free cycles.*

403 Before we can prove this theorem, we first need to prove the underlying property on hypermaps.  
 404 This is where the Jordan curve theorem for hypermaps (Theorem 19) is used.

405 ► **Lemma 26.** *Let  $\langle D, e, n, f \rangle$  be a plain loopless planar hypermap such that for all darts  
 406  $x, y, z$  with  $x, y \notin n^*(z)$  there exists an  $(n^{-1} \cup f)$ -path from  $x$  to  $y$  not containing any dart  
 407 in  $n^*(z)$ . Then there do not exist distinct darts  $x, y$  such that  $n^*xy$  and  $f^*xy$ .*

408 **Proof.** Assume there exist  $x \neq y$  such that  $n^*xy$  and  $f^*xy$ . We show that this contradicts  
 409 the planarity of  $G$ . Without loss of generality, we obtain a duplicate-free  $n^{-1}$ -path from  $y$  to  
 410  $x$  whose interior  $\pi$  is disjoint from  $f^*(x)$  (We make  $n^{-1}$ -steps starting at  $y$  and replace  $x$   
 411 with the first encountered dart in  $f^*(y)$ ). Now we can split the  $f$ -cycle containing  $x$  and  $y$   
 412 into two semi-cycles, one from  $x$  to  $y$  and another from  $y$  to  $x$ . We call their respective  
 413 interiors (which are both disjoint from  $\pi \cup \{x, y\}$ )  $\sigma_{x,y}$  and  $\sigma_{y,x}$ . By assumption, we can

414 obtain  $(n^{-1} \cup f)$ -paths avoiding  $n^*(x)$  and connecting any two darts outside of  $n^*(x)$ . Thus,  
 415 we obtain darts  $u \in \sigma_{y,x}$  and  $v \in \sigma_{x,y}$  and a duplicate-free  $(n^{-1} \cup f)$ -path from  $u$  to  $v$  disjoint  
 416 from the  $n$ -cycle containing both  $x$  and  $y$  whose interior we call  $\rho$ . Without loss of generality,  
 417 we can assume that  $\rho$  is also disjoint from  $\sigma_{x,y}$  and  $\sigma_{y,x}$  (otherwise we shorten  $\rho$ , possibly  
 418 changing the choice of  $u$  and  $v$ ). Finally set  $\sigma_{y,u}$  to be the part of  $\sigma_{y,x}$  before  $u$ . Thus,  
 419 we have that  $m := \pi \# [x] \# \sigma_{x,y} \# [y] \# \sigma_{y,u} \# u \# \rho$  is a duplicate-free  $(n^{-1} \cup f)$ -path.  
 420 Moreover, the first dart in  $m$  is  $n^{-1}(y)$  (which could be  $x$ ) and (since  $\sigma_{x,y}$  is an  $f$ -path) the  
 421 last dart is  $n(v)$  (cf. Figure 5). Since  $m$  visits  $v$  (which is in  $\sigma_{x,y}$ ) before  $y$ ,  $m$  is a “Moebius  
 422 contour” and Theorem 19 applies, contradicting the planarity of  $D$ . ◀

423 Now we can prove Theorem 25, justifying step (5) of the proof of Proposition 7.

424 **Proof of Theorem 25.** Let  $G$  be 2-connected and let  $g : \langle D, e, n, f \rangle \rightarrow G$  be a plane embed-  
 425 ding. Thus  $D$  is plain, loopless, and planar. Let  $s$  be a face of  $G$  under  $g$  arising as the  
 426 image of some  $f$ -cycle in  $D$ . It suffices to show that all the darts in this  $f$ -cycle belong to  
 427 different  $n$ -cycles. Since  $G$  is 2-connected, all vertices different from  $z$  can be connected using  
 428 paths that avoid  $z$ . These paths can be mapped to  $(n^{-1} \cup f)$ -paths in  $D$ . Hence, Lemma 26  
 429 applies, finishing the proof. ◀

430 The proof Theorem 25 exhibits a pattern that is repeated for various lemmas about plane  
 431 embeddings: we first show the underlying lemma for hypermaps and then lift the property  
 432 to the language of simple graphs and plane embeddings in order to use them in the proofs of  
 433 Proposition 7 and Theorem 8.

## 434 6 Modifying Plane Embeddings

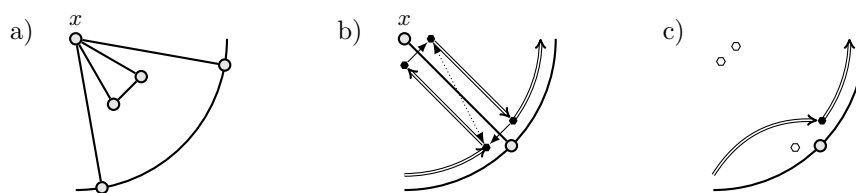
435 We now describe the operations on plane embeddings and their underlying hypermaps that  
 436 are required to carry out steps (4) and (7) of the proof of Proposition 7. That is, we show  
 437 how to remove a vertex from a plane embedding, obtaining a face containing all neighbors  
 438 of the removed vertex, and we show how to add a vertex, connecting it to an arbitrary  
 439 subsequence of a face-cycle.

440 We begin by showing that every subgraph of a plane graph has a plane embedding. While  
 441 this is intuitively obvious, the precise argument deserves some mention. Again, we need some  
 442 notation to express the underlying lemma about hypermaps:

443 Let  $T$  be a finite type and let  $f : T \rightarrow T$  be an injective function and let  $P$  be a subset of  $T$ .  
 444 We write  $\Sigma P$  for the *type* of elements of  $P$ , i.e., the type of dependent pairs  $\Sigma x : T. x \in P$ .  
 445 We define  $\text{skip}_P f : T \rightarrow T$  to be the function which for every  $x : T$  returns  $f^{n+1}(x)$  for the  
 446 least  $n$  such that  $f^{n+1}(x) \in P$  if such an  $n$  exists and  $x$  otherwise. Such an  $n$  always exists  
 447 when  $x \in P$ , so  $\text{skip}_P f$  can also be seen as a function  $\Sigma P \rightarrow \Sigma P$ . Finally, we write  $f \equiv g$ , to  
 448 denote that two functions agree on all arguments.

449 **► Lemma 27.** *Let  $\langle D, e, n, f \rangle$  be a hypermap, let  $P \subseteq D$ , and let  $\langle \Sigma P, e', n', f' \rangle$  be another hy-  
 450 permap such that  $e' \equiv \text{skip}_P e$  and  $n' \equiv \text{skip}_P n$ . Then  $\text{genus} \langle D, e, n, f \rangle \leq \text{genus} \langle \Sigma P, e', n', f' \rangle$ .*

451 **Proof.** By induction on  $|D|$ . If  $P$  is the full set, then the two hypermaps are isomorphic  
 452 and therefore have the same genus. Thus, we can assume there exists some  $z \notin P$ . Let  
 453  $H := \text{WalkupF } z$ . Since the Walkup operation does not increase the genus, it suffices to show  
 454  $\text{genus} \langle \Sigma P, e', n', f' \rangle \leq \text{genus } H$ . This follows by induction hypothesis since  $H$  is defined by  
 455 skipping over  $z$  in the edge and node permutations and, therefore,  $\langle \Sigma P, e', n', f' \rangle$  can be  
 456 obtained from  $H$ , again up to isomorphism, by skipping over the remaining elements of  $\bar{P}$ . ◀



■ **Figure 6** Removing a vertex from a 2-connected plane graph

457 Note that Lemma 27 applies to any hypermap, not just plain ones. This small generaliza-  
 458 tion allows us to prove the lemma by induction, removing a single dart at a time. This would  
 459 not work with plain maps, which always have an even number of darts. Also note that the  
 460 proof of the lemma above makes extensive use of isomorphisms for hypermaps, a notion that  
 461 is not defined in the formal development of the four-color theorem, where only an equivalence  
 462 for hypermaps with the same type of darts is defined. This turned out to be too restrictive  
 463 for our purposes. As we do for other types of graphs [8], we define isomorphisms between  
 464 hypermaps as bijections on the underlying type of darts that preserve the three permutations.

465 ► **Lemma 28.** *Let  $G$  be a 2-connected graph with vertex  $x$  and let  $g$  be a plane embedding.*  
 466 *Then there exists a plane embedding  $g'$  for  $G-x$  and a face of  $g'$  containing all vertices in  $N(x)$ .*

467 **Proof.** Let  $D = \langle D, e, n, f \rangle$  be the hypermap underlying  $g$ , and  $d_x : D$  such that  $g(d_x) = x$ .  
 468 Without loss of generality, we can assume that  $D$  is a simple hypermap. We set  $P :=$   
 469  $\overline{e^*(n^*(d_x))}$  and set  $D' = \langle \Sigma P, \text{skip}_P e, \text{skip}_P n, f' \rangle$  for some suitable  $f'$ , which amounts to  
 470 removing all  $e$ -cycles intersecting  $n^*(d_x)$ .  $D'$  is clearly plain, and by Lemma 27  $D'$  is also  
 471 planar. Since  $x \notin g(P)$ , the restriction of  $g$  to  $D'$  yields a plane embedding  $g' : D' \rightarrow (G-x)$ .  
 472 It remains to show that  $g'$  has a face containing  $N(x)$ . First, 2-connectedness of  $G$  rules  
 473 out the scenario depicted in Figure 6(a), where removing  $x$  would disconnect the graph.  
 474 Moreover, it ensures that every  $n$ -cycle (in  $D$ ) has at least size two. Together with  $D$  being  
 475 simple, this ensures that no  $n$ -cycle other than the one for  $x$  vanishes and that  $f'$  needs to  
 476 skip over at most one removed dart at a time (Figure 6(b-c)), allowing us to give a simple  
 477 explicit definition of  $f'$ :  $f'(z) :=$  if  $f(z) \in P$  then  $f(z)$  else  $n^{-1}(f(z))$

478 Moreover, we have that for all  $d \in n^*(d_x)$ ,  $f(d)$  is in  $P$  and on the same (original)  
 479  $n$ -cycle as  $e(d)$ , meaning every dart  $f(d)$  represents a neighbor of  $x$ . Thus, it suffices to show  
 480  $f'^*(fd_1)(fd_2)$  for  $d_1, d_2 \in n^*(d_x)$ . We prove this claim by induction on the  $n$ -path from  $d_1$   
 481 to  $d_2$ , reducing the problem to showing  $f'^*(fd)(fnd)$  for  $d \in n^*(d_x)$ . Since  $D$  is simple,  
 482 the  $f$ -orbit of  $f(d)$  as length at least 3 and therefore the shape  $[f(d)] \# o \# [(e(n(d)), d)]$ .  
 483 Moreover, since  $D$  is an embedding for a 2-connected graph, we can use Lemma 26 to show  
 484 that  $e(n(d))$  and  $d$  are the only darts from the  $f$ -orbit of  $f(d)$  that are not in  $P$ . Thus, the  
 485 claim follows from the definition of  $f'$  since  $n^{-1}(e(n(d))) = f(n(d))$ . ◀

486 Note that the proof above uses Lemma 26 for the second time. When we use the lemma in  
 487 step (4) of the proof of Proposition 7, we apply it to the 3-connected graph  $G/xy$ , exploiting  
 488 that  $G/xy - v_{xy}$  is still 2-connected, which in turn allows us to argue that the obtained face  
 489 containing all the neighbors is bounded by a duplicate-free cycle (cf. step (5) and Theorem 25).

490 Finally, we justify step (7) of Proposition 7, which amounts to two applications of the  
 491 lemma below, where  $G+(z, A)$  is the simple graph  $G$  extended with a new vertex  $z$  which  
 492 is made adjacent to all vertices in the set  $A$ .



493 ► **Lemma 29.** *Let  $g : D \rightarrow G$  be a plane embedding, let  $[x] \uplus p \uplus [y] \uplus q$  be a face of  $g$ , and*  
 494 *let  $\{x, y\} \subseteq A \subseteq \{x, y\} \cup p$ . Then there exists a plane embedding of  $G + (z, A)$  with a face*  
 495  *$[x, z, y] \uplus q$ .*

496 **Proof.** We first show that for every face  $[u] \uplus s$  under some embedding, one can add a  
 497 single vertex  $v$  and obtain an embedding of  $G + (v, \{u\})$  with face  $[u, v, u] \uplus s$ . Moreover,  
 498 one can always add an edge across a face, splitting a face  $[u] \uplus s_1 \uplus [v] \uplus s_2$  into two faces  
 499  $[v, u] \uplus s_1$  and  $[u, v] \uplus s_2$ . In each case, we show that the operation can be reversed by  
 500 a genus-preserving double Walkup operation, showing that the initial addition preserves  
 501 the genus. The claim then follows by first adding  $z$  and the  $xz$ -edge and then adding the  
 502 remaining edges in the order in which they appear in  $p \uplus [y]$ . ◀

503 This finishes the justification for the individual steps of the proof of Proposition 7. We remark  
 504 that Lemmas 28 and 29 are “lossy” in that we do not prove that the untouched part of the  
 505 embedding remains the same. This would only clutter the statements and is not needed  
 506 for our purposes. Should the need arise, it would be straightforward to turn the underlying  
 507 constructions into definitions and provide multiple lemmas, as we do with isomorphisms [8].

## 508 7 Combining Plane Embeddings

509 It remains to give a formal account of the combinations of plane embeddings performed in  
 510 the proof of Theorem 8. That is, we need to be able to glue two plane embeddings together,  
 511 either along a shared vertex or along a shared edge, the latter being used in the case outlined  
 512 in the informal proof sketch of Theorem 8 given in Section 3.

513 It is straightforward to show that disjoint unions of planar hypermaps are again planar.  
 514 As a consequence, both gluing operations can be reduced to obtaining a plane embedding  
 515 for  $G/xy$  from a plane embedding for  $G$ . Here, gluing along an edge amounts to merging  
 516 the respective ends of the two edges one by one. On hypermaps, merging two nodes only  
 517 changes the node and face permutations, leaving the type of darts and the edge permutation  
 518 unchanged. Moreover, both the change to the node permutation and the change for the face  
 519 permutation can be expressed in terms of a single successor-swapping operation.

520 Let  $f : T \rightarrow T$  be an injective function over a finite type  $T$  and let  $x \neq y$ .

$$521 \quad \text{switch}[x, y, f](z) := \begin{cases} fy & \text{if } z = x \\ fx & \text{if } z = y \\ fz & \text{otherwise} \end{cases} \quad \begin{array}{c} \text{Diagram showing two cycles sharing a vertex } x. \text{ The left cycle has vertices } fx \text{ and } x. \text{ The right cycle has vertices } y \text{ and } fy. \text{ Dashed lines indicate the mapping } x \rightarrow y \text{ and } fx \rightarrow fy. \end{array}$$

523 The behavior of  $\text{switch}[x, y, f]$  is to either link two  $f$ -cycles (if  $x$  and  $y$  are on different  
 524  $f$ -cycles, as in the drawing above) or to separate an  $f$ -cycle into two cycles (if  $x$  and  $y$  are  
 525 on the same  $f$ -cycle). Further, we have that

$$526 \quad \text{merge} \langle D, e, n, f \rangle d_1 d_2 := \langle D, e, \text{switch}[d_1, d_2, n], \text{switch}[f^{-1}d_2, f^{-1}d_1, f] \rangle$$

527 is a hypermap. If  $d_1$  and  $d_2$  are darts from different node cycles,  $\text{merge } D d_1 d_2$  merges said  
 528 node cycles, adapting the face cycles accordingly. In particular,  $\text{merge } D d_1 d_2$  preserves the  
 529 genus of  $D$  if either  $d_1$  and  $d_2$  are from separate components of  $D$  or if  $d_1$  and  $d_2$  lie on a  
 530 common face cycle. In the first case,  $N$  is decreased by one while  $F$  increases by one; in the  
 531 second case, both  $N$  and  $F$  are decreased by one, but so is  $C$ .

532 If  $g : D \rightarrow G$  is an embedding of some graph  $G$ , then for all  $x, y : G$  that are not adjacent,  
 533 and for all  $d_x$  and  $d_y$  such that  $g d_x = x$  and  $g d_y = y$ ,  $\text{merge } D, d_1 d_2$  can be used to embed



534  $G/xy$ . If  $x$  and  $y$  lie common face of  $g$ , then  $x$  and  $y$  are the images of two darts  $d_x$  and  $d_y$   
 535 that lie on a common face cycle in  $D$ , and  $\text{merge } D d_x d_y$  yields an embedding of  $G/xy$ .  
 536 If  $x$  and  $y$  are not connected in  $G$ , any choice of preimages of  $x$  and  $y$  will yield a plane  
 537 embedding of  $G/xy$ . Hence, for gluing two embeddings together on a single vertex, we can  
 538 make an arbitrary choice. For gluing along two edges  $x-x'$  and  $y-y'$  we know that there  
 539 must be two faces  $[x, x'] \dashv s_1$  and  $[y', y] \dashv s_1$ . Choosing  $d_x$  and  $d_y$  to be the preimages of  
 540  $x$  and  $y$  on the respective face cycles ensures that  $\text{merge } D d_x d_y$  has an  $f$ -cycle containing  
 541 preimages for  $x'$  and  $y'$ , allowing us to obtain a plane embedding for  $(G/xy)/x'y'$ , which  
 542 corresponds to gluing together two components of  $G$  along the edges  $x-x'$  and  $y-y'$ .

543 Note that, due to Definition 20 allowing parallel edges, we do not need to remove darts  
 544 when gluing along an edge. Putting everything together, we obtain the lemma used in the  
 545 proof of Theorem 8:

546 **► Lemma 30.** *Let  $G$  be a simple graph, and let  $(V_1, V_2)$  be a separation, such that  $V_1 \cap V_2 =$   
 547  $\{x, y\}$  and  $x-y$ . If there are plane embeddings for  $G[V_1]$  and  $G[V_2]$ , then there is also a plane  
 548 embedding for  $G$ .*

## 549 8 Main Results

550 Putting everything together, we obtain the following theorem, which corresponds exactly to  
 551 the theorem formalized in Coq.

552 **► Theorem 31.** *Let  $G$  be a  $K_5$ -free and  $K_{3,3}$ -free simple graph without isolated vertices. Then  
 553 there exists a (combinatorial) plane embedding for  $G$ .*

```
554 Theorem Wagner (G : sgraph) : no_isolated G ->
555   ~ minor G 'K_3,3 /\ ~ minor G 'K_5 -> inhabited (plane_embedding G).
```

556 Note that, compared with Theorem 8, we have the additional technical side condition that  
 557  $G$  may not have isolated vertices. As mentioned in Remark 21, this is necessary, because  
 558 hypermaps cannot represent isolated vertices. However, isolated vertices can often be treated  
 559 separately without too much effort as exemplified below.

560 **► Definition 32.** *A (loopless) hypermap  $\langle D, e, n, f \rangle$  is  $k$ -colorable if there is a coloring of its  
 561 darts using at most  $k$  colors, such that for all  $d : D$ , the color of  $e(d)$  is different from the color  
 562 of  $d$  and the color of  $n(d)$  is the same as the color of  $d$ . A simple graph is  $k$ -colorable, if there is  
 563 a coloring of its vertices using at most  $k$  colors such that adjacent vertices have different colors.*

564 **► Theorem 33** ([9, 10]). *Every planar loopless hypermap is 4-colorable*

565 **► Theorem 34.** *Let  $G$  be a  $K_5$ -free and  $K_{3,3}$ -free simple graph. Then  $G$  is four-colorable.*

566 **Proof.** Let  $V$  be the set of vertices with nonempty neighborhood. We obtain a 4-coloring of  
 567  $G[V]$  using Theorems 31 and 33. This coloring extends to a 4-coloring of  $G$  by picking an  
 568 arbitrary color for the isolated vertices. ◀

## 569 9 Conclusion and Future Work

570 We have introduced a combinatorial approximation of embeddings of graphs in the plane  
 571 and proved that, with respect to this notion of plane embedding, every  $K_5$ -free and  $K_{3,3}$ -free  
 572 graph without isolated vertices is planar. This corresponds to proving the mathematically  
 573 interesting direction of Wagner's theorem, and allows proving a structural variant of the

574 four-color theorem that does not make reference to hypermaps or plane embeddings. That  
575 is, we bridge the gap between simple graphs and hypermaps, making the four-color theorem  
576 available to the setting of a more standard representation of graphs.

577 The main focus of this work was to bridge the aforementioned gap rather than provide a  
578 faithful proof of the usual formulation of Wagner's theorem. Nevertheless, we argue that  
579 Theorem 8 and its proof are actually quite faithful to the usual formulation. First, it seems  
580 plausible that the notion of plane embedding can be adapted to allow for isolated vertices by  
581 relaxing the surjectivity requirement, allowing isolated vertices to not have a dart mapped  
582 to them. However, this would come at the cost of some (minor) complications, as one could  
583 no longer define a partial inverse for every embedding. More importantly, key arguments  
584 of the proof (e.g., Theorems 11 and 25 and Lemmas 12 and 28) closely correspond to what  
585 one would find in a detailed paper proof [1, 5]. The main difference is that arguments about  
586 modifications of plane embeddings, many of which are normally handled informally, either  
587 vanish completely or are replaced by rigorous machine-checked proofs on hypermaps. It  
588 should be said that finding these proofs took considerable effort. Hypermaps are complex  
589 objects and, apart from the work of Gonthier [9, 10], there is little material in the literature  
590 on how to reason efficiently using hypermaps on paper and in an interactive theorem prover.  
591 Combined with the fact that some of the proofs are quite technical (e.g. Lemma 26), the  
592 learning curve is fairly steep. I hope that this work will contribute to making hypermaps  
593 more accessible.

594 Standing at around 7000 lines (counting additions to the preexisting graph-theory library),  
595 the development is substantial, increasing the total size of the library by more than a third.  
596 Around half of these additions deal with operations on hypermaps and plane embeddings.  
597 Both the total size and the fraction dealing with hypermaps are bigger than originally  
598 envisioned, and I hope that both can still be improved.

599 As mentioned above, we have only proved one direction of Wagner's theorem. It remains  
600 to prove that graphs that can be represented using planar hypermaps have neither  $K_5$  nor  
601  $K_{3,3}$  as a minor. Compared to the effort required to prove Theorem 31, this should be a  
602 relatively straightforward extension. We have already proved that  $K_5$  and  $K_{3,3}$  do not have  
603 plane embeddings (cf. Proposition 23). Hence, it only remains to show that if  $G$  has a plane  
604 embedding, then so does every minor  $H$  of  $G$  that does not have isolated vertices. For this,  
605 one would need to decompose the minor map  $\phi : H \rightarrow 2^G$  into smaller steps, that can be  
606 followed by constructions on hypermaps. For instance, one could first remove, in a single  
607 step, all vertices of  $G$  that are not used (Lemma 27) and then, step by step, contract the  
608 edges within the sets  $\phi(x)$  for  $x : H$ . This would be a variation of the usual characterization  
609 of minors as a sequence of edge deletions, vertex deletions, and edge contractions.

610 Besides the converse direction of Wagner's theorem, there are many other related theorems  
611 that would make for interesting future work. It is well known that in the case of 3-connected  
612 planar graphs, all plane embeddings have the same structure [1, Theorem 10.28]. In our  
613 setting, this means that the embedding is unique up to isomorphisms of hypermaps. Further,  
614 a common strengthening of Proposition 7 is to show that one can obtain a plane embedding  
615 in which all inner faces are convex. This strengthening is not expressible using the hypermap  
616 model of plane embeddings, and this raises the question whether one could introduce an  
617 abstract notion of plane embedding and instantiate it with hypermaps as well as models  
618 based on axiomatic geometry or embeddings in the real plane. On the other hand, given that  
619 the (combinatorial) plane embedding of a 3-connected planar graph is unique, it should also  
620 be possible to directly construct a convex embedding in the real plane for this hypermap,  
621 separating the existence and convexity parts of the proof.

## References

- 622 1 Adrian Bondy and U.S.R. Murty. *Graph Theory*, volume 244 of *Graduate Texts in Mathematics*.  
623 Springer-Verlag London, 2008.
- 624 2 C.-T. Chou. Mechanical verification of distributed algorithms in higher-order logic\*. *The*  
625 *Computer Journal*, 38(2):152–161, 01 1995. URL: [https://doi.org/10.1093/comjnl/38.](https://doi.org/10.1093/comjnl/38.2.152)  
626 2.152, arXiv:[http://oup.prod.sis.lan/comjnl/article-pdf/38/2/152/1051283/380152.](http://oup.prod.sis.lan/comjnl/article-pdf/38/2/152/1051283/380152.pdf)  
627 pdf, doi:10.1093/comjnl/38.2.152.
- 628 3 Ching-Tsun Chou. A formal theory of undirected graphs in higher-order logic. In *TPHOL*,  
629 volume 859 of *LNCS*, pages 144–157. Springer, 1994. doi:10.1007/3-540-58450-1\_40.
- 630 4 Robert Cori. *Un code pour les graphes planaires et ses applications*. PhD thesis, Univ. Paris  
631 VII, 1973.
- 632 5 R. Diestel. *Graph Theory (2nd edition)*. Graduate Texts in Mathematics. Springer, 2000.
- 633 6 Christian Doczkal, Guillaume Combette, and Damien Pous. A formal proof of the  
634 minor-exclusion property for treewidth-two graphs. In Jeremy Avigad and Assia Mah-  
635 boubi, editors, *Interactive Theorem Proving (ITP 2018)*, volume 10895 of *LNCS*, pages  
636 178–195. Springer, 2018. URL: [https://doi.org/10.1007/978-3-319-94821-8\\_11](https://doi.org/10.1007/978-3-319-94821-8_11), doi:  
637 10.1007/978-3-319-94821-8\_11.
- 638 7 Christian Doczkal and Damien Pous. Completeness of an axiomatization of graph isomorphism  
639 via graph rewriting in Coq. In *Proc. of 9th ACM SIGPLAN International Conference on*  
640 *Certified Programs and Proofs (CPP '20), January 20–21, 2020, New Orleans, LA, USA*, 2020.  
641 doi:10.1145/3372885.3373831.
- 642 8 Christian Doczkal and Damien Pous. Graph theory in coq: Minors, treewidth, and iso-  
643 morphisms. *J. Autom. Reason.*, 64(5):795–825, 2020. URL: [https://doi.org/10.1007/](https://doi.org/10.1007/s10817-020-09543-2)  
644 [s10817-020-09543-2](https://doi.org/10.1007/s10817-020-09543-2), doi:10.1007/s10817-020-09543-2.
- 645 9 Georges Gonthier. A computer-checked proof of the four colour theorem, 2005. URL:  
646 <https://www.cl.cam.ac.uk/~lp15/Pages/4colproof.pdf>.
- 647 10 Georges Gonthier. Formal proof — the four-color theorem. *Notices Amer. Math. Soc.*,  
648 55(11):1382–1393, 2008.
- 649 11 Thomas Hales, Mark Adams, Gertrud Bauer, Tat Dat Dang, John Harrison, Le Truong Hoang,  
650 Cezary Kaliszyk, Victor Magron, Sean McLaughlin, and Tat Thang Nguyen. A formal proof of  
651 the kepler conjecture. *Forum of Mathematics, Pi*, 5, 2017. doi:10.1017/fmp.2017.1.
- 652 12 Y. Nakamura and P. Rudnicki. Euler circuits and paths. *Formalized Mathematics*, 6(3):417–425,  
653 1997.
- 654 13 Daniel E. Severín. Formalization of the domination chain with weighted parameters (short  
655 paper). In John Harrison, John O’Leary, and Andrew Tolmach, editors, *Interactive Theorem*  
656 *Provin (ITP 2019)*, volume 141 of *LIPICs*, pages 36:1–36:7. Schloss Dagstuhl - Leibniz-  
657 Zentrum für Informatik, 2019. URL: <https://doi.org/10.4230/LIPICs.ITP.2019.36>, doi:  
658 10.4230/LIPICs.ITP.2019.36.
- 659 14 Abhishek Kr Singh and Raja Natarajan. A constructive formalization of the weak perfect graph  
660 theorem. In *Proc. of 9th ACM SIGPLAN International Conference on Certified Programs and*  
661 *Proofs (CPP '20), January 20–21, 2020, New Orleans, LA, USA*, 2020.
- 662 15 Saul Stahl. A combinatorial analog of the jordan curve theorem. *J. Comb. Theory, Ser. B*,  
663 35(1):28–38, 1983. URL: [https://doi.org/10.1016/0095-8956\(83\)90078-3](https://doi.org/10.1016/0095-8956(83)90078-3), doi:10.1016/  
664 0095-8956(83)90078-3.
- 665 16 W. T. Tutte. Duality and trinity. In *Colloquium Mathematical Society Janos Bolyai 10*, pages  
666 1459–1472, 1975.
- 667 17 David W. Walkup. How many ways can a permutation be factored into two n-cycles? *Discret.*  
668 *Math.*, 28(3):315–319, 1979. URL: [https://doi.org/10.1016/0012-365X\(79\)90138-9](https://doi.org/10.1016/0012-365X(79)90138-9), doi:  
669 10.1016/0012-365X(79)90138-9.
- 670