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Dominance inequalities for scheduling around an unrestrictive common due date

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Abstract

The problem considered in this work consists in scheduling a set of tasks on a single machine, around an unrestrictive common due date to minimize the weighted sum of earliness and tardiness. This problem can be formulated as a compact mixed integer program (MIP). In this article, we focus on neighborhood-based dominance properties, where the neighborhood is associated to insert and swap operations. We derive from these properties a local search procedure providing a very good heuristic solution. The main contribution of this work stands in an exact solving context: we derive constraints eliminating the non locally optimal solutions with respect to the insert and swap operations. We propose linear inequalities translating these constraints to strengthen the MIP compact formulation. These inequalities, called dominance inequalities, are different from standard reinforcement inequalities. We provide a numerical analysis which shows that adding these inequalities significantly reduces the computation time required for solving the scheduling problem using a standard solver.

Keywords: scheduling, integer programming, common due date, dominance properties

1. Introduction

The scheduling problem studied in this work falls within the just-in-time scheduling field. In this framework, each task has a due date, and any deviation from this

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due date is penalized. From one hand, the tardiness is penalized to model the customer dissatisfaction, on the other hand, the earliness is penalized to model the induced storage costs. The reader can refer to the seminal surveys of Baker & Scudder (1990) and Kramer & Subramanian (2019) for the early results in this field.

We consider a set J of n tasks with fixed processing times $(p_j)_{j \in J} \in \mathbb{R}_+^J$, to be non-preemptively processed on a single machine. These tasks share a common *due date* d for which tasks should be preferably completed. In this work, we assume that the due date is *unrestrictive*, i.e. $d \geq \sum p_j$.

In a just-in-time framework, tasks completing before or after d will therefore incur penalties according to unit earliness (resp. tardiness) penalties $(\alpha_j)_{j \in J} \in \mathbb{R}_+^J$ (resp. $(\beta_j)_{j \in J} \in \mathbb{R}_+^J$). A schedule is defined by the task completion times denoted by $(C_j)_{j \in J}$. Using $[x]^+$ to denote the positive part of $x \in \mathbb{R}$, the *earliness* (resp. *tardiness*) of a task $j \in J$ is given by $[d - C_j]^+$ (resp. $[C_j - d]^+$). Given parameters d , p , α and β , the Unrestrictive Common Due Date Problem (UCDDP) aims at finding a schedule minimizing the total *penalty*:

$$\sum_{j \in J} \alpha_j [d - C_j]^+ + \beta_j [C_j - d]^+$$

This criterion is non-regular, i.e. is not a non-increasing function of completion time C_j for any $j \in J$. Moreover, the penalty function is not a linear function of the completion times.

If all task penalties are equal, i.e. $\alpha_j = \beta_j$ for any task $j \in J$, the UCDDP is solvable in polynomial time (Kanet, 1981). If task penalties are symmetric, i.e. $\alpha_j = \beta_j$ for all $j \in J$, the problem is NP-hard (Hall & Posner, 1991). Therefore, the problem that we consider with arbitrary α and β coefficients is also NP-hard.

A wide range of variables can be used to formulate a single machine scheduling problem as a Mixed Integer Program (MIP): completion time variables, time-indexed variables, linear ordering variables, positional date and assignment variables (Queyranne & Schulz, 1994). However, only few works based on linear programming are proposed for the UCDDP. Biskup & Feldmann (2001a) propose a compact MIP formulation based on disjunctive variables, which is only given to compare the proposed heuristic method to an exact method for small instances. van den Akker et al. (2002) propose a formulation based on an exponential number of binary variables using column generation and Lagrangian relaxation. In Falq et al. (2021), we propose a MIP based on natural variables, similar to completion time variables, together with a compact MIP based on partition variables.

Furthermore, the UCDDP have been solved through several other dedicated exact methods like branch-and-bound algorithms (*e.g.* Sourd (2009)) and dynamic programming methods (*e.g.* Hall & Posner (1991), Hoogeveen & van de Velde (1991), Tanaka & Araki (2013)). Moreover, a benchmark is provided in Biskup & Feldmann (2001a), together with a heuristic method. An important remark is that the efficiency of the latter approaches comes from the exploitation of dominance properties. In particular, the dedicated Branch-and-Bound proposed in Sourd (2009) for an exact solving of UCDDP, solves up to 1000-tasks instances of the benchmark provided in Biskup & Feldmann (2001a).

Among the dominance properties, we can distinguish the structural ones, which allow to restrict the solution set to the set of the schedules having a specific structure. These properties are already taken into account in the compact formulation that we propose in Falq et al. (2021). Indeed, thanks to their structure, the dominant schedules are encoded by only n binary variables resulting in a formulation whose size does not depend on the time horizon, in contrast with the time-indexed formulations. In the present work, we focus on another type of dominance property called neighborhood based dominance properties. We propose to model them by linear inequalities.

One important contribution of this article is to propose inequalities strengthening the linear formulation in a new way. They are different from standard reinforcement and symmetry-breaking inequalities. The standard reinforcement inequalities cut fractional points to improve the lower bound obtained from the linear relaxation, the symmetry-breaking inequalities cut integer points, - which may be optimal- to reduce the search space. In contrast, Dominance inequalities aims at cutting sub-optimal integer points.

This article is organized as follows. In Section 2, we recall structural dominance properties leading to reformulate the UCDDP into a partition problem; then we adapt some neighborhood based dominance properties used on schedules for partitions. Section 3 translates in a linear way the constraints proposed in Section 2, which leads to a new compact linear formulation, enriched with dominance inequalities. Section 4 presents some experimental results to show the contribution of considering dominance in an exact solving process, as in a heuristic approach.

2. Dominance properties

A subset of solutions is said *dominant* if it contains at least one optimal solution, and *strictly dominant* if it contains all the optimal solutions. For brevity, a

schedule will be said dominant (resp. strictly dominant) if it belongs to a dominant (resp. strictly dominant) set. In this article, we will only use dominance properties about solutions, even if there also exist dominance properties about instances or problems (Jouglet & Carlier, 2011).

2.1. Structural dominance properties

In order to describe dominant schedules, we first provide some useful definitions. A task $j \in J$ is said *early* (resp. *tardy*) if it completes at time $C_j \leq d$ (resp. $C_j > d$). Among the early tasks, the one completing at d (if it exists), is called the *on-time* task. The *early-tardy partition* of a schedule is the pair (E, T) where E (resp. T) is the early (resp. tardy) task subset. Moreover, we will use α -ratio (resp. β -ratio) to designate α_j/p_j (resp. β_j/p_j). We define a *block* as a feasible schedule without idle time between task execution, a *d-schedule* as a feasible schedule with an on-time task, and a *d-block* as a *d-schedule* which is also a block. A schedule is said *V-shaped* if early tasks are scheduled in non-decreasing order of their α -ratios and the tardy ones in non-increasing order of their β -ratios.

The following lemma gives dominance properties already known for the unrestrictive common due date problem with symmetric penalties (Hall & Posner, 1991). These results have been extended to asymmetric penalties in Falq et al. (2021), using the same task shifting and exchange proof arguments.

Lemma 1 (Falq et al. (2021))

The set of V-shaped schedules is strictly dominant for the UCDDP.
The set of *d*-blocks is dominant for the UCDDP. Moreover, if unit earliness and tardiness penalties are positive, i.e. $(\alpha, \beta) \in \mathbb{R}_+^{*J} \times \mathbb{R}_+^{*J}$, then the set of blocks is strictly dominant.

Thanks to the *d*-block dominance, we can make some assumptions on parameters p, α, β without loss of generality. A task having a zero processing time can be inserted between two tasks of a schedule without impacting other tasks. In particular in a *d*-block, such a task can be inserted at the due date, which incurs no penalty variation. Hence, we assume that $\forall j \in J, p_j \in \mathbb{R}_+^*$. Since the due date d is large enough, a task having a zero earliness (resp. tardiness) penalty can be added at the beginning (resp. at the end) of a *d*-block without incurring any cost variation. Hence, we assume that $\forall j \in J, (\alpha_j, \beta_j) \in (\mathbb{R}_+^*)^2$.

Since they form a dominant set, we only consider V-shaped *d*-blocks in the following. Note that in such schedules, tardy tasks are completely processed after d , since there is no straddling task, i.e. there is no task starting before d and completing after d . Therefore, the early (resp. tardy) task set can be referred to the left

(resp. right) side of d .

In general, a V-shaped d -block cannot be encoded by its early-tardy partition. Indeed, if two early (resp. tardy) tasks have the same α -ratio (resp. β -ratio), one cannot determine which one is processed first, they can be sequenced arbitrarily. However, swapping two such tasks in a V-shaped d -block results in another V-shaped d -block having the same penalty. Furthermore, all the V-shaped d -block having the same early-tardy partition have the same penalty. Based on this remark, two V-shaped d -blocks having the same early-tardy partition will be said *equivalent*. We denote by \sim this relation.

We define an *ordered bi-partition* of a set A as a couple (A^1, A^2) where $\{A^1, A^2\}$ is a partition of A . This is not only a partition into two subsets since the two subsets are not symmetric, *i.e.* $(A^1, A^2) \neq (A^2, A^1)$. Note that the early-tardy partition of a schedule is an ordered bi-partitions. More precisely, there is a one-to-one correspondence between equivalence classes for \sim and ordered bi-partitions (E, T) of J where $E \neq \emptyset$. Indeed, the set of the early tasks of a d -block cannot be empty since it contains at least the on-time task. Let $\vec{\mathcal{P}}_2^*(J)$ denote the set of such (E, T) , and $\vec{\mathcal{P}}_2(J)$ denote the set of any ordered bi-partition of J . In the sequel, we will only say *partition* to refer to an ordered bi-partition or to an early-tardy partition, when there is no ambiguity.

For any partition $(E, T) \in \vec{\mathcal{P}}_2^*(J)$, let $f(E, T)$ denote the penalty of the equivalence class (E, T) , that is the penalty of any V-shaped d -block of partition (E, T) . For sake of consistency, we extend the definition of f to $\vec{\mathcal{P}}_2(J)$ by setting $f(\emptyset, J)$ as the penalty of any V-shaped block starting at time d . From now on, our aim is to find a partition of J minimizing f , since the UCDDP can be formulated as follows.

$$(F^1) : \min_{(E, T) \in \vec{\mathcal{P}}_2^*(J)} f(E, T)$$

2.2. Neighborhood based dominance properties

Let us recall some definitions used in local search context (Aarts & Lenstra, 2003). A *neighborhood function* N is a function which associates to any solution S a subset of solutions $N(S)$, called the *neighborhood* of S . A solution of $N(S)$ is called a *neighbor* of S . Moreover, a solution S is *locally optimal* with respect to minimizing function ϕ if $\phi(S) \leq \phi(S')$, for any neighbor $S' \in N(S)$. If, on the contrary, there exists $S' \in N(S)$ such that $\phi(S') < \phi(S)$, S is *dominated* (by S').

Given a neighborhood function, the set of locally optimal solutions always contains all optimal solutions, and is therefore a strictly dominant set. This statement can be seen as a generic dominance property denoted \mathcal{G} . This kind of dominance property is commonly used in local search. Indeed, a step of a local search procedure consists in enumerating some of the neighbors of a given solution S , computing their values and, if a better solution is found, then moving to the best solution found at the current iteration, which is equivalent to discard S since it is dominated.

- *Operation-based neighborhoods*

We call *operation* any (eventually partial) function, which maps a solution to another solution. In this work, we will consider neighborhood functions based on a set of operations. The neighborhood of a solution S is then the set of the solutions obtained by applying to S any operation defined on S . This kind of neighborhood functions allows to use the generic dominance property \mathcal{G} in a different way. Instead of considering sequentially the neighborhood of each solution, we will consider sequentially each operation. For each one, any solution is compared to its neighbor obtained using this operation. This differentiates the solution-centered and the operation-centered point of view, as developed in the next section/point.

- *The solution-centered and the operation-centered points of view*

Figures 1 and 2 illustrate these two points of view on the same set of solutions $\{A, B, \dots, N\}$ represented by the blue points. We consider a given set of operations and the associated neighborhood function. An arrow is set from a solution X to a solution Y if X is compared to Y in order to determine whether X is dominated by Y .

In Figure 1, we focus on one solution, J , which is compared to all the solutions obtained by applying to J an operation defined on J . Since J is compared to all its neighborhood, *i.e.* $\{C, F, G, I, M, N\}$, one can determine if J is locally optimal.

In Figure 2, we focus on one operation. All the solutions where this operation can be applied are compared with the obtained neighbors. Solutions A, B, C, G and N are not compared to others solutions, since the considered operation cannot be applied on these solutions. Conversely, since solutions D, E, F, H, I, J, K, L and M are compared to one of their neighbors, they might be discarded. For example if E is better than H , H can be discarded, no matter whether H is better than K . However, we cannot say that the non-discarded solutions are locally optimal, since only one neighbor is taken into account. To this end, all the operations must be considered.

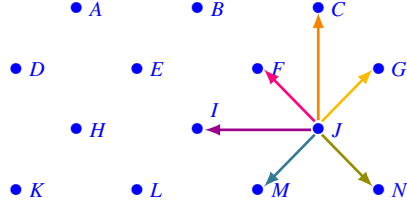


Figure 1: Illustration of the *solution-centered* point of view

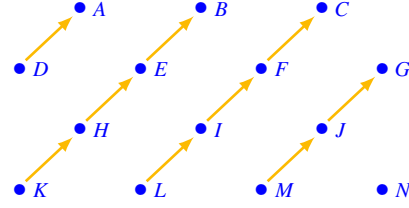


Figure 2: Illustration of the *operation-centered* point of view

Let us now present the main ideas of a generic method to handle operation-based dominance property. First, find operations inducing an objective function variation that can be explicitly expressed. Then derive, for each operation, a constraint ensuring that any solution is not dominated by applying this operation. Using all the obtained constraints, only the dominant set of property \mathcal{G} subsists as all the non-locally optimal solutions have been removed (where local is understood with respect to the neighborhood defined by the operations).

In Section 2.3, we introduce two families of operations over the partitions, and follow this method, resulting in constraints (I_u) , (I'_u) and $(S_{u,v})$ translating Property 3.

2.3. Operations for early-tardy partitions

In the case of problems where the solutions are partitions, two operations are commonly considered to define a neighborhood: the *insertion*, which consists in moving two elements, one in each subset and the *swap* which consists in swapping two elements of each subset. Let us define solutions which are locally optimal in the neighborhoods associated with these operations.

Definition 2

- Let $(E, T) \in \vec{\mathcal{P}}_2^*(J)$.
- (i) (E, T) is an *insert local optimum* if $\begin{cases} \forall v \in T, f(E, T) \leq f(E \cup \{v\}, T \setminus \{v\}), \\ \forall u \in E, f(E, T) \leq f(E \setminus \{u\}, T \cup \{u\}). \end{cases}$
 - (ii) (E, T) is a *swap local optimum* if $\forall u \in E, \forall v \in T, f(E, T) \leq f(E \setminus \{u\} \cup \{v\}, T \cup \{u\} \setminus \{v\})$.

Property 3

The set of insert locally optimal partitions, as well as the set of swap locally optimal partitions, is strictly dominant when minimizing f over $\vec{\mathcal{P}}_2^*(J)$.

In terms of scheduling, the insert operation consists in removing a task $j \in J$ from the early (resp. tardy) side and inserting j on the tardy (resp. early) side, as early (resp. as tardy) as possible according to its β -ratio (resp. α -ratio). The tasks scheduled before (resp. after) j are right-shifted (resp. left-shifted) by p_j time units. The swap operation consists in sequentially inserting an early task on the tardy side and inserting another tardy task on the early side, or vice-versa, as described above. Let us denote by $\text{insert}_u(\mathcal{S})$ (resp. $\text{swap}_{u,v}(\mathcal{S})$) the schedule obtained from a schedule \mathcal{S} by the insert (resp. swap) operation on task u (resp. on tasks u, v).

Since the insert and swap operations are fundamentally defined on partitions (*i.e.* over equivalence classes of V-shaped d -blocks), these operations preserve the equivalence. Therefore we have $\forall u \in J, \mathcal{S} \sim \mathcal{S}' \Rightarrow \text{insert}_u(\mathcal{S}) \sim \text{insert}_u(\mathcal{S}')$ and $\forall (u, v) \in J^2, \mathcal{S} \sim \mathcal{S}' \Rightarrow \text{swap}_{u,v}(\mathcal{S}) \sim \text{swap}_{u,v}(\mathcal{S}')$, assuming that u and v are not on the same side in these schedules.

In order to provide an expression of the penalty variation induced by such an operation, it is convenient to choose a specific V-shaped d -block as representative of an equivalence class. More precisely, the chosen representative will depend on the inserted task (resp. on the swapped tasks). Let us introduce some notations, related to a given task $u \in J$. To describe the early side of a V-shaped d -block with regard to u , the set of remaining tasks $J \setminus \{u\}$ can be split according to their α -ratio into two subsets $A(u)$ and $\bar{A}(u)$ defined as follows.

$$A(u) = \left\{ i \in J \mid \frac{\alpha_i}{p_i} > \frac{\alpha_u}{p_u} \right\} \quad \text{and} \quad \bar{A}(u) = \left\{ i \in J \setminus \{u\} \mid \frac{\alpha_i}{p_i} \leq \frac{\alpha_u}{p_u} \right\}$$

Similarly, we introduce the two following subsets to describe the tardy side.

$$B(u) = \left\{ i \in J \mid \frac{\beta_i}{p_i} > \frac{\beta_u}{p_u} \right\} \quad \text{and} \quad \bar{B}(u) = \left\{ i \in J \setminus \{u\} \mid \frac{\beta_i}{p_i} \leq \frac{\beta_u}{p_u} \right\}$$

Note that if u is early in a V-shaped d -block, early tasks belonging to $A(u)$ are necessarily scheduled after u , because of their α -ratio. To illustrate, we can observe in Figure 3 that the tasks of $A(u) \cap E$ are scheduled after u in the schedule \mathcal{S} . Conversely, an early task of $\bar{A}(u)$ is not necessarily scheduled before u , when its α -ratio is the same as u . However, we will consider a representative where u is placed after all the early tasks of $\bar{A}(u)$, that is as tardy as possible according to its α -ratio. Similarly, in case where u is tardy, we will consider a representative where u is scheduled before all tardy tasks of $\bar{B}(u)$. Such a representative will be called a *u-canonical* V-shaped d -block. The schedule \mathcal{S} (resp. \mathcal{S}') in Figure 3 represents the shape of a *u-canonical* V-shaped d -block when u is early (resp. tardy).

- *Penalty variation induced by an insert operation*

Given $u \in J$, let (E, T) be a partition such that $u \in E$. We aim to express the variation of f induced by the insertion of u in T , i.e. between (E, T) and $(E \setminus \{u\}, T \cup \{u\})$. To this end, we consider a u -canonical representative \mathcal{S} of (E, T) , and the V-shaped d -block \mathcal{S}' obtained from \mathcal{S} by inserting u in T , as early as possible, that is just after tardy tasks of $B(u)$. Note that \mathcal{S}' is thus a u -canonical representative of $(E \setminus \{u\}, T \cup \{u\})$. Let (e, t) (resp. (e', t')) denote the earliness and tardiness vector of tasks in \mathcal{S} (resp. \mathcal{S}').

As we can observe in Figure 3, early tasks of $A(u)$ and tardy tasks of $B(u)$ are identically scheduled in \mathcal{S} and \mathcal{S}' . The penalty variation induced by the insertion of u is then only due to the move of u and to the right-shifting of early tasks of $\bar{A}(u)$ and tardy tasks of $\bar{B}(u)$.

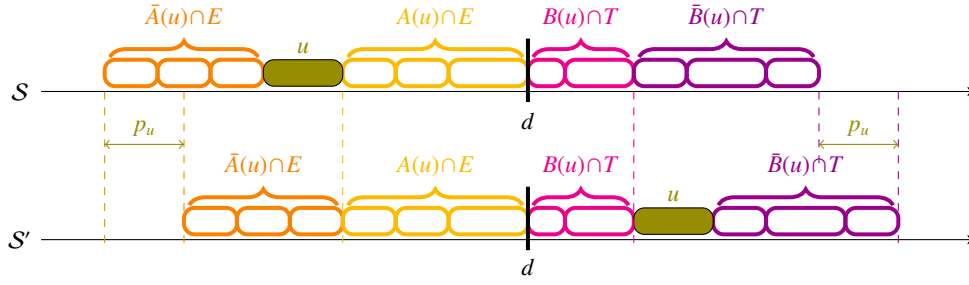


Figure 3: Illustration of the insert operation of an early task u on the tardy side

Each task $j \in \bar{A}(u) \cap E$ of \mathcal{S} is postponed p_u time units later in \mathcal{S}' , while staying early. We then have $e'_j = e_j - p_u$. The earliness penalty of task j in \mathcal{S}' is therefore $\alpha_j e'_j = \alpha_j e_j - \alpha_j p_u$, which represents a reduction of $\alpha_j p_u$ compared to its earliness penalty in \mathcal{S} . Summing up over $\bar{A}(u) \cap E$, we obtain a reduction of the earliness penalties of $p_u \alpha(\bar{A}(u) \cap E)$, where $x(I) = \sum_{i \in I} x_i$ for any $x \in \mathbb{R}^n$ and any $I \subseteq [1..n]$. Similarly, the right-shifting of tasks in $\bar{B}(u) \cap T$ induces an increase of the tardiness penalties of $p_u \beta(\bar{B}(u) \cap T)$, since for each $j \in \bar{B}(u) \cap T$ we have $\beta_j t'_j = \beta_j (t_j + p_u)$.

Moreover, since $e_u = p(A(u) \cap E)$, removing u from the early side induces a reduction of its earliness penalty of $\alpha_u p(A(u) \cap E)$. Similarly, introducing u on the tardy side induces an increase of its tardiness penalty of $\beta_u t'_u = \beta_u (p(B(u) \cap T) + p_u)$ since $t_u = 0$.

Finally, the penalty variation between \mathcal{S} and \mathcal{S}' , is given by the following expression.

$$\Delta_u(E, T) = -\alpha_u p(A(u) \cap E) + \beta_u (p(B(u) \cap T) + p_u) + p_u (\beta(\bar{B}(u) \cap T) - \alpha(\bar{A}(u) \cap E))$$

Since \mathcal{S} and \mathcal{S}' are representative of (E, T) and $(E \setminus \{u\}, T \cup \{u\})$ respectively, $\Delta_u(E, T)$ is also the variation of f induced by the insertion of u in T . Property 4.(i) follows.

The insert operation of the tardy task u on the early side applied on schedule \mathcal{S}' provides schedule \mathcal{S} . Note that this statement requires that \mathcal{S} is u -canonical. This observation allows to establish that the penalty variation induced by inserting u on the early side in \mathcal{S}' is simply $-\Delta_u(E, T)$, and results in Property 4.(ii).

Property 4

- Let (E, T) be a partition.
- (i) For any $u \in E$, $f(E \setminus \{u\}, T \cup \{u\}) = f(E, T) + \Delta_u(E, T)$.
 - (ii) For any $v \in T$, $f(E \cup \{v\}, T \setminus \{v\}) = f(E, T) - \Delta_v(E, T)$.

Let us introduce, for a given task $u \in J$, the two following constraints.

$$u \in E \Rightarrow \Delta_u(E, T) \geq 0 \quad (I_u)$$

$$u \in T \Rightarrow \Delta_u(E, T) \leq 0 \quad (I'_u)$$

Thanks to Property 4, (I_u) (resp. (I'_u)) discards every partition where u is early (resp. tardy), but which would have a lower penalty if u were tardy (resp. early). Note that the constraint $\Delta_u(E, T) \geq 0$ (resp. $\Delta_u(E, T) \leq 0$) might be not satisfied by an optimal solution where u is tardy (resp. early).

Moreover, only considering constraints (I_u) and (I'_u) for a given u is not sufficient to discard all insert-dominated partitions. Indeed, these constraints do not take into account the whole insert-neighborhood of the solutions as each one is only compared to its neighbor obtained by an insert operation on task u . It is thus needed to consider constraints (I_u) and (I'_u) for every $u \in J$ to translate the dominance of the insert local optimal solutions. In Section 3, we will explain how these constraints can be used in a linear formulation.

• *Penalty variation induced by a swap operation*

This section follows the same organization as the previous one. The objective is to obtain constraints translating the dominance of the swap locally optimal solutions.

Given $(u, v) \in J^2$ such that $u \neq v$, let (E, T) be a partition such that $u \in E$ and $v \in T$, and \mathcal{S} be a u and v -canonical representative of (E, T) . We denote by \mathcal{S}' the schedule obtained from \mathcal{S} by swapping u and v so that \mathcal{S}' is also both u and v -canonical, that is by scheduling u after all the early tasks having the same α -ratio α_u/p_u and v before all the tardy tasks having the same β -ratio β_v/p_v . \mathcal{S}' is a representative of $(E \setminus \{u\} \cup \{v\}, T \setminus \{v\} \cup \{u\})$. Let (e, t) (resp. (e', t')) denote the earliness

and tardiness vector of tasks in \mathcal{S} (resp. \mathcal{S}').

If $\frac{\alpha_v}{p_v} < \frac{\alpha_u}{p_u}$, all the early tasks of $\bar{A}(v)$ are scheduled before u in \mathcal{S} , since the V-shaped property holds, but the early tasks of $A(v) \setminus \{u\}$ can be scheduled before or after u . This case is illustrated in Figure 4. Conversely, if $\frac{\alpha_v}{p_v} \geq \frac{\alpha_u}{p_u}$, all the early tasks of $A(v)$ are scheduled after u in \mathcal{S} , but those of $\bar{A}(v) \setminus \{u\}$ can be scheduled before or after u . This case is illustrated in Figure 5.

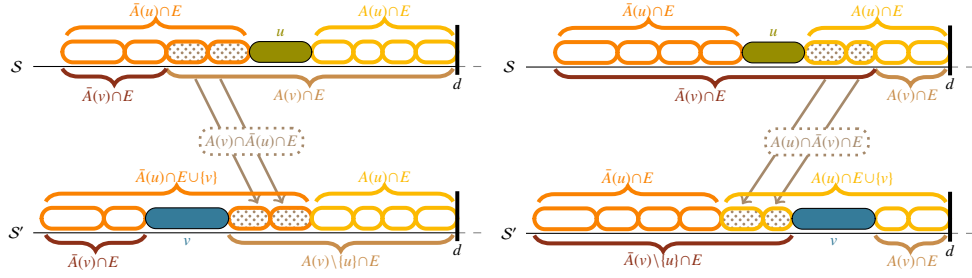


Figure 4: Early side variation induced by swapping u and v when $\frac{\alpha_v}{p_v} < \frac{\alpha_u}{p_u}$

Figure 5: Early side variation induced by swapping u and v when $\frac{\alpha_v}{p_v} \geq \frac{\alpha_u}{p_u}$

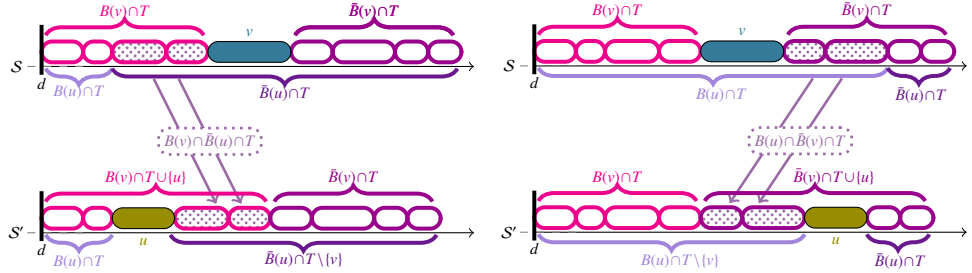


Figure 6: Tardy side variation induced by swapping u and v when $\frac{\beta_v}{p_v} \leq \frac{\beta_u}{p_u}$

Figure 7: Tardy side variation induced by swapping u and v when $\frac{\beta_v}{p_v} > \frac{\beta_u}{p_u}$

Similarly, we can distinguish two cases depending on the relative order of the β -ratios of u and v , as illustrated in Figures 6 and 7.

As shown in Figures 4 and 5, the earliness of u in \mathcal{S} is $e_u = p(A(u) \cap E)$, while the tardiness of u in \mathcal{S}' is $e'_u = p(B(u) \setminus \{v\} \cap T) + p_u$ (Cf. Figures 6 and 7). Note that v is removed from $B(u)$ since v is not tardy in \mathcal{S}' , and therefore cannot contribute to the tardiness of u . Similarly, we have $t_v = p(B(v) \cap T) + p_v$ and $e'_v = p(A(v) \setminus \{u\} \cap E)$ since u is not early in \mathcal{S}' .

Moreover, the tasks of $A(u) \cap E$ are identically scheduled in \mathcal{S} and \mathcal{S}' only if $\frac{\alpha_v}{p_v} < \frac{\alpha_u}{p_u}$. In this case, tasks of $\bar{A}(u) \cap E$ are not consecutive in \mathcal{S}' since v separates them into two blocks: tasks of $\bar{A}(v) \cap E$ which have been left-shifted by $p_v - p_u$ time units, and tasks of $A(v) \cap \bar{A}(u) \cap E$ which have been right-shifted by p_u time units (Cf. Figure 4).

In the opposite case, *i.e.* if $\frac{\alpha_v}{p_v} \geq \frac{\alpha_u}{p_u}$, tasks of $A(u) \cap E$ are not consecutive in \mathcal{S}' . Indeed, v separates them into two blocks: tasks of $A(v) \cap E$ which are identically scheduled in \mathcal{S} and \mathcal{S}' ; and tasks of $A(u) \cap \bar{A}(v) \cap E$ which are left-shifted by p_v time units. Moreover, in that case tasks of $A(v) \cap E$ are left-shifted by $p_v - p_u$ time units (Cf. Figure 5).

Note that, in the two previous paragraphs, as in Figures 4 to 7, we assumed that $p_v - p_u \geq 0$. In the contrary case, *i.e.* if $p_v - p_u < 0$, tasks are not left-shifted by $p_v - p_u$ time units but rather right-shifted by $p_u - p_v$ time units.

From these observations, we can express the earliness penalty variation of all the tasks in $E \setminus \{u\}$ by expressing, in each case, the penalty variation induced by a block shifting as described for the insert operation. The same method can be applied for the tasks of $T \setminus \{v\}$. The reader can refer to Figures 6 and 7 for illustration. Finally, the penalty variation between \mathcal{S} and \mathcal{S}' is given by the following expression.

$$\begin{aligned} \Delta_{u,v}(E, T) = & -\alpha_u p(A(u) \cap E) + \beta_u (p(B(u) \setminus \{v\} \cap T) + p_u) \\ & - \beta_v (p(B(v) \cap T) + p_v) + \alpha_v p(A(v) \setminus \{u\} \cap E) \\ & + \begin{cases} (-p_u + p_v) \alpha(\bar{A}(v) \cap E) - p_u \alpha(A(v) \cap \bar{A}(u) \cap E) & \text{if } \frac{\alpha_v}{p_v} < \frac{\alpha_u}{p_u} \\ (-p_u + p_v) \alpha(\bar{A}(u) \cap E) + p_v \alpha(A(u) \cap \bar{A}(v) \cap E) & \text{otherwise} \end{cases} \\ & + \begin{cases} (p_u - p_v) \beta(\bar{B}(v) \cap T) + p_u \beta(B(v) \cap \bar{B}(u) \cap T) & \text{if } \frac{\beta_v}{p_v} \leq \frac{\beta_u}{p_u} \\ (p_u - p_v) \beta(\bar{B}(u) \cap T) - p_v \beta(B(u) \cap \bar{B}(v) \cap T) & \text{otherwise} \end{cases} \end{aligned}$$

Property 5

Let (E, T) be a partition.

For any $(u, v) \in E \times T$, $f(E \setminus \{u\} \cup \{v\}, T \setminus \{v\} \cup \{u\}) = f(E, T) + \Delta_{u,v}(E, T)$.

For a given pair of tasks $(u, v) \in J^2$ such that $u \neq v$, let us introduce the following constraint.

$$(u, v) \in E \times T \Rightarrow \Delta_{u,v}(E, T) \geq 0 \quad (S_{u,v})$$

Thanks to Property 5, $(S_{u,v})$ discards every partition where u is early, v is tardy and which would have a lower penalty in the contrary case. As for the insert operation, constraint $\Delta_{u,v}(E, T) \geq 0$ might be not satisfied by optimal partitions where u is not early or v is not tardy. Moreover, it is needed to consider constraints $(S_{u,v})$ for every $(u, v) \in J^2$ such that $u \neq v$ to translate the dominance of swap locally optimal solutions. In Section 3, we will explain how these constraints can be used in a linear formulation.

3. Neighborhood based dominance in linear programming

The dominance properties described in Section 2 can be used in a linear programming framework. In this section, we provide linear inequalities translating constraints (I_u) , (I'_u) and $(S_{u,v})$ for all tasks u and v .

3.1. A compact MIP translating F^1

We first recall the linear compact MIP formulation for the UCDDP given in Falq et al. (2021). In this formulation denoted by F^2 , a partition is encoded by a vector δ of n binary variables. Given $\delta \in \{0, 1\}^J$, the partition encoded by δ is $(\{j \in J \mid \delta_j = 1\}, \{j \in J \mid \delta_j = 0\})$. In other words, for each $j \in J$, δ_j indicates whether j is early or not.

Moreover, F^2 also uses X , a vector of binary variables indexed by the set $J^< = \{(i, j) \in J^2 \mid i < j\}$, in order to linearize products of type $\delta_i \delta_j$ as proposed by Fortet (1959).

Lemma 6 (Fortet (1959))

Let $(\delta, X) \in \mathbb{R}^J \times \mathbb{R}^{J^<}$.

If $\delta \in \{0, 1\}^J$ and (δ, X) satisfies the following inequalities:

$$\forall (i, j) \in J^<, X_{i,j} \geq \delta_i - \delta_j \quad (1) \qquad \forall (i, j) \in J^<, X_{i,j} \leq \delta_i + \delta_j \quad (3)$$

$$\forall (i, j) \in J^<, X_{i,j} \geq \delta_j - \delta_i \quad (2) \qquad \forall (i, j) \in J^<, X_{i,j} \leq 2 - (\delta_i + \delta_j) \quad (4)$$

then $\forall (i, j) \in J^<, X_{i,j} = \begin{cases} 1 & \text{if } \delta_i \neq \delta_j \\ 0 & \text{otherwise} \end{cases}$, $(1 - \delta_i)(1 - \delta_j) = \frac{2 - (\delta_i + \delta_j) - X_{i,j}}{2}$ and $\delta_i \delta_j = \frac{\delta_i + \delta_j - X_{i,j}}{2}$.

Let us consider the polyhedron $P = \{(\delta, X) \in \mathbb{R}^J \times \mathbb{R}^{J^<} \mid (1-4)\}$ and the set of its integer points $\text{int}_\delta(P) = P \cap \{0, 1\}^J \times \{0, 1\}^{J^<}$. From Lemma 6, if a partition (E, T) is encoded by δ , there exists a unique X such that $(\delta, X) \in \text{int}_\delta(P)$. We will say that (δ, X) encodes (E, T) .

To obtain an expression of the penalty of a partition from its encoding (δ, X) , two orders on J are introduced. Let ρ and σ be two functions from $[1..n]$ to J , such that

$$\left(\frac{\alpha_{\rho(k)}}{p_{\rho(k)}} \right)_{k \in [1..n]} \quad \text{and} \quad \left(\frac{\beta_{\sigma(k)}}{p_{\sigma(k)}} \right)_{k \in [1..n]} \quad \text{are non-increasing}$$

A d -block is said ρ - σ -*shaped* if the early (resp. tardy) tasks are processed in decreasing order of ρ^{-1} (resp. increasing order of σ^{-1}). Each equivalence class of V-shaped d -blocks admits a unique ρ - σ -shaped representative. This representative is used to provide the following expression of the penalty of the partition encoded by $(\delta, X) \in \{0, 1\}^J \times \{0, 1\}^{J^<}$.

$$\begin{aligned} g(\delta, X) = & \sum_{j \in J} \alpha_j \left(\sum_{k=1}^{\rho^{-1}(j)-1} p_{\rho(k)} \frac{\delta_j + \delta_{\rho(k)} - X_{j, \rho(k)}}{2} \right) \\ & + \beta_j \left(\sum_{k=1}^{\sigma^{-1}(j)-1} p_{\sigma(k)} \frac{2 - (\delta_j + \delta_{\sigma(k)}) - X_{j, \sigma(k)}}{2} + p_j(1 - \delta_j) \right) \end{aligned}$$

Finally, the compact formulation for the UCDDP provided in Falq et al. (2021) is the following.

$$(F^2) : \quad \min_{(\delta, X) \in \text{int}_\delta(P)} g(\delta, X)$$

Formulation F^2 is a direct linear translation of F^1 . Indeed, there is a one to one correspondence between their solution sets, *i.e.* between $\text{int}_\delta(P)$ and $\vec{\mathcal{P}}_2(J)$, and $g(\delta, X) = f(E, T)$ for any (δ, X) encoding (E, T) .

3.2. Linear inequalities translating constraints (I_u) , (I'_u) and $(S_{u,v})$

- Linear inequalities translating constraints (I_u) and (I'_u) for any $u \in J$

Let $u \in J$. If $\delta \in \{0, 1\}^J$ encodes a partition (E, T) , the penalty variation $\Delta_u(E, T)$ can be expressed linearly from δ as follows.

$$\Delta_u(\delta) = -\alpha_u \sum_{i \in A(u)} p_i \delta_i + \beta_u \left(\sum_{i \in B(u)} p_i(1 - \delta_i) + p_u \right) + p_u \left(\sum_{i \in \bar{B}(u)} \beta_i(1 - \delta_i) - \sum_{i \in \bar{A}(u)} \alpha_i \delta_i \right)$$

Moreover, if $\delta \in \{0, 1\}^J$ encodes (E, T) , we can translate constraint (I_u) as follows.

$$\begin{aligned} (E, T) \text{ satisfies } (I_u) & \Leftrightarrow (u \in E \text{ and } \Delta_u(E, T) \geq 0) \text{ or } u \in T \\ & \Leftrightarrow (\delta_u = 1 \text{ and } \Delta_u(E, T) \geq 0) \text{ or } \delta_u = 0 \end{aligned}$$

To unify these two cases into one inequality, we introduce the following constant which is an upper bound of $-\Delta_u(\delta)$ for any $\delta \in \{0, 1\}^J$.

$$M_u = \alpha_u p(A(u)) - \beta_u p_u + p_u \alpha(\bar{A}(u))$$

Since we have $\Delta_u(\delta) \geq -M_u$ for any $\delta \in \{0, 1\}^J$, the following inequality is satisfied by every $\delta \in \{0, 1\}^J$ such that $(1 - \delta_u) = 1$.

$$\Delta_u(\delta) \geq -M_u(1 - \delta_u) \quad (5)$$

Conversely, for every $\delta \in \{0, 1\}^J$ such that $(1 - \delta_u) = 0$, inequality (5) is satisfied if and only if $\Delta_u(\delta) \geq 0$.

Considering these two cases, $\delta \in \{0, 1\}^J$ satisfies (5) if and only if the partition encoded by δ satisfies (I_u) . Property 7.(i) follows.

Similarly, to translate constraint (I'_u) , we introduce the following constant which is an upper bound of $-\Delta_u(\delta)$ for any $\delta \in \{0, 1\}^J$.

$$M'_u = \beta_u p(B(u)) + \beta_u p_u + p_u \beta(\bar{B}(u))$$

Since we have $\forall \delta \in \{0, 1\}^J$, $-\Delta_u(\delta) \geq -M'_u$, the following inequality is satisfied by every $\delta \in \{0, 1\}^J$ such that $\delta_u = 1$.

$$-\Delta_u(\delta) \geq -M'_u \delta_u \quad (6)$$

Conversely, if $\delta_u = 0$, δ satisfies (6) if and only if $-\Delta_u(\delta) \geq 0$.

Considering these two cases, $\delta \in \{0, 1\}^J$ satisfies (6) if and only if the partition encoded by δ satisfies (I_u) . Property 7.(ii) follows.

Property 7

- Let $\delta \in \{0, 1\}^J$ and let (E, T) be the partition encoded by δ . For any $u \in J$,
- (i) δ satisfies inequality (5) **if and only if** (E, T) satisfies constraint (I_u) .
 - (ii) δ satisfies inequality (6) **if and only if** (E, T) satisfies constraint (I'_u) .

- *Linear inequalities translating constraint $(S_{u,v})$ for any $(u, v) \in J^2$*

Let $(u, v) \in J^2$ such that $u \neq v$. If $\delta \in \{0, 1\}^J$ encodes a partition (E, T) , the penalty variation $\Delta_{u,v}(E, T)$ can be expressed linearly from δ as follows.

$$\begin{aligned}
\Delta_{u,v}(\delta) = & -\alpha_u \sum_{i \in A(u)} p_i \delta_i + \beta_u \sum_{i \in B(u) \setminus \{v\}} p_i (1 - \delta_i) + \beta_u p_u - \beta_v \sum_{i \in B(v)} p_i (1 - \delta_i) - \beta_v p_v + \alpha_v \sum_{i \in A(v) \setminus \{u\}} p_i \delta_i \\
& + \begin{cases} (-p_u + p_v) \sum_{i \in \bar{A}(v)} \alpha_i \delta_i - p_u \sum_{i \in A(v) \cap \bar{A}(u)} \alpha_i \delta_i & \text{if } \frac{\alpha_v}{p_v} < \frac{\alpha_u}{p_u} \\ (-p_u + p_v) \sum_{i \in \bar{A}(u)} \alpha_i \delta_i + p_v \sum_{i \in A(u) \cap \bar{A}(v)} \alpha_i \delta_i & \text{otherwise} \end{cases} \\
& + \begin{cases} (p_u - p_v) \sum_{i \in \bar{B}(v)} \beta_i (1 - \delta_i) + p_u \sum_{i \in B(v) \cap \bar{B}(u)} \beta_i (1 - \delta_i) & \text{if } \frac{\beta_v}{p_v} \leq \frac{\beta_u}{p_u} \\ (p_u - p_v) \sum_{i \in \bar{B}(u)} \beta_i (1 - \delta_i) - p_v \sum_{i \in B(u) \cap \bar{B}(v)} \beta_i (1 - \delta_i) & \text{otherwise} \end{cases}
\end{aligned}$$

Moreover, if $\delta \in \{0, 1\}^J$ encodes (E, T) , we can translate constraint $(S_{u,v})$ as follows.

$$\begin{aligned}
(E, T) \text{ satisfies } (S_{u,v}) & \Leftrightarrow (u, v) \notin E \times T \text{ or } \Delta_{u,v}(E, T) \geq 0 \\
& \Leftrightarrow (u \notin E \text{ or } v \notin T) \text{ or } (u \in E \text{ and } v \in T \text{ and } \Delta_{u,v}(E, T) \geq 0) \\
& \Leftrightarrow ((1 - \delta_u) = 1 \text{ or } \delta_v = 1) \text{ or } (((1 - \delta_u) = 0 \text{ and } \delta_v = 0) \text{ and } \Delta_{u,v}(\delta) \geq 0) \\
& \Leftrightarrow (\delta_v + (1 - \delta_u)) \in \{1, 2\} \text{ or } ((\delta_v + (1 - \delta_u)) = 0 \text{ and } \Delta_{u,v}(\delta) \geq 0)
\end{aligned}$$

To unify these cases into one inequality, we introduce the following constant $\widetilde{M}_{u,v}$ which is an upper bound on $-\Delta_{u,v}(\delta)$ for any $\delta \in \{0, 1\}^J$.

$$\begin{aligned}
\widetilde{M}_{u,v} = & \alpha_u p(A(u)) - \beta_u p_u + \beta_v p(B(v)) + \beta_v p_v \\
& + \begin{cases} [p_u - p_v]^+ \alpha(\bar{A}(v)) + p_u \alpha(A(v) \cap \bar{A}(u)) & \text{if } \frac{\alpha_v}{p_v} < \frac{\alpha_u}{p_u} \\ [p_u - p_v]^+ \alpha(\bar{A}(u)) & \text{if } \frac{\alpha_v}{p_v} \geq \frac{\alpha_u}{p_u} \end{cases} \\
& + \begin{cases} [p_v - p_u]^+ \beta(\bar{B}(v)) & \text{if } \frac{\beta_v}{p_v} \leq \frac{\beta_u}{p_u} \\ [p_v - p_u]^+ \beta(\bar{B}(u)) + p_v \beta(B(u) \cap \bar{B}(v)) & \text{if } \frac{\beta_v}{p_v} > \frac{\beta_u}{p_u} \end{cases}
\end{aligned}$$

Since we have $\Delta_{u,v}(\delta) \geq -\widetilde{M}_{u,v}$ for any $\delta \in \{0, 1\}^J$, the inequality $\Delta_{u,v}(\delta) \geq -\widetilde{M}_{u,v}(\delta_v + (1 - \delta_u))$ is satisfied for every $\delta \in \{0, 1\}^J$ such that $(\delta_v + (1 - \delta_u)) = 1$. In particular, this inequality is satisfied for every $\delta \in \{0, 1\}^J$ such that $(\delta_u, \delta_v) = (0, 0)$ or $(1, 1)$. For

$\delta \in \{0, 1\}^J$ such that $(\delta_v + (1 - \delta_u)) = 2$, *i.e.* such that $(\delta_u, \delta_v) = (0, 1)$, this inequality might not be satisfied if $\tilde{M}_{u,v} \leq 0$, since $-\tilde{M}_{u,v} \not\geq -2\tilde{M}_{u,v}$ in this case. To provide an inequality satisfied by every $\delta \in \{0, 1\}^J$ such that $(\delta_u, \delta_v) \neq (1, 0)$ we introduce the following constant.

$$M_{u,v} = \begin{cases} \tilde{M}_{u,v} & \text{if } \tilde{M}_{u,v} \geq 0 \\ \tilde{M}_{u,v}/2 & \text{otherwise} \end{cases}$$

We then have $-\tilde{M}_{u,v} \geq -2M_{u,v}$ and $-\tilde{M}_{u,v} \geq -M_{u,v}$. Therefore, the following inequality is satisfied by every $\delta \in \{0, 1\}^J$ such that $(\delta_v + (1 - \delta_u)) \in \{1, 2\}$, *i.e.* such that $(\delta_u, \delta_v) \neq (1, 0)$.

$$\Delta_{u,v}(\delta) \geq -M_{u,v}(\delta_v + (1 - \delta_u)) \quad (7)$$

Conversely, for every $(\delta_v + (1 - \delta_u)) = 0$, $\delta \in \{0, 1\}^J$ such that *i.e.* such that $(\delta_u, \delta_v) = (1, 0)$, inequality (7) is satisfied if and only if $\Delta_{u,v}(\delta) \geq 0$. Finally, $\delta \in \{0, 1\}^J$ satisfies (7) if and only if the partition encoded by δ satisfies $(S_{u,v})$. Property 8 follows.

Property 8

Let $\delta \in \{0, 1\}^J$ and let (E, T) be the partition encoded by δ . For any $(u, v) \in J^2$ such that $u \neq v$, δ satisfies (7) **if and only if** (E, T) satisfies constraint $(S_{u,v})$.

In the sequel, insert inequalities (5) and (6) and swap inequalities (7) will be called *dominance inequalities*.

3.3. Link between dominance properties and operations

In this section, we show how to benefit from the fact that each dominance inequality is based on an operation. When a vector $\delta \in \{0, 1\}^J$ encoding a partition (E, T) does not satisfy a dominance inequality, applying the corresponding operation to (E, T) provides a partition with a strictly lower penalty. The following property, resulting from properties 4, 5, 7 and 8, formally states this result.

Property 9

Let $\delta \in \{0, 1\}^J$ and let (E, T) be the partition encoded by δ . For any $u \in J$,
 (i) δ does not satisfy (5) for u **if and only if** $u \in E$ and $f(E \setminus \{u\}, T \cup \{u\}) < f(E, T)$,
 (ii) δ does not satisfy (6) for u **if and only if** $u \in T$ and $f(E \cup \{u\}, T \setminus \{u\}) < f(E, T)$.
 (iii) Moreover, for any $(u, v) \in J^2$ such that $u \neq v$, δ does not satisfy (7) for (u, v) **if and only if** $(u, v) \in E \times T$ and $f(E \setminus \{u\} \cup \{v\}, T \setminus \{v\} \cup \{u\}) < f(E, T)$.

Proof: Let us fix $u \in J$. From Property 7, δ does not satisfy inequality (5) if and only if (E, T) does not satisfy constraints (I_u) , which is equivalent to $u \in E$ and $\Delta_u(E, T) < 0$.

Using Property 4, it is equivalent to $u \in E$ and $f(E \setminus \{u\} \cup \{v\}, T \setminus \{v\} \cup \{u\}) - f(E, T) < 0$, which proves (i). The proofs of (ii) and (iii) follow the same scheme. \square

Property 9 will be used in Section 4 to propose a local search procedure. The following corollary, which directly derives from the negation of statements (i), (ii) and (iii), ensures that the solution provided by this local search procedure is an insert and swap local optimum.

Corollary 10

Let $\delta \in \{0, 1\}^J$ and let (E, T) be the partition encoded by δ .

- (i) (E, T) is an insert local optimum **if and only if** δ satisfies inequalities (5) and (6) for all $u \in J$.
- (ii) (E, T) is a swap local optimum **if and only if** δ satisfies inequality (7) for all $(u, v) \in J^2$ such that $u \neq v$.

The following section presents experimental results to assess the practical relevance of the dominance inequalities.

4. Numerical results

All experiments are carried out using a single thread with Intel(R) Xeon(R) X5677, @ 3.47GHz, and 144Gb RAM. Linear programs (LP) and MIP are solved with Cplex 12.6.3.0.

The numerical experiments are performed on the instance benchmark proposed by Biskup & Feldmann (2001a), available online on OR-Library (Biskup & Feldmann, 2001b). For each number of tasks $n \in \{10, 20, 50, 100, 200\}$, ten triples (p, α, β) of $(\mathbb{N}^{*n})^3$ are given. For each one, we assume that $d = p(J)$, so that the due date is unrestrictive. For the sake of comparison, we additionally construct instances with $n \in \{60, 80\}$ (resp. $n \in \{120, 150, 180\}$) by only considering the first n tasks of the previous 100-task (resp. 200-task) instances. Unless otherwise specified, the gap, time and number of nodes presented in the following tables are average values over the ten instances for a given n and the time limit is set to 3600 seconds.

To measure the improvement induced by the insert or swap inequalities, we compare the four following formulations.

- F^2 : the formulation defined in Section 3, only with inequalities (1-4)
- F^i : the formulation obtained from F^2 by adding (5) and (6) for all $u \in J$
- F^s : the formulation obtained from F^2 by adding (7) for all $(u, v) \in J^2$ s.t. $u \neq v$
- F^{i+s} : the formulation obtained from F^s by adding (5) and (6) for all $u \in J$

For a given formulation F , we distinguish two settings: a setting with all available features, that is using Cplex default, denoted by F_d , and a setting with less Cplex features, denoted by F_l . Two types of features are disabled in this setting: the cut generation, which produces reinforcement inequalities and adds them to the formulation, and the primal heuristic procedures. The cut generation is disabled in order to measure the impact of the dominance inequalities on the linear relaxation value of the formulation F^2 , rather than their impact on the linear relaxation value of a strengthened formulation. The primal heuristic procedures have been disabled to focus on the lower bound since we have other methods to quickly obtain good feasible solutions (Cf. Section 4.3).

This results in eight formulation settings: $F_l^2, F_l^i, F_l^s, F_l^{i+s}, F_d^2, F_d^i, F_d^s$ and F_d^{i+s} . For each one, inequalities (1-4), as well as inequalities (5-7) when included, are added initially.

Let us recall that only the δ variables need to be integer in F^2 . Indeed, from Lemma 6, if $\delta \in \{0, 1\}^J$, inequalities (1-4) ensures that $X \in \{0, 1\}^{J^<}$. It is also the case for F^i, F^s and F^{i+s} . Therefore, unless otherwise specified, δ variables are set as binary variables, while X variables are set as continuous variables. Consequently, the branching decisions only involve δ .

4.1. Solving MIP formulations to optimality

Table 1 provides the results obtained by solving MIP to optimality, using the eight formulation settings. Each line corresponds to the ten instances of a given size n . More precisely, Table 1 entries are the following.

- #opt : the number of instances solved to optimality within the time limit
- time : the average running time in seconds over the instances solved to optimality
- #nd : the average number of nodes, except the root node, in the search tree, over the instances solved to optimality

For a given formulation setting, we choose to stop the run at a line of the table if less than 5 over ten instances are solved to optimality. For the subsequent lines, we report a "-" in the table.

Using formulation setting F_l^2 , the ten n -task instances are solved to optimality within the time limit for n up to 50. In contrast, using F_l^i , it is the case for n up to 60, using F_l^s for n up to 120 and using F_l^{i+s} for n up to 150. Within approximately

5 minutes, F_l^2 solves 50-task instances, F_l^i 60-task instances, F_l^s 100-task instances and F_l^{i+s} 120-task instances. This computation time decrease is due to a drastic reduction in the number of nodes. For example, for $n=50$ the number of nodes goes from more than 53 000 for F_l^2 to only 31 for F_l^{i+s} . With this latter formulation setting, the number of nodes is low, it is at most 200, even for large size instances. However, the time limit is reached for some 180- and 200-task instances, since the size of the linear program solved at each node is large.

In light of the four first columns of Table 1, we can conclude that, with less Cplex features, adding insert and swap inequalities significantly reduces the number of nodes and hence the computation time. More precisely, adding only swap inequalities is better than adding only insert inequalities, but adding both of them provides the best performance.

The four last columns show the same improvement in terms of computation time and number of nodes when Cplex default features are used.

Let us now focus on the 4th and 5th columns to compare the impact of the dominance inequalities and the impact of Cplex default features. For small instances, *i.e.* $n \in \{10, 20\}$, Cplex default features allow to solve the problem at the root node (*Cf.* F_d^2 columns). However, from $n = 50$, the number of nodes grows fast, so that no 60-task instance can be solved within 3600 seconds. Conversely, we already noticed that adding swap and insert inequalities limits the number of nodes (*Cf.* F_l^{i+s} columns), so that the ten 150-task instances are solved within 3600 seconds. Finally, adding the swap and insert inequalities provides better results than adding Cplex default features.

Up to size 60, F_d^{i+s} solves all instances at the root node, and is faster than F_l^{i+s} . For larger instances, except 200-task instances, F_l^{i+s} and F_d^{i+s} solves the problem in similar computation times, even if F_l^{i+s} explores a smaller number of nodes: for example, it is two times smaller for $n = 150$. For $n = 200$, F_d^{i+s} solves 4 over 10 instances, while F_l^{i+s} only solves 1 over 10 instances. To conclude, F_l^{i+s} and F_d^{i+s} offer comparable performances, so that, for both settings, the formulations providing the best results are the ones with insertion and swap inequalities.

4.2. Lower bound obtained at the root node

To further investigate the impact of dominance inequalities, we focus in this section on the root node of the search tree for different formulation settings. More precisely, we compare the different lower bounds obtained at the root node.

In the Cplex framework, setting the node limit to 0 allows to only solve the root node of a MIP: the branch-and-bound algorithm is stopped before the first branching. If Cplex default features are activated, the preprocessing is applied and the cuts are added before the algorithm stops. For a given formulation setting F , the corresponding run with the node limit set to 0 is denoted by F -RN. This results in eight runs: F_l^2 -RN, F_l^i -RN, F_l^s -RN, F_l^{i+s} -RN, F_d^2 -RN, F_d^i -RN, F_d^s -RN, F_d^{i+s} -RN.

Note that, in the Cplex framework, solving F -RN is different from solving the linear relaxation of F , denoted by F -LP. Indeed, F -LP is obtained by setting δ variables as continuous variables, which desactivates most of the Cplex features. In particular, the reinforcement cuts cannot be added since they are not valid for the relaxed formulation. Similarly, the inference procedure on the binary variables cannot be applied. We run the four linear relaxations F^2 -LP, F^i -LP, F^s -LP and F^{i+s} -LP. Surprisingly, the obtained values are the same for these four relaxations. In other words, adding insert and swap inequalities does not improve the linear relaxation value. Therefore, we only present in Table 2 the results for F^2 -LP.

To measure the quality of the nine different lower bounds obtained, we compute, when it is possible, the optimality gap, *i.e.* $(OPT - LB)/OPT$ where OPT denotes the optimal value and LB the lower bound. When the optimal value is not known, we compute a gap using the best upper bound that we get UB , *i.e.* $(UB - LB)/UB$. Such gaps are indicated with a "*" in Table 2. For each of the nine runs, the entries of Table 2 are the following.

L-gap : the average optimality gap of the lower bound obtained at the root node
time : the average running time in seconds over the ten instances

The obtained lower bound is exactly the same using F^2 -LP, F_l^2 -RN or F_l^s -RN. We deduce that with less Cplex features and without insert inequalities, setting the δ variables as binary or continuous variables, provides the same lower bound. Moreover, this lower bound is quite weak, since the average optimality gap is larger than 40% even for the 10-task instances. The computation times using F^2 -LP and F_l^2 -RN are similar: 2 seconds for the 100-task instances and about 12 minutes for the 500-task instances. The computation time required for F_l^s -RN is larger: almost 20 seconds for the 100-task instances and 47 minutes for the 500-task instances.

The lower bound obtained when only considering the insert inequalities is slightly better when the δ variables are set as binary variables for $n \in \{10, 20\}$. Indeed, the average optimality gap is 33% instead of 41% when $n = 10$, and 66% instead of 68% when $n = 20$ (*Cf.* F^2 -LP and F_l^i -RN columns). The computation time using F_l^i -RN is comparable to the computation time using F^2 -LP and F_l^2 -RN.

The lower bound obtained when considering both insert and swap inequalities, is significantly better when the δ variables are set as binary variables. Indeed, the average optimality gap is smaller than 39% for any value of n , and it is equal to 0 for $n = 10$ (Cf. F_l^{i+s} -RN column). The computation time for F_l^{i+s} -RN is between those for F_l^i -RN and F_l^s -RN: 14 seconds for the 100-task instances and about 30 minutes for the 500-task instances.

The lower bound provided using F_d^2 -RN, is better than the one obtained using F_l^2 -RN, that is with less Cplex features. Indeed, the average optimality gap is 7% instead of 41% when $n = 10$, and 46% instead of 94% when $n = 120$. However, the lower bound is weaker than the one obtained for F_l^{i+s} , whose optimality gap is 0 for $n = 10$ and 38% for $n = 120$. Moreover, the computation times using F_d^2 -RN increases fast with the increase of n so that the root node cannot be solved within one hour for sizes larger than 120.

Combining Cplex features with insert inequalities gives almost the same results (Cf. F_d^i -RN column). Conversely, combining Cplex features with swap inequalities gives better results (Cf. F_d^s -RN column). In particular, the average computation time is reduced so that instances up to size 200 can be solved. Moreover, the optimality gap is less than 22% for all solved instances. Finally, using F_d^{i+s} -RN gives even better results, the average optimality gap does not exceed 15%, even for 200-task instances, which are solved in 418 seconds, instead of 1200 using F_d^s -RN.

In a nutshell, combining insert and swap inequalities is the best to obtain a lower bound at the root node. Not using Cplex features allows its fast computation (Cf. F_l^{i+s} -RN column). Conversely, using them allows to obtain a better lower bound at the expense of the computation time (Cf. F_d^{i+s} -RN column).

4.3. Using swap and insert inequalities to obtain an upper bound

In this section, we propose two upper bounds on the optimal value. The first one is derived from the fractional solution obtained at the root node by a simple rounding procedure. The second one is obtained by applying in addition a local search procedure.

We derive an integer solution (δ, X) by rounding a fractional solution $(\tilde{\delta}, \tilde{X})$, as follows.

$$\forall j \in J, \delta_j = \begin{cases} 0 & \text{if } \widetilde{\delta}_j < \frac{1}{2} \text{ or } (\widetilde{\delta}_j = \frac{1}{2} \text{ and } \alpha_j < \beta_j) \\ 1 & \text{otherwise} \end{cases} \text{ and } \forall (i, j) \in J^<, X_{i,j} = \begin{cases} 1 & \text{if } \delta_i \neq \delta_j \\ 0 & \text{otherwise} \end{cases}$$

By construction, (δ, X) satisfies inequalities (1-4) (Cf. Lemma 6). It is thus a solution of F^2 , and $g(\delta, X)$ is an upper bound of the optimal value. However, it is not necessarily a solution for F^i , F^s and F^{i+s} formulations, since (δ, X) does not necessarily satisfy the insert and swap inequalities.

In order to transform such a solution into a solution satisfying the dominance inequalities, we can iteratively apply the operation associated to each violated dominance inequality, until all of them are satisfied. Algorithm 1 presents a way to implement this procedure that we call `Insert_swap_improvement`. From Property 9, if an insert (resp. a swap) inequality is not satisfied, applying the appropriate insert (resp. swap) operation provides a strictly better solution. Therefore, each solution is considered at most once in this procedure. Since the number of solutions is finite, the `Insert_swap_improvement` procedure finishes.

The returned solution is an insert and swap local optimum, since it satisfies all dominance inequalities (Cf. Corrolary 10).

`Insert_swap_improvement`

input: $\delta \in \{0, 1\}^J$

output: δ' encoding an insert and swap local optimum

```

 $\delta' \leftarrow \delta$ ; is_locally_opt  $\leftarrow$  false
while (not is_locally_opt)
    is_locally_opt  $\leftarrow$  true
    for  $u \in J$ 
        if  $\delta'_u = 1$  and  $\Delta_u(\delta') < 0$  //  $\delta'$  does not satisfy (5)
             $\delta'_u \leftarrow 0$ ; is_locally_opt  $\leftarrow$  false
        if  $\delta'_u = 0$  and  $\Delta_u(\delta') > 0$  //  $\delta'$  does not satisfy (6)
             $\delta'_u \leftarrow 1$ ; is_locally_opt  $\leftarrow$  false
    for  $v \in J \setminus \{u\}$ 
        if  $\delta'_u = 1, \delta'_v = 0$  and  $\Delta_{u,v}(\delta') < 0$  //  $\delta'$  does not satisfy (7)
             $\delta'_u \leftarrow 0$ ;  $\delta'_v \leftarrow 1$ ; is_locally_opt  $\leftarrow$  false
return  $\delta'$ 

```

Algorithm 1: the improvement procedure by insert and swap operations

Note that this algorithm can be seen as a local search procedure for the neighborhood associated to the insert and swap operations. Moreover, this procedure can be applied to any integer solution. Particularly, by sake of comparison we apply it to the solutions obtained by the heuristic "Heur II" provided by Biskup & Feldmann (2001a).

We finally compare the upper bounds given by the six following heuristic solutions.

BF : the solution obtained by the Biskup and Feldmann heuristic

BF+ : the solution obtained by applying Insert_swap_improvement to *BF*

R1 : the solution obtained by rounding the fractional solution of $F^2\text{-LP}$

R1+ : the solution obtained by applying Insert_swap_improvement to *R1*

R2 : the solution obtained by rounding the fractional solution of $F_d^{i+s}\text{-RN}$

R2+ : the solution obtained by applying Insert_swap_improvement to *R2*

In the sequel, we will use the same notation for both a heuristic solution and its value, which provides an upper bound on the optimal value. To measure the quality of these upper bounds, Table 3 presents their optimality gap denoted by U-gap, *i.e.* $(UB - OPT)/OPT$ where *OPT* denotes the optimal value and *UB* the upper bound. The Biskup and Feldmann heuristic provides a solution in less than 1 second. Applying rounding and Insert_swap_improvement to a fractional solution provides a solution in less than 1 second for instances up to size 200. Therefore, the time needed to obtain *R1* and *R1+* (resp. *R2* and *R2+*) is essentially the computation time required to solve $F^2\text{-LP}$ (resp. $F_d^{i+s}\text{-RN}$) given in Table 2.

As shown in Table 3, *BF* is a good upper bound. Indeed, its optimality gap is smaller than 0.35% for instance sizes larger than 50. However, this bound is improved by Insert_swap_improvement: the optimality gap of *BF+* is smaller than 0.02% for all the instances. With an optimality gap larger than 170%, *R1* is a very weak upper bound, while *R1+*, with an optimality gap smaller than 0.01%, is very good, and even slightly better than *BF+*. With an optimality gap smaller than 17%, *R2* is a better upper bound than *R1*, and *R2+* is exactly the same as *R1+*.

Finally, *BF+*, *R1+* and *R2+* are very good upper bounds. However it is worth noticing that even if the computation time to obtain *BF+* is about 1 second, the bound is obtained without any guarantee, since no lower bound is provided. Conversely, the computation time to obtain *R2+* is larger: 25 seconds for $n = 100$ and about 7 minutes for $n = 200$, but a lower bound is provided. *R2+* is then guaranteed to be at 14% of the optimal value for $n = 100$, and at 15% for $n = 200$ (*Cf.* L-gap of $F_d^{i+s}\text{-RN}$ in Table 2). *R1+* is a compromise between *BF+* and *R2+*. Indeed, for

instances up to size 200, $R1+$ is provided in less than 20 seconds together with a lower bound, but the guarantee obtained from this lower bound is quite weak (97% for $n=200$, Cf. F_l^2 -RN in Table 2).

4.4. Insert and swap operations use cases

Insert and swap operations can be used in different ways. Table 4 presents the best way to use them depending on the expected solution quality.

- To obtain an upper bound: apply rounding and `Insert_swap_improvement` to the fractional solution given by F^2 -LP. (Cf. $R1+$ column in Table 4).
- To obtain an upper bound with a better guarantee than the one obtained with $R1+$: apply rounding and `Insert_swap_improvement` to the fractional solution given by F_d^{i+s} -RN. (Cf. $R2+$ column in Table 4).
- To obtain a 5%-approached solution: use F_d^{i+s} , setting the gap limit to 5%. (Cf. F_d^{i+s} -5% column in Table 4).
- To obtain an exact solution: use F_d^{i+s} . (Cf. F_d^{i+s} column in Table 4).

Table 4 sums up the performance of the four above mentioned use cases. To measure the performances on the 200-task instances, no time limit is fixed. The entries of Table 4 are the following.

L-gap : the average optimality gap of the provided lower bound

U-gap : the average optimality gap of the provided upper bound

time : the average running time in seconds

#nd : the average number of nodes except the root node

New experiments are conducted for the results reported in F_d^{i+s} -5% and F_d^{i+s} columns when $n = 200$. These results are gathered with the previously obtained results in Table 4 to offer an overview.

Table 4 shows that the number of nodes is lowered by 37.0% while the computation time is only lowered by 10.8% on average for $n = 200$. In addition, for the six 200-task instances where F_d^{i+s} reaches the time limit, only less than 100 nodes are explored. The limit for solving F_l^{i+s} is thus the size or the difficulty of the LPs solved at each node, rather than the number of nodes.

Trying to address this issue, we implemented a separation algorithm for the insert and swap inequalities using a callback function. The time needed to solve 50-task instances using this separation algorithm and Cplex features was 1513 seconds with 925 nodes in average. We observe that 98% of the computation time is

used by the UserCut Callback to add 71 inequalities in average. This is not surprising since the separation algorithm consists in simply evaluating the terms of inequality (5) and (6) for the n possible tasks u , and the terms of inequality (7) for the n^2 possible couples (u, v) , which results in an $O(n^3)$ procedure.

Providing a faster separation algorithm could reduce the computation time, but the branching scheme, and then the number of nodes, would be the same. Since this number of nodes is quite large compared to the performance of F_d^{i+s} (which solves all 50-task instances at the root node), we conclude that adding dominance inequalities through a separation procedure reduces their impact.

Indeed, when initially added, the dominance inequalities allow to the Cplex pre-solve phase to fix some variables to 0 or 1. The number of LPs variables is then reduced and the value obtained at each node is improved. When the δ variables are set as continuous variables, this presolve is not executed. It is then consistent with the observation that adding dominance inequalities in this latter case has no impact (Cf. Section 4.2).

5. Conclusion

In this work, we propose a new way to use neighborhood-based dominance properties, which results in a new kind of reinforcement inequalities. In contrast with the commonly used reinforcement inequalities, which cut fractional points, these inequalities cut non locally optimal solutions. In particular, for the compact formulation F^2 , we provide linear inequalities cutting all the solutions which are not insert and swap locally optimal.

From a practical point of view, we show that adding insert and swap inequalities greatly improves performances of F^2 . Indeed, instead of 50-task instances, we can now solve up to 150-task instances to optimality within one hour.

Insert and swap inequalities can also be used to provide a heuristic solution which is slightly better than the one proposed by Biskup & Feldmann (2001a). For instances up to size 200, this heuristic solution is obtained in less than 20 seconds. A lower bound providing a 15% gap can also be obtained in less than 420 seconds.

We observe that insert and swap inequalities do not improve the linear relaxation value of the compact formulation F^2 . However, used in conjunction with Cplex features, they allow to improve the lower bound obtained at the root node. Two issues follow. Firstly, for a version of F^2 reinforced by cuts or by branching decisions, do dominance properties improve the linear relaxation value? Secondly, which procedure implemented in the Cplex features take advantage of the insert

and swap inequalities? Addressing these issues requires an appropriate experimental framework.

Moreover, this work could be extended to other problems where the solutions can be encoded by partitions (any kind of partition, not necessarily ordered bipartitions). For instance, inequalities similar to insert and swap inequalities could be used in a generalization of UCDDP to a parallel machine framework. Indeed, if the tasks share a common due date, the dominant schedules can be encoded by ordered $2m$ -partitions, where m is the number of machines. This is true even if the common due date, the processing times and the unit earliness and tardiness penalties depend on the machine. Beyond the scheduling field, such inequalities could also be used in the maximum cut problem (Karp, 1972) or in a maximum k -cut problem (Frieze & Jerrum, 1997).

For other combinatorial problems where solutions do not have a partition structure; some neighborhood-based dominance inequalities could also be designed using appropriate operations.

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n	F_l^2			F_l^i			F_l^s			F_l^{i+s}			F_d^2			F_d^i			F_d^s			F_d^{i+s}		
	#opt	time	#nd	#opt	time	#nd	#opt	time	#nd	#opt	time	#nd	#opt	time	#nd	#opt	time	#nd	#opt	time	#nd	#opt	time	#nd
10	10	29	11	10	34	10	10	32	7	10	0	0	10	26	0	10	22	0	10	3	0	10	0	0
20	10	51	162	10	63	91	10	63	25	10	42	11	10	44	0	10	54	0	10	41	0	10	10	0
50	10	311	53596	10	76	2101	10	90	56	10	67	31	10	1310	24725	10	156	1293	10	15	0	10	13	0
60	5	2078	228193	10	186	8063	10	74	83	10	58	41	0	-	-	5	439	2904	10	93	66	10	15	0
80	0	-	-	9	815	17604	10	137	138	10	77	70	-	-	-	2	2823	1402	10	219	322	10	79	73
100	-	-	-	4	2800	23965	10	291	215	10	109	75	-	-	-	-	-	-	10	529	542	10	165	141
120	-	-	-	-	-	-	10	728	269	10	219	122	-	-	-	-	-	-	10	1578	779	10	363	181
150	-	-	-	-	-	-	8	2532	410	10	786	201	-	-	-	-	-	-	2	3172	660	10	1011	481
180	-	-	-	-	-	-	1	3514	285	6	2460	194	-	-	-	-	-	-	-	-	-	5	1537	284
200	-	-	-	-	-	-	-	-	-	1	1929	127	-	-	-	-	-	-	-	-	-	4	2524	710

Table 1: Effect of adding insert and swap inequalities on exact solving

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n	F^2 -LP		F_l^2 -RN		F_l^i -RN		F_l^s -RN		F_l^{i+s} -RN		F_d^2 -RN		F_d^i -RN		F_d^s -RN		F_d^{i+s} -RN	
	L-gap	time	L-gap	time	L-gap	time	L-gap	time	L-gap	time	L-gap	time	L-gap	time	L-gap	time	L-gap	time
10	41%	0	41%	0	33%	0	41%	0	0%	0	7%	1	5%	1	0%	1	0%	0
20	68%	0	68%	0	66%	0	68%	0	12%	1	28%	2	28%	2	6%	1	2%	0
50	86%	0	86%	1	86%	1	86%	6	28%	6	42%	27	41%	31	17%	5	11%	3
60	89%	0	89%	1	89%	1	89%	7	36%	7	41%	91	41%	95	22%	9	16%	5
80	92%	1	92%	1	92%	1	92%	11	34%	8	43%	345	43%	359	21%	28	15%	10
100	93%	2	93%	2	93%	2	93%	19	35%	14	45%	1091	44%	1152	21%	62	14%	25
120	94%	3	94%	4	94%	11	94%	31	38%	15	46%	3189	46%	3192	22%	133	16%	52
150	96%	6	96%	13	96%	15	96%	60	34%	29	-	-	-	-	22%	352	15%	130
180	96%	12	96%	19	96%	23	96%	98	34%	49	-	-	-	-	22%	766	15%	274
200	97%	19	97%	25	97%	31	97%	126	39%	72	-	-	-	-	22%	1204	15%	418
500	99%*	722	99%*	698	99%*	742	99%*	2820	36%*	1870	-	-	-	-	-	-	-	-

Table 2: Effect of adding insert and swap inequalities on lower bounds

n	BF	$BF+$	$R1$	$R1+$	$R2$	$R2+$
	U-gap	U-gap	U-gap	U-gap	U-gap	U-gap
10	2.04%	0.00%	170%	0.00%	0.00%	0.00%
20	0.95%	0.00%	196%	0.00%	1.33%	0.00%
50	0.35%	0.02%	203%	0.00%	13.83%	0.00%
60	0.26%	0.01%	170%	0.01%	16.80%	0.01%
80	0.22%	0.01%	172%	0.00%	16.36%	0.00%
100	0.18%	0.00%	174%	0.00%	15.72%	0.00%
120	0.10%	0.00%	170%	0.00%	15.77%	0.00%
150	0.10%	0.00%	171%	0.00%	15.27%	0.00%
180	0.10%	0.00%	171%	0.00%	16.09%	0.00%
200	0.10%	0.01%	171%	0.01%	16.28%	0.00%

Table 3: Comparison of different heuristics providing an upper bound

to obtain:	an upper bound			a lower bound			a 5%-approximation		an exact solution	
use:	$R1+$			$R2+$			F_d^{i+s} -5%		F_d^{i+s}	
n	L-gap	U-gap	time	L-gap	U-gap	time	time	#nd	time	#nd
50	86%	0.00%	<1	11%	0.00%	3	8	0	4	34
100	93%	0.00%	2	14%	0.00%	25	160	114	165	141
200	97%	0.01%	20	15%	0.01%	418	7420	928	8317	1474
500	-	-(99%)	778	-	-	-	-	-	-	-

Table 4: Different ways of using insert and swap inequalities