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Ground state solitary waves local controllability for the nonlinear focusing Schrödinger equation in the mass critical and slightly mass subcritical regime

Ludovick Gagnon *

February 14, 2021

Abstract

In this paper, we prove the local controllability around different ground state solitary wave for the slightly subcritical mass and mass critical nonlinear Schrödinger equation. More precisely, if Q_{c_1} and Q_{c_2} denotes the ground states with two different scaling, we prove the local controllability from Q_{c_1} to Q_{c_2} in a minimal time depending on c_1 and c_2 . The results presented relies on the blow-up profile in the mass critical case and on the recent self-similar blow-up result obtained by [1] in the slightly intercritical case.

Keywords : Nonlinear focusing Schrödinger equation, controllability, solitary waves

MSC2020 : 93B05 35Q51 35Q55 93C10

1 Introduction

We study in this article the controllability between different ground state solitary waves in the mass critical and slightly subcritical mass regime of the nonlinear focusing Schrödinger equation,

$$\begin{cases} i\psi_t + \Delta\psi + \psi|\psi|^{p-1} = 0, & (x, t) \in \mathbb{R}^d \times \mathbb{R}^+, \\ \psi(x, 0) = \psi_0, & x \in \mathbb{R}^d, \end{cases} \quad (1)$$

with $\psi_0 \in H^1(\mathbb{R}^d)$ and $p > 1$. Equation (1) has many physical applications for $p = 3$ and $d = 1, 2$, as it serves as a model for signal propagation in nonlinear optic for optic fibers and self-focusing laser beams in hollow core fibers ([15, 41]). The nonlinear focusing Schrödinger equation (1) is also completely integrable if $d = 1$ and $p = 3$ ([7]).

The solutions to (1) conserve three quantities, the mass,

$$\|\psi(\cdot, t)\|_{L^2(\mathbb{R}^d)} = \|\psi_0\|_{L^2(\mathbb{R}^d)},$$

the energy,

$$E(\psi(\cdot, t)) := \frac{1}{2} \|\nabla\psi(\cdot, t)\|_{L^2}^2 - \frac{1}{p+1} \|\psi(\cdot, t)\|_{L^{p+1}}^{p+1} = E(\psi_0),$$

and momentum,

$$\operatorname{Im} \left(\int \nabla\psi(x, t) \bar{\psi}(x, t) dx \right) = \operatorname{Im} \left(\int \nabla\psi_0(x) \bar{\psi}_0(x) dx \right).$$

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Moreover, solutions to (1) are invariant with respect to the scaling

$$\psi_\lambda(x, t) = \lambda^{\frac{2}{p-1}} \psi(\lambda x, \lambda^2 t), \quad \lambda > 0, \quad (2)$$

and the scaling is preserved for the homogeneous Sobolev norm $\|\psi_\lambda(t)\|_{\dot{H}^{s_c}} = \|\psi(\cdot, t)\|_{\dot{H}^{s_c}}$ with

$$s_c := \frac{d}{2} - \frac{2}{p-1}.$$

Equation (1) is said to be mass critical if $s_c = 0$, mass subcritical if $s_c < 0$ and mass supercritical if $s_c > 0$. Likewise, it is said to be energy subcritical if $s_c < 1$, energy critical if $s_c = 1$ and energy supercritical if $s_c > 1$.

1.1 Bound states

The bound states, or solitary waves, are stationary solutions to (1) of the form $\psi(x, t) = e^{i\lambda t} W_\lambda(x)$ where $W_\lambda(x) = \lambda^{\frac{1}{p-1}} W(\sqrt{\lambda}x)$ is solution to the nonlinear elliptic equation,

$$\Delta W_\lambda + |W_\lambda|^{p-1} W_\lambda = \lambda W_\lambda, \quad x \in \mathbb{R}^d. \quad (3)$$

Solutions to (3) exist if the energy is subcritical and if and only if $\lambda > 0$. The solutions W_λ to (3) belongs to $W^{3,q}(\mathbb{R}^d)$ for any $2 \leq q < \infty$ and numerous properties of bound states have been established. It is well-known that the solution to (3) is not unique, but unicity is recovered under the additional constraint of seeking a positive and radially symmetric solution to (3) [42, 43]. This unique solution, the ground state solitary wave, denoted $Q_\lambda(x) = \lambda^{\frac{1}{p-1}} Q(\sqrt{\lambda}x)$, belongs to the Schwarz class $\mathcal{S}(\mathbb{R}^d)$ and has an explicit representation in dimension one,

$$Q(x) = \left(\frac{p+1}{2} \operatorname{sech}^2 \left(\frac{p-1}{2} x \right) \right)^{1/p-1}.$$

In higher dimensions, the exponential decay of the ground state is characterized by the following estimate for $r \geq 1$ ([7, 41]),

$$\left| Q(r) - \kappa r^{-(d-1)/2} e^{-r} \right| + \left| Q'(r) - \kappa r^{-(d-1)/2} e^{-r} \right| \leq C r^{-(d+1)/2} e^{-r}. \quad (4)$$

where $r = |x|$ and where $\kappa, C > 0$ are two constants, depending on $d \geq 1$. Here and below, the constant κ will always refer to the constant appearing in the inequality (4). Finally, Q_λ satisfies $E(Q_\lambda) = \lambda E(Q) = 0$ in the mass critical case thanks to Pohozaev's identity [7]. We refer to [42, Appendix B] for additional properties of the ground state.

1.2 Well-posedness in the different regimes in \mathbb{R}^d

The ground state Q plays an important role in the description of the global well-posedness of (1). The local well-posedness of (1) with initial data $\psi_0 \in H^1(\mathbb{R}^d)$ is established by the Cauchy theory when the energy is subcritical for $d \geq 3$, and without any restriction on $p > 1$ for $d = 1, 2$ ([7, 13]). Such solutions belongs to $\psi \in C([0, T_{\psi_0}); H^1(\mathbb{R}^d))$ and there is a blow-up alternative ([7]), that is to say, either $T_{\psi_0} = \infty$ and the solution is defined globally, or $\lim_{t \rightarrow T_{\psi_0}^-} \|\psi(t)\|_{H^1(\mathbb{R}^d)} = +\infty$. In the mass subcritical regime $s_c < 0$, the conserved quantities as well as the Gagliardo-Nirenberg inequality allows to extend the time existence T_{ψ_0} of all solutions to (1), implying the global well-posedness in $H^1(\mathbb{R}^d)$ ([7]).

The mass critical regime $s_c = 0$ is the first regime to exhibit blow-up phenomenon. Weinstein proved in [43] that the estimate $\|\psi_0\|_{L^2(\mathbb{R}^d)} < \|Q\|_{L^2(\mathbb{R}^d)}$ ensures the global existence of solutions to (1), thanks to the conservation of mass, energy and the Gagliardo-Nirenberg estimate

$$E(\psi) \geq \frac{1}{2} \|\psi\|_{H^1(\mathbb{R}^d)}^2 \left(1 - \left(\frac{\|\psi\|_{L^2(\mathbb{R}^d)}^2}{\|Q\|_{L^2(\mathbb{R}^d)}^2} \right)^{4/d} \right), \quad \forall \psi \in H^1(\mathbb{R}^d).$$

This estimate was proved to be sharp by Merle [25] in the following sense. Assume $\|\psi_0\|_{L^2(\mathbb{R}^d)} = \|Q\|_{L^2(\mathbb{R}^d)}$ and assume that the solution $\psi(\cdot, t)$ of (1) blows up in finite time $T > 0$. Then the solution ψ is given, up to the translations of the equation, by the pseudo conformal transformation profile

$$S(x, t) = \frac{1}{|t|^{d/2}} Q\left(\frac{x}{t}\right) e^{-i\frac{|x|^2}{4t} + \frac{i}{t}}. \quad (5)$$

This profile lies in the set $\Sigma = H^1(\mathbb{R}^d) \cap \{x\psi_0 \in L^2(\mathbb{R}^d)\}$, invariant by the flow, and blows up at the speed $\|\nabla S(\cdot, t)\|_{L^2(\mathbb{R}^d)} \simeq 1/|t|$. A general blow-up phenomenon occurs in Σ in the range $0 \leq s_c < 1$ due to the so-called virial identity. Indeed, assume $\psi_0 \in \Sigma = H^1(\mathbb{R}^d) \cap \{x\psi_0 \in L^2(\mathbb{R}^d)\}$ and $E(\psi_0) < 0$. Then, the associated solution $\psi(\cdot, t)$ of (1) belongs to Σ and

$$\frac{d^2}{dt^2} \|x\psi(\cdot, t)\|_{L^2(\mathbb{R}^d)}^2 = 4d(p-1)E(\psi_0) < 0,$$

leading to a contradiction on the infinite time of existence of the solution.

The blow-up phenomenon in the mass critical regime was further investigated in a serie of works [26, 27, 28, 29, 30, 36]. The blow-up phenomenon was studied in the set

$$\mathcal{B}_{\alpha^*} = \left\{ \psi_0 \in H^1(\mathbb{R}^d) \mid \|Q\|_{L^2}^2 < \|\psi_0\|_{L^2}^2 < \|Q\|_{L^2}^2 + \alpha^* \right\}$$

for $\alpha^* > 0$ sufficiently small. Assuming the *spectral property*¹, it was shown that if $E(\psi_0) > 0$, then blow-up occurs at rate $1/|t|$, the same rate as the pseudo conformal transformation (5) and if $E(\psi_0) < 0$, then blow-up occurs at the rate

$$\|\nabla \psi(\cdot, t)\|_{L^2(\mathbb{R}^d)} \simeq \sqrt{\frac{\log |\log(T-t)|}{T-t}}.$$

We shall denote here the concentration factor in the mass critical regime

$$\lambda^*(t) = \sqrt{\frac{T-t}{\log |\log(T-t)|}}.$$

This rate is not self-similar as

$$\lim_{t \rightarrow T^-} \frac{\lambda^*(t)}{\sqrt{T-t}} = 0.$$

In fact, more can be said about the blow-up profile in the case $E(\psi_0) < 0$. Indeed, in [28], it is shown that the solution $\psi(\cdot, t)$ coming from the initial data $\psi_0 \in \mathcal{B}_{\alpha^*}$ such that $E(\psi_0) < 0$ decomposes as

$$\psi(x, t) = \frac{e^{i\gamma(t)}}{\lambda(t)} (Q + \epsilon) \left(\frac{x - x(t)}{\lambda(t)}, t \right),$$

¹Without going into details, the spectral property is related to the coercivity of linear form coming from the linearization around state Q along with H^1 orthogonality properties. It was proven in dimension $d = 1$ in [28] and in dimensions $d = 2, 3, 4$ using rigorously verified computing [11]. It is conjectured that the spectral property holds in any dimension. We refer to these works for a precise definition.

with $\|\epsilon\|_{H^1(\mathbb{R}^d)} \rightarrow 0$ as $\alpha^* \rightarrow 0$. An important remark that we shall use in the present article is that, if the initial data ψ_0 is radial, then translation parameter $x(t)$ is equal to 0 ([28]).

Let us underline here that in the mass subcritical case $s_c < 0$, not only solutions are always globally defined in $H^1(\mathbb{R}^d)$, but the ground state is in fact orbitally stable in $H^1(\mathbb{R}^d)$, that is stable up to translation in space and phase shift [8]. More precisely, for any $\epsilon > 0$, there exists $\delta > 0$ such that if,

$$\inf_{(\gamma, x_0) \in [0, 2\pi] \times \mathbb{R}} \|\psi_0(\cdot - x_0) - Q(\cdot)e^{i\gamma}\|_{H^1(\mathbb{R}^d)} < \delta,$$

then

$$\inf_{(\gamma, x_0) \in [0, 2\pi] \times \mathbb{R}} \|\psi(\cdot - x_0, t) - e^{i(t+\gamma)}Q(\cdot)\|_{H^1(\mathbb{R}^d)} < \epsilon.$$

Finally, the mass intercritical case $0 < s_c < 1$ is conjectured ([40, 44]) to exhibit self-similar blow-up in finite time for initial data close to the ground state Q in $\dot{H}^1(\mathbb{R}^d)$. The existence and stability of the self-similar blow-up was proved in [31] for $0 < s_c \ll 1$ and a precise description of the blow-up profile was obtained in a recent work of Bahri, Martel and Raphaël [1].

Before stating our main results, we present a short overview of the controllability of solitary waves as well as the controllability of the Schrödinger equation, as it will ease the presentation and allow us to state more precisely the nature of our result.

1.3 Literature overview

Despite the extensive literature on solitary waves, it seems that little is known so far on its controllability properties. If the solitary waves are defined on the whole space, there seems to be only one result in the literature dealing with the controllability of the ground state solution to (3) of the generalized KdV equation due to Muñoz. In [33], considering the ground state on the whole line and using a moving distributed control exponentially small away from the ground state, Muñoz proved the approximate controllability in large time from Q_{c_1} to Q_{c_2} , $c_1, c_2 \in \mathbb{R}^+$. His proof relies heavily on the nonlinear interaction between the control and the solitary wave.

Other results are found in the literature on the controllability when the ground state is defined on bounded domains. Lange and Teismann [18], the 1-D nonlinear focusing Schrödinger equation was considered on a bounded domain with homogeneous Dirichlet boundary conditions and with a distributed control. In [18], the ground state is defined as the unique positive solution in $x \in (0, 1)$ of

$$\begin{cases} \psi''(x) + \psi^3(x) = \psi(x), & x \in (0, 1), \\ \psi(0) = \psi(1) = 0, \end{cases} \quad (6)$$

and the local controllability around the ground state was obtained using the HUM method [23] and a fine spectral analysis of the underlying linearized equation. A similar question was addressed by Castelli and Teismann in [6], but this time with the size $L(t)$ of the 1-D domain having the role of the control.

We also find few results on the stabilization of the solitary waves. In [32], the approximate stability of the bilinear Schrödinger equation in \mathbb{R}^n , modelling the action of a laser on a quantum particle, around ground state solutions to

$$(\Delta + V(x))\psi(x) = 0, \quad x \in \mathbb{R}^n$$

was proven by Mirrahimi under spectral hypothesis of the operator $\Delta + V$. In the particular case of a dipolar function $\mu(x) = x$ and with a domain equal to the interval $(-1/2, 1/2)$, a global practical stabilisation was obtained by Beauchard and Mirrahimi through a Lyapunov analysis

in [3]. Finally, we note that the N -solitons solution was used by the author as a trajectory in [12] to achieve small-time Lagrangian controllability for the Korteweg-de Vries equation.

Regarding the controllability properties of the nonlinear Schrödinger equation, we emphasize on the result of Rosier and Zhang [37], as part of the present work is the extension of their results. They have proved for $d \geq 1$ the exact local controllability of

$$\begin{cases} i\psi_t + \Delta\psi + |\psi|^2\psi = 0, & (x, t) \in \Omega \times (0, T), \\ \psi(x, 0) = \psi_0(x), & x \in \Omega, \end{cases} \quad (7)$$

for $\Omega \subset \mathbb{R}^d$ bounded and with a boundary $\partial\Omega$ of class C^2 and of outward normal ν , with a boundary control v for the Dirichlet or Neumann boundary condition,

$$\psi(x, t) = v(x, t), \quad (x, t) \in \partial\Omega \times (0, T), \quad \text{or} \quad \frac{\partial\psi}{\partial\nu}(x, t) = v(x, t), \quad (x, t) \in \partial\Omega \times (0, T). \quad (8)$$

around smooth trajectories of (1).

Theorem 1 (Theorem 1.1, [37]). *Let $T > 0$ be given and let $w \in C^\infty([-\epsilon, T + \epsilon]; S(\mathbb{R}))$ be a solution of the nonlinear Schrödinger equation, with $\lambda \in \mathbb{C}^*$,*

$$iw_t + \Delta w + \lambda|w|^2w = 0,$$

for any $(x, t) \in \Omega_1 \times (-\epsilon, T + \epsilon)$ where $\epsilon > 0$ and Ω_1 is a bounded domain in \mathbb{R}^d with $\bar{\Omega} \subset \Omega_1$. Assume

$$s > \frac{d}{2}, \quad \text{or} \quad 0 \leq s < \frac{d}{2} \quad \text{with} \quad 1 \leq d \leq 2 + 2s, \quad \text{or} \quad s = 0, 1 \quad \text{with} \quad d = 2.$$

Then there exists a $\delta > 0$ such that for any $\psi_0, \psi_1 \in H^s(\Omega)$ satisfying

$$\|\psi_0 - w(\cdot, 0)\|_{H^s(\Omega)} \leq \delta, \quad \|\psi_1 - w(\cdot, T)\|_{H^s(\Omega)} \leq \delta,$$

one can find an appropriate boundary control function $v(x, t)$ such that (7)-(8) admits a solution $\psi \in C([0, T]; H^s(\Omega))$ such that

$$\psi(x, 0) = \psi_0(x), \quad \psi(x, T) = \psi_1(x), \quad \text{in } \Omega.$$

The control strategy deployed in [37] consists to extend the initial data ψ_0 to \mathbb{R}^d and to consider the control problem in the whole space

$$\begin{cases} i\psi_t + \Delta\psi + |\psi|^2\psi = \varphi(x)h(x, t), & (x, t) \in \mathbb{R}^d \times \mathbb{R}^+, \\ \psi(x, 0) = \psi_0(x), & x \in \mathbb{R}^d, \end{cases} \quad (9)$$

where $\varphi \in C^\infty(\mathbb{R}^d; [0, 1])$ is a cut-off function such that

$$\varphi(x) = \begin{cases} 1, & \text{if } |x| \geq R + 1 \\ 0, & \text{if } |x| \leq R \end{cases} \quad (10)$$

with $R > 0$ sufficiently large so that $\Omega \subset B_R(0)$. This strategy of extending the solution to the whole space to deduce boundary controllability property is attributed to Russell who first used it for the wave equation in [38, 39]. Littman and Taylor later on gave a general principle for the boundary controllability of a linear partial differential equation, that is reversibility, smoothing properties and uniqueness usually leads to controllability [24]. In [37], the smoothing properties come from the Strichartz estimates available in \mathbb{R}^d , hence the domain extension. The

controllability of the linearized equation associated to (9) around the smooth trajectory w is obtained in [37] using a Carleman estimate for $C^\infty([-\epsilon, T + \epsilon]; \mathcal{S}(\mathbb{R}))$ potentials. The local exact controllability of (9) around smooth trajectories is deduced by a fixed point argument, and Theorem 1 is obtained by constructing the control v in (8) as the trace of (9), and the controlled solution of (7) is obtained as the restriction of the solution of (9) to Ω . Such solutions may be defined in a weak sense (see [23]). For instance, for the Dirichlet boundary conditions, we say that ψ is the solution to (7)-(8) if $\forall \tau \in [0, T]$,

$$i \int_{\Omega} \psi(x, \tau) \bar{\phi}(x, \tau) - \psi_0(x) \bar{\phi}(x, 0) dx - \int_0^\tau \int_{\Omega} \nabla \psi(x, t) \nabla \bar{\phi}(x, t) dx dt + \int_0^\tau \int_{\Omega} \psi |\psi|^2 \phi dx dt = \int_0^\tau \int_{\partial\Omega} v(x, t) \partial_\nu \phi(x, t) dx dt \quad (11)$$

for any $\phi \in C([0, T]; H^1(\Omega) \cap \{\partial_\nu \phi \in C([0, T]; L^2(\partial\Omega))\}$. We shall underline here that this procedure provide the existence of a solution to (7)-(8). However, the well-posedness of (7)-(8) is not ensured, especially for the energy supercritical case. We shall come back to this point after stating our main results.

Other controllability results were obtained for the nonlinear Schrödinger equation. The controllability of the nonlinear defocusing Schrödinger equation on a compact manifold without boundary was proved by Dehman, Gérard and Lebeau [10] using the exponential decay of the nonlinear problem and local controllability, assuming that the support of the internal control satisfies the Geometric Control Condition [2], with a control time depending on the size of the initial data. This argument of first stabilizing the solution and then using local controllability around zero was also used by Laurent in [19, 20]. In the latter paper, one of the unsolved question about the controllability for the linear Schrödinger equation is highlighted, that is, what is the link between the Geometric Control Condition (GCC) and the unique continuation? It was shown by Lebeau that the GCC is sufficient for the controllability of the linear Schrödinger equation [22], but unlike the linear wave equation where GCC is necessary and sufficient (see [4]), GCC is not always necessary, as illustred by Jaffard [16] and Burq and Zworski [5]. Extensive study was done on the controllability of the linear and bilinear Schrödinger equation and with boundary or internal control. We refer the reader to the surveys [21, 45] on the subject

1.4 Main results

We present here the two main results of exact local controllability around ground states with different scaling in the mass critical and slightly subcritical regime. Let $T > 0$ and $\Omega \subset \mathbb{R}^d, d \geq 1$ be an open and bounded domain with a boundary $\partial\Omega$ of class C^2 . In the spirit of [37], we consider the boundary controllability of the focusing nonlinear Schrödinger equation

$$\begin{cases} i\psi_t + \Delta\psi + \psi|\psi|^{p-1} = 0, & (x, t) \in \Omega \times (0, T), \\ \psi(x, t) = v(x, t), & (x, t) \in \partial\Omega \times (0, T), \\ \psi(x, 0) = \psi_0(x), & x \in \Omega, \end{cases} \quad (12)$$

where $v \in L^2((0, T); L^2(\partial\Omega))$ is the boundary control. We begin by stating the result in the mass critical setting $p - 1 = 4/d$. This result covers the physically relevant case $p = 3$ in dimension $d = 2$.

Theorem 2. *Let $d \geq 1$ and $\Omega \subset \mathbb{R}^d$ be an open bounded domain with a boundary $\partial\Omega$ of class C^2 . Let $c_1, c_2 \in \mathbb{R}^+$ and $c_1 < c_2$. For $d \geq 5$, assume that the spectral property ([28]) holds. Then there exists $\epsilon > 0$ such that, for every $\psi_0, \psi_1 \in H^1(\Omega)$ such that*

$$\inf_{\gamma \in [0, 2\pi]} \|\psi_0 - e^{i\gamma} Q_{c_1}\|_{H^1(\Omega)} < \epsilon, \quad \inf_{\gamma \in [0, 2\pi]} \|\psi_1 - e^{i\gamma} Q_{c_2}\|_{H^1(\Omega)} < \epsilon,$$

there exists a control $v \in C([0, T]; H^{1/2^-}(\partial\Omega))$ such that (12) admits a solution $\psi \in C([0, T]; H^1(\Omega))$ satisfying

$$\psi(x, 0) = \psi_0(x), \quad \psi(x, T) = \psi_1(x), \quad \text{in } \Omega,$$

for any $T > T_1^* - T_2^* > 0$, where

$$\frac{\log |\log(T_1^*)|}{T_1^*} = c_1, \quad \frac{\log |\log(T_1^* - T_2^*)|}{T_1^* - T_2^*} = c_2.$$

Our second result yields that the local exact controllability around two different ground states in the slightly mass subcritical regime with a different minimal time.

Theorem 3. *Let $d \geq 1$ and $\Omega \subset \mathbb{R}^d$ be an open bounded domain with a boundary $\partial\Omega$ of class C^2 . Let $c_1, c_2 \in \mathbb{R}^+$ with $c_2 > c_1$. Then, there exists $1 < \bar{p} < p^*$ such that for every $p \in (\bar{p}, p^*)$, there exists $\epsilon > 0$ such that, for every $\psi_0, \psi_1 \in H^1(\Omega)$ such that*

$$\inf_{\gamma \in [0, 2\pi]} \|\psi_0 - e^{i\gamma} Q_{c_1}\|_{H^1(\Omega)} < \epsilon, \quad \inf_{\gamma \in [0, 2\pi]} \|\psi_1 - e^{i\gamma} Q_{c_2}\|_{H^1(\Omega)} < \epsilon,$$

there exists a control $v \in C([0, T]; H^{1/2^-}(\partial\Omega))$ such that (12) admits a solution $\psi \in C([0, T]; H^1(\Omega))$ satisfying

$$\psi(x, 0) = \psi_0(x), \quad \psi(x, T) = \psi_1(x), \quad \text{in } \Omega,$$

in time $T > T_{c_1, c_2} > 0$, where

$$T_{c_1, c_2} := C \frac{c_2 - c_1}{c_1 c_2},$$

with $C(p) > 0$ independent of c_1, c_2 .

The main idea behind Theorem 2 and Theorem 3 is to use the blow-up profile as a trajectory of controllability. More precisely, we use blow-up profiles that remain close to the ground state for any time $t > 0$, allowing to exploit the regularity of the ground state to use local exact controllability around the ground state. To this end, we extend the local controllability around smooth trajectories of (1) for $p = 3$ of Rosier and Zhang [37, Theorem 1.1] to the local controllability around the ground state in the mass subcritical and mass critical regime. Then, in the mass critical case, we use the trajectory in the set \mathcal{B}_{α^*} such that $E(\psi_0) < 0$. Using the $C^\infty(\mathbb{R}; \mathcal{S}(\mathbb{R}^d))$ regularity of the ground state, we are able to drive the initial data to the blow-up profile in arbitrarily small time. The blow-up profile is then used to reach the neighborhood of Q_{c_2} . The local controllability around Q_{c_2} is then used once again to drive the solution to the final state. Let us highlight that a direct application of Theorem 6 only yields the local controllability around one ground state Q_c . The minimal time of controllability in both theorems comes from the speed of concentration of the blow-up profile.

To prove Theorem 3, we prove that the blow-up profile in the mass slightly supercritical case of [1] actually holds in the mass slightly subcritical case. We moreover prove that this profile is close to the ground state in $H_{loc}^1(\mathbb{R}^d)$. Then, using the same control strategy as in Theorem 2, we control a neighborhood of Q_{c_1} to a neighborhood of Q_{c_2} .

The result here relies on the blow-up profiles for (12). In turn, the global $H^1(\Omega)$ well-posedness of (12) fails to hold, even in the mass subcritical case. Even if Strichartz estimates are available in bounded domain or on the exterior of a bounded domain, it is not sufficient to deduce the global well-posedness. This is not that surprising, as it was shown in [35] that (12) exhibits blow-up phenomenon in the mass critical case, even with $v \equiv 0$. In this case, the ground state solitary wave is defined as in (6). In this sense, and as in [37], the solution of (12) described in Theorem 2, Theorem 3 and Theorem 6 are to be understood as satisfying (11), that is, as the restriction of the solution defined on the whole space, and the control as the trace of the solution.

1.5 Structure of the article

In Section 2, we recall the well-posedness results for the focusing nonlinear Schrödinger equation. In Section 3, we extend the results of Rosier and Zhang [37] to deduce the exact local controllability around the ground state for the mass critical and subcritical regime. Section 4 is dedicated to the various results on the blow-up profiles that we shall need for the proof of the main results. Finally, Section 5 is reserved for the proof of Theorem 2 and 3.

2 Well-posedness

2.1 Functional framework

We consider, for Ω an open subset of \mathbb{R}^d , the space $L^p(\Omega)$ of measurable complex-valued functions $u : \Omega \rightarrow \mathbb{C}$ such that $\|u\|_{L^p(\Omega)} < \infty$ with the norm

$$\|u\|_{L^p(\Omega)} = \begin{cases} (\int_{\Omega} |u(x)|^p dx)^{1/p}, & \text{if } p < \infty, \\ \text{ess sup}_{x \in \Omega} |u(x)|, & \text{if } p = \infty. \end{cases}$$

The space $L^2(\Omega)$ is a real Hilbert space equipped with the scalar product

$$(u, v)_{L^2(\Omega)} = \text{Re} \left(\int_{\Omega} u(x) \overline{v(x)} dx \right), \quad \forall u, v \in L^2(\Omega).$$

Likewise, $H^m(\Omega)$, $m \in \mathbb{N}$ is a real Hilbert space with the scalar product

$$(u, v)_{H^m(\Omega)} = \sum_{0 \leq |\alpha| \leq m} \text{Re} \left(\int_{\Omega} D^{\alpha} u(x) \overline{D^{\alpha} v(x)} dx \right), \quad \forall u, v \in H^m(\Omega),$$

where $\alpha \in \mathbb{N}^d$ is a multi-index and $D^{\alpha} = D^{(\alpha_1, \dots, \alpha_d)} = \partial_{x_1}^{\alpha_1} \dots \partial_{x_d}^{\alpha_d}$.

2.2 Linear Schrödinger equation

We now turn to the well-posedness of the linear Schrödinger equation. For any $s \in \mathbb{R}$, denote $A\psi = i\Delta\psi$, $\forall \psi \in D(A)$ with $A : D(A) = H^{s+2}(\mathbb{R}^d) \rightarrow H^s(\mathbb{R}^d)$ the infinitesimal generator of the group of isometry $S(t)$ on $H^s(\mathbb{R}^d)$. The solution of

$$\begin{cases} i\psi_t + \Delta\psi = 0, & (x, t) \in \mathbb{R}^d \times \mathbb{R}, \\ \psi(x, 0) = \psi_0(x), & x \in \mathbb{R}^d, \end{cases}$$

for $\psi_0 \in H^s(\mathbb{R}^d)$, is given by $\psi(\cdot, t) = S(t)\psi_0$ and satisfies $\psi(\cdot, t) \in C(\mathbb{R}; H^s(\mathbb{R}^d))$. A classical semigroup result [7, 34] also yields that the unique solution to the inhomogeneous problem

$$\begin{cases} i\psi_t + \Delta\psi = f, & (x, t) \in \mathbb{R}^d \times \mathbb{R}, \\ \psi(x, 0) = \psi_0(x), & x \in \mathbb{R}^d, \end{cases}$$

is $\psi(\cdot, t) \in C(\mathbb{R}; H^s(\mathbb{R}^d))$ for any $s \in \mathbb{R}$, $\psi_0 \in H^s(\mathbb{R}^d)$ and $f \in L^1_{loc}(\mathbb{R}; H^s(\mathbb{R}^d))$. The solution is also given by the Duhamel formula : for any $t \in \mathbb{R}$,

$$\psi(t) = S(t)\psi_0 - i \int_0^t S(t-\tau)f(\tau)d\tau, \quad \text{in } H^s(\mathbb{R}^d).$$

2.3 Strichartz estimates

Let us now introduce the definition of admissible pair for the Strichartz estimates. Let $1 \leq q \leq \infty$ and $r \in \mathbb{R}^+$. We say that the pair (q, r) is admissible if and only if

$$\frac{2}{q} = d \left(\frac{1}{2} - \frac{1}{r} \right), \quad (13)$$

and if $2 \leq r \leq \infty$ for $d = 1$, $2 \leq r < \infty$ for $d = 2$ and

$$2 \leq r \leq \frac{2d}{d-2},$$

if $d \geq 3$. We denote q' the conjugate of q such that $\frac{1}{q} + \frac{1}{q'} = 1$. Let us recall the Strichartz estimates that we shall use in this paper.

Lemma 1 (Strichartz's estimates, [7]). *For any $s \in \mathbb{R}$, the following holds*

- If (q, r) is an admissible pair, then there exists a constant $C > 0$ such that, for every $\psi \in H^s(\mathbb{R}^d)$,

$$\|S(t)\psi\|_{L^q(\mathbb{R}; W^{s,r}(\mathbb{R}^d))} \leq C \|\psi\|_{H^s(\mathbb{R}^d)},$$

- Let $I \subset \mathbb{R}$, be an interval, bounded or not, $J = \bar{I}$ satisfying $0 \in J$. If (γ, ρ) and (q, r) are two admissible pairs, then there exists a constant $C > 0$ such that for every $f \in L^{\gamma'}(I, W^{s,\rho'}(\mathbb{R}^d))$

$$\left\| \int_0^t S(t-\tau)f(\tau)d\tau \right\|_{L^q(I; W^{s,r}(\mathbb{R}^d))} \leq C \|f\|_{L^{\gamma'}(I, W^{s,\rho'}(\mathbb{R}^d))}.$$

The above Strichartz estimates allows us to deduce the well-posedness of the linearized equation of (9) around smooth trajectories of (1). Indeed, consider

$$\begin{cases} i\psi_t + \Delta\psi + a(x, t)\psi + b(x, t)\bar{\psi} = f(x, t), & (x, t) \in \mathbb{R}^d \times (0, T), \\ \psi(x, 0) = \psi_0(x), & x \in \mathbb{R}^d, \end{cases} \quad (14)$$

with $\psi_0 \in H^s(\mathbb{R}^d)$, $f \in L^1_{loc}(\mathbb{R}; H^s(\mathbb{R}^d))$ and a, b complex-valued functions belonging to $C^\infty(\mathbb{R}; \mathcal{S}(\mathbb{R}^d))$. The solution to (14) is given by the Duhamel formula,

$$\psi(t) = S(t)\psi_0 - i \int_0^t S(t-\tau)(a\psi + b\bar{\psi} + f)(\tau)d\tau, \quad \text{in } H^s(\mathbb{R}^d).$$

The well-posedness of (14) in $C([0, \infty); H^s(\mathbb{R}^d))$ was proved [37] by proving that the Strichartz estimates of Lemma 1 also holds for the flow map $S_L(t)$ associated to

$$L\psi := i\psi_t + \Delta\psi + a(x, t)\psi + b(x, t)\bar{\psi}.$$

Proposition 1 (Proposition 2.4, [37]). *For any $s \in \mathbb{R}$ and any an admissible pair (q, r) such that $q > 2$, then the following holds:*

- There exists a constant $C > 0$ such that, for every $\psi \in H^s(\mathbb{R}^d)$,

$$\|S_L(t)\psi\|_{L^q(\mathbb{R}; W^{s,r}(\mathbb{R}^d))} \leq C \|\psi\|_{H^s(\mathbb{R}^d)},$$

- Let $I \subset \mathbb{R}$, be an interval, bounded or not, $J = \bar{I}$ satisfying $0 \in J$. Let (γ, ρ) be another admissible pair, then there exists a constant $C > 0$ such that for every $f \in L^{\gamma'}(I, W^{s, \rho'}(\mathbb{R}^d))$,

$$\left\| \int_0^t S_L(t - \tau) f(\tau) d\tau \right\|_{L^q(I; W^{s, r}(\mathbb{R}^d))} \leq C \|f\|_{L^{\gamma'}(I, W^{s, \rho'}(\mathbb{R}^d))}.$$

These estimates together with standard semigroup arguments [34] allows to prove the well-posedness of

$$\begin{cases} i\psi_t + \Delta\psi + a(x, t)\psi + b(x, t)\bar{\psi} = f(x, t), & (x, t) \in \mathbb{R}^d \times \mathbb{R}^+, \\ \psi(x, 0) = \psi_0(x), & x \in \mathbb{R}^d. \end{cases} \quad (15)$$

Proposition 2. Let $\psi_0 \in H^1(\mathbb{R}^d)$, $f \in L^1(\mathbb{R}^+; H^1(\mathbb{R}^d))$. Then equation (15) is well-posed with $\psi \in C(\mathbb{R}^+; H^1(\mathbb{R}^d))$.

2.4 Well-posedness of the nonlinear equation on \mathbb{R}^d

We now turn to the well-posedness of the nonlinear equation

$$\begin{cases} i\psi_t + \Delta\psi + |\psi|^{p-1}\psi = 0, & (x, t) \in \mathbb{R}^d \times \mathbb{R}^+, \\ \psi(x, 0) = \psi_0(x), & x \in \mathbb{R}^d, \end{cases} \quad (16)$$

where $\psi_0 \in H^1(\mathbb{R}^d)$. The local well-posedness holds using Kato's argument [17] (see also [7, Section 4.4]).

Theorem 4 ([17]). Let $j \in C(\mathbb{C}; \mathbb{C})$ such that $j(0) = 0$,

$$|j(u) - j(v)| \leq L(K)|u - v|,$$

for all $u, v \in \mathbb{C}$ such that $|u|, |v| \leq K$ for $K > 0$ with

$$\begin{cases} L(t) \in C([0, \infty)), & d = 1, \\ L(t) \leq C(1 + t^\alpha), & \text{with } 0 \leq \alpha < \frac{4}{d-2}, \text{ if } d \geq 2. \end{cases}$$

Set $g(u)(x) = j(u(x))$ for all measurable $u : \mathbb{R}^d \rightarrow \mathbb{C}$ almost everywhere in \mathbb{R}^d . Assume j , considered as a function of $\mathbb{R}^2 \rightarrow \mathbb{R}^2$, is of class C^1 . Let $\delta > 0$. Then, for every $\psi_0 \in H^1(\mathbb{R}^d)$ such that $\|\psi_0\|_{H^1} < \delta$, there exists T_{ψ_0} such that $\psi(\cdot, t) \in C([0, T_{\psi_0}); H^1(\mathbb{R}^d))$. Moreover, either $T_{\psi_0} = \infty$ and the solution is globally well-posed, or $\lim_{t \rightarrow T_{\psi_0}^-} \|\nabla\psi(\cdot, t)\|_{L^2(\mathbb{R}^d)} = \infty$.

The same proof also implies the local existence of the solution to

$$\begin{cases} i\psi_t + \Delta\psi + |\psi|^{p-1}\psi = f(x, t), & (x, t) \in \mathbb{R}^d \times \mathbb{R}^+, \\ \psi(x, 0) = \psi_0(x), & x \in \mathbb{R}^d, \end{cases} \quad (17)$$

with $f \in L^1(\mathbb{R}; H^1(\mathbb{R}^d))$ assuming a smallness condition on the source term, $\|f\|_{L^1(\mathbb{R}; H^1(\mathbb{R}^d))} < \delta$. In fact, one deduces the local well-posedness of (17) around any smooth trajectory w of (1) from [7, Theorem 4.4.6],

$$\begin{cases} i\psi_t + \Delta\psi + |w + \psi|^{p-1}(w + \psi) - |w|^{p-1}w = f(x, t), & (x, t) \in \mathbb{R}^d \times \mathbb{R}^+, \\ \psi(x, 0) = \psi_0(x), & x \in \mathbb{R}^d. \end{cases} \quad (18)$$

Proposition 3. *Let $\psi_0 \in H^1(\mathbb{R}^d)$, $f \in L^1(\mathbb{R}; H^1(\mathbb{R}^d))$, $w \in C^\infty(\mathbb{R}; \mathcal{S}(\mathbb{R}^d))$ solution to (1) and $\delta > 0$. Then, for every ψ_0 and f such that $\|\psi_0\|_{H^1(\mathbb{R}^d)} < \delta$ and $\|f\|_{L^1(\mathbb{R}; H^1(\mathbb{R}^d))} < \delta$, there exists $T_{\psi_0, f} > 0$ such that the solution to (18) satisfies $\psi(\cdot, t) \in C([0, T_{\psi_0, f}]; H^1(\mathbb{R}^d))$.*

Indeed, [7, Theorem 4.4.6] is stated as follow for general nonlinear equation of the form

$$\begin{cases} i\psi_t + \Delta\psi + g(\psi) = 0, & (x, t) \in \mathbb{R}^d \times \mathbb{R}^+, \\ \psi(x, 0) = \psi_0(x), & x \in \mathbb{R}^d, \end{cases} \quad (19)$$

with the following assumption on the non-linearity $g \in C(H^1(\mathbb{R}^d); H^{-1}(\mathbb{R}^d))$: suppose there exist $2 \leq r, \rho < \frac{2d}{d-2}$ ($2 \leq r, \rho < \infty$ if $d = 1$) such that

$$\|g(u) - g(v)\|_{L^{\rho'}(\mathbb{R}^d)} \leq C(\delta)\|u - v\|_{L^r(\mathbb{R}^d)}, \quad (20)$$

for all $u, v \in H^1(\mathbb{R}^d)$ such that $\|u\|_{H^1(\mathbb{R}^d)} < \delta$, $\|v\|_{H^1(\mathbb{R}^d)} < \delta$ and

$$\|g(u)\|_{W^{1, \rho'}(\mathbb{R}^d)} \leq C(\delta)(1 + \|u\|_{W^{1, r}(\mathbb{R}^d)}), \quad (21)$$

for all $u \in H^1(\mathbb{R}^d) \cap W^{1, r}(\mathbb{R}^d)$, such that $\|u\|_{H^1(\mathbb{R}^d)} < \delta$. Then, (we refer to [7] for a more precise statement)

Theorem 5 (Theorem 4.4.6 [7]). *Let $g = g_1 + g_2 + \dots + g_k$ such that each $g_j, j = 1, \dots, k$ satisfies (20) and (21) for some exponent r_j, ρ_j . Then, for every $\psi_0 \in H^1(\mathbb{R}^d)$, there exists a unique strong H^1 solution of (19) defined on a maximal time interval $(0, T_{\psi_0})$.*

Proposition 3 follows then easily. Indeed, define

$$g(\psi) = |w + \psi|^{p-1}(w + \psi) - |w|^{p-1}w,$$

$\theta(z) = 1$ if $|z| \leq 1$ and

$$g_1(\psi) = \theta(\psi)g(\psi), \quad g_2(\psi) = (1 - \theta(\psi))g(\psi).$$

Then g_1, g_2 satisfies (20) and (21) (see the proof of Proposition 5). Hence the local well-posedness follows for $f = 0$, and the case with $f \neq 0$ with $\|f\|_{L^1(\mathbb{R}; H^1(\mathbb{R}^d))} < \delta$ is dealt classically by including f in the Duhamel formula and using the Strichartz estimates.

3 Local exact controllability around the ground state

The goal of this section is to extend the local exact controllability around smooth trajectories of (1) obtained by Rosier and Zhang [37] for $p = 3$ and $d \geq 1$ to the local exact controllability around the ground state for the mass critical and mass subcritical regime $0 < p - 1 \leq 4/d$ with $d \geq 1$.

Theorem 6. *Let $T > 0$, $p \in [1, p^*]$ and $\lambda > 0$. Then there exists a $\delta > 0$ such that for any $\psi_0, \psi_T \in H^1(\Omega)$ satisfying*

$$\|\psi_0 - \psi_\lambda(\cdot, 0)\|_{H^1(\Omega)} \leq \delta, \quad \|\psi_T - \psi_\lambda(\cdot, T)\|_{H^1(\Omega)} \leq \delta,$$

one can find an appropriate boundary control function $v(x, t)$ such that (12) admits a solution $\psi \in C([0, T]; H^1(\Omega))$ such that

$$\psi(x, 0) = \psi_0(x), \quad \psi(x, T) = \psi_T(x), \quad \text{in } \Omega.$$

We begin by recalling their exact controllability result for the linear equation.

3.1 Controllability of the linearized equation

Let $T > 0$ and consider

$$\begin{cases} i\psi_t + \Delta\psi + a(x, t)\psi + b(x, t)\bar{\psi} = \varphi(x)h(x, t), & (x, t) \in \mathbb{R}^d \times (0, T), \\ \psi(x, 0) = \psi_0(x), & x \in \mathbb{R}^d, \end{cases} \quad (22)$$

with $\psi_0 \in H^s(\mathbb{R}^d)$, $a, b \in C^\infty((0, T); \mathcal{S}(\mathbb{R}^d))$, $h \in L^2((0, T); H^s(\mathbb{R}^d))$ and $\varphi \in C^\infty(\mathbb{R}^d; [0, 1])$ defined by (10).

Theorem 7 ([37, Theorem 3.1]). *Let $T > 0$ and $s \geq 0$ be given and assume $a, b \in C^\infty((0, T); \mathcal{S}(\mathbb{R}^d))$. There exists a bounded linear operator*

$$G : H^s(\mathbb{R}^d) \times H^s(\mathbb{R}^d) \rightarrow L^2((0, T); H^s(\mathbb{R}^d)),$$

such that for any $\psi_0, \psi_T \in H^s(\mathbb{R}^d)$, if one chooses $h = G(\psi_0, \psi_T)$ as a control input, then the system (22) admits a solution $\psi \in C([0, T]; H^s(\mathbb{R}^d))$ satisfying

$$\psi(\cdot, T) = \psi_T, \text{ in } H^s(\mathbb{R}^d).$$

3.2 Local exact controllability around smooth trajectories

We now turn to the local exact controllability of,

$$\begin{cases} i\psi_t + \Delta\psi + |\psi|^{p-1}\psi = \varphi(x)h(x, t), & (x, t) \in \mathbb{R}^d \times (0, T), \\ \psi(x, 0) = \psi_0(x), & x \in \mathbb{R}^d, \end{cases} \quad (23)$$

around ground state solitary wave $\psi_\lambda(x, t) = e^{i\lambda t}Q_\lambda(x)$. We recall that, in this case, $\psi_\lambda \in C^\infty(\mathbb{R}; \mathcal{S}(\mathbb{R}^d))$ and that Q_λ is positive. We first write solution to (23) under the form $\psi = \psi_\lambda + y$, where y solves,

$$\begin{cases} iy_t + \Delta y + \frac{p+1}{2}Q_\lambda^{p-1}y + \frac{p-1}{2}e^{i2\lambda t}Q_\lambda^{p-1}\bar{y} + g(y) = \varphi(x)h(x, t), & (x, t) \in \mathbb{R}^d \times (0, T), \\ y(x, 0) = y_0(x), & x \in \mathbb{R}^d, \end{cases} \quad (24)$$

where,

$$g(y) = |e^{i\lambda t}Q_\lambda + y|^{p-1}(e^{i\lambda t}Q_\lambda + y) - e^{i\lambda t}Q_\lambda^p - \frac{p+1}{2}Q_\lambda^{p-1}y - \frac{p-1}{2}e^{i2\lambda t}Q_\lambda^{p-1}\bar{y}. \quad (25)$$

Denote $S_L(t)$ the flow map associated to (22) with

$$a(x, t) = \frac{p+1}{2}Q_\lambda^{p-1}(x), \quad b(x, t) = \frac{p-1}{2}e^{2i\lambda t}Q_\lambda^{p-1}(x). \quad (26)$$

The following result give the sufficient functional framework to obtain the local exact controllability of (24).

Proposition 4. *Let $s \geq 0$, $T > 0$ and g defined by (25). If the following holds in a Banach space $X_{s,T} \subset C([0, T]; H^s(\mathbb{R}^d))$: there exists $C > 0$ such that $\phi \in H^s(\mathbb{R}^d)$,*

$$\|S_L(t)\phi\|_{X_{s,T}} \leq C\|\phi\|_{H^s(\mathbb{R}^d)},$$

for every $f \in L^2((0, T); H^s(\mathbb{R}^d))$,

$$\left\| \int_0^t S_L(t-\tau)f(\cdot, \tau) d\tau \right\|_{X_{s,T}} \leq C\|f\|_{L^2((0, T); H^s(\mathbb{R}^d))},$$

for every $z_1, z_2 \in X_{s,T}$,

$$\left\| \int_0^t S_L(t-\tau)g(z_1)(\tau) d\tau \right\|_{X_{s,T}} \leq C(1 + \|z_1\|_{X_{s,T}}) \|z_1\|_{X_{s,T}}^{p-1},$$

and

$$\left\| \int_0^t S_L(t-\tau)(g(z_1) - g(z_2))(\tau) d\tau \right\|_{X_{s,T}} \leq C(\|z_1\|_{X_{s,T}} + \|z_2\|_{X_{s,T}} + \|z_1\|_{X_{s,T}}^{p-1} + \|z_2\|_{X_{s,T}}^{p-1}) \|z_1 - z_2\|_{X_{s,T}},$$

then (24) is locally exactly controllable in $H^s(\mathbb{R}^d)$, that is, there exists $\delta > 0$ such that for every $y_0, y_1 \in H^s(\mathbb{R}^d)$ such that,

$$\|y_0\|_{H^s(\mathbb{R}^d)} < \delta, \quad \|y_1\|_{H^s(\mathbb{R}^d)} < \delta,$$

then one can find a control $h \in L^2((0, T); H^s(\mathbb{R}^d))$ such that the solution $y \in C([0, T]; H^s(\mathbb{R}^d))$ of (24) satisfies,

$$y(\cdot, T) = y_1, \text{ in } H^s(\mathbb{R}^d).$$

Proof. The proof is very similar to [37, Theorem 4.1]. First, notice that (24) linearized around zero is given by (22), with a and b are defined by (26). Since a, b defined by (26) belong to $C^\infty(\mathbb{R}; \mathcal{S}(\mathbb{R}^d))$, the exact controllability given by Theorem 7 holds.

The solution to (24) is given by the Duhamel formula,

$$y(t) = S_L(t)y_0 - \int_0^t S_L(t-\tau)g(y)(\tau)d\tau + \int_0^t S_L(t-\tau)\varphi(x)h(x, \tau)d\tau.$$

Using the time-reversibility, we can assume $y_1 = 0$. The proof of Proposition 4 is obtained by a classical Fixed Point Theorem. Indeed, let

$$h = G \left(y_0, \int_0^t S_L(t-\tau)g(u)(\tau)d\tau \right)$$

for every $u \in C([0, T]; H^s(\mathbb{R}))$. Then, let

$$\Gamma(u)(t) = WS_L(t)y_0 - \int_0^t S_L(t-\tau)g(y)(\tau)d\tau + \int_0^t S_L(t-\tau)\varphi(x)h(x, \tau)d\tau.$$

We easily see that

$$\Gamma(u)(0) = y_0, \quad \Gamma(u)(T) = 0.$$

It therefore suffices to prove that $\Gamma(u)(t)$ is a contraction in $X_{s,T}$ thanks to the various estimates in hypothesis of Proposition 4. From now on, the proof is the same as in [37, Theorem 4.1] and we therefore omit it here. □

It remains to define the proper space $X_{s,T}$ and to prove the estimates of Proposition 4.

Proposition 5. *Let $T > 0$, $d \geq 1$, $s = 1$ and $r = p + 1$ for $p \in (1, p^*]$. Then the estimates of Proposition 4 hold in $X_T := C([0, T]; H^1(\mathbb{R}^d)) \cap L^q((0, T); W^{1,r}(\mathbb{R}^d))$ endowed with the norm $\|\cdot\|_{X_{s,T}} := \|\cdot\|_{L^\infty((0,T); H^1(\mathbb{R}^d))} + \|\cdot\|_{L^q((0,T); W^{1,r}(\mathbb{R}^d))}$.*

Proof. Let $r := p + 1$ and q defined by (13). The pair (q, r) is admissible pair since $r \geq 2$, r is always finite and satisfies

$$2 \leq r \leq \frac{2d}{d-2},$$

if $d \geq 3$. Following [37], let $\xi \in C_0^\infty(\mathbb{C})$ be such that $\xi(z) = 1$ for $|z| \leq 1$ and set

$$g_1(z) = \xi(z)g(z), \quad g_2(v) = (1 - \xi(z))g(z)$$

Since $p > 1$, we have the following for $C_i > 0$,

$$\begin{aligned} \|g_1(z)\|_{L^2(\mathbb{R}^d)} &\leq C\|z\|_{L^2(\mathbb{R}^d)} + \|\xi(z) \left((1 + |\psi_c|^{p-1})(z + \psi_c) - e^{ict} Q_c^p \right)\|_{L^2(\mathbb{R}^d)} \\ &\leq C_1(C_2 + \|z\|_{L^2(\mathbb{R}^d)}) \\ &\leq C_3\|z\|_{L^2(\mathbb{R}^d)} \end{aligned}$$

and

$$\begin{aligned} \|g_2(z)\|_{L^{r'}(\mathbb{R}^d)} &\leq C\|z\|_{L^{r'}(\mathbb{R}^d)} + \|\xi(z) \left((1 + |\psi_c|^{p-1})(z + \psi_c) - e^{ict} Q_c^p \right)\|_{L^{r'}(\mathbb{R}^d)} \\ &\leq C_1(C_2 + \|z\|_{L^{r'}(\mathbb{R}^d)}^{p-1}) \\ &\leq C_3\|z\|_{L^{r'}(\mathbb{R}^d)}^{p-1} \end{aligned}$$

A straightforward estimation yields

$$\|g_1(z_1) - g_1(z_2)\|_{L^2(\mathbb{R}^d)} \leq C\|z_1 - z_2\|_{L^2(\mathbb{R}^d)}$$

and using the inequality $\|u|^{p-1}u - v|^{p-1}v\| \leq C(|u|^{p-1} + |v|^{p-1})|u - v|$ for $p > 1$ ([42]), Hölder and Minkowski inequalities, we deduce

$$\begin{aligned} \|g_2(z_1) - g_2(z_2)\|_{L^{r'}} &\leq C\|Q_c^{p-1}(z_1 - z_2)\|_{L^{r'}} + C\left(\|z_1 + \psi_c\|_{L^{r'}}^{p-1} + \|z_2 + \psi_c\|_{L^{r'}}^{p-1}\right)\|z_1 - z_2\|_{L^{r'}} \\ &\leq C\|z_1 - z_2\|_{L^r} + C\left(\|z_1 + \psi_c\|_{L^r}^{p-1} + \|z_2 + \psi_c\|_{L^r}^{p-1}\right)\|z_1 - z_2\|_{L^r} \\ &\leq C\left(\|z_1\|_{L^r}^{p-1} + \|z_2\|_{L^r}^{p-1}\right)\|z_1 - z_2\|_{L^r}. \end{aligned}$$

where r' denotes the conjugate of r . Moreover, using the diamagnetic inequality ([42]) : for all $f \in H^1(\mathbb{R}^d)$

$$\nabla|f| \leq |\nabla f|,$$

in the sense of distribution, we obtain

$$\|\nabla g_1(z)\|_{L^2(\mathbb{R}^d)} \leq C\|\nabla z\|_{L^2(\mathbb{R}^d)}$$

and

$$\|\nabla g_2(z)\|_{L^{r'}(\mathbb{R}^d)} \leq C\|z\|_{L^r(\mathbb{R}^d)}^{p-1}\|\nabla z\|_{L^r(\mathbb{R}^d)}$$

Since $p \in (1, p^*]$, we have $q > 2$ (recall (13)) and therefore from [37, Proposition 2.4] (or see also [7]), we have that for every admissible pair (q, r) , the following holds

$$\begin{aligned} \|S_L(t)\psi_0\|_{X_T} &\leq C\|\psi_0\|_{H^1(\mathbb{R}^d)}, \\ \left\| \int_0^t S_L(t-\tau)\varphi(\cdot)h(\cdot, \tau)d\tau \right\|_{X_{s,T}} &\leq C\|\varphi h\|_{L^1((0,T);H^1(\mathbb{R}^d))} \leq C\|\varphi h\|_{L^2((0,T);H^1(\mathbb{R}^d))}, \\ \left\| \int_0^t S_L(t-\tau)(g(z))(\tau)d\tau \right\|_{X_T} &\leq C\left(\|g_1(z)\|_{L^1((0,T);H^1(\mathbb{R}^d))} + \|g_2(z)\|_{L^{q'}((0,T);W^{1,r'}(\mathbb{R}^d))}\right), \\ \left\| \int_0^t S_L(t-\tau)(g(v_1) - g(v_2))(\tau)d\tau \right\|_{X_T} &\leq C\left(\|g_1(z_1) - g_1(z_2)\|_{L^1((0,T);H^1(\mathbb{R}^d))}\right) \\ &\quad + \|g_2(z_1) - g_2(z_2)\|_{L^{q'}((0,T);W^{1,r'}(\mathbb{R}^d))}. \end{aligned}$$

In a similar fashion as above, one deduces that for $z_1, z_2 \in X_T$, $g_1(z_1) \in L^\infty((0, T); H^1(\mathbb{R}^d))$ and $g_2(z_1) \in L^q((0, T); W^{1, r'}(\mathbb{R}^d))$ with the following estimates

$$\begin{aligned} \|g_1(z_1)\|_{L^\infty((0, T); H^1(\mathbb{R}^d))} &\leq C \|z_1\|_{L^\infty((0, T); H^1(\mathbb{R}^d))} \\ \|g_2(z_1)\|_{L^{q'}((0, T); W^{1, r'}(\mathbb{R}^d))} &\leq C \|z_1\|_{L^\infty((0, T); L^{1, r}(\mathbb{R}^d))}^{p-1} \|z_1\|_{L^q((0, T); W^{1, r}(\mathbb{R}^d))} \\ \|g_1(z_1) - g_1(z_2)\|_{L^\infty(H^1)} &\leq C (\|z_1\|_{L^\infty(H^1)} + \|z_2\|_{L^\infty(H^1)}) \|z_1 - z_2\|_{L^\infty(H^1)} \\ \|g_2(z_1) - g_2(z_2)\|_{L^{q'}(W^{1, r'})} &\leq C \left(\|z_1\|_{L^\infty(L^{1, r})}^{p-1} + \|z_2\|_{L^\infty(L^{1, r})}^{p-1} \right) \|z_1 - z_2\|_{L^\infty(W^{1, r})} \end{aligned}$$

Consequently,

$$\|g_1(z)\|_{L^1((0, T); H^1(\mathbb{R}^d))} + \|g_2(z)\|_{L^{q'}((0, T); W^{1, r'}(\mathbb{R}^d))} \leq C(1 + \|z\|_{X_T}) \|z\|_{X_T}^{p-1}$$

and

$$\begin{aligned} \|g_1(z_1) - g_1(z_2)\|_{L^1((0, T); H^1(\mathbb{R}^d))} + \|g_2(z_1) - g_2(z_2)\|_{L^{q'}((0, T); W^{1, r'}(\mathbb{R}^d))} \\ \leq C \left(\|z_1\|_{X_T} + \|z_2\|_{X_T} + \|z_1\|_{X_T}^{p-1} + \|z_2\|_{X_T}^{p-1} \right) \|z_1 - z_2\|_{X_T}, \end{aligned}$$

that is, the desired estimates. □

We finally obtain the local exact controllability of (12) around smooth trajectories of (1) by taking the trace of the solution of (23) as the control. The well-posedness and the properties of the control operator G ensures that the solution ψ defined this way belongs to $C([0, T]; H^1(\mathbb{R}^d))$.

4 Properties of the blow-up trajectory

We present here the properties of the blow-up trajectory in the mass critical and mass slightly subcritical regime that shall be needed for the proof of Theorem 3. We begin by recalling the blow-up profile close in $\dot{H}^1(\mathbb{R}^d)$ to the ground state in the mass slightly supercritical regime obtained in [1]. Using this construction, we prove first in Theorem 9 that this construction yields a blow-up profile close in $\dot{H}^1(\mathbb{R}^d)$ to the ground state in the slightly subcritical regime. The second part of the proof of Theorem 9 lies in the proof that the blow-up profile is close to the ground state in $H_{loc}^1(\mathbb{R}^d)$. This closeness to the ground state shall be used in the proof of Theorem 3 to drive the initial or final data to the blow-up trajectory using the local controllability around the ground state. We recall that one cannot expect the blow-up profile to be close to the ground state in $H^1(\mathbb{R}^d)$ as the blow-up profile does not belong to $L^2(\mathbb{R}^d)$ [1].

4.1 Blow-up profile in the slightly mass supercritical regime

The blow-up profile constructed in [1] is based on the ansatz

$$\psi(x, t) = \frac{1}{\lambda^{\frac{p-1}{2}}(t)} e^{i\theta(t)} \tilde{\psi} \left(\frac{x}{\lambda(t)} \right), \quad (27)$$

where $\tilde{\psi}$ is assumed to be radially symmetric, $\tilde{\psi}(x) = \tilde{\psi}(|x|) = \tilde{\psi}(r)$, $r := |x|$ and where, for a given $T \in \mathbb{R}$ and $b > 0$, $\lambda(t)$ and $\theta(t)$ are defined for $t \in [0, T)$ by

$$\lambda(t) := \sqrt{2b(T-t)}, \quad \theta(t) := -\frac{\log(T-t)}{2b}. \quad (28)$$

The parameter b , that shall be small, is used as a bifurcation parameter. Indeed, for ψ defined by (45) to be a solution to (1), $\tilde{\psi}$ has to satisfy

$$\Delta\tilde{\psi} - \tilde{\psi} + ib \left(\frac{2}{p-1}\psi + x \cdot \nabla\psi \right) + |\tilde{\psi}|^{p-1}\tilde{\psi} = 0, \quad x \in \mathbb{R}^d. \quad (29)$$

Notice that taking $b = 0$ in (29) is equivalent to (3), which makes the bifurcation parameter b apparent. Bahri, Martel and Raphaël are able to give a precise description of the finite energy self-similar blow-up according to the ansatz (45).

Theorem 8. [1, Theorem 1] *Let $d \geq 1$, $0 < s_c \ll 1$ and $p^* = 1 + 4/d$ be the mass critical exponent. There exists $\epsilon > 0$ such that for any p satisfying*

$$0 < p - p^* < \epsilon,$$

there exists $b(p) > 0$ and a non-zero radially symmetric solution $\tilde{\psi}$ to (29) such that

$$\tilde{\psi} \in \dot{H}^1(\mathbb{R}^d) \cap C^2(\mathbb{R}^d), \quad E(\tilde{\psi}) = 0.$$

Moreover, as $p \rightarrow p^{+}$, the following hold*

- *Law for the nonlinear eigenvalue : $b = b_{s_c}(1 + o(1))$ where b_{s_c} is defined by*

$$s_c = \frac{\kappa^2}{N_c b_{s_c}} \exp\left(-\frac{\pi}{b_{s_c}}\right), \quad N_c = \int_0^\infty Q^2(r)r^{d-1}dr, \quad (30)$$

and κ is defined as in (4);

- *Bifurcation from the soliton profile : $\|\tilde{\psi} - Q\|_{\dot{H}^1(\mathbb{R}^d)} = o(1)$;*
- *Non-oscillatory behaviour for the outgoing wave:*

$$\lim_{r \rightarrow \infty} r^{\frac{2}{p-1}} |\tilde{\psi}(r)| = \rho_{s_c}(1 + o(1)), \quad \limsup_{r \rightarrow \infty} r^{\frac{p+1}{p-1}} |\tilde{\psi}'(r)| < \infty,$$

where

$$\rho_{s_c} = \sqrt{2N_c s_c}.$$

Based on the construction of [1], we shall prove that there exists a blow-up profile in the mass slightly subcritical case satisfying the following.

Theorem 9. *Let $d \geq 1$, $-1 \ll s_c < 0$ and $p^* = 1 + 4/d$ be the mass critical exponent. Then there exists $\epsilon > 0$ such that for any p satisfying*

$$-\epsilon < p - p^* < 0$$

there exists $b(p) > 0$ and a non-zero radially symmetric solution $\tilde{\psi}$ to (29) such that

$$\tilde{\psi} \in \dot{H}^1(\mathbb{R}^d) \cap C^2(\mathbb{R}^d), \quad E(\tilde{\psi}) = 0.$$

Moreover, as $p \rightarrow p^{-}$, the following hold*

- *Law for the nonlinear eigenvalue : $b = b_{s_c}(1 + o(1))$ where b_{s_c} is defined by*

$$|s_c| = \frac{\kappa^2}{N_c b_{s_c}} \exp\left(-\frac{\pi}{b_{s_c}}\right), \quad N_c = \int_0^\infty Q^2(r)r^{d-1}dr;$$

- *Bifurcation from the soliton profile* : $\|\tilde{\psi} - Q\|_{H_{loc}^1(\mathbb{R}^d)} = o(1)$;
- *Non oscillatory behaviour for the outgoing wave*:

$$\lim_{r \rightarrow \infty} r^{\frac{2}{p-1}} |\tilde{\psi}(r)| = \rho_{s_c}(1 + o(1)), \quad \limsup_{r \rightarrow \infty} r^{\frac{p+1}{p-1}} |\tilde{\psi}'(r)| < \infty,$$

where

$$\rho_{s_c} = \sqrt{2N_c |s_c|}.$$

For the controllability purposes of Theorem 3, we shall prove that the blow-up profile satisfies $\|\tilde{\psi} - Q\|_{H_{loc}^1(\mathbb{R}^d)} = o(1)$. Compared to Theorem 8, this requires to prove the additional convergence $\|\tilde{\psi} - Q\|_{L_{loc}^2(\mathbb{R}^d)} = o(1)$. We therefore recall below the main results behind Theorem 8 to collect the estimates of the blow-up profile in the different regions.

Let $|x| = r \in [0, \infty)$ and consider the change of variables

$$\tilde{\psi}(x) = \exp\left(-i\frac{br^2}{4}\right) P(r). \quad (31)$$

Then $\tilde{\psi}$ is a radial symmetric solution to (29) if we are able to find $P : [0, \infty) \rightarrow \mathbb{C}$ solution to

$$\begin{cases} P'' + \frac{d-1}{r}P' + \left(\frac{b^2r^2}{4} - 1 - ibs_c\right)P + |P|^{p-1}P = 0, & r > 0, \\ P'(0) = 0. \end{cases} \quad (32)$$

The dependence between b and s_c in (32) is relaxed by looking for a solution P_σ of

$$\begin{cases} P'' + \frac{d-1}{r}P' + \left(\frac{b^2r^2}{4} - 1 - ib\sigma\right)P + |P|^{p-1}P = 0, & r > 0, \\ P'(0) = 0, \end{cases} \quad (33)$$

with $\sigma > 0$ small. Clearly, if P_σ is solution to (33) for any $\sigma > 0$ small, then taking $\sigma = s_c$ yields P_{s_c} solution to (32) and therefore a radially symmetric profile $\tilde{\psi}$ satisfying (29). Additional parameters $(\sigma, \rho, \gamma, \theta) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}$ are introduced in order to construct the profile P_σ , with an a priori control with respect to σ :

$$b \in \left[b_\sigma - \frac{1}{2}b_\sigma^{\frac{13}{6}}, b_\sigma + \frac{1}{2}b_\sigma^{\frac{13}{6}} \right], \quad (34)$$

where $b_\sigma > 0$ is defined by

$$\sigma = \frac{\kappa^2}{N_c b_\sigma} \exp\left(-\frac{\pi}{b_\sigma}\right), \quad N_c = \int_0^\infty Q^2(r)r^{d-1}dr,$$

and

$$\rho \in \left[\frac{1}{2}\rho_\sigma, \frac{3}{2}\rho_\sigma \right], \quad \gamma \in \left[-\frac{1}{2}\gamma_\sigma, \frac{1}{2}\gamma_\sigma \right], \quad \theta \in \left[-\frac{1}{2}\theta_\sigma, \frac{1}{2}\theta_\sigma \right], \quad (35)$$

where

$$\rho_\sigma = \sqrt{2N_c^{\frac{1}{2}}}\sqrt{\sigma}, \quad \gamma_\sigma = b_\sigma^{\frac{1}{6}} \exp\left(-\frac{2}{\sqrt{b_\sigma}}\right), \quad \theta_\sigma = b_\sigma^{\frac{1}{6}} \exp\left(-\frac{\pi}{b_\sigma}\right) \exp\left(\frac{2}{\sqrt{b_\sigma}}\right).$$

In particular, there exists $C > 0$,

$$\left| \exp\left(\frac{\pi}{b} - \frac{\pi}{b_\sigma}\right) - 1 \right| \leq C \left| \frac{\pi}{b} - \frac{\pi}{b_\sigma} \right| \leq C b_\sigma^{\frac{1}{6}}, \quad \sigma \leq C b^{-1} \exp\left(-\frac{\pi}{b}\right). \quad (36)$$

These parameters allows are used to define the solutions of (33) over three intervals, $K := [0, r_K]$, $J = [r_K, r_I]$ and $I = [r_I, \infty)$, where $r_K = b^{-1/2}$ and $r_I = b^{-2}$. The three intervals take into account three different dynamic of the solution near the ground state : in K the non-linearity plays an important role, while in I , the ground state is exponentially small and therefore the equation is essentially linear. The interval J includes the so-called turning point, where the real part of the zeroth order operator vanishes². In the proof of Theorem 8, the solution in I is extended to J , and is denoted P_{ext} . It is then matched to the solution P_{int} in K at $r = r_K$ to yield the profile P_σ .

4.1.1 Solution P_{int} of (33) in K

In K , the solution to P_{int} of (33) on K satisfies the following

Proposition 6 (Proposition 4.1, [1]). *For $\sigma > 0$ small enough and for any b, γ satisfying (34) and (35), there exists a solution $P_{int} = P_{int}[\sigma, b, \gamma]$ of (33) on K satisfying*

$$\begin{aligned}\operatorname{Re}(P_{int}(r_K)) &= \kappa b^{\frac{d-1}{4}} \exp\left(-\frac{1}{\sqrt{b}}\right) (1 + O(b^{\frac{1}{3}})) + \kappa_A \gamma b^{\frac{d-1}{4}} \exp\left(\frac{1}{\sqrt{b}}\right), \\ \operatorname{Re}(P'_{int}(r_K)) &= -\kappa b^{\frac{d-1}{4}} \exp\left(-\frac{1}{\sqrt{b}}\right) (1 + O(b^{\frac{1}{3}})) + \kappa_A \gamma b^{\frac{d-1}{4}} \exp\left(\frac{1}{\sqrt{b}}\right), \\ \operatorname{Im}(P_{int}(r_K)) &= \kappa_B \sigma b^{\frac{d+3}{4}} \exp\left(\frac{1}{\sqrt{b}}\right) (1 + O(b^{\frac{1}{4}})), \\ \operatorname{Im}(P'_{int}(r_K)) &= \kappa_B \sigma b^{\frac{d+3}{4}} \exp\left(\frac{1}{\sqrt{b}}\right) (1 + O(b^{\frac{1}{4}})),\end{aligned}$$

and there exists $C > 0$ such that

$$\|P_{int} - Q\|_{\dot{H}^1(K)} \leq C b^{\frac{1}{12}}.$$

Moreover, the map $(\sigma, b, \gamma) \mapsto (P_{int}[\sigma, b, \gamma](r_K), P'_{int}[\sigma, b, \gamma](r_K))$ is continuous.

The solution P_{int} is obtained from the decomposition

$$P = (Q + \gamma A + \phi_+) + i(b\sigma B + \phi_-) \quad (37)$$

by a fixed point argument on $(\phi_+, \phi_-) \in E_K$ where E_K is the complete metric space

$$E_K := \{(\phi_+, \phi_-) : K \rightarrow \mathbb{R}^2 \text{ is continuous and satisfies } \|(\phi_+, \phi_-)\|_K \leq 1\} \quad (38)$$

endowed with the norm

$$\|(\phi_+, \phi_-)\|_K := \max(b^{-1/3} \mathcal{N}_+(\phi_+), b^{-5/4} \sigma^{-1} \mathcal{N}_-(\phi_-)),$$

where

$$\mathcal{N}_+(\phi_+) = \|\phi_+/Q\|_{L^\infty(K)}, \quad \mathcal{N}_-(\phi_-) = \|H\phi_-\|_{L^\infty(K)}, \quad H(r) = (1+r)^{d-1}Q(r), \quad r \geq 0.$$

Moreover, the functions A and B are associated with the linearized operators around the ground state, defined by

$$L_+ = -\partial_{rr} - \frac{d-1}{r} \partial_r + 1 - pQ^{p-1}, \quad (39)$$

$$L_- = -\partial_{rr} - \frac{d-1}{r} \partial_r + 1 - Q^{p-1}, \quad (40)$$

and satisfies, by collecting the results from Lemma 4.1 and Lemma 4.4 of [1],

²it is in fact on (42) that this turning point is seen, and it depends on the dimension d (see [1, 14])

Lemma 2 ([1]). *There exist C^2 functions $A : [0, \infty) \rightarrow \mathbb{R}$ and $B : [0, \infty) \rightarrow \mathbb{R}$ solutions respectively of $L_+A = 0$ and $L_-B = -Q$ on $(0, \infty)$, such that $A(0) = 1$, $A'(0) = B(0) = B'(0) = 0$ and*

$$\begin{aligned} A(r) &= \kappa_A r^{-\frac{d-1}{2}} e^r (1 + O(r^{-1})), & r \in [1, \infty), \\ A'(r) &= \kappa_A r^{-\frac{d-1}{2}} e^r (1 + O(r^{-1})), & r \in [1, \infty), \\ B(r) &= \kappa_B r^{-\frac{d-1}{2}} e^r (1 + O(r^{-1})), & r \in [1, \infty), \\ B'(r) &= \kappa_B r^{-\frac{d-1}{2}} e^r (1 + O(r^{-1})), & r \in [1, \infty), \end{aligned}$$

for a constant $\kappa_A \neq 0$ and $\kappa_B = N_c/(2\kappa) > 0$.

Proposition 6 is obtained in part with the following estimates on the decomposition of P_{int} (37) which relates to the closeness to the ground state with respect to the parameter b (see for instance eq (4.15) in [1]). We shall use these estimates for the closeness of the ground state to the blow-up profile in $H_{loc}^1(\mathbb{R}^d)$.

Lemma 3 ([1]). *There exists $C > 0$ such that*

$$|\gamma A| \leq C b^{\frac{1}{6}} Q, \quad |\phi_+| \leq C b^{\frac{1}{3}} Q, \quad b|\sigma B| + |\phi_-| \leq C \exp\left(-\frac{\pi}{b} + \frac{2}{\sqrt{b}}\right) Q.$$

Proof. These estimates come from the a priori bounds on the parameters σ, b, γ , (4), (34), (35) and (36) that, A and B given by Lemma 2 and for every $(\phi_+, \phi_-) \in E_K$, using the definition of the norm \mathcal{N}_\pm . Indeed, one has for $r \in K$, since Q is positive,

$$\left| \frac{\phi_+}{Q} \right| \leq \left\| \frac{\phi_+}{Q} \right\|_{L^\infty(K)} \leq b^{1/3},$$

from the definition of \mathcal{N}_+ . Moreover, from the definition A and the a priori bound on γ , we have

$$|\gamma A| \leq \frac{1}{2} \gamma_\sigma |A| \leq C b_\sigma^{1/6} \exp\left(-\frac{2}{\sqrt{b_\sigma}}\right) r^{-\frac{d-1}{2}} e^r,$$

Using (4) we have $\kappa r^{-(d-1)/2} e^{-r} \leq Q(r) + C r^{-(d+1)/2} e^{-r}$,

$$\begin{aligned} |\gamma A| &\leq C b_\sigma^{1/6} \exp\left(-\frac{2}{\sqrt{b_\sigma}}\right) e^{2r} \left(Q(r) + C r^{-(d+1)/2} e^{-r} \right) \\ &\leq C b_\sigma^{1/6} \exp\left(\frac{2}{\sqrt{b}} - \frac{2}{\sqrt{b_\sigma}}\right) (Q(r) + C) \\ &\leq C b_\sigma^{1/6} \left(Q(r) + \left| \frac{2}{\sqrt{b}} - \frac{2}{\sqrt{b_\sigma}} \right| Q(r) \right) \\ &\leq C b_\sigma^{1/6} Q(r). \end{aligned}$$

The bound on $b\sigma B$ and ϕ_- are obtained similarly. □

4.1.2 Solution of (33) in $I \cup J$

The profile in P_{ext} in $I \cup J$ is constructed in two main steps. First, the solution is defined in $I = [b^{-2}, \infty)$ by letting

$$P(r) = r^{-\frac{d-1}{2}} U(r), \tag{41}$$

where

$$U'' + \left(\frac{b^2 r^2}{4} - 1 - \frac{(d-1)(d-3)}{4r^2} - ib\sigma \right) U + r^{-\frac{1}{2}(d-1)(p-1)} |U|^{p-1} U = 0 \quad (42)$$

The equation (42) is essentially linear near the profile Q in the region I as profile Q is exponentially decreasing.

Proposition 7 (Proposition 2.1, [1]). *For $\sigma > 0$ small enough and for any b, ρ satisfying (34), (35), there exists a C^2 solution U of (42) on I satisfying*

$$U(r) = \rho r^{-\frac{1}{2}+\sigma} \exp\left(\frac{r^2}{4}\right) \exp\left(-i\frac{\ln r}{b}\right) (1 + O(b^{-3}r^{-2})) \quad (43)$$

$$U'(r) = i\frac{b}{2}\rho r^{\frac{1}{2}+\sigma} \exp\left(\frac{r^2}{4}\right) \exp\left(-i\frac{\ln r}{b}\right) (1 + O(b^{-3}r^{-2})), \quad (44)$$

and

$$U' - i\frac{br}{2}U = O(b^{-1}r^{-1}|U|) = O(\rho b^{-1}r^{-\frac{3}{2}+\sigma}).$$

Moreover, the map $(\sigma, b, \rho) \mapsto (U[\sigma, b, \rho](b^{-2}), U'[\sigma, b, \rho](b^{-2}))$ is continuous.

As pointed out in [1, Remark 2.3], the profile P given by (41) in I belongs to $\dot{H}^1(I)$ but not in $L^2(I)$ due to the asymptotic given by (43), (44). However, (43) is sufficient to recover $L^2_{loc}(I)$ for the profile P given by (41), which shall be used to deduce $\|P - Q\|_{L^2_{loc}(I)} = o(1)$.

The solution U on I is then extended to $I \cup J$, the region including the turning point, which complicates significantly the analysis.

Proposition 8 (Proposition 3.1, [1]). *For $\sigma > 0$ small enough and for any b, ρ , satisfying (34), (35), the solution U of (42) on I constructed in Proposition 7 extends to a solution of (42) on $J \cup I$. Moreover, there exists a real $\theta_{ext} \in [0, 2\pi)$ such that the function P_{ext} defined by*

$$P_{ext}(r) = e^{i\theta_{ext}} r^{-\frac{d-1}{2}} U(r), r \in J \cup I,$$

is a solution of (33) on $J \cup I$ and satisfies

$$\begin{aligned} \operatorname{Re}(P_{ext}(r_K)) &= \frac{\rho b^{\frac{d+1}{4}}}{\sqrt{2}} \exp\left(\frac{\pi}{2b} - \frac{1}{\sqrt{b}}\right) (1 + O(b^{\frac{1}{4}})) \\ \operatorname{Re}(P'_{ext}(r_K)) &= -\frac{\rho b^{\frac{d+1}{4}}}{\sqrt{2}} \exp\left(\frac{\pi}{2b} - \frac{1}{\sqrt{b}}\right) (1 + O(b^{\frac{1}{4}})) \\ \operatorname{Im}(P_{ext}(r_K)) &= \frac{\rho b^{\frac{d+1}{4}}}{2\sqrt{2}} \exp\left(\frac{1}{\sqrt{b}} - \frac{\pi}{2b}\right) (1 + O(b^{\frac{1}{4}})) \\ \operatorname{Im}(P'_{ext}(r_K)) &= \frac{\rho b^{\frac{d+1}{4}}}{2\sqrt{2}} \exp\left(\frac{1}{\sqrt{b}} - \frac{\pi}{2b}\right) (1 + O(b^{\frac{1}{4}})) \end{aligned}$$

Moreover, the map $(\sigma, b, \rho) \mapsto (P_{ext}[\sigma, b, \rho](r_K), P'_{ext}[\sigma, b, \rho](r_K))$ is continuous

4.1.3 Matching asymptotic

The solution P_{int} , defined over K , and P_{ext} , defined over $J \cup I$, are matched at r_K using a fixed-point argument on the parameters.

Theorem 10 (Theorem 2, [1]). *Let $d \geq 1$ and $p_* < \bar{p}$. There exists $\sigma_0 > 0$ such that for any $\sigma \in (0, \sigma_0)$, and $p \in [p_*, \bar{p}]$, there exists b, ρ, γ, θ satisfying (34), (35) such that the solution $P_{ext}[\sigma, b, \rho]$ of (33) on $I \cup J$ given by Proposition 3.1 and $P_{int}[\sigma, b, \gamma]$ of (33) on K given by Proposition 4.1 satisfy the following matching conditions*

$$P_{int}(r_K) = P_{ext}(r_K), \quad P'_{int}(r_K) = P'_{ext}(r_K).$$

In particular, the function P defined on $[0, \infty)$ by

$$P(r) = \begin{cases} P_{int}(r), & r \in K, \\ P_{ext}(r), & r \in I \cup J, \end{cases}$$

is a C^2 solution of (33) on $[0, \infty)$ satisfying the asymptotic: for r large,

$$\begin{aligned} |P(r)| &= |\rho| r^{-\frac{d}{2} + \sigma} (1 + O(b^{-3} r^{-2})) \\ P'(r) - i \frac{br}{2} P(r) &= O(\rho b^{-1} r^{-\frac{d}{2} - 1 + \sigma}) \end{aligned}$$

Theorem 8 is obtained from Theorem 10 by choosing $\sigma = s_c = \frac{d}{2} - \frac{2}{p-1}$.

4.2 Blow-up profile for the mass slightly subcritical case

We deduce Theorem 9 in the mass slightly subcritical regime $-1 \ll s_c < 0$.

Proof. Consider ψ according to the ansatz

$$\psi(x, t) = \frac{1}{\lambda^{\frac{2}{p-1}}(t)} e^{i\theta(t)} \tilde{\psi} \left(\frac{x}{\lambda(t)} \right), \quad (45)$$

where $\tilde{\psi}$ is assumed radially symmetric and $\lambda(t), \theta(t)$ defined by (28). Then, if $\tilde{\psi}$ satisfies

$$\Delta \tilde{\psi} - \tilde{\psi} + ib \left(\frac{2}{p-1} \psi + x \cdot \nabla \psi \right) + |\tilde{\psi}|^{p-1} \tilde{\psi} = 0, \quad x \in \mathbb{R}^d. \quad (46)$$

Then ψ defined by (45) satisfies (1). Now, let

$$\tilde{\psi}(x) = \exp \left(-i \frac{br^2}{4} \right) P(r).$$

If P is a solution to

$$\begin{cases} P'' + \frac{d-1}{r} P' + \left(\frac{b^2 r^2}{4} - 1 - ibs_c \right) P + |P|^{p-1} P = 0, & r > 0, \\ P'(0) = 0. \end{cases} \quad (47)$$

then $\tilde{\psi}$ is a solution to (46). The proof of Theorem 8 is obtained from Proposition 6, Proposition 7 and Proposition 8 by relaxing the dependence of (47) with respect to $s_c > 0$ by considering the family of equations,

$$\begin{cases} P'' + \frac{d-1}{r} P' + \left(\frac{b^2 r^2}{4} - 1 - ib\sigma \right) P + |P|^{p-1} P = 0, & r > 0, \\ P'(0) = 0. \end{cases} \quad (48)$$

with respect to the parameter $\sigma > 0$. To prove that the conclusion of Theorem 8 holds in the mass slightly subcritical case $-1 \ll s_c < 0$, we seek a profile P solution to

$$\begin{cases} P'' + \frac{d-1}{r}P' + \left(\frac{b^2r^2}{4} - 1 + ib\sigma\right)P + |P|^{p-1}P = 0, & r > 0, \\ P'(0) = 0. \end{cases} \quad (49)$$

where the sign of $ib\sigma P$ is chosen so that $b > 0$ and $\sigma > 0$ (recall that in the mass subcritical case, $\text{Im}(-ibs_c) > 0$). Therefore, if P denotes the solution of (48) in the mass supercritical case, then \bar{P} is solution to (49). Indeed, taking the complex conjugate of (49) yields,

$$\begin{cases} \bar{P}'' + \frac{d-1}{r}\bar{P}' + \left(\frac{b^2r^2}{4} - 1 + ib\sigma\right)\bar{P} + |\bar{P}|^{p-1}\bar{P} = 0, & r > 0, \\ \bar{P}'(0) = 0. \end{cases}$$

with $b > 0$ and $\sigma > 0$. Hence, Proposition 6, Proposition 7 and Proposition 8 hold, and taking $\sigma = s_c = \frac{d}{2} - \frac{2}{p-1}$, the conclusion of Theorem 8 holds in the mass slightly subcritical case.

To conclude the proof of Theorem 9, it remains to prove $\|\tilde{\psi} - Q\|_{L^2_{loc}(\mathbb{R}^d)} = o(1)$. Let C be a compact subset of \mathbb{R}^d . Denote first $C_K = B_{r_K}(0) \cap C$, where $B_{r_K}(0)$ is the ball of \mathbb{R}^d centered at the origin and of radius $r_K = b^{-1/2}$. We have,

$$\begin{aligned} \|\tilde{\psi} - Q\|_{L^2(C_K)}^2 &\leq \|\tilde{\psi} - Q\|_{L^2(B_{r_K}(0))}^2 \\ &= \int_{0 \leq r \leq r_K} \left| \exp\left(-i\frac{br^2}{4}\right) P(r) - Q(r) \right|^2 r^{d-1} dr \\ &= \int_{0 \leq r \leq r_K} \left| \left(1 - \exp\left(-i\frac{br^2}{4}\right)\right) Q(r) \right|^2 r^{d-1} dr + \int_{0 \leq r \leq r_K} |P(r) - Q(r)|^2 r^{d-1} dr. \end{aligned}$$

We divide the region of integration of the first integral,

$$\begin{aligned} \int_{r \leq r_K} \left| \left(1 - \exp\left(-i\frac{br^2}{4}\right)\right) Q(r) \right|^2 r^{d-1} dr &= \int_{0 \leq r \leq b^{-1/4}} \left| \left(1 - \exp\left(-i\frac{br^2}{4}\right)\right) Q(r) \right|^2 r^{d-1} dr \\ &\quad + \int_{b^{-1/4} \leq r \leq r_K} \left| \left(1 - \exp\left(-i\frac{br^2}{4}\right)\right) Q(r) \right|^2 r^{d-1} dr. \end{aligned}$$

Then, on one hand

$$\begin{aligned} \int_{0 \leq r \leq b^{-1/4}} \left| \left(1 - \exp\left(-i\frac{br^2}{4}\right)\right) Q(r) \right|^2 r^{d-1} dr &\leq \sup_{0 \leq r \leq b^{-1/4}} \left| 1 - \exp\left(-i\frac{br^2}{4}\right) \right| \int_{0 \leq r < \infty} Q^2(r) r^{d-1} dr \\ &\leq N_c \sup_{0 \leq r \leq b^{-1/4}} \left| 1 - \exp\left(-i\frac{br^2}{4}\right) \right| \\ &\leq Cb^{1/2}. \end{aligned}$$

On the other hand, using the exponential decay of the ground state (4)

$$\int_{b^{-1/4} \leq r \leq r_K} \left| \left(1 - \exp\left(-i\frac{br^2}{4}\right)\right) Q(r) \right|^2 r^{d-1} dr \leq 2 \int_{b^{-1/4} \leq r \leq r_K} |Q(r)|^2 r^{d-1} dr \leq Ce^{-b^{-1/4}} \quad (50)$$

To deal with the profile P , we use the profile decomposition of P in K , the bounds of Lemma 3. Indeed, there exists $(\phi_+, \phi_-) \in E_K$ (recall (38)) such that

$$P = (Q + \gamma A + \phi_+) + i(b\sigma B + \phi_-)$$

and such that Lemma 3 holds. Then,

$$\begin{aligned}
\int_{0 \leq r \leq r_K} |P(r) - Q(r)|^2 r^{d-1} dr &= \int_{0 \leq r \leq r_K} |P(r) - Q(r)|^2 r^{d-1} dr \\
&= \int_{0 \leq r \leq r_K} |\gamma A + \phi_+ + i(b\sigma B + \phi_-)|^2 r^{d-1} dr \\
&\leq C b^{1/6} \int_{0 \leq r \leq r_K} |Q(r)|^2 r^{d-1} dr \\
&\leq C b^{1/6} \int_{0 \leq r < \infty} |Q(r)|^2 r^{d-1} dr \\
&\leq C b^{1/6}.
\end{aligned}$$

Hence, we deduce $\|\tilde{\psi} - Q\|_{L^2(C_K)} \rightarrow 0$ as $b \rightarrow 0$. Second, we deal with $C_{I \cup J} := C \setminus C_K$. Denote $r_C > 0$, the radius of $B_{r_C}(0)$ such that $C \subset B_{r_C}(0)$. Since the profile P_{int} on $I \cup J$ satisfies (43), (44) of Proposition 7, we have, using again (4)

$$\begin{aligned}
\|\tilde{\psi} - Q\|_{L^2(C_{I \cup J})}^2 &\leq \|\tilde{\psi} - Q\|_{L^2(B_{r_C}(0) \setminus B_{r_K}(0))}^2 \\
&\leq \int_{r_K \leq r \leq r_C} |P(r) - Q(r)|^2 r^{d-1} dr \\
&\quad + \int_{r_K \leq r \leq r_C} \left| \left(1 - \exp\left(-i \frac{br^2}{4}\right) \right) Q(r) \right|^2 r^{d-1} dr
\end{aligned}$$

Dealing with the second integral as for (50), we deduce

$$\int_{r_K \leq r \leq r_C} \left| \left(1 - \exp\left(-i \frac{br^2}{4}\right) \right) Q(r) \right|^2 r^{d-1} dr \leq C e^{-b^{-1/2}}.$$

We are left with the first integral. We use the exponential decay of the ground state given by (43) to obtain

$$\begin{aligned}
\int_{r_K \leq r \leq r_C} |P(r) - Q(r)|^2 r^{d-1} dr &\leq \int_{r_K \leq r \leq r_C} \left| r^{-(d-1)/2} U(r) - \kappa r^{-(d-1)/2} e^{-r} \right|^2 r^{d-1} dr \\
&\quad + C \int_{r_K \leq r \leq r_C} e^{-2r} r^{-1} dr \\
&\leq \int_{r_K \leq r \leq r_C} |U(r)|^2 dr + \kappa \int_{r_K \leq r \leq r_C} e^{-2r} dr \\
&\quad + C \int_{r_K \leq r \leq r_C} e^{-2r} r^{-1} dr.
\end{aligned}$$

The last two integrals are bounded by $C e^{-2b^{-1/2}}$. It remains to bound the first integral. In order to do so, we use (50), and the bound on ρ with respect to b given by (35) and (36) to deduce

$$\begin{aligned}
\int_{r_K \leq r \leq r_C} |U(r)|^2 dr &\leq (r_C - b^{-1/2}) \|U\|_{L^\infty(r_K \leq r \leq r_C)}^2 \\
&\leq |r_C - b^{-1/2}| b^{-5} e^{-\pi/b}
\end{aligned}$$

which allows to finally prove that $\|\tilde{\psi} - Q\|_{L^2(C_{I \cup J})} \rightarrow 0$ as $b \rightarrow 0$. Combining this fact with $\|\tilde{\psi} - Q\|_{\dot{H}^1(\mathbb{R}^d)} \rightarrow 0$ as $b \rightarrow 0$, we finally deduce $\|\tilde{\psi} - Q\|_{H_{loc}^1(\mathbb{R}^d)} \rightarrow 0$ as $b \rightarrow 0$, which ends the proof. \square

5 Controllability to the blow-up trajectory

We are now in position to prove Theorem 2 and Theorem 3. We begin by the proof of Theorem 3.

Proof. Let $d \geq 1$. Consider for now $p \in [1, p^*)$. Denote ψ_b the blow-up profile

$$\psi_b(x, t) = \frac{1}{\lambda(t)^{\frac{2}{p-1}}} e^{i\theta(t)} \tilde{\psi} \left(\frac{x}{\lambda(t)} \right).$$

where $\tilde{\psi}$ is given by Theorem 9 and where

$$\lambda(t) = \sqrt{2b(T-t)}, \quad \theta(t) = -\frac{\log(T-t)}{2b}, \quad t < T,$$

where $T > 0$ is fixed the following way. For $c_1 < c_2 \in \mathbb{R}^+$, let

$$T = \frac{1}{2bc_1}, \quad T - T_{c_1, c_2} = \frac{1}{2bc_2}. \quad (51)$$

Hence,

$$\psi_b(x, 0) = e^{i\theta(0)} c_1^{\frac{2}{p-1}} \tilde{\psi}(c_1 x), \quad \psi_b(x, T_{c_1, c_2}) = e^{i\theta(T_{c_1, c_2})} c_2^{\frac{2}{p-1}} \tilde{\psi}(c_2 x)$$

We deduce the minimal time of controllability

$$T_{c_1, c_2} = \frac{c_2 - c_1}{2bc_1 c_2}. \quad (52)$$

Now, for $p \in (1, p^*]$, denote $\delta_1(p), \delta_2(p) \in \mathbb{R}^+$ such that Theorem 6 apply for the smooth trajectory ψ_{c_1} and ψ_{c_2} . Define $\tilde{\delta} := \min_{p \in [1, p^*]} \{\delta_1(p), \delta_2(p)\}$. We easily see that $\tilde{\delta} > 0$ since Proposition 4 holds for $p \in [1, p^*]$. We now fix $\underline{p} \in (1, p^*)$ such that

$$\|Q_{c_1} - c_1^{\frac{2}{\underline{p}-1}} \tilde{\psi}(c_1 x)\|_{H^1(\Omega)} < \tilde{\delta}, \quad \|Q_{c_2} - c_2^{\frac{2}{\underline{p}-1}} \tilde{\psi}(c_2 x)\|_{H^1(\Omega)} < \tilde{\delta}.$$

This is always possible since $\|Q - \tilde{\psi}\|_{H_{loc}^1(\mathbb{R}^d)} \rightarrow 0$ as $b \rightarrow 0$.

We can now prove Theorem 2 the following way. By hypothesis, we choose the control given by Proposition 4 for the smooth trajectory $e^{ic_1 t} Q_{c_1}$ driving the initial ψ_0 to $u(0)$ in the time interval $(\epsilon, 0]$, we follow the trajectory $u(t)$ for $t \in [0, T]$ and use the control given by Proposition 4 for the smooth trajectory $e^{ic_2 t} Q_{c_2}$ to drive the state $\psi(T)$ to ψ_T . The boundary control is given by the trace of the $C([- \epsilon, T + \epsilon]; H^1(\mathbb{R}^d))$ on $\partial\Omega$

$$v(x, t) = \begin{cases} v_1(x, t), & t \in (0, \epsilon), \\ v_2(x, t), & t \in (\epsilon, T - \epsilon), \\ v_3(x, t), & t \in (0, \epsilon). \end{cases}$$

The solution (12) defined this way belongs to $C([- \epsilon, T + \epsilon]; H^1(\mathbb{R}^d))$, hence the result. □

We now turn to the proof of Theorem 2.

Proof. The proof of Theorem 2 is very similar to the proof above. Indeed, the only modification needed in the proof is the profile to go from one scaled ground state from another. In order to do so, we use the simple remark that the solution to (1) with the initial data $\psi_0 = (1 + \epsilon)Q_\lambda$ satisfies the requirement for $\epsilon > 0$ sufficiently small. Indeed, we have $\psi_0 \in \mathcal{B}_{\alpha^*}$ using that the $L^2(\mathbb{R}^d)$ -norm is invariant under the scaling in the mass critical case. Moreover, using Pohozaev's identity $E(Q_\lambda) = \lambda E(Q) = 0$,

$$\begin{aligned} E(\psi_0) &= \frac{(1 + \epsilon)^2}{2} \int |\nabla Q_\lambda|^2 - \frac{(1 + \epsilon)^{2+d/4}}{2 + d/4} \int |Q_\lambda|^{2+d/4} \\ &= E(Q_\lambda) + \frac{(1 + \epsilon)^2 - (1 + \epsilon)^{2+d/4}}{2 + d/4} \int |Q_\lambda|^{2+d/4} \\ &< 0 \end{aligned}$$

Therefore the solution ψ to (1) starting from $\psi_0 = (1 + \epsilon)Q_\lambda$ blow-up in finite time and belongs to $C([0, T); H^1(\mathbb{R}^d))$. We use the exact local controllability to reach this blow-up profile from the initial and to the final data in arbitrarily small time. □

6 Conclusion

In lights of the results presented here, we obtained the controllability of initial and final states close to ground state solitary waves with different scaling. In some sense, this strategy is close to the return method, as a trajectory with "good control properties" [9] was used to connect two different states. However, the small-time global controllability of (12) remains an open question.

Moreover, using the results presented here, we are able to address the question of finite time blow-up in the mass subcritical case on bounded domain. Indeed, consider

$$\begin{cases} i\psi_t + \Delta\psi + |\psi|^{p-1}\psi = 0, & (x, t) \in \Omega \times \mathbb{R}^+ \\ \psi(x, t) = v(x, t), & (x, t) \in \partial\Omega \times \mathbb{R}^+ \\ \psi(x, 0) = \psi_0, & x \in \Omega, \end{cases} \quad (53)$$

with $\psi_0 \in H^1(\Omega)$ and $v \in L^2(\mathbb{R}^+; H^{1/2}(\partial\Omega))$. Then

Theorem 11. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and $p < p^*$ for $0 < p^* - p \ll 1$. Then there exists $\psi_0 \in H^1(\Omega)$ close to Q in $H^1(\Omega)$ and $v \in L^2(\mathbb{R}^+; H^{1/2}(\partial\Omega))$ such that the solution blow-up in finite time with*

$$\|\nabla\psi\|_{L^2(\Omega)} \simeq \frac{1}{\sqrt{b(T-t)}}, \quad t \rightarrow T^-.$$

The framework given by (53) allows to properly define blow-up solutions in $H^1(\Omega)$ in the mass subcritical regime, that is, without having solutions not belonging to $L^2(|x| > R)$ for $R > 0$ large. We highlight that Theorem 11 is close to the blow-up phenomenon exhibited in for (53) in the mass critical case with $v \equiv 0$ where the ground state was defined as the unique positive steady state.

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