Brenier-Schrödinger problem on compact manifold with boundary

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Abstract

We consider the Brenier-Schrödinger problem on compact manifolds with boundary. In the spirit of a work by Arnaudon, Cruzeiro, Léonard and Zambrini, we study the kinetic property of regular solutions and obtain a link to the Navier-Stokes equations with an impermeability condition. We also enhance the class of models for which the problem admits a unique solution. This involves a method of taking quotients by reflection groups for which we give several examples.

Keywords: Brenier-Schrödinger, entropy, manifold with boundary, reflected Brownian motion, Navier-Stokes equations.

1 Introduction

In mechanics, there are two classical dual descriptions of every phenomenon. The first one is Newton's laws of motion (or Hamilton's equations). They characterise the evolution of a system by differential equations. The second one is the principle of least action. It characterises the motion as the minimiser of a functional constructed from the kinetic and the potential energy. Applied to the evolution of perfect fluids, the first approach leads to the Euler equations while the second approach sees the evolution as a geodesic in the space of volume preserving diffeomorphisms and it was developed by Arnold [5]. A relaxation of this problem was proposed by Brenier [10] where, instead of seeking a flow, he looks for a measure on the space of trajectories. His new problem is the minimisation of an average kinetic energy. The incompressibility constraint (i.e., volume preserving condition) becomes a constraint on the marginals and the final endpoint condition becomes an endpoints measure constraint. Brenier showed the accuracy of his problem by relating the solutions of the Euler equations with the solutions of his problem.

The problem treated in this article is the *Brenier-Schrödinger problem*. This has been introduced in [1] as a perturbation of Brenier's problem where the kinetic energy to be minimised is defined using a stochastic notion of velocity. This notion of velocity allows us to think the problem as an entropy minimisation under marginal and endpoint constraints and allows us to use convex optimisation approaches. It has been studied by several authors [2, 3, 6, 7, 8, 9, 19]. In the present article, we study this problem on compact manifolds with boundary and we work on the following two questions.

- The first one is the kinetics of the solutions. While Brenier's problem is linked to the Euler equations, the Brenier-Schrödinger problem is linked to the Navier-Stokes equations for which the viscosity term is a perturbation of the Euler equations. We prove that the backward stochastic velocity of a regular solution of the Brenier-Schrödinger problem

is a solution of the Newtonian part of the Navier-Stokes equations. This generalises to compact manifolds with boundary the result of [3] on the Euclidean space and the tori. The main difference in our framework is the behavior of the solutions at the boundary. This can be found in Section 2.

- The second question is about the existence of a solution. We give a necessary and sufficient condition for the existence of a unique solution to the incompressible Brenier-Schrödinger problem on homogeneous spaces. This generalises the result given for the tori in [3]. Moreover, we develop a method to transport this result to quotients by reflection groups. Finally, we mention an additional example on the Euclidean space in a non-incompressible setting. These results can be found in Section 3.

Let us describe the model. From now on, we fix a compact Riemannian manifold M with boundary ∂M, interior $\mathring{M}$ and normalised Riemannian volume measure vol, i.e., such that $vol(M) = 1$. In the present article, we are interested in a minimisation problem in the set $\mathcal{P}(\Omega)$ of probability measures on the path space

$$\Omega = \{\omega \in M^{[0,1]} : \omega \text{ is continuous}\},$$

which is endowed with the compact-open topology. This minimisation problem will be related to the *reflected Brownian motion* on M (restricted to the time interval $[0,1]$). This is the Markov process $(\beta_t)_{t\in[0,1]}$ whose generator is the Laplacian on M with a properly chosen domain. More precisely, it satisfies that, for every C^2 function $f : M \to \mathbb{R}$ such that $\mathrm{d}f_x \cdot \nu_x = 0$ at every $x \in \partial M$,

$$f(\beta_t) - \frac{1}{2}\int_0^t \Delta f(\beta_s)\mathrm{d}s$$

is a martingale with respect to the filtration $\sigma\Big((\beta_s)_{s\in[0,t]}\Big)$. See also [4] for an equivalent formulation, also recalled in Section 4. Let us call $R \in \mathcal{P}(\Omega)$, the law of the reflected Brownian motion on M whose initial position follows the law vol and $R^x \in \mathcal{P}(\Omega)$, the law of the reflected Brownian motion on M whose initial position is $x \in \mathring{M}$.

The object to be minimised is the so-called relative entropy whose general definition is the following. For any measurable space E, the *relative entropy* of a probability measure $\mu \in \mathcal{P}(E)$ with respect to a probability measure $\nu \in \mathcal{P}(E)$ is

$$H(\mu|\nu) = \int_E \rho \log \rho \,\mathrm{d}\nu$$

if $\mathrm{d}\mu = \rho\,\mathrm{d}\nu$ and $H(\mu|\nu) = \infty$ if μ is not absolutely continuous with respect to ν.

Now, let $\mathcal{T}$ be a measurable subset of $[0,1]$, $(\mu_t)_{t\in\mathcal{T}}$ a family of probability measures on M indexed by $\mathcal{T}$ and $\pi \in \mathcal{P}(M \times M)$. We are interested in minimising $H(Q|R)$ among all $Q \in \mathcal{P}(\Omega)$ with the following constraints. For any $t \in \mathcal{T}$, we ask that $Q_t = \mu_t$, where Q_t is the image measure (or pushforward) of Q by the canonical map $X_t : \omega \in \Omega \mapsto \omega(t) \in M$. Additionally, we ask that $Q_{01} = \pi$, where Q_{01} is the image measure of Q by the endpoints map $(X_0, X_1) : \omega \in \Omega \mapsto \big(\omega(0), \omega(1)\big) \in M \times M$. This minimisation problem is called the *Brenier-Schrödinger (or Brödinger, or Bredinger) problem*, it will be denoted by (BS) in this article and can be summarised as follows.

$$H(Q|R) \to \min,\ Q \in \mathcal{P}(\Omega),\ [Q_t = \mu_t,\ \forall t \in \mathcal{T}],\ Q_{01} = \pi. \tag{BS}$$

It is a strictly convex problem with convex constraints. Then, the problem (BS) admits a unique solution if and only if there exists $Q \in \mathcal{P}(\Omega)$ such that $Q_t = \mu_t$ for all $t \in \mathcal{T}$, $Q_{01} = \pi$ and $H(Q|R) < \infty$.

A particular case is the *incompressible Brenier-Schrödinger problem*, denoted by (iBS). This is the case where $\mathcal{T} = [0,1]$, $\mu_t = vol$ for every $t \in [0,1]$ and π has both marginals equal to vol.

$$H(Q|R) \to \min,\ Q \in \mathcal{P}(\Omega),\ [Q_t = vol,\ \forall t \in [0,1]],\ Q_{01} = \pi. \tag{iBS}$$

We give some examples where a solution to (iBS) exists in Section 3.

A particular class of solutions to (BS) are introduced in [3]. By using the dual maximisation problem, they have shown that if $P \in \mathcal{P}(\Omega)$ can be written as

$$\mathrm{d}P(X) = \exp\left(\eta(X_0, X_1) + \sum_{s\in\mathcal{S}} \theta_s(X_s) + \int_{\mathcal{T}} p_t(X_t)\,\mathrm{d}t\right)\mathrm{d}R(X)$$

for some bounded measurable $\eta : M \times M \to \mathbb{R}$, $p : \mathcal{T} \times M \to \mathbb{R}$ and $\theta : \mathcal{S} \times M \to \mathbb{R}$, then P is a solution of (BS) for the set $\mathcal{T} \cup \mathcal{S}$ and the marginals $\mu_t = P_t$. In fact, [3] describes the general form of a solution by relating the dual maximisation problem and the primal minimisation problem. The existence of such solutions is proved in [8] in the particular case of discrete problems, i.e for $\mathcal{T} = \emptyset$. In Section 2, we show that solutions of this form that are regular, in a sense to be precised there, give rise to solutions of the Navier-Stokes equations.

Let us summarise this article. In Section 2, we present the result on the description of regular solutions of (BS) via the Navier-Stokes equations. In Section 3, we present the results on the existence of solution of (iBS) on compact manifolds and a non-incompressible version (BS_γ) on $\mathbb{R}^n$ defined there. In Section 4, we develop the Girsanov theory to define the velocity of solutions. This makes a link between entropy minimisation and kinetic energy minimisation, see Remark 4.4. In Section 5, we give the proof of the kinetic results from Section 2. Finally, Section 6, Section 7 and Section 8 are dedicated to the proofs of the results about the existence from Section 3.

2 Results on the kinetics behavior

Let $\mathcal{T}$ be an open subset of $[0,1]$ which is a finite union of intervals and let $\mathcal{S}$ be a finite subset of $(0,1)$ such that $\mathcal{T} \cap \mathcal{S} = \emptyset$. Following [3], we say that $P \in \mathcal{P}(\Omega)$ is a *regular solution* of (BS) if it can be written as

$$\mathrm{d}P(X) = \exp\left(\eta(X_0, X_1) + \sum_{s\in\mathcal{S}} \theta_s(X_s) + \int_{\mathcal{T}} p_r(X_r)\,\mathrm{d}r\right)\mathrm{d}R(X). \tag{2.1}$$

for some regular enough functions $\eta : M \times M \to \mathbb{R}$, $p : \mathcal{T} \times M \to \mathbb{R}$ and $\theta : \mathcal{S} \times M \to \mathbb{R}$. The regularity is such that all equations in the theorem below makes sense (C^2 would be enough, for instance, but we are not interested in attaining the least possible regularity). As already explained, from a dual-primal problem argument explained in [3], a regular solution is an actual solution of (BS).

Now, let us introduce some further notation and comments before stating the results.

Forward regular solution. We will say that a regular solution P is a *forward regular*

solution of (BS) if, for every $x \in M$, the function $\psi^x : [0,1] \times M \to \mathbb{R}$ given by

$$\psi_t^x(z) = \log \mathbb{E}_{R^x}\left[\exp\left(\eta(x, X_1) + \sum_{s \in \mathcal{S} \cap (t,1]} \theta_s(X_s) + \int_{\mathcal{T} \cap (t,1]} p_r(X_r)\,\mathrm{d}r\right) \middle| X_t = z\right] \tag{2.2}$$

is C^2 in $z \in M$ and C^1 in $t \in [0,1] \setminus \mathcal{S}$ such that the function and its space first derivatives are (jointly) càdlàg in t. The times in $\mathcal{S}$ are sometimes called *shock times*. These are the times where $t \mapsto \psi_t^x(z)$ is discontinuous. On the other hand, the times in $\mathcal{T}$ are called *regular times*. The function p can be thought of as a *pressure field* while the functions θ_s can be thought of as *shock potentials*.

Backward regular solution. We will say that a regular solution P is a *backward regular solution* of (BS) if the time reversal of P is a forward regular solution. Equivalently, P is a backward regular solution of (BS) if, for every $y \in M$, the function $\varphi^y : [0,1] \times M \to \mathbb{R}$ given by

$$\varphi_t^y(z) = \log \mathbb{E}_{R^y}\left[\exp\left(\eta(X_1, y) + \sum_{s \in \mathcal{S} \cap [0,1-t)} \theta_s(X_{1-s}) + \int_{\mathcal{T} \cap [0,1-t)} p_r(X_{1-r})\,\mathrm{d}r\right) \middle| X_t = z\right] \tag{2.3}$$

is C^2 in $z \in M$ and C^1 in $t \in [0,1] \setminus \mathcal{S}$ such that the function and its space first derivatives are (jointly) càdlàg in t.

Disintegration by the final position. We will be interested in the family $(\overleftarrow{P}^y)_{y \in M}$ of probability measures on Ω that satisfy

$$P = \int_M \overleftarrow{P}^y \mathrm{d}P_1(y) \quad \text{and} \quad \overleftarrow{P}^y\left(\{\omega \in \Omega : \omega(1) = y\}\right) = 1 \text{ for every } y \in M. \tag{2.4}$$

These can be thought of as the conditional laws of P given the final position and are uniquely defined except for x on a set of P_0-measure zero.

Logarithm. Suppose that P has finite entropy with respect to R. We will use the notion of *stochastic velocity* (or mean derivative) of P introduced initially by Nelson in [18] for real processes. A presentation of the generalisation to manifold can be found in [14]. Recall that there exists an open set $\mathring{\mathcal{N}}$ of $T\mathring{M}$ that contains the zero section and such that the exponential map $\exp : \mathring{\mathcal{N}} \to \mathring{M}$ is well-defined and its restriction $\exp_x : \mathring{\mathcal{N}} \cap T_xM \to \mathring{M}$ is a diffeomorphism onto an open subset $U(x) \subset M$. Then, for $x \in M$ and $y \in U(x)$, define

$$\overrightarrow{xy} = \log_x(y)$$

as the unique element of $\mathring{\mathcal{N}} \cap T_xM$ such that $\exp(\overrightarrow{xy}) = y$. Since P is absolutely continuous with respect to R, we can see that, for any t, $X_t \notin \partial M$ for P-almost every X since this also happens for R-almost every X. The random times

$$\overleftarrow{\tau}_t = \frac{1}{2} \inf\left\{h \leq 0 : X_{t-s} \in U(X_t) \text{ for every } s \in [h, 0]\right\} \tag{2.5}$$

are strictly negative. They allow us to define mean derivatives in a manifold.

Covariant derivative and Laplacian. We denote by ∇_u the covariant derivative in the direction of u and by $\square$ the (negative definite) de Rham-Hodge-Laplace operator. More precisely, if δ the adjoint of the exterior differential from the space of one-forms to the space of two-forms, then the de Rham-Hodge-Laplace operator on one forms is defined

as $-(\mathrm{d}\delta + \delta\mathrm{d})$. The operator $\Box$ we need is obtained if we think vector fields as one-forms by using the metric.

Navier-Stokes equations. By the *Navier-Stokes equations* on M, we refer to the differential system of unknown (v, p), together with an initial condition v_0,

$$\begin{cases} (\partial_t + \nabla_{v_t})\, v_t = \dfrac{1}{2}\Box v_t - \nabla p_t, & t \in [0,1), \\ \operatorname{div}(v_t) = 0, & t \in [0,1), \\ \langle v(z), \nu_z \rangle = 0, & z \in \partial M, \end{cases}$$

where ν_z is the inward-pointing unit normal to z. Remark that our viscosity $\frac{1}{2}\Box$ differs from [2] where the Laplace operator on vector field is the divergence of the deformation tensor. However, in flat spaces, where Ricci curvature vanishes, both Laplace operators coincide.

Now, we are ready to state one of our main results, which is a generalisation of the Euclidean and tori results [3, Theorem 5.4].

Theorem 2.1 (Backward stochastic velocity and the Navier-Stokes equations)**.** *Suppose P is a backward regular solution of* (BS) *with associated function φ. Then, for P_1-almost every y, there exists a measurable function called the* backward stochastic velocity

$$\overleftarrow{v}^y \colon [0,1] \times \Omega \to TM$$

such that $t \mapsto \overleftarrow{v}^y_t(\omega)$ is left-continuous and has right limits for every $\omega \in \Omega$ and such that for every $t \in [0,1]$ we have that, for P-almost every X,

$$\lim_{h \to 0^+} \frac{1}{h} \mathbb{E}^{\leftarrow}_{P^y} \left[-\overrightarrow{X_t X_{t-h \wedge \overleftarrow{\tau}_t}} \middle| X_{[t,1]} \right] = \overleftarrow{v}^y_t(X).$$

Let $U^y_t(z) = -\nabla \varphi^y_{1-t}(z)$. Then, for P_1-almost every $y \in M$,

$$\overleftarrow{v}^y{}_t = U^y_t(X_t) \quad P\text{-almost surely.}$$

Moreover, for P_1 almost every $y \in M$, the time-dependent vector field U^y satisfies

$$\begin{cases} \left(\partial_t + \nabla_{U^y_t}\right) U^y_t = \dfrac{1}{2}\Box U^y_t - \mathbb{1}_{\mathcal{T}}(t) \nabla p_t, & t \in [0,1) \setminus \mathcal{S}, \\ U^y_{t^+} - U^y_t = \nabla \theta_t, & t \in S, \\ \langle U^y(z), \nu_z \rangle = 0, & z \in \partial M, \\ U^y_0 = -\nabla\big(\eta(\cdot, y)\big), & t = 0. \end{cases} \tag{2.6}$$

The first equation in (2.6) is the Newtonian part of the Navier-Stokes equations while the second equation describes the evolution at the shock times. The third equation tells us the behavior at the boundary of the domain, it says that the stochastic velocity satisfies the impermeability condition. The fourth is the initial condition of the problem. Nevertheless, the velocity does not seem to satisfy any continuity equation. The same approach on forward velocity results on a time reversed Navier-Stokes equation (or Navier-Stokes equation with negative viscosity) which is stated in Corollary 5.3.

A continuity equation is satisfied for a combination of averaged forward and backward velocities, $\overleftrightarrow{v}$ and $\overleftrightarrow{v}$. These are defined by

$$\overleftrightarrow{v}_t(z) = \mathbb{E}_P \left[\overrightarrow{v}^{X_0}_t \,\middle|\, X_t = z \right] \quad \text{and} \quad \overleftrightarrow{v}_t(z) = \mathbb{E}_P \left[\overleftarrow{v}^{X_1}_t \,\middle|\, X_t = z \right], \tag{2.7}$$

where $\overset{x\rightharpoonup}{v}_t$ and $\overset{\leftharpoonup y}{v}_t$ are defined in Theorem 2.1 and Corollary 5.3. They are also shown to be measurable in x and y respectively in these results. The *current velocity* is defined as

$$v_{cu} = \frac{1}{2}\Big(\overset{\rightharpoonup}{v}_t + \overset{\leftharpoonup}{v}_t\Big), \forall t \in [0,1] \tag{2.8}$$

and it satisfies the following continuity equation which generalise [3, Theorem 5.4].

Theorem 2.2 (Continuity equation)**.** *Assume that (BS) admits a forward and backward regular solution P. Then v_{cu} satisfies*

$$\partial_t \mu_t + \operatorname{div}(\mu_t v_{cu}) = 0, \forall t \in \mathcal{T}. \tag{2.9}$$

Actually, this result is not particular to bi-regular solutions, nor to solution. In fact, the proof uses only that P is a semi-martingale measure with finite entropy with respect to the reflected Brownian motion. There, the average velocity shall be defined using the drift given by Girsanov theorem, more precisely, by applying Theorem 4.1 to P and R instead of P^x and R^x. This equation has to be understood in distribution sense, i.e., for all $f \in \mathcal{C}^\infty(M)$ and every $t \in [0,1]$,

$$\mu_t(f) + \int_0^t \mu_s(\langle \mathrm{d}f, v_{cu}\rangle)\mathrm{d}s = 0. \tag{2.10}$$

In the incompressible case (iBS), where $\mu_t = vol$, the continuity equation becomes the incompressibility condition $\operatorname{div}(v_{cu}) = 0$. Nevertheless, v_{cu} does not satisfy the Navier-Stokes equations.

3 Results on the existence of solutions

Let M be a homogeneous compact Riemannian manifold, i.e., one where the isometries act transitively. We consider the incompressible Brenier-Schrödinger problem (iBS). We may write explicitly the dependence on π and M by $(\mathrm{iBS})_{M,\pi}$.

Theorem 3.1 (Existence for homogeneous spaces)**.** *The $(\mathrm{iBS})_{M,\pi}$ problem on a homogeneous compact manifold M admits a unique solution if and only if $H(\pi|vol \otimes vol) < \infty$.*

We will show that the property of existence of solutions is preserved under nice quotients. Our setting will be the following. Suppose that M is a connected compact Riemannian manifold (without boundary) and that G is a finite group of isometries of M. For $x \in M$, consider the stabiliser group $G_x = \{g \in G : g(x) = x\}$, and the induced subgroup $\mathbb{G}_x = \{\mathrm{d}g_x \in O(T_xM) : g \in G_x\}$ of the orthogonal group of T_xM. Let R_x be the set of reflections in $\mathbb{G}_x$, i.e., $T \in \mathbb{G}_x$ belongs to R_x if and only if $\{u \in T_xM : Tu = u\}$ has codimension one as a subspace of T_xM.

Definition 3.2 (Reflection group)**.** *We shall say that G is a reflection group (of isometries) if $\mathbb{G}_x$ is the group generated by R_x for every $x \in M$.*

We will be interested in the set $N = M/G$ which has a topological structure induced by the quotient map $q : M \to N$. Suppose that G is a reflection group. We shall make of N a manifold with corners. But first, let us recall the definition.

Definition 3.3 (Manifold with corners)**.** *Let N be a Hausdorff second countable topological space and let $n > 0$ be a positive integer. Suppose that we have a family $\{\varphi_\lambda\}_{\lambda\in\Lambda}$ of homeomorphisms*

$$\varphi_\lambda : U_\lambda \subset N \to V_\lambda \subset [0,\infty)^n$$

where U_λ (respectively V_λ) is an open subset of N (respectively of $[0,\infty)^n$). We say that the family $(\varphi_\lambda)_{\lambda\in\Lambda}$ is a smooth atlas with corners *if*

$$\bigcup_{\lambda\in\Lambda} U_\lambda = N$$

and, for every $\mu,\nu \in \Lambda$,

$$\varphi_\mu \circ \varphi_\nu^{-1} : \varphi_\nu\left(U_\mu \cap U_\nu\right) \to \varphi_\mu\left(U_\mu \cap U_\nu\right)$$

has a smooth extension to an open subset of $\mathbb{R}^n$. We will refer to $(N,(\varphi_\lambda)_{\lambda\in\Lambda})$ as a manifold with corners*.*

It will be useful to have in mind triangles (and squares) as the prototypical examples, the smooth atlas being given by the set of all diffeomorphisms from an open subset of the triangle (or square) to the open subsets of $[0,\infty)^n$. The notions of tangent bundle, Riemannian metric, Levi-Civita connection and stochastic differential equations can be carried over to manifolds with corners.

Lemma 3.4 (Quotient differentiable structure)**.** *Suppose that G is a reflection group of isometries of M. Then,*

$N = M/G$ has a (unique) structure of a Riemannian manifold with corners

such that, for every $x \in M$, there exists a neighborhood $U \subset N$ of $q(x)$ together with an isometric immersion $s : U \to M$ that is a local inverse of q, i.e., such that

$$q \circ s(x) = x \text{ for every } x \in U.$$

We fix some notation about the boundary of N. The set of points that, by some chart φ_λ, correspond to points of the (topological) boundary of $[0,\infty)^n \subset \mathbb{R}^n$ will be called the *boundary* and will be denoted by ∂N. The points that correspond to the singular points of the boundary of $[0,\infty)^n$ will be called the *corner points* and the set consisting of them will be denoted by $\mathcal{C}N$. A boundary point x that is not a corner point will be called a *regular boundary point* and there is a well-defined unit inward-pointing normal vector $\nu_x \in T_xN$ at x. The complement of ∂N, called the interior of N, will be denoted by $\mathring{N}$.

We will be interested on the reflected Brownian motion on these manifolds. Similarly to the case of manifolds with boundary, *the reflected Brownian motion* on N is a continuous stochastic process $(\beta_t)_{t\in[0,1]}$ on $N \setminus \mathcal{C}N$ that satisfies the following condition. For every C^2 function $f : N \to \mathbb{R}$ such that $\mathrm{d}f_x \cdot \nu_x = 0$ at every regular boundary point x, we have that

$$f(X_t) - \int_0^t \Delta f(X_s)\mathrm{d}s$$

is a martingale with respect to the filtration $\sigma((\beta_s)_{s\in[0,t]})$.

Theorem 3.5 (Existence for quotients)**.** *Let N be the quotient of M by a reflection group and denote by $q : M \to N$ its quotient map. Let π be a probability measure on $M \times M$ with both marginals equal to vol and such that $H(\pi|vol \otimes vol) < \infty$. Then,*

$$(\mathrm{iBS})_{M,\pi} \text{ admits a solution} \Rightarrow (\mathrm{iBS})_{N,(q\times q)_*\pi} \text{ admits a solution.}$$

Moreover, if $(\mathrm{iBS})_{M,\pi}$ *admits a solution for every* π *with finite entropy, then* $(\mathrm{iBS})_{N,\widetilde{\pi}}$ *admits a solution for every* $\widetilde{\pi}$ *with finite entropy.*

We end this section with a more exotic example. Let Ω the space of paths from $[0,1]$ to $M = \mathbb{R}^n$. Choose any probability measure $\chi \in \mathcal{P}(\mathbb{R}^n)$ and let R be the Brownian motion whose initial position has law χ. We are looking to the following problem

$$H(P|R) \to \min; \Big[P_t = \mathcal{N}(0, 1/4\,\mathrm{id}), \forall t \in [0,1]\Big], P_{01} = \pi, \tag{BS_γ}$$

where π is a probability measure on M^2. Through this example, we intend to challenge the assumptions of compactness and incompressibility.

Theorem 3.6 (Existence for Gaussian marginals)**.** *The Brenier-Schrödinger problem* BS_γ *admits a unique solution if and only if* $H(\pi|R_{01}) < \infty$.

4 Girsanov theorem

This section contains a version of Girsanov theorem which is a translation to a manifold setting of the results from [17]. This will make a link between entropy and kinetic energy and will be useful for the proof of Theorem 2.1 and Theorem 2.2.

We need to use a different but equivalent description of the reflected Brownian motion. Consider any vector field $\nu : M \to TM$ such that $\nu|_{\partial M}$ is the inward-pointing unit normal vector field. By using an embedding of M into an Euclidean space, we may construct a smooth family $(\sigma_x)_{x\in M}$ of linear maps $\sigma_x : \mathbb{R}^p \to T_xM$ such that $\sigma_x\sigma_x^* = \mathrm{id}_{T_xM}$. By smooth we mean that the map $\sigma : M \times \mathbb{R}^p \to TM$ defined by $\sigma(x,w) = \sigma_x(w)$ is smooth. The reflected Brownian motion on M can be defined as a semi-martingale $(\beta_t)_{t\in[0,1]}$ on M that solves the following Skorokhod problem. There exists a Brownian motion in $\mathbb{R}^p$ and a non-decreasing process $(L_s)_{s\in[0,1]}$ such that

$$\mathrm{d}\beta_t = \sigma(\beta_t)\mathrm{d}W_t + \nu_{\beta_t}\mathrm{d}L_t \quad \text{ and } \int_0^1 \mathbb{1}_{\mathring{M}}(\beta_s)\mathrm{d}L_s = 0.$$

More information can be found in [4]. We may notice that the process L is the local time of β at ∂M and that β is a reflected Brownian motion in the sense defined in the introduction, Section 1.

Now, we are interested in the description of solutions whenever they exist. Recall that R denotes the law of $(\beta_t)_{t\in[0,1]}$ whose initial position follows the law *vol*. Since the reference measure R is a semi-martingale measure, the classical Girsanov theory implies that a solution P of (BS) will also be a semi-martingale. Moreover, using the finite entropy condition, we have stronger boundedness properties on the Girsanov velocity vector field. Theorem 4.1 and Theorem 4.3 below are adaptations of results from [17] to a manifold setting. They use a variational viewpoint of the entropy to improve Girsanov theorem under a finite entropy condition. We give here sketches of the proofs and objects in a manifold language. For a wider view, see [17] for the $\mathbb{R}^n$ setting and [15] for manifolds. Let B be the drift, defined on 1-form valued processes by

$$B_t(\alpha,\omega) = \int_0^t \langle \alpha_s(\omega_s), \nu_{\omega_s}\rangle \,\mathrm{d}L_s(\omega), \tag{4.1}$$

where $\alpha_t \in \Gamma(T^*M)$ for all $t \in [0,1]$ and $\omega \in \Omega$. Notice that, since L is a function of bounded variation uniquely defined except for a set of R-measure zero, B from (4.1) is also uniquely defined except for a set of R-measure zero. Let A be the quadratic variation defined on bilinear form valued processes by

$$A_t(h,\omega) = \int_0^t \left\langle \mathrm{Tr}\Big(h_s(\omega_s)\Big)\right\rangle \mathrm{d}s, \tag{4.2}$$

where $h_t \in \Gamma(T^*M \otimes T^*M)$ for all $t \in [0,1]$. Here $\langle \mathrm{Tr}(h)\rangle$ denotes the contraction of h using the Riemannian metric, so that (4.2) is well-defined everywhere and not just almost everywhere. With this notation, the measure R satisfies the martingale problem $\mathcal{MP}(B,A)$, i.e., R is the unique probability measure such that

$$M_t^f := f(X_t) - f(X_0) - B_t(\mathrm{d}f) - \frac{1}{2}A_t(\mathrm{Hess}f),$$

is an R-local martingale for every $f \in \mathcal{C}^\infty(M)$. We denote by $\mathrm{d}X$ its Itô derivative and by $\mathrm{d}_m^R X$ the martingale part of $\mathrm{d}X$ with respect to R (see [12, Definition 7.33]). Both are infinitesimal vector fields. The problem $\mathcal{MP}(B,A)$ implies that

$$\mathrm{d}X_t = \mathrm{d}_m^R X_t + \mathrm{d}B_t,\ R\text{-almost surely},$$

and

$$\mathrm{d}[X,X]_t = \mathrm{d}A_t,\ R\text{-almost surely}.$$

For the version of Girsanov theorem we are interested in, we will use the space $\mathcal{G}$ of measurable functions $g : [0,1]\times\Omega \to T^*M$ such that $g_t(\omega) \in T^*_{\omega_t}M$ for every $t \in [0,1]$ and for every $\omega \in \Omega$. For any probability measure Q on Ω, we define the semi-norm on $\mathcal{G}$

$$\|g\|_Q = \mathbb{E}_Q\left[\int_0^1 \|g_t\|^2_{T^*M}\mathrm{d}t\right]^{1/2} = \mathbb{E}_Q\left[A_1(g\otimes g)\right]^{1/2}.$$

Identifying functions by using the semi-norm $\|.\|_Q$, we define the Hilbert spaces

$$\mathcal{G}(Q) = \{g \in \mathcal{G} : \|g\|_Q < +\infty\} \quad \text{and} \quad \mathcal{H}(Q) = \{g \in \mathcal{G}(Q) : g \,\text{adapted}\}.$$

Adapted means here that, for every $t \in [0,1]$, the map $\omega \mapsto g(t,\omega)$ is measurable with respect to the completion of $\sigma((X_s)_{s\in[0,t]})$ using the Brownian motion law R, where the map $X_s : \Omega \to M$ is the projection map $X_s(\omega) = \omega_s$.

The following result is Girsanov theorem for the family $(P^x)_{x\in M}$ of probability measures on Ω that satisfy

$$P = \int_M P^x \mathrm{d}P_0(x) \quad \text{and} \quad P^x\left(\{\omega \in \Omega : \omega(0) = x\}\right) = 1 \text{ for every } x \in M.$$

By taking a time reversal, it would tell us something about the family $(\overleftarrow{P^y})_{y\in M}$, defined in (2.4), but we will not use this until later.

Theorem 4.1 (Girsanov theorem)**.** *Let P be such that $H(P|R) < \infty$. Then, for P_0-almost every $x \in M$, the probability measure P^x is the law of a semi-martingale and there exists an adapted process $\zeta^x \in \mathcal{H}(P^x)$ such that*

$$P^x \in \mathcal{MP}(B + \hat{B}^x, A) \quad \textit{and} \quad \hat{B}^x = A(\zeta^x \otimes \cdot).$$

Remark 4.2. *In other words, P^x-almost surely, $\mathrm{d}X_t = \mathrm{d}_m^{P^x} X_t + \mathrm{d}A_t(\zeta^x \otimes \cdot) + \nu_{X_t}\mathrm{d}L_t(X)$, where the P^x-martingale part is equal to $\mathrm{d}_m^{P^x} X_t = \mathrm{d}_m^{R^x} X_t - \mathrm{d}A_t(\zeta^x \otimes \cdot)$.*

Proof. Due to the chain rule for the entropy [11, Theorem C.3.1]

$$H(P|R) = H(P_0|R_0) + \int_M H(P^x|R^x)\mathrm{d}P_0(x),$$

we have that $H(P^x|R^x) < \infty$ for P_0-almost every $x \in M$.

For $h \in \mathcal{H}(P^x)$, we would like to define the processes N^h by

$$N_t^h = \int_0^t \langle h_s, \mathrm{d}_m^{R^x} X_s\rangle,\ 0 \le t \le 1. \tag{4.3}$$

If $h \in \mathcal{H}(P^x) \cap \mathcal{H}(R^x)$, the process N^h can be defined by (4.3). Its stochastic exponential $\mathcal{E}(N^h) = \exp(N^h - \frac{1}{2}[N^h, N^h])$ is a positive local martingale, so that it is a supermartingale and

$$0 \le \mathbb{E}_{R^x}[\mathcal{E}(N^h)_1] \le 1. \tag{4.4}$$

For $h \in \mathcal{H}(P^x) \cap \mathcal{H}(R^x)$, let u be the function $u : \omega \in \Omega \mapsto N_1^h - \frac{1}{2}[N^h, N^h]_1$. The variational definition of the entropy, known as the Donsker-Varadhan variational formula, implies that

$$\mathbb{E}_{P^x}[u] - \log \mathbb{E}_{R^x}[e^u] \le H(P^x|R^x) < +\infty.$$

Using (4.4), we have that

$$\mathbb{E}_{P^x}[u] \le H(P^x|R^x).$$

Then, since $\mathbb{E}_{P^x}\left[[N^h, N^h]_1\right] = \|h\|^2_{\mathcal{G}(P^x)}$ is finite, we have

$$\mathbb{E}_{P^x}[N_1^h] \le H(P^x|R^x) + \frac{1}{2}\|h\|^2_{\mathcal{G}(P^x)}.$$

Repeating the same calculation with $-h$ and λh for $\lambda > 0$, for all $h \in \mathcal{H}(P^x) \cap \mathcal{H}(R^x)$

$$\lambda\left|\mathbb{E}_{P^x}[N_1^h]\right| \le H(P^x|R^x) + \frac{\lambda^2}{2}\|h\|^2_{\mathcal{G}(P^x)}. \tag{4.5}$$

If $\|h\|_{\mathcal{G}(P^x)} \neq 0$, we can take $\lambda = \sqrt{2H(P^x|R^x)}\|h\|^{-1}_{\mathcal{G}(P^x)}$ and obtain

$$\left|\mathbb{E}_{P^x}[N_1^h]\right| \le \sqrt{2H(P^x|R^x)}\|h\|_{\mathcal{G}(P^x)}.$$

Letting $\lambda \to \infty$ in (4.5), this inequality remains valid if $\|h\|_{\mathcal{G}(P^x)} = 0$. So the linear form $h \mapsto \mathbb{E}_{P^x}[N_1^h]$ is continuous on $\mathcal{H}(P^x) \cap \mathcal{H}(R^x)$. This set is dense in $\mathcal{H}(P^x)$ since it contains the dense set of stair processes

$$h : (t, \omega) \in [0,1] \times \Omega \mapsto \sum_{i=1}^k \sigma(X_t) h_i \mathbb{1}_{]S_i, T_i]},$$

with $k \in \mathbb{N}$, $(h_i)_{1 \le i \le k} \in \mathbb{R}^n$ and $S_i < T_i \le S_{i+1}$ stopping times. So, $h \mapsto \mathbb{E}_{P^x}[N_1^h]$ extends linearly in a unique continuous way to $\mathcal{H}(P^x)$. By Riesz representation theorem, there exists a process $\zeta^x \in \mathcal{H}(P^x)$ dual to this linear form, i.e

$$\mathbb{E}_{P^x}\left[\int_0^1 \langle \mathrm{d}f, \mathrm{d}_m^{R^x} X_t\rangle\right] = \mathbb{E}_{P^x}\left[\int_0^1 \langle \mathrm{d}f, \mathrm{d}A_t(\zeta_t^x \otimes \cdot)\rangle\right].$$

In conclusion, under P, X is a semi-martingale with quadratic variation A and drift $B + \hat{B}$. □

We remark that using the classical Girsanov theory we could only have proved that, P^x-almost surely, $A_t(\zeta^x \otimes \cdot) < +\infty$. From now on, ζ_t^x will be identified, as a vector field, with the drift $\hat{B}_t = A_t(\zeta^x \otimes \cdot)$. We show in Section 5 that it is the Nelson forward stochastic velocity of P^x.

Léonard's approach to Girsanov theory also gives us an expression of the density of P^x with respect to R^x in terms of ζ^x. This will be essential for the proof of Theorem 2.1.

Theorem 4.3 (Density in terms of velocity)**.** *With the notation of Theorem 4.1, for P_0-almost every $x \in M$, the density of P^x is given by*

$$\frac{\mathrm{d}P^x}{\mathrm{d}R^x} = \mathbb{1}_{\left\{\frac{\mathrm{d}P^x}{\mathrm{d}R^x} > 0\right\}} \exp\left(\int_0^1 \langle \zeta_t^x, \mathrm{d}_m^{P^x} X_t\rangle - \frac{1}{2}\int_0^1 \|\zeta_s^x\|^2 \,\mathrm{d}s\right).$$

Sketch of the proof. The proof is divided in three parts. Firstly, we prove a change of measure formula for stopped processes. This is the following well-known argument. We define the sequence $(\sigma_k)_{k\geq 1}$ of stopping times by

$$\sigma_k = \inf\left\{t \in [0,1] : A_t(\zeta^x \otimes \zeta^x) \geq k\right\},$$

where, since we are thinking on subsets of $[0,1]$, we use the convention that the infimum of the empty set is 1. These stopping times localise the semi-martingale

$$N_t = \int_0^t \langle \zeta_s^x, \mathrm{d}_m^{R^x} X_s\rangle,\ 0 \leq t \leq 1.$$

Let R^{σ_k} denote the law of $X_{\cdot\wedge\sigma_k}$ when X follows the law R^x and let $\mathcal{E}(N)_{\sigma_k}$ denote the stochastic exponential of N at the time σ_k. Hence, the measure $Q_k = \mathcal{E}(N)_{\sigma_k} R^{\sigma_k}$ is a probability measure satisfying the martingale problem $\mathcal{MP}\Big((B+\hat{B})_{\cdot\wedge\sigma_k}, A_{\cdot\wedge\sigma_k}\Big)$. As a second step, using the additional assumption that P is equivalent to R, we prove the theorem. Here, the key argument is a uniqueness property satisfied by the reflected Brownian motion: R^x is the unique measure in $\mathcal{MP}(B,A)$ absolutely continuous with respect to R^x starting from R_0. Property gives us the density of P^{σ_k}. The equivalence assumption is used to have $\sigma_k \to +\infty$ R^x-a.s and obtain the density of P^x. We finish with a regularisation argument. The measure $P_n^x = (1-\frac{1}{n})P^x + \frac{1}{n}R^x$ is equivalent to R^x and converge to P^x in a sufficiently strong sense to obtain the result at the limit. □

Remark 4.4 (Entropy and kinetic energy)**.** *The proof of Theorem 4.3 also works for P and R instead of P^x and R^x. As a consequence, we would obtain that if $H(P|R) < \infty$,*

$$H(P|R) = H(P_0|R_0) + \frac{1}{2}\mathbb{E}_P\left[\int_0^1 \|\zeta_t\|^2 \,\mathrm{d}t\right],$$

where ζ can be seen as a forward stochastic velocity as in (5.2) *by using P instead of P^x. This formula for the entropy makes a parallel between the Brenier problem, as the minimisation of a* classical *kinetic energy, and Brenier-Schrödinger problem, as the minimisation of a* stochastic *kinetic energy in Nelson's sense. The advantage of an entropy formulation of the problem is the convex optimisation tools.*

5 Proof of the Navier-Stokes equations and the continuity equation

This section will be devoted to the proof of Theorem 2.1 and Theorem 2.2. To prove Theorem 2.1 we will first prove its 'forward velocity' counterpart in Corollary 5.3. Following [3],

the idea is to compare the density obtained from Theorem 4.3 and the density from the definition of a regular solution (2.1). This is done in the following lemma. We recall that ζ^x is the one obtained in Theorem 4.1.

Lemma 5.1 (Comparison of densities)**.** *Suppose that P is a regular solution of* (BS)*. Then, for P_0-almost every $x \in M$ and for every $t \in [0,1]$,*

$$\langle \zeta_t^x, \mathrm{d}_m^{R^x} X_t \rangle - \frac{1}{2} \|\zeta_t^x\|^2 \, \mathrm{d}t = \mathbb{1}_{\mathcal{S}}(t)\theta_t + p_t \mathrm{d}t + \mathrm{d}\psi_t^x(X_t), \; P^x\text{-almost surely.}$$

Proof. The idea of the proof is to compare two expressions of the density of P^x with respect to R^x, the first given by Theorem 4.3 and the second by the definition of a regular solution. On the one hand, from Theorem 4.3, the density is

$$\frac{\mathrm{d}P^x}{\mathrm{d}R^x} = \exp\left(\int_{[0,1]} \langle \zeta_t^x, \mathrm{d}_m^{P^x} X_t \rangle - \frac{1}{2} \int_{[0,1]} \|\zeta_t^x\|^2 \, \mathrm{d}t \right), \; P^x\text{-a.s.}$$

Then we restrict the density to $\mathcal{F}_t = \sigma((X_u)_{u \in [0,t]})$ by using that

$$\frac{\mathrm{d}P^x_{[0,t]}}{\mathrm{d}R^x_{[0,t]}} = \mathbb{E}_{R^x}\left[\frac{\mathrm{d}P^x}{\mathrm{d}R^x} \middle| \mathcal{F}_t \right].$$

For all $t \in [0,1]$ we have

$$\frac{\mathrm{d}P^x_{[0,t]}}{\mathrm{d}R^x_{[0,t]}} = \exp\left(\int_0^t \langle \zeta_s^x, \mathrm{d}_m^{P^x} X_s \rangle - \frac{1}{2} \int_0^t \|\zeta_s^x\|^2 \, \mathrm{d}s \right), \; P^x\text{-a.s.}$$

On the other hand, from the definition of a regular solution, we know that P has the form (2.1). By disintegration, for R_0-almost every $x \in M$,

$$\frac{\mathrm{d}P^x}{dR^x} = \exp\left(\eta(x, X_1) + \sum_{s \in \mathcal{S}} \theta_s(X_s) + \int_{\mathcal{T}} p_t(X_t) \, \mathrm{d}t \right), \; R^x\text{-a.s.}$$

Then, conditioning with respect to $\mathcal{F}_t$ and using the Markov property of R^x,

$$\begin{aligned}
\frac{\mathrm{d}P^x_{[0,t]}}{\mathrm{d}R^x_{[0,t]}} &= \mathbb{E}_{R^x}\left[\exp\left(\eta(x, X_1) + \sum_{s \in \mathcal{S}} \theta_s(X_s) + \int_{\mathcal{T}} p_r(X_r) \, \mathrm{d}r \right) \middle| \mathcal{F}_t \right] \\
&= \exp\left(\sum_{s \le t} \theta_s(X_s) + \int_{\mathcal{T} \cap [0,t]} p_r(X_r) \, \mathrm{d}r \right) \\
&\qquad \times \mathbb{E}_{R^x}\left[\exp\left(\eta(x, X_1) + \sum_{s > t} \theta_s(X_s) + \int_{\mathcal{T} \cap]t,1]} p_r(X_r) \, \mathrm{d}r \right) \middle| \mathcal{F}_t \right] \\
&= \exp\left(\sum_{s \in \mathcal{S}, s \le t} \theta_s(X_s) + \int_{\mathcal{T} \cap [0,t]} p_r(X_r) \, \mathrm{d}r + \psi_t^x(X_t) \right), \; R^x\text{-a.s.}
\end{aligned}$$

We confront both expressions for $\frac{\mathrm{d}P^x_{[0,t]}}{\mathrm{d}R^x_{[0,t]}}$ and conclude.

□

Theorem 5.2 (Hamiltonian equation for the potential)**.** *Assume that P is a forward regular solution of* (BS)*. Then, for P_0-almost every $x \in M$, the function ψ^x defined by* (2.2) *is a classical solution of*

$$\begin{cases} \partial_t \psi_t^x - \frac{1}{2}\Delta\psi_t^x + \|\nabla\psi_t^x\|^2 + \mathbb{1}_{\mathcal{T}}(t)p_t = 0, & t \in [0,1) \setminus \mathcal{S}, \\ \psi_t^x - \psi_{t^-}^x = -\theta_t, & t \in \mathcal{S}, \\ \langle \nabla\psi_t^x(z), \nu_z \rangle = 0, & z \in \partial M, \\ \psi_1^x = \eta(x,\cdot), & t = 1, \end{cases} \tag{5.1}$$

which is a second-order Hamiltonian equation.

Moreover, for P_0-almost every $x \in M$,

$$\zeta_t^x = \nabla\psi_t^x(X_t),\ \mathrm{d}t \otimes P^x\text{-almost surely.}$$

Proof. According to Theorem 4.1, we have $\mathrm{d}_m^{P^x} X_t = \mathrm{d}_m^{R^x} X_t - \zeta_t \mathrm{d}t$, P^x-almost surely. By Lemma 5.1, for P_0-almost every $x \in M$ and for all $t \in [0,1]$,

$$\mathrm{d}\psi_t^x(X_t) = \langle \zeta_t^x, \mathrm{d}_m^{P^x} X_t \rangle + \left(\frac{1}{2}\|\zeta_t^x\|^2 - p_t(X_t)\right)\mathrm{d}t - \mathbb{1}_{\mathcal{S}}(t)\theta_t(X_t),\ P^x\text{-almost surely.}$$

On the other hand, since ψ^x is regular enough, the semi-martingale $(\psi^x(X_t))_{t\in[0,1]}$ satisfies the Itô formula. For all $t \in [0,1]$, we have

$$\begin{aligned} \mathrm{d}\psi_t^x(X_t) =& [\psi_t^x - \psi_{t^-}^x](X_t) + \left\langle \nabla\psi_t^x(X_t), \mathrm{d}_m^{P^x} X_t \right\rangle + \langle \nabla\psi_t^x(X_t), \zeta_t^x \rangle \,\mathrm{d}t \\ &+ \langle \nabla\psi_t^x(X_t), \nu_{X_t} \rangle \,\mathrm{d}L_t + \left(\frac{1}{2}\Delta + \partial_t\right)\psi_t^x(X_t)\mathrm{d}t,\ P^x\text{-a.s.} \end{aligned}$$

The Doob-Meyer decomposition of a semi-martingale allows the following identifications using the previous equations.

$$\begin{cases} \zeta_t^x = \nabla\psi_t^x(X_t), & \mathrm{d}t \otimes \mathrm{d}P^x(X)\text{-a.s.} \\ -\mathbb{1}_{\mathcal{S}}(t)\theta_t(X_t) = [\psi_t^x - \psi_{t^-}^x](X_t), & \mathrm{d}P^x(X)\text{-a.s.} \\ \frac{1}{2}\|\zeta_t^x\|^2 - p_t(X_t) = \langle \nabla\psi_t^x(X_t), \zeta_t^x \rangle + \left(\frac{1}{2}\Delta + \partial_t\right)\psi_t^x(X_t), & \mathrm{d}t \otimes \mathrm{d}P^x\text{-a.s.} \\ \langle \nabla\psi_t^x(X_t), \nu_{X_t} \rangle \mathbb{1}_{\partial M}(X_t) = 0, & \mathrm{d}L_t(X) \otimes \mathrm{d}P^x(X)\text{-a.s.} \end{cases}$$

We complete the proof by using the covering property of X under P^x so that

$$\begin{cases} \zeta_t^x = \nabla\psi_t^x(X_t), & \mathrm{d}t \otimes \mathrm{d}P^x(X)\text{-a.s.}, \\ -\theta_t(z) = [\psi_t^x - \psi_{t^-}^x](z), & t \in \mathcal{S}, z \in M, \\ \left(\frac{1}{2}\Delta + \partial_t\right)\psi_t^x(z) + \frac{1}{2}\|\nabla\psi_t^x(z)\|^2 + \mathbb{1}_{\mathcal{T}}(t)p_t(z) = 0, & t \in [0,1) \setminus \mathcal{S}, z \in M, \\ \langle \nabla\psi^x(z), \nu_z \rangle = 0, & z \in \partial M. \end{cases}$$

□

The function ψ^x plays the role of a scalar potential of ζ^x. The previous theorem tells us that, in fact, the randomness for ζ^x can be thought of as coming only from the position X_t. Recall that ∇ denotes the covariant derivative and $\Box$ denotes the de Rham-Hodge-Laplace operator $-(\mathrm{d}\delta + \delta\mathrm{d})$ by identifying vector fields with one-forms. Using the notation of Section 2, define the random times

$$\tau_t = \frac{1}{2}\sup\{h \geq 0 : X_{t+s} \in U(X_t) \text{ for every } s \in [0,h]\}$$

which are strictly positive.

Corollary 5.3 (Forward stochastic velocity)**.** *Suppose that P is a forward regular solution of* (BS)*. Then, for P_0-almost every $x \in M$, there exists a measurable function*

$$\overset{x\rightharpoonup}{v} \colon [0,1] \times \Omega \to TM$$

such that $t \mapsto \overset{x\rightharpoonup}{v}_t(\omega)$ is right-continuous and has left limits for every $\omega \in \Omega$ and such that for every $t \in [0,1]$ we have that, for P^x-almost every X,

$$\lim_{h \to 0^+} \frac{1}{h} \mathbb{E}_{P^x} \left[\overrightarrow{X_t X_{t+h\wedge\tau_t}} \middle| X_{[0,t]} \right] = \overset{x\rightharpoonup}{v}_t(X). \tag{5.2}$$

Let $V_t^x = \nabla \psi_t^x$. Then, for P_0-almost every $x \in M$,

$$\overset{x\rightharpoonup}{v}_t = V_t^x(X_t), \quad P^x\text{-almost surely.}$$

Moreover, for P_0 almost all $x \in M$, the time-dependent vector field V^x satisfies

$$\begin{cases} (\partial_t + \nabla_{V^x}) V^x = -\dfrac{1}{2} \Box V^x - \mathbb{1}_{\mathcal{T}}(t) \nabla p, & t \in [0,1) \setminus \mathcal{S}, \\ V_t^x - V_{t^-}^x = -\nabla \theta_t, & t \in S, \\ \langle V^x(z), \nu_z \rangle = 0, & z \in \partial M, \\ V_1^x = \nabla \Big(\eta(\cdot, x) \Big), & t = 1. \end{cases}$$

Proof. Theorem 4.1 tells us that, P^x-almost surely, $\mathrm{d}X_t = \mathrm{d}_m^{P^x} X_t + \zeta_t^x \mathrm{d}t + \nu_{X_t} \mathrm{d}L_t$. Moreover, we know by Theorem 5.2 that we may choose a version of ζ^x that is càdlàg in t. Applying Itô's formula to the logarithm log, and taking the limit by using the right-continuity of ζ^x we show that the limit defining $\overset{x\rightharpoonup}{v}_t$ exists and that $\overset{x\rightharpoonup}{v}(X) = \zeta^x = \nabla \psi_t^x(X_t)$. The rest is a consequence of Theorem 5.2 by taking the gradient of (5.1) and using that $(\mathrm{d}\delta + \delta \mathrm{d})$ commutes with the exterior derivative d. □

Now we are ready to give the proof of Theorem 2.1.

Proof of Theorem 2.1. There is a strong link between $\overset{x\rightharpoonup}{v}_t$, the forward stochastic velocity obtained in Corollary 5.3, and the backward stochastic velocity to be obtained here. This is achieved through the time reversal transformation. Let $\mathrm{rev} : \Omega \to \Omega$ be the time reversal transformation defined by $\mathrm{rev}(\omega)_t = \omega_{1-t}$. Let $P^* = \mathrm{rev}_* P$ be the pushforward of P by the map rev. Then, the limit defining the forward stochastic velocity is related to the sought limit for the backward stochastic velocity by

$$\overset{\leftharpoondown y}{v}_t = - \overset{y\rightharpoonup P^*}{v}_{1-t} \circ \mathrm{rev}, \tag{5.3}$$

where $\overset{y\rightharpoonup P^*}{v}$ denotes the forward stochastic velocity associated to P^*. We just need to notice that P^* satisfy the requirements of Corollary 5.3. Since the reference measure R is reversible, we have $\mathrm{rev}_* R = R$ so that (2.1) becomes

$$\mathrm{d}P^*(X) = \exp\left(\eta^*(X_0, X_1) + \sum_{s \in \mathcal{S}*} \theta_s^*(X_s) + \int_{\mathcal{T}^*} p_t^*(X_t) \, \mathrm{d}t \right) \mathrm{d}R(X),$$

where $\eta^*(x,y) = \eta(y,x)$, $\mathcal{S}^* = \{1 - s : s \in \mathcal{S}\}$, $\theta_s^* = \theta_{1-s}$, $\mathcal{T}^* = \{1 - t : t \in \mathcal{T}\}$ and $p_t^* = p_{1-t}$. The function ψ from (2.2) for P^* equals the function φ from (2.3) for P or, more precisely,

$$\varphi_t^y(z) = \log \mathbb{E}_{R^y} \left[\exp\left(\eta^*(y, X_1) + \sum_{s \in \mathcal{S}^*, s > t} \theta_s^*(X_s) + \int_{\mathcal{T}^* \cap]t,1]} p_r^*(X_r) \, \mathrm{d}r \right) \middle| X_t = z \right].$$

We may conclude by Corollary 5.3 and (5.3).

□

Now we give the proof of Theorem 2.2. Notice that we will not use the regularity of the solution. In particular, there are no η, p and θ involved.

Proof of Theorem 2.2. Let $f \in \mathcal{C}^\infty(M)$, and $0 \leq t \leq 1$. On one hand, we have that

$$\begin{aligned}
P_t(f) &= \mathbb{E}_P[f(X_t)] \\
&= \int_M \mathbb{E}_{P^x}[f(X_t)]\,\mathrm{d}P_0(x) \\
&= \int_M \mathbb{E}_{P^x}\left[f(x) + \int_0^t \langle \mathrm{d}f, \overset{x\rightharpoonup}{v}_s\rangle\,\mathrm{d}s + \frac{1}{2}\int_0^t \Delta f(X_s)\,\mathrm{d}s + \int_0^t \langle \mathrm{d}f, \nu_{X_s}\rangle\,\mathrm{d}L_s\right]\mathrm{d}P_0(x) \\
&= P_0(f) + \int_0^t P_s\left(\langle \mathrm{d}f, \overset{\leftrightarrow}{v}_s\rangle + \frac{1}{2}\Delta f\right)\mathrm{d}s + \mathbb{E}_P\left[\int_0^t \langle \mathrm{d}f, \nu_{X_s}\rangle\,\mathrm{d}L_s\right].
\end{aligned}$$

On the other hand, for P_1-almost every $y \in M$, under the reversed law P^{y*}, X is a semi-martingale with drift $\overset{y\rightharpoonup P^*}{v}\,\mathrm{d}t + \nu\mathrm{d}L^*$ where the relation between $\overset{\leftharpoonup y}{v}$ and $\overset{y\rightharpoonup P^*}{v}$ is given by (5.3) and $L_t(X) = L^*_{1-t}(X^*)$ for all $t \in [0,1]$, where $X^* = \mathrm{rev}(X)$. We have

$$\begin{aligned}
&\mathbb{E}_{P^{y*}}[f(X_{1-t})] \\
&\quad= \mathbb{E}_{P^{y*}}\left[f(y) + \int_0^{1-t} \langle \mathrm{d}f, \overset{y\rightharpoonup P^*}{v}_s(X)\rangle\,\mathrm{d}s + \frac{1}{2}\int_0^{1-t} \Delta f(X_s)\,\mathrm{d}s + \int_0^{1-t} \langle \mathrm{d}f, \nu_{X_s}\rangle\,\mathrm{d}L^*_s(X)\right] \\
&\quad= \mathbb{E}_{P^{y*}}\left[f(y) + \int_t^1 \langle \mathrm{d}f, \overset{y\rightharpoonup P^*}{v}_{1-s}(X)\rangle\,\mathrm{d}s + \frac{1}{2}\int_t^1 \Delta f(X^*_s)\,\mathrm{d}s + \int_t^1 \langle \mathrm{d}f, \nu_{X_s}\rangle\,\mathrm{d}L^*_{1-s}(X)\right] \\
&\quad= \mathbb{E}_{P^{y*}}\left[f(y) - \int_t^1 \langle \mathrm{d}f, \overset{\leftharpoonup y}{v}_s(X^*)\rangle\,\mathrm{d}s + \frac{1}{2}\int_t^1 \Delta f(X^*_s)\,\mathrm{d}s + \int_t^1 \langle \mathrm{d}f, \nu_{X_s}\rangle\,\mathrm{d}L_s(X^*)\right].
\end{aligned}$$

Hence, by disintegration along P_1, we have that

$$\begin{aligned}
P_t(f) &= \mathbb{E}_{P^*}[f(X_{1-t})] \\
&= \int_M \mathbb{E}_{P^{y*}}[f(X_{1-t})]\,\mathrm{d}P_1(y) \\
&= P_1(f) + \int_t^1 P_s\left(\langle \mathrm{d}f, -\overset{\leftrightarrow}{v}_s(X)\rangle + \frac{1}{2}\Delta f\right)\mathrm{d}s + \mathbb{E}_P\left[\int_t^1 \langle \mathrm{d}f, \nu_{X_s}\rangle\,\mathrm{d}L_s\right].
\end{aligned}$$

Before differentiating, we need to show that the terms with local time are regular enough. For $\varepsilon > 0$, we denote $\partial^\varepsilon M$ the ε-tubular neighbourhood of ∂M. We have

$$\int_0^t \langle \mathrm{d}f, \nu_{X_s}\rangle\,\mathrm{d}L_s = \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon}\int_0^t \langle \mathrm{d}f, \nu_{X_s}\rangle \mathbb{1}_{X_s \in \partial^\varepsilon M}\,\mathrm{d}s.$$

Then, we obtain

$$\begin{aligned}
\mathbb{E}_P\left[\int_0^t \langle \mathrm{d}f, \nu_{X_s}\rangle\,\mathrm{d}L_s\right] &= \frac{1}{2}\int_0^t \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\mathbb{E}_P\left[\langle \mathrm{d}f, \nu_{X_s}\rangle \mathbb{1}_{X_s\in\partial^\varepsilon M}\right]\mathrm{d}s \\
&= \frac{1}{2}\int_0^t \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}P_s(\langle \mathrm{d}f, \nu\rangle \mathbb{1}_{\partial^\varepsilon M})\,\mathrm{d}s \\
&= \frac{1}{2}\int_0^t \underline{P}_s(\langle \mathrm{d}f, \nu\rangle),
\end{aligned}$$

where $\underline{P}_s$ denotes the normalised surface measure associated to P_s. It follows that, for all $t \in [0,1]$,

$$\partial_t P_t(f) = P_t(\langle \overset{\leftrightarrow}{v}_t, \mathrm{d}f\rangle + \frac{1}{2}\Delta f) + \underline{P}_t(\langle \nu, \mathrm{d}f\rangle) = P_t(\langle \overset{\leftrightarrow}{v}_t, \mathrm{d}f\rangle - \frac{1}{2}\Delta f) - \underline{P}_t(\langle \nu, \mathrm{d}f\rangle). \quad (5.4)$$

Since P is a satisfies $P_t = \mu_t$ for every $t \in \mathcal{T}$, the proof is complete. □

6 Proof of the existence on homogeneous spaces

In this section we prove Theorem 3.1. The proof is inspired by [3] which is, in turn, inspired by [10]. The idea is to find a path measure Q of finite relative entropy and satisfying the marginal conditions. The candidate for such a measure is

$$Q = \int_{M^3} R(\cdot|X_0 = x, X_{1/2} = z, X_1 = y)\, \sigma(\mathrm{d}x\mathrm{d}z\mathrm{d}y), \tag{6.1}$$

with $\sigma(\mathrm{d}x\mathrm{d}z\mathrm{d}y) = \pi(\mathrm{d}x\mathrm{d}y)vol(\mathrm{d}z)$ belongs to $\mathcal{P}(M^3)$. It extends the result in [3] of existence on the torus, using the same property of invariance of the Brownian motion and the Riemannian volume, under isometries.

Proposition 6.1 (Constraints and entropy)**.** *The path measure Q satisfies the marginal and endpoint constraints*

$$P_t = vol, \forall t \in [0,1] \quad \text{and} \quad P_{01} = \pi.$$

In addition, if $H(\pi|vol \otimes vol) < \infty$ then $H(Q|R) < \infty$.

Proof. First, remark that, since R is a Markov measure, we have

$$\begin{aligned} R(\cdot|X_0 = x, X_{1/2} = z, X_1 = y) = R(X_{[0,1/2]} \cdot |X_0 = x, X_{1/2} = z) \\ \times R(X_{[1/2,1]} \in \cdot|X_{1/2} = z, X_1 = y). \end{aligned}$$

Now, let us check the endpoint constraints. For measurable subsets A and B of M,

$$\begin{aligned} Q_{01}(A \times B) &= Q(X_0 \in A, X_1 \in B) \\ &= \int_{M^3} R(X_0 \in A|X_0 = x, X_{1/2} = z) R(X_1 \in B|X_{1/2} = z, X_1 = y)\, \sigma(\mathrm{d}x\mathrm{d}z\mathrm{d}y) \\ &= \int_{M^3} \mathbb{1}_A(x)\mathbb{1}_B(y)\, \sigma(\mathrm{d}x\mathrm{d}z\mathrm{d}y) \\ &= \sigma(A \times M \times B) \\ &= \pi(A \times B). \end{aligned}$$

So $Q_{01} = \pi$. Then, we prove that Q_t is invariant under isometries for all t. Let $t \in [0, 1/2]$ and let f be a bounded measurable function on M. We have

$$\begin{aligned} \int_M f\, \mathrm{d}Q_t &= \int_{M^3} \mathbb{E}_R\left[f(X_t) \,\middle|\, X_0 = x, X_{1/2} = z\right] \sigma(\mathrm{d}x\mathrm{d}z\mathrm{d}y) \\ &= \int_{M^2} \mathbb{E}_R\left[f(X_t) \,\middle|\, X_0 = x, X_{1/2} = z\right] vol(\mathrm{d}x)vol(\mathrm{d}z). \end{aligned}$$

For every isometry g of M, using the invariance in law of the Brownian motion and the invariance of the Riemannian volume measure under isometry, we have that

$$\begin{aligned} \int_M f \circ g\, \mathrm{d}Q_t &= \int_{M^2} \mathbb{E}_R\left[f \circ g(X_t) \,\middle|\, X_0 = x, X_{1/2} = z\right] vol(\mathrm{d}x)vol(\mathrm{d}z) \\ &= \int_{M^2} \mathbb{E}_R\left[f(X_t) \,\middle|\, X_0 = g(x), X_{1/2} = g(z)\right] vol(\mathrm{d}x)vol(\mathrm{d}z) \\ &= \int_{M^2} \mathbb{E}_R\left[f(X_t) \,\middle|\, X_0 = x, X_{1/2} = z\right] vol(\mathrm{d}x)vol(\mathrm{d}z) \\ &= \int_M f\, \mathrm{d}Q_t. \end{aligned}$$

Since vol is the unique probability measure on M that is invariant under isometries (see, for instance, [13, Proposition 476C]), we obtain $Q_t = vol$. The result is obtained, mutatis mutandis, for $t \in [1/2, 1]$. Then, Q satisfies the marginal constraint.

We have now to prove that $H(Q|R) < \infty$. Denote

$$Q_{0,1/2,1} = Q(X_0 \in \cdot, X_{1/2} \in \cdot, X_1 \in \cdot)$$

and $Q^{xzy} = Q(\cdot|X_0 = x, X_{1/2} = z, X_1 = y)$, and similarly for R. We have, by using the chain rule for the entropy [11, Theorem C.3.1],

$$\begin{aligned} H(Q|R) &= H(Q_{0,1/2,1}|R_{0,1/2,1}) + \int_{M^3} H(Q^{xzy}|R^{xzy})\, Q_{0,1/2,1}(\mathrm{d}x\mathrm{d}z\mathrm{d}y) \\ &= H(\sigma|R_{0,1/2,1}) \\ &= H(\sigma_{01}|R_{01}) + \int_{M^2} H(\sigma^{xy}|R^{xy}_{1/2})\, \sigma_{01}(\mathrm{d}x\mathrm{d}y) \\ &= H(\pi|R_{01}) + \int_{M^2} H(vol|R^{xy}_{1/2})\, \pi(\mathrm{d}x\mathrm{d}y), \end{aligned}$$

where $R^{xy}_{1/2}$ is the law at time $t = 1/2$ of the Brownian bridge between x at time $t = 0$ and y at time $t = 1$. By definition of the relative entropy, we have

$$H(vol|R^{xy}_{1/2}) = \int_M \log\left(\frac{\mathrm{d}vol}{\mathrm{d}R^{xy}_{1/2}}\right) vol(\mathrm{d}z).$$

We denote by p the heat kernel on M. We have

$$\frac{\mathrm{d}R^{xy}_{1/2}}{\mathrm{d}vol}(z) = \frac{p_{1/2}(x,z)p_{1/2}(z,y)}{p_1(x,y)}.$$

This quantity is continuous in x, y and z. As M is compact, the density can be bounded uniformly in the three variables. So the relative entropy $H(Q|R)$ is finite if and only if $H(\pi|R_{01})$ is finite which, since the density of R_{01} with respect to $vol \otimes vol$ is continuous and strictly positive, is equivalent to $H(\pi|vol \otimes vol) < \infty$. □

The homogeneity of M seems to be important to show that, at a fixed time, the law of the Brownian bridge between two independent uniformly distributed random variables is the uniform measure vol. It is not clear and it would be interesting to understand if this holds or not on a non-homogeneous space.

Proof of Theorem 3.1. By Proposition 6.1, if the entropy of π is finite, there exists a measure Q that satisfies the constraints of the problem (BS) and has finite entropy with respect to H. We conclude by the strict convexity of the entropy and the convex constraints.

On the other hand, if Q is the unique solution then, in particular, Q_{01} has finite entropy with respect to R_{01}. Since R_{01} has a continuous and strictly positive density with respect to $vol \otimes vol$, we also have that $H(Q_{01}|vol \otimes vol) < \infty$. □

7 Proof of the existence for quotient spaces

The goal of this section is to prove Theorem 3.5 and to give some examples of the existence of solutions to the Brenier-Schrödinger problem. Theorem 3.5 describes a relation between the Brenier-Schrödinger problem on compact Riemannian manifolds and on some quotients

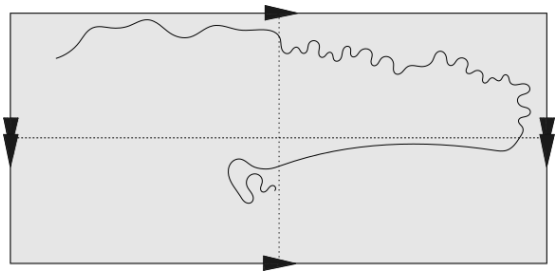

Figure 1: A path in the torus.

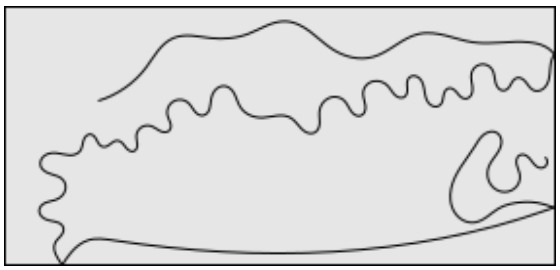

Figure 2: The projected path in the rectangle.

of these. For instance, we want to see the *n-hypercube* as a quotient of a *flat n-dimensional torus* (see Figure 1 and 2 where the reflections are along the dotted lines) or a positively curved *n-ball* as a quotient of the *n-sphere*.

We begin by giving a proof of the existence of the Riemannian structure on a quotient by a reflection group. This is similar to what happens on $\mathbb{R}^n$ where the theory of reflection groups is well-known (see, for instance, [16]).

Proof of Lemma 3.4. The topological structure of $N = M/G$ is induced by the quotient map $q : M \to N$. Given a reflection group G, we shall make of N a manifold with corners in the following way. Let $y \in N$ and take any $x \in M$ with $q(x) = y$.

If $G_x = \{e\}$ then there exists an open neighborhood V of x such that $gV \cap hV = \emptyset$ for every $g \neq h$ in G. Since q is open and $q|_V$ is injective and continuous we have that $q|_V : V \to q(V)$ is an homeomorphism and we can assume that (by taking a smaller V if necessary) V is diffeomorphic to an open subset of $(0, \infty)^n$. This gives an atlas to the open set of points that can be written as $q(x)$ with $G_x = \{e\}$. We can even define a metric on this open set with the help of these $q|_V$.

If $G_x \neq \{e\}$ we consider the exponential map

$$\exp_x : W \subset T_xM \to V \subset M$$

on an open neighborhood W of $0 \in T_xM$ invariant under $\mathbb{G}_x$ such that $\exp_x|_W$ is a diffeomorphism onto its image V. Moreover, by choosing V small enough we assume that $gV \cap V = \emptyset$ for every $g \notin G_x$. Since

$$g \exp_x(w) = \exp_x(\mathrm{d}g_x w) \tag{7.1}$$

for $g \in G_x$ and $w \in T_xM$, the open set V is invariant under G_x. Equation (7.1) tells us that the action of G_x on W (as $\mathbb{G}_x$) is isomorphic to the action of G_x on V. Then, we only need to understand

$$W/\mathbb{G}_x.$$

But, since $\mathbb{G}_x$ is a reflection group, we know that $T_xM/\mathbb{G}_x$ can be identified with a particular fundamental domain of the action of $\mathbb{G}_x$ on T_xM, called closed chamber (see [16, Section 1.12]), and, in particular, it has a structure of a manifold with corners so that $W/\mathbb{G}_x$ inherits this structure. Using $\exp_x$ we have given to the open set $V/G_x \simeq q(V)$ the structure of a manifold with corners. In fact, if $C \subset T_xM$ is a closed chamber, we have identified $\exp_x(C \cap W)$ with $q(V)$. The latter identification gives a Riemannian metric to $q(V)$ which is completely characterised by the isometric properties required for q.

□

We will also need the following standard lemma whose proof we recall.

Lemma 7.1 (Fundamental domain)**.** *Let G be a finite group of isometries of M. Then, there exists an open subset V of M such that*

- *$gV \cap hV = \emptyset$ for every $g \neq h$ in G and*
- *$vol\left(M \setminus \bigcup_{g\in G} gV\right) = 0$.*

Proof. Let $x \in M$ such $G_x = \{e\}$ and define the set

$$V = \{y \in M : \forall g \in G \setminus \{e\},\, d(x,y) < d(gx,y)\},$$

where d is the distance function on the Riemannian manifold M. Since G is a group of isometries we have that

$$hV = \{y \in M : \forall g \in G \setminus \{h\},\, d(hx,y) < d(gx,y)\}.$$

We only need to see that, for $\xi \neq \zeta$ in M,

$$vol\,\{y \in M : d(\xi,y) = d(\zeta,y)\} = 0.$$

This is true since the map $y \mapsto d(\xi,y) - d(\zeta,y)$ is smooth and regular outside the cutlocus of ξ and ζ and since every cutlocus has *vol*-measure zero.

□

Notice that, in particular, for every $g \in G$ and $x \in gV$ the group G_x contains only the identity so that $q|_{gV}$ is an isometry onto its image. There is an intuitive relation between a Brownian motion on M and on its quotient by a reflection group.

Lemma 7.2 (Brownian motion under quotients)**.** *Suppose that G is a reflection group of isometries of M. Let $\{B_t^x\}_{t\geq 0}$ be a Brownian motion on M starting at $x \in M$. Then $\{q(B_t^x)\}_{t\geq 0}$ does not touch the corner points almost surely and*

$$\{q(B_t^x)\}_{t\geq 0} \text{ is a reflected Brownian motion on } M/G.$$

Proof. The fact that $q(B_t^x)$ does not touch the corner points is a result of the following facts. The set $q^{-1}(\mathcal{C}N)$ is a finite union of submanifolds of dimension less or equal than $n-2$ and the Brownian motion B_t^x almost surely does not touch submanifolds of dimension less or equal than $n-2$.

Now, for every $\varepsilon > 0$, we consider the ε-neighborhood of the corner points,

$$\mathcal{N}_\varepsilon = \{x \in N : d(x,y) < \varepsilon \text{ for some } y \in \mathcal{C}N\}.$$

Let $f : N \to \mathbb{R}$ be a smooth map such that $\mathrm{d}f_x\nu_x = 0$ at every regular boundary point x and consider

$$F = f \circ q$$

which can be seen to be C^2 on $M \setminus q^{-1}(\mathcal{C}N)$. Let $\varepsilon > 0$ and let $F^\varepsilon : M \to \mathbb{R}$ be a C^2 function on M that coincide with F outside of $\mathcal{N}_\varepsilon$. Then,

$$F^\varepsilon(B_t^x) - \int_0^t \Delta F^\varepsilon(B_s^x)\mathrm{d}s \text{ is a martingale}$$

with respect to the filtration $(B^x_s)_{s\in[0,t]}$ so that, if

$$T_\varepsilon = \inf\left\{t \geq 0 :\ B^x_t \in q^{-1}\left(\mathcal{N}_\varepsilon\right)\right\},$$

we have that

$$F(B^x_{t\wedge T_\varepsilon}) - \int_0^{t\wedge T_\varepsilon} \Delta F(B^x_s)\mathrm{d}s \text{ is also a martingale.}$$

By using that

$$\Delta F = (\Delta f)\circ q$$

we have proved that

$$f(q(B^x_{t\wedge T_\varepsilon})) - \int_0^{t\wedge T_\varepsilon} \Delta f(q(B^x_s))\mathrm{d}s \text{ is a martingale}$$

with respect to the filtration given by $\mathcal{G}_t = \sigma((B^x_s)_{s\in[0,t]})$. In particular, since it is adapted to the filtration given by $\mathcal{F}_t = \sigma((q(B^x_s))_{s\in[0,t]})$ and since $\mathcal{F}_t \subset \mathcal{G}_t$, it is also a martingale with respect to this filtration. Finally, since $q(B^x_t)$ does not touch $\mathcal{C}N$, we can see that $T_\varepsilon \uparrow \infty$ as $\varepsilon \downarrow 0$ which completes the proof.

□

In the rest of this section and for notational simplicity we denote by σ, instead of vol, the normalised volume measure on M, and by $\tilde{\sigma}$, the normalised volume measure on $N = M/G$. Let R be the law of the Brownian motion on M whose initial position has law σ and let $\tilde{R}$ be the law of the reflected Brownian motion on N whose initial position has law $\tilde{\sigma}$. We have the following result.

Lemma 7.3 (Image of the reversible Wiener measure)**.** *Denote by $q(R)$ the image measure of R by the map induced by q from $C([0,1],M)$ to $C([0,1],N)$. Then,*

$$q(R) = \tilde{R}.$$

Proof. By Lemma 7.2, $q(R)$ is the law of the Brownian motion on N whose initial position is distributed according to $q_*\sigma$, the image measure of σ by q. It is enough, then, to notice that $q_*\sigma = \tilde{\sigma}$. By Lemma 7.1, $\sigma = \sum_{g\in G}\sigma|_{gV}$, so that

$$q_*\sigma = q_*\left(\sum_{g\in G}\sigma|_{gV}\right) = \sum_{g\in G} q_*\left(\sigma|_{gV}\right).$$

We have that

$$\tilde{\sigma}\left(q(M\setminus \cup_{g\in G}\, gV)\right) = 0. \tag{7.2}$$

since the measure of ∂N is zero and, on the complement of $q^{-1}(\partial N)$, the map q is smooth so that the image of a set of measure zero has also measure zero. Since $q|_{gV}$ is an isometry onto its image we have that

$$q_*\left(\sigma|_{gV}\right) = \text{ volume measure on } N,$$

where we have used (7.2) which says that $\tilde{\sigma}\left(N\setminus q(gV)\right) = 0$. We obtain

$$q_*\sigma = \mathrm{card}(G)\,(\text{volume measure on } N)$$

which, after normalising, concludes the proof. □

We are ready to give the proof of Theorem 3.5.

Proof of Theorem 3.5. As in the previous lemma, we use the notation σ, instead of vol, for the normalised volume measure on M, and denote by $\tilde{\sigma}$, the normalised volume measure on $N = M/G$.

Let us prove the first assertion of Theorem 3.5. Let Q be a probability measure on $C([0,1], M)$ such that $Q_{01} = \pi$, $Q_t = \sigma$ for every $t \in [0,1]$ and $H(Q|R) < \infty$. We need to find a probability measure $\tilde{Q}$ on $C([0,1], N)$ such that $\tilde{Q}_{01} = (q \times q)_* \pi$, $\tilde{Q}_t = \tilde{\sigma}$ for every $t \in [0,1]$ and $H(\tilde{Q}|\tilde{R}) < \infty$. Notice that

$$H(q(Q)|q(R)) \leq H(Q|R) < \infty.$$

Since $q(Q)$ satisfies the marginal assumptions and since $q(R) = \tilde{R}$, the proof is completed by taking $\tilde{Q} = q(Q)$.

Now, to prove the second assertion we need to write $\tilde{\pi}$ as $q_*\pi$ for some nice π. For this, we shall use Lemma 7.1. Since $H(\tilde{\pi}|\tilde{R}_{01}) < \infty$ we have that $H(\tilde{\pi}|\tilde{\sigma} \otimes \tilde{\sigma}) < \infty$. In particular, $\tilde{\pi}$ gives measure zero to $N \times N \setminus q(U) \times q(U)$. For every $(g,h) \in G \times G$, consider the map

$$(q|_{gU} \times q|_{hU})^{-1} : q(U) \times q(U) \to gU \times hU$$

and consider the measure

$$\pi_{g,h} = (q|_{gU} \times q|_{hU})^{-1}_* \, \tilde{\pi}$$

which satisfies

$$(q \times q)_* \pi_{g,h} = \tilde{\pi},$$

Nevertheless, it does not satisfy the marginal conditions. Notice that, if

$$\alpha_{g,h} = (q|_{gU} \times q|_{hU})^{-1}_* \, (\tilde{\sigma} \times \tilde{\sigma}) \quad \text{and} \quad \sigma_g = (q|_{gU})^{-1}_* \, \tilde{\sigma}$$

then

$$\alpha_{g,h} = |G|^2 \, (\sigma \times \sigma) \, |_{gU \times hU} \quad \text{and} \quad \sigma_g = |G| \, \sigma|_{gU}.$$

Moreover, the first marginal of $\pi_{g,h}$ is σ_g and its second marginal is σ_h. Then, if we define

$$\pi = \frac{1}{|G|^2} \sum_{(g,h) \in G \times G} \pi_{g,h}$$

we may notice that the first and second marginals of π are σ and that

$$(q \times q)_* \pi = \tilde{\pi}.$$

We can also find its entropy by integrating and obtain that

$$H(\pi|\sigma \otimes \sigma) = H(\tilde{\pi}|\tilde{\sigma} \otimes \tilde{\sigma}).$$

Since $H(\pi|\sigma \otimes \sigma) < \infty$ if and only if $H(\pi|R_{01}) < \infty$ and $H(\tilde{\pi}|\tilde{\sigma} \otimes \tilde{\sigma}) < \infty$ if and only if $H(\tilde{\pi}|\tilde{R}_{01}) < \infty$ we may conclude.

□

We consider now some simple examples of quotient spaces where Theorem 3.5 holds. Almost all of these will be quotients of the flat two-dimensional torus which we define now. Let u and v be two linearly independent vectors of $\mathbb{R}^2$. We will denote by $\mathbb{T}_{u,v}$ the manifold

$$\mathbb{T}_{u,v} = \mathbb{R}^2 / \{au + bv : a, b \in \mathbb{Z}\}$$

endowed with the Riemannian metric induced by $\mathbb{R}^2$. We begin by describing two examples that are actual two-dimensional manifolds with boundary (without corners).

Example 7.4 (Cylinder)**.** *Suppose that u and v are orthogonal. The map*

$$\{\alpha u + \beta v : (\alpha, \beta) \in [0,1]^2\} \to \{\alpha u + \beta v : (\alpha, \beta) \in [0,1]^2\}$$
$$xu + yv \mapsto xu + (1-y)v$$

induces an isometry of $\mathbb{T}_{u,v}$ and the quotient space is isometric to the cylinder

$$\{z \in \mathbb{C} : 2\pi|z| = |u|\} \times [0, |v|/2].$$

Example 7.5 (Flat Möbius strip)**.** *Suppose that $|u| = |v|$. The map*

$$\{\alpha u + \beta v : (\alpha, \beta) \in [0,1]^2\} \to \{\alpha u + \beta v : (\alpha, \beta) \in [0,1]^2\}$$
$$xu + yv \mapsto yu + xv$$

induces an isometry of $\mathbb{T}_{u,v}$ and the quotient space is isometric to the flat Möbius strip

$$[0, \|u+v\|/2] \times [0, \|u-v\|/2] / \sim$$

where $\sim$ is the identification of the vertical sides in opposite directions. Figure 3 shows a representation of the torus and the considered isometry is the reflection along the dotted diagonal. Figure 4 shows the canonical representation of the flat Möbius strip as part of (four times) the representation of the torus.

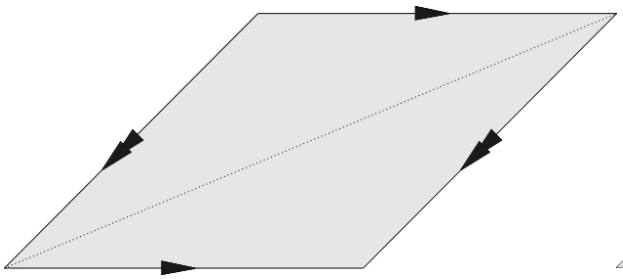

Figure 3: The Möbius strip as a quotient.

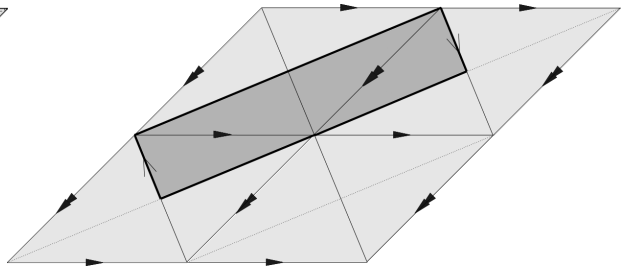

Figure 4: The Möbius strip.

The next four examples are two-dimensional manifolds with corners.

Example 7.6 (Rectangle)**.** *Suppose that u and v are orthogonal. The maps*

$$\{\alpha u + \beta v : (\alpha, \beta) \in [0,1]^2\} \to \{\alpha u + \beta v : (\alpha, \beta) \in [0,1]^2\}$$
$$xu + yv \mapsto xu + (1-y)v$$

and

$$\{\alpha u + \beta v : (\alpha, \beta) \in [0,1]^2\} \to \{\alpha u + \beta v : (\alpha, \beta) \in [0,1]^2\}$$
$$xu + yv \mapsto (1-x)u + yv$$

generate a reflection group of isometries of $\mathbb{T}_{u,v}$ and the quotient space is isometric to

$$[0, |u|/2] \times [0, |v|/2].$$

Example 7.7 (Isosceles right triangle)**.** *A $45°$ right triangle can be seen as a quotient of a square by a reflection along its diagonal. Using the previous example, we can also see it as a quotient of a torus (see Figure 6).*

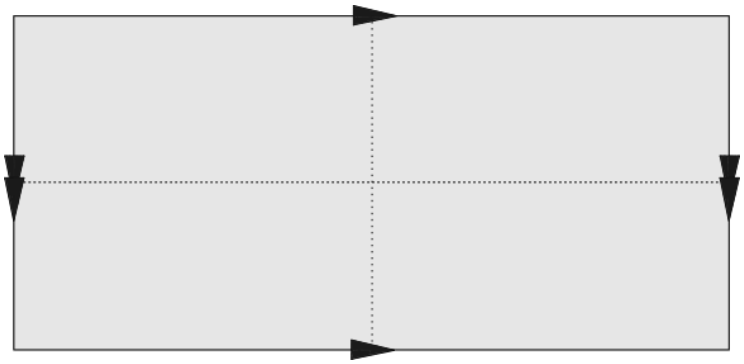

Figure 5: A rectangle as a quotient of the torus.

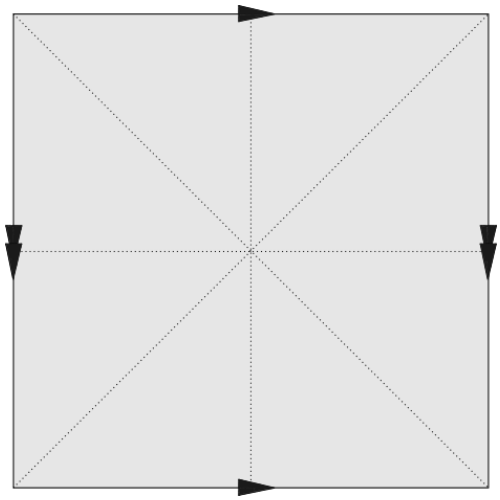

Figure 6: A 45° triangle rectangle as a quotient.

Example 7.8 (Equilateral triangle)**.** *If $2u \cdot v = \|u\|\|v\|$, the torus $\mathbb{T}_{u,v}$ can be seen as a quotient of an hexagon identifying opposite sides as in Figure 7. Then, if we consider the group generated by the reflections along the dotted lines in Figure 7 we can obtain an equilateral triangle as a quotient space.*

Example 7.9 (60° right triangle)**.** *A 60° right triangle can be seen as a quotient of the equilateral triangle by a reflection. Using the previous example we can see it also as a quotient of a torus.*

Finally, as n-dimensional cases we consider the following examples.

Example 7.10 (n-hyperrectangle)**.** *Let $a_1, \dots, a_n > 0$ and let $u_1, \dots, u_n$ be orthogonal vectors in $\mathbb{R}^n$ such that $\|u_i\| = a_i$ for any $i \in \{1, \dots, n\}$. We may consider the flat n-dimensional torus*

$$\mathbb{T}^n = \mathbb{R}^n / \{m_1 u_1 + \cdots + m_n u_n : m_1, \dots, m_n \in \mathbb{Z}\}$$

and the group generated by the reflections induced by the family (indexed by $i \in \{1, \dots, n\}$) of maps

$$\{\alpha_1 u_1 + \cdots + \alpha_n u_n : \alpha_i \in [0,1]\} \to \{\alpha_1 u_1 + \cdots + \alpha_n u_n : \alpha_i \in [0,1]\}$$
$$\sum_{k=1}^{n} x_k u_k \mapsto \sum_{k \neq i} x_k u_k + (1 - x_i) u_i.$$

The quotient of $\mathbb{T}^n$ by this group is a $n-$hyperrectangle with lengths $a_1/2, \dots, a_n/2$.

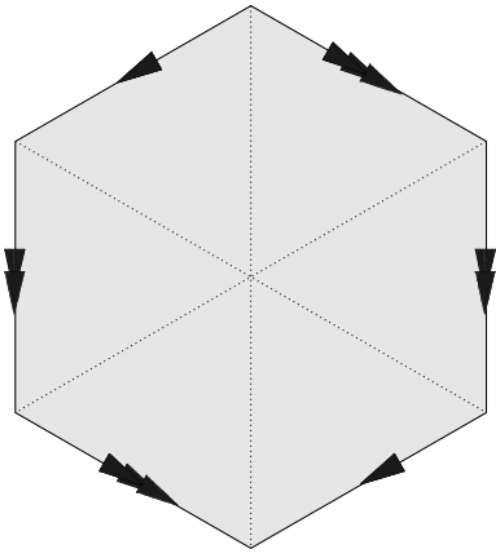

Figure 7: An equilateral triangle as a quotient of the torus.

Example 7.11 (Curved n-ball)**.** *Consider the n-dimensional sphere*

$$\mathbb{S}^n = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^n : |x_1|^2 + \cdots + |x_{n+1}|^2 = 1\}.$$

The quotient of $\mathbb{S}^n$ *by the map*

$$\begin{aligned} \mathbb{S}^n &\to \mathbb{S}^n \\ (x_1, \dots, x_n, x_{n+1}) &\mapsto (x_1, \dots, x_n, -x_{n+1}) \end{aligned}$$

is a curved n-ball.

8 Proof of the existence for the Gaussian case

We consider the following path measure

$$Q = \int_{M^3} R(\cdot | X_0 = x, X_{1/2} = z, X_1 = y)\, \pi(\mathrm{d}x\mathrm{d}y)\gamma_{1/4}(\mathrm{d}z),$$

where γ_{σ^2} denotes the density of $\mathcal{N}(0, \sigma^2\,\mathrm{id})$. This measure is the analogue of (6.1).

Proposition 8.1 (Constraints and entropy: Gaussian case)**.** *The measure Q satisfies the endpoints and marginal constraints* $Q_{01} = \pi$ *and* $Q_t = \mathcal{N}(0, 1/4\,\mathrm{id})$ *for every* $t \in [0, 1]$*. If* $H(\pi|R_{01}) < \infty$ *then* $H(Q|R) < \infty$*.*

Proof. The steps and arguments of the proof are the same as in Section 6. Firstly, as in the proof of Proposition 6.1, the endpoint condition $Q_{01} = \pi$ is obviously satisfied. Then, for $t \in [0, 1/2]$, we have

$$Q_t = \int_{M^2} R_t(\cdot | X_0 = x, X_{1/2} = z)\, \gamma_{1/4}(dx)\gamma_{1/4}(dz)$$

where $R_t(\cdot|X_0 = x, X_{1/2} = z)$ is the law, at time t of a Brownian bridge on $[0, 1/2]$ between x and z. It is a normal distribution $\mathcal{N}\left((1-2t)x + 2tz, t(1-2t)\right)$. So Q_t is a normal distribution and we have that

$$(1-2t)Y + 2tZ + \sqrt{t(1-2t)}W \sim Q_t,$$

where $Y,Z \sim \mathcal{N}(0, 1/4\,\mathrm{id})$ and $W \sim \mathcal{N}(0, \mathrm{id})$ are independent random variables. It follows that $Q_t = \mathcal{N}(0, 1/4\,\mathrm{id})$ for all $t \in [0, 1/2]$ and for all $t \in [0, 1]$ with the same argument.

It remain to verify the entropy condition. As in the symmetric space case, we have

$$H(Q|R) = H(\pi|R_{01}) + \int_M H(\gamma_{1/4}|R^{xy}_{1/2})\,\pi(dxdy).$$

Using the heat kernel in M, we have :

$$\frac{dR^{xy}_{1/2}}{d\gamma_{1/4}}(z) = e^{2\langle z, x+y\rangle - \frac{1}{2}|x-y|^2}.$$

And then, the entropy is

$$H(\gamma_{1/4}|R^{xy}_{1/2}) = \frac{1}{2}|x-y|^2$$

So we have that

$$\begin{aligned} H(Q|R) &\leq H(\pi|R_{01}) + \int_{M^2} \frac{1}{2}|x-y|^2\,\pi(dxdy) \\ &\leq H(\pi|R_{01}) + \int_{M^2} (x^2+y^2)\,\pi(dxdy) \\ &\leq H(\pi|R_{01}) + 2\int_M x^2\,\gamma_{1/4}(dx) \\ &\leq H(\pi|R_{01}) + \frac{n}{2} \end{aligned}$$

which completes the proof □

Proof of Theorem 3.6. It follows the proof of Theorem 3.1 but now using Proposition 8.1 instead of Proposition 6.1. □

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