An urn model with random multiple drawing and random addition

Irene Crimaldi, Pierre-Yves Louis, Ida Germana Minelli

URN MODELS WITH RANDOM MULTIPLE DRAWING AND RANDOM ADDITION

Abstract. We consider a two-color urn model with multiple drawing and random time-dependent addition matrix. The model is very general with respect to previous literature: the number of sampled balls at each time-step is *random*, the addition matrix is *not balanced* and it has general *random entries*. For the proportion of balls of a given color, we prove almost sure convergence results. In particular, in the case of equal reinforcement means, we prove fluctuation theorems (through CLTs in the sense of stable convergence and of almost sure conditional convergence, which are stronger than convergence in distribution) and we give asymptotic confidence intervals for the limit proportion, whose distribution is generally unknown.

Irene Crimaldi[1], Pierre-Yves Louis[23], Ida G. Minelli[4]

Keywords. Hypergeometric Urn; Multiple drawing urn; Pólya urn; Random process with reinforcement; Randomly reinforced urn; Central limit theorem; Stable convergence; Opinion dynamics; Epidemic models

MSC2010 Classification. Primary: 60B10; 60F05; 60F15; 60G42
Secondary: 62P25; 91D30 ; 92C60

Contents

[1]IMT School for Advanced Studies Lucca, Piazza San Ponziano 6, 55100 Lucca, Italy, irene.crimaldi@imtlucca.it

[2]PAM UMR 02.102, Université Bourgogne Franche-Comté, AgroSup Dijon, 1 esplanade Erasme, F-21000, Dijon, France, pierre-yves.louis@agrosupdijon.fr

[3]Institut de Mathématiques de Bourgogne, UMR 5584 CNRS, Université Bourgogne Franche-Comté, F-21000, Dijon, France, pierre-yves.louis@math.cnrs.fr

[4]Dipartimento di Ingegneria e Scienze dell'Informazione e Matematica, Università degli Studi dell'Aquila, Via Vetoio (Coppito 1), 67100 L'Aquila, Italy, idagermana.minelli@univaq.it

1. Introduction

Reinforcement (see [34] for a review) means the tendency of a stochastic evolution to increase (or sometimes decrease, so called, negative reinforcement) the occurrence of an event in relationship with the number of time this event took place in the past. The Pólya urn stochastic process is the fundamental and paradigmatic example. It led to several generalizations.

The original evolution rule of the Pólya urn is based on picking one ball in an urn filled with colored balls and replacing that ball in the urn together with one or more balls, according to some "updating matrix". More generalized samples have been considered, leading to *multi-drawing* based updating rules. In these models, many balls are selected at each time and returned before adding some new ones according to a reinforcement rule. Bi-color and multi-color models have been considered, as well as models where the extraction of the balls is with or without replacement. The number of sampled balls is always a fixed constant and the "replacement matrix" is in general assumed to be balanced, that is, the number of added balls to the urn is constant along time (*e.g.* [9,10,19,21,23,25,28,31]). In particular, in [20,28,31] the number of added balls is a deterministic function of the composition of the extracted sample. Results deal with the asymptotic behavior, evolution of moments, almost sure convergence and Central Limit Theorems (CLTs) for the fraction of balls of a given color in the urn. In the model considered in [29], m balls are sampled at a time, with replacement, and the distribution of the increment of one color follows, given the past, a binomial distribution with parameters m and p, where p depends on weights associated to the drawn colors. Results mainly deal with regimes where "fixation" happens, which is more interesting for reinforced random walks applications. Moreover, different urn models with multi-drawing were considered in relationship with some specific applications. See for instance [15,24,26,27].

Other urn models merge multi-drawing and random replacement matrix. The paper [2] is a generalization of [1] and it deals with a constant sample size and a random replacement matrix. This matrix can be of Pólya (diagonal) or Friedman (anti-diagonal, reinforcement of the non chosen color) type and its entries have time-homogenous distribution. In particular, we point out that CLTs are not proven for the Pólya type case. As we will see later on, we here fill in this gap.

The papers [3,12] study a multi-drawing model (called HRRU, hypergeometric randomly reinforced urn model) with a *random* number N_n of sampled balls and a random replacement matrix of rank 1 (bicolor case). The number of added balls of a given color is proportional to the number of balls of the same color in the sample, but the *random reinforcement* factor is the same for both colors. Note that this model generalizes the one recently given in [8]. The almost sure convergence of the color proportions toward a non degenerate random variable is proven. Necessary and sufficient conditions for no-atoms in the limiting distribution are given.

In this paper, we consider a two-color urn model, with *multiple drawing and random time-dependent addition matrix*. The model is very general with respect to previous literature: the number of sampled balls at each step is random, the addition matrix, defining the number of additional balls, has general random entries. More precisely, for both colors, the random number of added balls is proportional to the number of balls of the same color in the sample, with *possibly different random* coefficients A_n, B_n (which may be *correlated* and their distribution may *depend on time* n). The model studied in [3,12] corresponds to the particular case $A_n = B_n$. The reinforcement rule we consider is *not balanced* (thus the long-run behavior of the total number S_n of balls in the urn at time n needs to be studied). We prove almost sure convergence results for the proportion as well as fluctuation results, through central limit theorems in the sense of stable convergence and of almost sure conditional convergence, by suitably extending some approaches employed in the urn model literature without multi-drawing (see [4,5,32]). Specifically, we consider two cases. If the factors A_n and B_n have the same mean (equal reinforcement means case), the limit proportion Z is

random without atoms. In the case of unequal limit reinforcement means, the proportion converges almost surely to 1 (or 0). When the limit proportion Z is random, the proven central limit theorems are employed in order to obtain asymptotic confidence intervals.

Some applications of the urn models with multi-drawing are described in [26]. Moreover, like explained in [3,12], the present model may be applied in the context of *technology adoption* to model, for example, the evolution of the choice between different operative systems by companies. Below we illustrate other possible interpretations in the contexts of *opinion dynamics* and propagation of contagious diseases (*epidemic models*).

Applications to opinion dynamics could be developed as follows. Assume to be before an election between two candidates. People decide who they are going to vote for. People who have already decided are represented as the colored balls already in the urn, the color meaning the choice for one candidate. One assume this is a not evolving choice. At each iteration, a group (with random size N_n) of people is sampled (without replacement) and each one is given the opportunity to convince a group of other people. The new-comers will adopt the same choice as the person who convinced them. The heterogeneity of this reproduction mechanism is modeled through the time-dependent randomness of the factors A_n and B_n. The assumption of equal reinforcement means would mean that in the long-run no advantage is given to any party. We can also consider the evolution of the diffusion of a binary opinion through social networks, like *Twitter*. Each agent inside a connected community has an un-changing opinion (for instance, a vote or a purchased product). This community will grow dynamically through immigration of followers. At each step, a subset (with random size N_n) of agents is chosen. Each agent of this committee is allowed to call into the community of followers sharing their opinion. Once again, the heterogeneity of this growth mechanism is modeled by allowing the multiplying factors A_n and B_n for each opinion to be random. Correlation between these growth coefficients are possible. If one of these coefficients is eventually larger in mean, then the associated opinion will dominate eventually (but may take some time). If both coefficients are equal in mean then some random equilibrium takes place.

In the original paper [17], where the Pólya urn model was first defined, smallpox epidemy was the context it was applied to (see for instance [22, 30] and references therein). Therefore, a second application of our model one could have in mind is the diffusion of genetic variants of viruses (see for instance [33] for a review on epidemic models on networks). We do not pretend to do any modeling study here but want to illustrate the potentialities of our model as a "toy model". Assume one want to model the propagation of a virus, existing in two forms. Assume to consider a time scale such that there are infinitely many persons to be possibly contaminated and that once a person is contaminated, he/she remains contagious "for ever" (no recovering, no dying). Balls in the urn represent the contaminated persons by one of the two variants of the virus (corresponding to the two possible colors of the balls). We do have in mind the initial exponential regime of the propagation of two competing variants of one virus. Each discrete time-step of the urn's evolution means a contagion step. People that are contaminating are assimilated to the sample made without replacement in the urn. This is a random number N_n and this randomness may depend on time and on the total number of contaminated persons. One chosen contaminating person diffuse the same variant. Each variant has its own amplifying factor A_n (resp. B_n): one assume that each selected person, contaminated by a given variant, is contaminating the same number of people. This somewhat unrealistic hypothesis is compensated by the fact that the number of individuals infected by one person is random, with a time-dependent and variant-dependent randomness. Moreover, A_n and B_n could be correlated. This model gives insights: if the limit means (time-asymptotic reproduction means of each variant in this context) are unequal, one kind of virus will eventually dominate. If they are equal, there is a limiting genuinely random proportion, for which we provide confidence intervals.

Finally, another application context could be population dynamics in case of competitive or cooperative growth. As before, the flexibility of the model lies in the choice of N_n, A_n and B_n. The joint distribution of $[A_n, B_n]$ is important to model competition or cooperation. One may think to bacterial populations and the evolution of their respective proportions in the microbial gut.

The paper is organized as follows. In Section 2 we formally define the model. In Section 3 we state and prove the main results. In Subsection 3.1 we prove the almost sure convergence towards a limit proportion Z. Different behaviors occur according to equality/unequality of the limit reinforcement means. In particular, in the case of equal reinforcement means, we provide precise asymptotic rates: indeed, in Subsection 3.2 we establish central limit theorems for the proportion Z_n of the balls of a given color in the urn and for the empirical mean M_n of the proportion of the balls of a given color in the samples. Moreover, in the case of equal reinforcement means, in Subsection 3.3, we prove that the distribution of the limit proportion Z has no atoms and, in Subsection 3.4, we provide asymptotic confidence intervals for Z, centered in Z_n and M_n. We then present in Section 4 more specific examples, illustrated with some numerical simulations. The paper is enriched with an appendix in three parts which collects some more technical lemmas and general results, in particular about stable convergence and its variants.

2. The model

An urn contains $a \in \mathbb{N} \setminus \{0\}$ balls of color A and $b \in \mathbb{N} \setminus \{0\}$ balls of color B. At each discrete time $n \geq 1$, we simultaneously (*i.e.* without replacement) draw a random number N_n of balls. Let X_n be the number of extracted balls of color A. Then we return the extracted balls in the urn together with other $A_n X_n$ balls of color A and $B_n(N_n - X_n)$ balls of color B. More precisely, we take a probability space $(\Omega, \mathcal{A}, P)$ and, on it, some random variables N_n, X_n, A_n, B_n such that, for each $n \geq 1$, we have:

(A1) The conditional distribution of the random variable N_n given

$$[N_1, X_1, A_1, B_1 \ldots, N_{n-1}, X_{n-1}, A_{n-1}, B_{n-1}]$$

is concentrated on $\{1, \ldots, S_{n-1}\}$ where S_{n-1} is the total number of balls in the urn at time $n-1$, that is

$$S_{n-1} = a + b + \sum_{j=1}^{n-1} A_j X_j + \sum_{j=1}^{n-1} B_j (N_j - X_j). \tag{1}$$

(A2) The conditional distribution of the random variable X_n given

$$[N_1, X_1, A_1, , B_1 \ldots, N_{n-1}, X_{n-1}, A_{n-1}, B_{n-1}, N_n]$$

is hypergeometric with parameters N_n, S_{n-1} and H_{n-1}, where H_{n-1} is the total number of balls of color A at time $n-1$, that is

$$H_{n-1} = a + \sum_{j=1}^{n-1} A_j X_j. \tag{2}$$

(A3) The random vector $[A_n, B_n]$ takes values in $\mathbb{N} \setminus \{0\} \times \mathbb{N} \setminus \{0\}$ and it is independent of

$$[N_1, X_1, A_1, B_1, \ldots, N_{n-1}, X_{n-1}, A_{n-1}, B_{n-1}, N_n, X_n]\,.$$

According to the above notation, the random variable X_n corresponds to the number of balls having the color A in a random sample without replacement of size N_n from an urn with H_{n-1} balls of color A and $K_{n-1} = (S_{n-1} - H_{n-1})$ balls of color B. The reinforcement rule is of the "multiplicative" type: indeed, each time n, we add to the urn $A_n X_n$ balls of color A and $B_n(N_n - X_n)$ balls of color B. Therefore, the total number of added balls to the urn, that is $A_n X_n + B_n(N_n - X_n)$,

is random and depends on n.

Note that we do not specify the conditional distribution of the random variable N_n (the sample size) given the past

$$[N_1, X_1, A_1, B_1 \ldots, N_{n-1}, X_{n-1}, A_{n-1}, B_{n-1}]$$

nor the distribution of $[A_n, B_n]$ (the random reinforcement factors A_n and B_n may have different distributions, they may be correlated and their joint and marginal distributions may vary with n).

It is worthwhile to remark that this model include the Hypergeometric Randomly Reinforced Urn (HRRU) studied in [3, 12] (take $A_n = B_n$ for all n), which in turn include the model recently given in [8]. In particular, two special cases are the classical Pólya urn (the case with $N_n = 1$ and $A_n = B_n = k \in \mathbb{N} \setminus \{0\}$ for each n) and the 2-colors randomly reinforced urn with the reinforcements for the two colors equal or different in mean (the case with $N_n = 1$ for each n and $[A_n, B_n]$ arbitrarily random in $\mathbb{N}\setminus\{0\} \times \mathbb{N}\setminus\{0\}$). Moreover, as told in Section 1, previous literature (we refer to the quoted papers in Sec. 1) deals with the case when the sample size N_n is a fixed constant, not depending on n, and/or the balanced case (constant number of added balls to the urn each time).

We set Z_n equal to the proportion of balls of color A in the urn (immediately after the updating of the urn at time n and immediately before the $(n+1)$-th extraction), that is $Z_0 = a/(a+b)$ and

$$Z_n = \frac{H_n}{S_n} \quad \text{for } n \geq 1.$$

Moreover we set

$$\mathcal{F}_0 = \{\emptyset, \Omega\}, \quad \mathcal{F}_n = \sigma\big(N_1, X_1, A_1, B_1, \ldots, N_n, X_n, A_n, B_n\big) \quad \text{for } n \geq 1\,,$$

and

$$\mathcal{G}_n = \mathcal{F}_n \vee \sigma(N_{n+1}), \quad \mathcal{H}_n = \mathcal{G}_n \vee \sigma(A_{n+1}, B_{n+1}) \quad \text{for } n \geq 0.$$

By the above assumptions and notation, we have

$$E[A_{n+1} \,|\, \mathcal{G}_n] = E[A_{n+1}]\,, \qquad E[B_{n+1} \,|\, \mathcal{G}_n] = E[B_{n+1}] \tag{3}$$

and

$$\begin{aligned} E[X_{n+1} \,|\, \mathcal{H}_n] = E[X_{n+1} \,|\, \mathcal{G}_n] = N_{n+1} Z_n\,, \\ E[N_{n+1} - X_{n+1} \,|\, \mathcal{H}_n] = E[N_{n+1} - X_{n+1} \,|\, \mathcal{G}_n] = N_{n+1}(1 - Z_n)\,. \end{aligned} \tag{4}$$

Finally, we set $\mathcal{X}_n = \{0 \vee N_n - (S_{n-1} - H_{n-1}), \ldots, N_n \wedge H_{n-1}\}$ and, for each $k \in \mathcal{X}_n$,

$$p_{n,k} = p_k(N_n, S_{n-1}, H_{n-1}) = \frac{\binom{H_{n-1}}{k}\binom{S_{n-1}-H_{n-1}}{N_n-k}}{\binom{S_{n-1}}{N_n}}\,. \tag{5}$$

3. Asymptotic results

In this section we prove some convergence results for the model described in Section 2 by suitably extending some approaches employed in the urn model literature without multi-drawing (see [4, 5, 32]).

Set $E[A_n] = m_{A,n}$ and $E[B_n] = m_{B,n}$ for all n. We will assume that the two sequences $(m_{A,n})_n$ and $(m_{B,n})_n$ respectively converge to $m_A \in (0, +\infty)$ and $m_B \in (0, +\infty)$. Moreover, we will consider the following cases:

1) $m_A > m_B$.
2) $m_{A,n} = m_{B,n} = m_n$ and so $m_A = m_B = m \in (0, +\infty)$.

For simplicity, throughout the paper, we will assume

$$A_n \vee B_n \vee N_n \leq C \qquad \text{for some (integer) constant } C.$$

We will signal when this assumption can be easily removed. Sometimes it may be replaced by an assumption of uniformly integrability, but we will not focus on this fact.

We start with proving a result valid for both cases.

Lemma 3.1. *We have*

$$H_n \xrightarrow{a.s.} +\infty \qquad \textit{and} \qquad K_n = (S_n - H_n) \xrightarrow{a.s.} +\infty\,.$$

As a consequence, we obviously have $S_n \xrightarrow{a.s.} +\infty$.

Proof. First suppose $a \wedge b \geq C$ so that $N_i \leq H_{i-1}$ for each n. Let $T = \inf\{n : X_n \neq N_n\} = \inf\{n : (N_n - X_n) > 0\}$. For each $k \geq 1$, we have

$$t_k = P\{T > k\} = P(X_i = N_i\,, i = 1, \ldots, k) = E\left[\prod_{i=1}^{k} \frac{H_{i-1}}{S_{i-1}} \times \cdots \times \frac{H_{i-1} - (N_i - 1)}{S_{i-1} - (N_i - 1)}\right]$$

$$= E\left[\prod_{i=1}^{k} \prod_{j=0}^{N_i - 1} \frac{a - j + \sum_{h=1}^{i-1} A_h N_h}{a + b - j + \sum_{h=1}^{i-1} A_h N_h}\right].$$

We recall that, given $c_1, c_2, c_3 > 0$, we have

$$x \leq c_1 \Leftrightarrow \frac{c_2 + x}{c_2 + c_3 + x} \leq \frac{c_1 + c_2}{c_1 + c_2 + c_3}\,.$$

Therefore, applying the above inequality with $x = \sum_{h=1}^{i-1} A_h N_h \leq (i-1)C^2 = c_1$, $c_2 = a - j$, $c_3 = b$, we get

$$t_k \leq E\left[\prod_{i=1}^{k} \prod_{j=0}^{N_i - 1} \frac{a - j + (i-1)C^2}{a + b - j + (i-1)C^2}\right] \leq E\left[\prod_{i=1}^{k} \left(\frac{a + (i-1)C^2}{a + b - N_i + 1 + (i-1)C^2}\right)^{N_i}\right] \leq$$

$$\prod_{i=1}^{k} \frac{a + (i-1)C^2}{a + b - C + 1 + (i-1)C^2} = \exp\left(\sum_{i=1}^{k} \ln(1 - (b - C)/(a + b - C + 1 + (i-1)C^2))\right) \longrightarrow 0 \quad \text{as } k \to +\infty\,.$$

This fact means that $P(T = +\infty) = \lim_k t_k = 0$, *i.e.* $P(T < +\infty) = 1$. By the strong Markov's property, we can conclude that $P(N_n - X_n > 0\, i.o.) = 1$, *i.e.* $\sum_n (N_n - X_n) = +\infty$ almost surely. Since $K_n = S_n - H_n = b + \sum_{i=1}^{n} B_i(N_i - X_i) \geq \sum_{i=1}^{n}(N_i - X_i)$, we get $K_n = S_n - H_n \to +\infty$ almost surely. Similarly, we can obtain that $H_n \to +\infty$ almost surely.

In the general case, we have

$$t_k = P(T > k) = P(X_i = N_i\,, i = 1, \ldots, k)$$
$$= P(X_i = N_i\,, i = 1, \ldots, k \,|\, N_i \leq H_{i-1}\,, i = 1, \ldots, k) P(N_i \leq H_{i-1}\,, i = 1, \ldots, k)\,,$$

where $P(X_i = N_i\,, i = 1, \ldots, k \,|\, N_i \leq H_{i-1}\,, i = 1, \ldots, k)$ is equal to the product studied before and so it converges to 0. ∎

3.1. **Almost sure convergence.**

Theorem 3.2. *Assume to be in case 1) (i.e. $m_A > m_B$). Then $Z_n \xrightarrow{a.s.} Z = 1$.*

Proof. Let $e \in (m_B/m_A,\, 1)$ and set $Q_n = K_n/H_n^e$ for all n. Then, using that $(1-x)^e \le 1 - ex$ for $0 \le x \le 1$, $H_n \le H_{n+1} \le H_n + C^2$ and (4), we have:

$$\begin{aligned}
E[Q_{n+1}/Q_n - 1 \,|\, \mathcal{H}_n] &= E\left[\frac{K_n + B_{n+1}(N_{n+1} - X_{n+1})}{K_n}\left(\frac{H_n}{H_{n+1}}\right)^e \,|\, \mathcal{H}_n\right] - 1 \\
&= E\left[\left(\frac{H_n}{H_{n+1}}\right)^e - 1 \,|\, \mathcal{H}_n\right] + E\left[\frac{B_{n+1}(N_{n+1} - X_{n+1})}{K_n}\left(\frac{H_n}{H_{n+1}}\right)^e \,|\, \mathcal{H}_n\right] \\
&\le -eE\left[\frac{A_{n+1}X_{n+1}}{H_{n+1}} \,|\, \mathcal{H}_n\right] + E\left[\frac{B_{n+1}(N_{n+1} - X_{n+1})}{K_n} \,|\, \mathcal{H}_n\right] \\
&\le -eE\left[\frac{A_{n+1}X_{n+1}}{H_n + C^2} \,|\, \mathcal{H}_n\right] + E\left[\frac{B_{n+1}(N_{n+1} - X_{n+1})}{K_n} \,|\, \mathcal{H}_n\right] \\
&= -e\frac{A_{n+1}N_{n+1}}{S_n}\frac{H_n}{H_n + C^2} + \frac{B_{n+1}N_{n+1}}{S_n}\,.
\end{aligned}$$

Taking the conditional expectation with respect to $\mathcal{G}_n$ and using (3), we get

$$E[Q_{n+1}/Q_n - 1 \,|\, \mathcal{G}_n] \le \frac{N_{n+1}}{S_n}\left(m_{B,n+1} - e\, m_{A,n+1}\frac{H_n}{H_n + C^2}\right).$$

Since H_n goes to $+\infty$ (see Lemma 3.1), $\lim_n m_{A,n+1} = m_A > m_B = \lim_n m_{B,n+1}$ and $e \in (m_B/m_A,\, 1)$, we obtain that the above conditional expectation is smaller or equal than zero for n large enough. It follows that, for large n, we have

$$E[Q_{n+1} - Q_n \,|\, \mathcal{G}_n] = Q_n E[Q_{n+1}/Q_n - 1 \,|\, \mathcal{G}_n] \le 0$$

This means that $(Q_n)_n$ is eventually a positive (*i.e.* non-negative) $\mathcal{G}$-supermartingale and so it converges almost surely to a finite random variable. In order to conclude, it is enough to observe that, since $H_n \le S_n$, $S_n \xrightarrow{a.s.} +\infty$ and $e < 1$, we have

$$1 - Z_n = \frac{K_n}{S_n} = Q_n\frac{H_n^e}{S_n} \le Q_n S_n^{-(1-e)} \xrightarrow{a.s.} 0\,,$$

that is $Z_n \xrightarrow{a.s.} 1$. ∎

Theorem 3.3. *Assume to be in case 2). Then, we have*

$$|\, E[Z_{n+1}|\mathcal{G}_n] - Z_n \,| \le E[(A_{n+1} + B_{n+1})^2]\frac{N_{n+1}^2}{n^2} \tag{6}$$

and so the process (Z_n) is a $\mathcal{G}$-quasi-martingale and it almost surely converges to a random variable Z taking values in $[0, 1]$.

It is easy to see that, in order that (Z_n) is $\mathcal{G}$-quasi-martingale, it is enough to require the condition

$$\sum_n E[(A_{n+1} + B_{n+1})^2]\frac{E[N_{n+1}^2]}{n^2} < +\infty\,, \tag{7}$$

which is obviously satisfied when $A_n \vee B_n \vee N_n \le C$ for some constant C. Moreover, as we will see, for the proof of the above lemma it is sufficient to assume only $m_{A,n} = m_{B,n} = m_n$ for all n (it is not necessary to have (m_n) convergent).

Proof. After some computations, we get

$$Z_{n+1} - Z_n = \frac{(1 - Z_n)A_{n+1}X_{n+1} - Z_n B_{n+1}(N_{n+1} - X_{n+1})}{S_{n+1}}\,. \tag{8}$$

Therefore, by the model assumptions, the conditional expectation $E[Z_{n+1} - Z_n|\mathcal{H}_n]$ is equal to

$$\sum_{k\in\mathcal{X}_{n+1}} [(1 - Z_n)\frac{A_{n+1}k}{S_n + A_{n+1}k + B_{n+1}(N_{n+1} - k)} - Z_n \frac{B_{n+1}(N_{n+1} - k)}{S_n + A_{n+1}k + B_{n+1}(N_{n+1} - k)}]p_{n+1,k}\,,$$

where $\mathcal{X}_{n+1} = \{0 \vee N_{n+1} - (S_n - H_n), \ldots, N_{n+1} \wedge H_n\}$ and $p_{n+1,k} = p_k(N_{n+1}, S_n, H_n)$ is given by (5). We observe that $\mathcal{X}_{n+1}$ and $p_{n+1,k}$ are $\mathcal{G}_n$-measurable and so the conditional expectation $E[Z_{n+1} - Z_n|\mathcal{G}_n]$ is equal to

$$\sum_{k\in\mathcal{X}_{n+1}} \left\{(1 - Z_n)E\left[\frac{A_{n+1}k}{S_n + A_{n+1}k + B_{n+1}(N_{n+1} - k)}|\mathcal{G}_n\right] - Z_n E\left[\frac{B_{n+1}(N_{n+1} - k)}{S_n + A_{n+1}k + B_{n+1}(N_{n+1} - k)}|\mathcal{G}_n\right]\right\} p_{n+1,k}\,.$$

Now, we consider the above quantity and we add and subtract the quantity $A_{n+1}k/S_n$ in the first conditional expectation and the quantity $B_{n+1}(N_{n+1}-k)/S_n$ in the second conditional expectation, so that the two conditional expectations can be rewritten respectively as

$$E\left[\frac{-A_{n+1}^2k^2 - A_{n+1}B_{n+1}k(N_{n+1} - k)}{S_n[S_n + A_{n+1}k + B_{n+1}(N_{n+1} - k)]}|\mathcal{G}_n\right] + \frac{m_n k}{S_n}$$
$$E\left[\frac{-B_{n+1}^2(N_{n+1} - k)^2 - A_{n+1}B_{n+1}k(N_{n+1} - k)}{S_n[S_n + A_{n+1}k + B_{n+1}(N_{n+1} - k)]}|\mathcal{G}_n\right] + \frac{m_n(N_{n+1} - k)}{S_n}\,,$$

where we have used (3) and the fact that $m_{A,n} = m_{B,n} = m_n$. Finally, we observe that

$$\sum_{k\in\mathcal{X}_{n+1}} \frac{(1 - Z_n)m_n k - Z_n m_n(N_{n+1} - k)}{S_n} p_{n+1,k} = \frac{m_n}{S_n} \sum_{k\in\mathcal{X}_{n+1}} (k - N_{n+1}Z_n)p_{n+1,k} = 0,$$

because $\sum_{k\in\mathcal{X}_{n+1}} kp_{n+1,k}$ is the mean value of the hypergeometric distribution with parameters N_{n+1}, S_n, H_n and so it is equal to $N_{n+1}H_n/S_n = N_{n+1}Z_n$. Summing up, the conditional expectation $E[Z_{n+1} - Z_n|\mathcal{G}_n]$ is equal to

$$\sum_{k\in\mathcal{X}_{n+1}} E\left[\frac{Z_n B_{n+1}^2(N_{n+1} - k)^2 - (1 - Z_n)A_{n+1}^2k^2 + (2Z_n - 1)A_{n+1}B_{n+1}k(N_{n+1} - k)}{S_n[S_n + A_{n+1}k + B_{n+1}(N_{n+1} - k)]}|\mathcal{G}_n\right] p_{n+1,k}\,.$$

Therefore, using assumption (A3), we have

$$|\,E[Z_{n+1}|\mathcal{G}_n] - Z_n\,| \le E\left[\frac{(A_{n+1} + B_{n+1})^2 N_{n+1}^2}{S_n^2}|\mathcal{G}_n\right] = E[(A_{n+1} + B_{n+1})^2]\frac{N_{n+1}^2}{S_n^2}$$

and, since $A_n \wedge B_n \wedge N_n \ge 1$ by definition, we finally get (6). When condition (7) is satisfied (as when $A_n \vee B_n \vee N_n \le C$ for some constant C), the process (Z_n) is a $\mathcal{G}$-martingale taking values in $[0, 1]$ and, hence, it almost surely converges to some random variable Z taking values in $[0, 1]$. ∎

Remark 3.4. From (8), we immediately get that, if $A_n = B_n$ for all n, then

$$Z_{n+1} - Z_n = \frac{A_{n+1}(X_{n+1} - Z_n N_{n+1})}{S_n + A_{n+1}N_{n+1}}$$

and so (Z_n) is an $\mathcal{H}$-martingale, because of assumptions (A1) and (A2). Therefore, for its almost sure convergence, it is not necessary condition (7). This is the case considered in [3, 12].

Remark 3.5. Lemma B.1 (with $Y_n = X_n/N_n$) immediately implies that, in both cases 1) and 2), the sequence

$$M_n = \frac{1}{n}\sum_{j=1}^{n}\frac{X_j}{N_j}, \tag{9}$$

which is the empirical mean of the proportion, in the samples, of balls of color A, also converges almost surely to Z.

Proposition 3.6. *Assume to be in one of the previous two cases 1) and 2) and let* $Z \overset{a.s.}{=} \lim_n Z_n$. *Moreover, assume*

$$E[N_n|\mathcal{F}_{n-1}] \xrightarrow{a.s.} N\,, \tag{10}$$

where N *is a (strictly positive finite) random variable.*

Then

$$\frac{H_n}{n} \xrightarrow{a.s.} m_A N Z\,, \qquad \frac{K_n}{n} = \frac{S_n - H_n}{n} \xrightarrow{a.s.} m_B N(1-Z).$$

and so

$$\frac{S_n}{n} \xrightarrow{a.s.} m_A N Z + m_B N(1-Z).$$

Proof. It is enough to apply Lemma B.1 with $Y_j = A_j X_j$ (resp. $Y_j = B_j(N_j - X_j)$. Indeed, we have $Y_j \le A_j N_j$ (resp. $Y_j \le B_j N_j$) for each j and so $E[Y_j^2] \le E[(A_j + B_j)^2]E[N_j^2]$. Moreover

$$\begin{aligned} E[A_j X_j|\mathcal{F}_{j-1}] &= E\left[E[\,E[A_j X_j|\mathcal{H}_{j-1}]\,|\mathcal{G}_{j-1}]\,|\mathcal{F}_{j-1}\right] = E\left[\,E[A_j N_j Z_{j-1}|\mathcal{G}_{j-1}]|\mathcal{F}_{j-1}\right] \\ &= E[m_{A,j} N_j Z_{j-1}|\mathcal{F}_{j-1}] = m_{A,j} E[N_j|\mathcal{F}_{j-1}] Z_{j-1} \xrightarrow{a.s.} m_A N Z \end{aligned}$$

and

$$\begin{aligned} E[B_j(N_j - X_j)|\mathcal{F}_{j-1}] &= E\left[E[\,E[B_j(N_j - X_j)|\mathcal{H}_{j-1}]\,|\mathcal{G}_{j-1}]\,|\mathcal{F}_{j-1}\right] \\ &= E\left[\,E[B_j N_j(1 - Z_{j-1})|\mathcal{G}_{j-1}]|\mathcal{F}_{j-1}\right] \\ &= E[m_{B,j} N_j(1 - Z_{j-1})|\mathcal{F}_{j-1}] \\ &= m_{B,j} E[N_j|\mathcal{F}_{j-1}](1 - Z_{j-1}) \xrightarrow{a.s.} m_B N(1-Z)\,. \end{aligned}$$

Therefore, we have $H_n/n \xrightarrow{a.s.} m_A N Z$ and $K_n/n \xrightarrow{a.s.} m_B N(1-Z)$ and so $S_n/n = H_n/n + K_n/n \xrightarrow{a.s.} m_A N Z + m_B N(1-Z)$. ∎

Remark 3.7. When we are in case 1), then $Z = 1$ almost surely and so we have H_n and S_n go to $+\infty$ with rate n. Moreover, we observe that, for each $e \in (m_B/m_A,\, 1)$, we have

$$n^{1-e}(1 - Z_n) = n^{1-e}\frac{K_n}{S_n} = \left(\frac{n}{S_n}\right)^{1-e}\left(\frac{H_n}{S_n}\right)^{e} Q_n\,,$$

where Q_n is defined as in the proof of Theorem 3.2. Since n/S_n, H_n/S_n and Q_n converge almost surely to suitable finite random variables, we get that $n^{1-e}(1 - Z_n)$ converges almost surely to a finite random variable. Since e is arbitrary, we necessarily have $n^{1-e}(1 - Z_n) \xrightarrow{a.s.} 0$, that is, for all $e \in (m_B/m_A,\, 1)$, we have $1 - Z_n \overset{a.s.}{=} o(n^{-(1-e)})$ and so $K_n = S_n(1 - Z_n) = o(n^e)$.

When we are in case 2), since $mN > 0$ almost surely, the above limit result implies that S_n goes to $+\infty$ with rate n; while it is not sufficient in order to get some information on the asymptotic behavior of H_n and K_n, because Z may assume the value 0 or 1. In the sequel, we will prove that both H_n and K_n go to $+\infty$ at rate n.

Theorem 3.8. *Assume to be in case 2) and assume condition* (10). *Then we have* $P(Z = 0) + P(Z = 1) = 0$. *(Consequently the rate at which* H_n *and* K_n *go to* $+\infty$ *is equal to* n.*)*

Proof. Set $Y_n = \ln(H_n/K_n)$, $\Delta_n = E[Y_{n+1} - Y_n|\mathcal{G}_n]$ and $Q_n = E[(Y_{n+1} - Y_n)^2]$. If we prove $\sum_n \Delta_n < +\infty$ and $\sum_n Q_n < +\infty$ almost surely, then Y_n converges almost surely to a finite random variable (see Lemma 3.2 in [35]). This fact implies that H_n/K_n converges to a random variable Y with values in $(0, +\infty)$. It follows that $Z_n = \frac{H_n}{S_n} = \frac{H_n/K_n}{H_n/K_n+1}$ converges almost surely to $Y/(Y+1)$, which is a random variable with values in $(0,1)$. Then $P(Z=0) + P(Z=1) = 0$.

The rest of the proof is devoted to verify that $\sum_n \Delta_n < +\infty$ and $\sum_n Q_n < +\infty$ almost surely. To this regard, we recall that, by Lemma A.3, we have $1/K_n = O(1/n^\gamma)$ and $1/H_n = O(1/n^\gamma)$ with $\gamma > 0$. Moreover, using the notation (5), we have

$$E[\ln(H_{n+1}) - \ln(H_n)|\mathcal{H}_n] - E[\ln(K_{n+1}) - \ln(K_n)|\mathcal{H}_n] =$$
$$\sum_{k\in\mathcal{X}_{n+1}} \{(\ln(H_n + A_{n+1}k) - \ln(H_n)) - (\ln(K_n + B_{n+1}(N_{n+1}-k)) - \ln(K_n))\}\, p_{n+1,k} =$$
$$\sum_{k\in\mathcal{X}_{n+1}} \left\{ \int_0^{A_{n+1}k} \frac{1}{H_n + t}\, dt - \int_0^{B_{n+1}(N_{n+1}-k)} \frac{1}{K_n + t}\, dt \right\} p_{n+1,k}$$

Since $1/(H_n + t) \le 1/H_n$ and $1/(K_n + t) \ge 1/K_n - t/K_n^2$ for each $t \ge 0$ and each n, the last term of the above equalities is eventually smaller or equal than

$$\sum_{k\in\mathcal{X}_{n+1}} \left\{ \frac{A_{n+1}k}{H_n} - \frac{B_{n+1}(N_{n+1}-k)}{K_n} + c\frac{B_{n+1}^2(N_{n+1}-k)^2}{2K_n^2} \right\} p_{n+1,k}\,.$$

Now, we observe that

$$E[\sum_{k\in\mathcal{X}_{n+1}} \left(\frac{A_{n+1}k}{H_n} - \frac{B_{n+1}(N_{n+1}-k)}{K_n} \right) p_{n+1,k}\,|\,\mathcal{G}_n] = \frac{m_{n+1}N_{n+1}}{S_n} - \frac{m_{n+1}N_{n+1}}{S_n} = 0\,.$$

Therefore, we have for n large enough (using $(1 - Z_n) = K_n/S_n$)

$$\Delta_n \le \frac{cC^2}{2K_n^2} \left\{ Z_n(1-Z_n)N_{n+1}\frac{S_n - N_{n+1}}{S_n - 1} + N_{n+1}^2(1-Z_n)^2 \right\} = O(1/(K_nS_n)) = O(1/n^{1+\gamma})\,.$$

Similarly, we have

$$E[(\ln(H_{n+1}) - \ln(H_n) - \ln(K_{n+1}) + \ln(K_n))^2|\mathcal{H}_n] \le$$
$$2\left\{E[(\ln(H_{n+1}) - \ln(H_n))^2|\mathcal{H}_n] + E[(\ln(K_{n+1}) - \ln(K_n))^2|\mathcal{H}_n]\right\} \le$$
$$2\sum_{k\in\mathcal{X}_{n+1}} \left(\frac{A_{n+1}^2k^2}{H_n^2} + \frac{B_{n+1}^2(N_{n+1}-k)^2}{K_n^2} \right) p_{n+1,k} = O(1/(H_nS_n)) + O(1/K_nS_n) = O(1/n^{1+\gamma})\,.$$

The last statement (into the brackets) immediately follows from Proposition 3.6. ∎

3.2. **Central limit theorems for the case of equal reinforcement means.** Since in case 2), the limit proportion is a random variable Z, in the sequel we provide results in order to get some information on it.

Theorem 3.9. *Assume to be in case 2) and assume condition* (10)*. Moreover, suppose to have*

$$E[N_n^2|\mathcal{F}_{n-1}] \xrightarrow{a.s.} Q\,, \tag{11}$$

where Q is a (strictly positive finite) random variable, and

$$q_{A,n} = E[A_n^2] \to q_A\,, \quad q_{B,n} = E[B_n^2] \to q_B\,, \quad q_{AB,n} = E[A_nB_n] \to q_{AB}\,, \tag{12}$$

where q_A, q_B and q_{AB} are (strictly positive finite) constants.

Then $\sqrt{n}(Z_n - Z)$ converges in the sense of the almost sure conditional convergence with respect to $\mathcal{F} = (\mathcal{F}_n)$ to the Gaussian kernel $\mathcal{N}(0, V)$, where

$$\begin{aligned} V &= Z(1-Z)\frac{(1-Z)q_A[(1-Z)N + ZQ] + Zq_B[ZN + (1-Z)Q] - 2Z(1-Z)q_{AB}(Q-N)}{(mN)^2} \\ &= Z(1-Z)\frac{N[(1-Z)^2 q_A + Z^2 q_B + 2Z(1-Z)q_{AB}] + Z(1-Z)Q[q_A + q_B - 2q_{AB}]}{(mN)^2}\,. \end{aligned} \tag{13}$$

Before the proof, we premise some remarks.

Remark 3.10. Regarding formula (13), recall that $N \geq 1$ a.s., $Q \geq 1$ a.s., $q_A \geq 1\, q_B \geq 1$, $q_{AB} \geq 1$ and $q_A + q_B - 2q_{AB} = \lim_n E[(A_n - B_n)^2] \geq 0$. Moreover, we have proven that $P(Z = 0) = P(Z = 1) = 0$ (see Theorem 3.8). Therefore, we have $P(V > 0) = 1$. In addition, we note that V is not degenerate provided $P(Z = z) < 1$ for all $z \in (0, 1)$. For this last fact, we refer to the next Theorem 3.15, which states that we also have $P(Z = z) = 0$ for all $z \in (0, 1)$.

Remark 3.11. When $A_n = B_n$ for all n, we have $q_A = q_B = q_{AB} = q$ and so we get $V = Z(1-Z)q/(m^2 N)$, that does not depend on Q. Indeed, in this case the above assumption (11) can be deleted (see [12]).

Remark 3.12. When $N_n = k$ for each n, with k a fixed constant, we have

$$\begin{aligned} V &= kZ(1-Z)\frac{(1-Z)^2 q_A + Z^2 q_B + 2Z(1-Z)q_{AB} + Z(1-Z)k(q_A + q_B - 2q_{AB})}{(mk)^2} \\ &= Z(1-Z)\frac{(1-Z)^2 q_A + Z^2 q_B + Z(1-Z)[k(q_A + q_B) - 2q_{AB}(k-1)]}{m^2 k}\,. \end{aligned} \tag{14}$$

In particular, for $k = 1$, we observe that V does not depend on q_{AB}.

Proof. Setting $X'_n = X_n/N_n$ for each n, the sequence (X'_n) is $\mathcal{G}$-adapted and bounded. Moreover, we have

$$E[X'_{n+1}|\mathcal{G}_n] = E[N_{n+1}^{-1} X_{n+1}|\mathcal{G}_n] = N_{n+1}^{-1} E[X_{n+1}|\mathcal{G}_n] = N_{n+1}^{-1} N_{n+1} Z_n = Z_n\,. \tag{15}$$

We want to apply Theorem C.2 to $Y_n = X'_n$. By Theorem 3.3, we have

$$n^3 E\left[\,(E[Z_{n+1}|\mathcal{G}_n] - Z_n)^2\,\right] \longrightarrow 0.$$

Therefore, in order to prove Theorem 3.9, it suffices to prove that the following conditions are satisfied

c1) $E[\sup_{j\geq 1} \sqrt{j}|Z_{j-1} - Z_j|\,] < +\infty$;
c2) $n\sum_{j\geq n}(Z_{j-1} - Z_j)^2 \xrightarrow{a.s.} V$.

In the following we verify the above conditions.
Condition c1). We observe that, by (8) and recalling that $A_j \wedge B_j \wedge N_j \geq 1$ and $A_j \vee B_j \vee N_j \leq C$, we have

$$|Z_{j-1} - Z_j| \leq \frac{(A_j + B_j)N_j}{j} \leq \frac{2C^2}{j}\,. \tag{16}$$

Therefore condition c1) is obviously verified.
Condition c2). We want to apply Lemma B.1 with $Y_j = j^2(Z_{j-1} - Z_j)^2$. By the assumptions and inequality (16), we have $\sum_{j\geq 1} j^{-2} E[Y_j^2] < +\infty$. Moreover, by equality (8), we have

$$(Z_{j-1} - Z_j)^2 = \frac{(1-Z_{j-1})^2 A_j^2 N_j^2 (X'_j)^2}{S_j^2} + \frac{Z_{j-1}^2 B_j^2 N_j^2 (1-X'_j)^2}{S_j^2} - 2\frac{Z_{j-1}(1-Z_{j-1})A_j B_j N_j^2 X'_j(1-X'_j)}{S_j^2}\,.$$

Therefore, we study the convergence of the following three terms:

- $T_{1,j-1} = j^2 E\left[\frac{(1-Z_{j-1})^2 A_j^2 N_j^2 (X'_j)^2}{S_j^2}|\mathcal{F}_{j-1}\right]$,
- $T_{2,j-1} = j^2 E\left[\frac{Z_{j-1}^2 B_j^2 N_j^2 (1-X'_j)^2}{S_j^2}|\mathcal{F}_{j-1}\right]$,
- $T_{3,j-1} = j^2 E\left[\frac{Z_{j-1}(1-Z_{j-1}) A_j B_j N_j^2 X'_j (1-X'_j)}{S_j^2}|\mathcal{F}_{j-1}\right]$.

Consider the first term $T_{1,j-1}$. By assumption (A3), we get the two inequalities:

$$T_{1,j-1} \geq \frac{j^2}{(S_{j-1}+C^2)^2}(1-Z_{j-1})^2 E[A_j^2]E[N_j^2(X'_j)^2|\mathcal{F}_{j-1}]$$
$$T_{1,j-1} \leq \frac{j^2}{S_{j-1}^2}(1-Z_{j-1})^2 E[A_j^2]E[N_j^2(X'_j)^2|\mathcal{F}_{j-1}].$$

Since $S_n/n \xrightarrow{a.s.} Nm > 0$, $Z_{j-1} \xrightarrow{a.s.} Z$ and $E[A_j^2] \to q_A$, it is enough to verify the almost sure convergence of $E[N_j^2(X'_j)^2|\mathcal{F}_{j-1}]$. To this purpose, we observe that we can write

$$E[N_j^2(X'_j)^2|\mathcal{F}_{j-1}] = E\left[N_j^2 E[(X'_j)^2|\mathcal{G}_{j-1}]\,|\,\mathcal{F}_{j-1}\right]$$

and, by (A2), the conditional expectation $E[(X'_j)^2|\mathcal{G}_{j-1}]$ coincides with

$$\begin{aligned} N_j^{-2}E[X_j^2|\mathcal{G}_{j-1}] &= N_j^{-2}\left[Z_{j-1}(1-Z_{j-1})(S_{j-1}-1)^{-1}N_j(S_{j-1}-N_j) + Z_{j-1}^2 N_j^2\right] \\ &= Z_{j-1}(1-Z_{j-1})(S_{j-1}-1)^{-1}N_j^{-1}\left(S_{j-1}-N_j\right) + Z_{j-1}^2. \end{aligned}$$

Therefore we obtain

$$E[N_j^2(X'_j)^2|\mathcal{F}_{j-1}] = Z_{j-1}(1-Z_{j-1})(S_{j-1}-1)^{-1}\left(S_{j-1}E[N_j|\mathcal{F}_{j-1}] - E[N_j^2|\mathcal{F}_{j-1}]\right) + Z_{j-1}^2 E[N_j^2|\mathcal{F}_{j-1}],$$

which converges almost surely to $Z(1-Z)N + Z^2Q$ (since $E[N_j^2|\mathcal{F}_{j-1}]$ is bounded by C^2 and $S_{j-1} \xrightarrow{a.s.} +\infty$). Hence $T_{1,j-1}$ converges almost surely to $T_1 = Z(1-Z)^2 q_A (mN)^{-2}[(1-Z)N + ZQ]$. Similarly, we get

$$\begin{aligned} E[N_j^2(1-X'_j)^2|\mathcal{F}_{j-1}] &= E[N_j^2|\mathcal{F}_{j-1}] + E[N_j^2(X'_j)^2|\mathcal{F}_{j-1}] - 2E[N_j^2 X'_j|\mathcal{F}_{j-1}] \\ &= E[N_j^2|\mathcal{F}_{j-1}] + E[N_j^2(X'_j)^2|\mathcal{F}_{j-1}] - 2Z_j E[N_j^2|\mathcal{F}_{j-1}] \\ &\longrightarrow Q + Z(1-Z)N + Z^2Q - 2ZQ = Z(1-Z)N + (1-Z)^2 Q. \end{aligned}$$

and so $T_{2,j-1}$ converges almost surely to $T_2 = Z^2(1-Z)q_B(mN)^{-2}[ZN + (1-Z)Q]$. Finally, we have

$$\begin{aligned} E[N_j^2 X'_j(1-X'_j)|\mathcal{F}_{j-1}] &= E[N_j^2 X'_j|\mathcal{F}_{j-1}] - E[N_j^2(X'_j)^2|\mathcal{F}_{j-1}] \\ &= Z_{j-1}E[N_j^2|\mathcal{F}_{j-1}] - E[N_j^2(X'_j)^2|\mathcal{F}_{j-1}] \\ &\longrightarrow ZQ - Z(1-Z)N - Z^2Q = Z(1-Z)(Q-N). \end{aligned}$$

and so $T_{3,j-1}$ converges almost surely to $T_3 = Z^2(1-Z)^2 q_{AB}(mN)^{-2}(Q-N)$. By Lemma B.1, condition c2) is satisfied with $V = T_1 + T_2 - 2T_3$. The proof is so concluded. ∎

Theorem 3.13. *Under the assumptions of Theorem 3.9, suppose also that*

$$E[N_n^{-1}|\mathcal{F}_{n-1}] \xrightarrow{a.s.} L\,, \tag{17}$$

where L is a (positive bounded) random variable.

Then

$$[\sqrt{n}(M_n - Z_n), \sqrt{n}(Z_n - Z)] \xrightarrow{stably} \mathcal{N}(0,U) \otimes \mathcal{N}(0,V),$$

where M_n is defined in (9)*, V is defined in* (13) *and $U = V + Z(1-Z)[L - 2N^{-1}]$.*

In particular, we have that $\sqrt{n}(M_n - Z_n)$ converges stably to $\mathcal{N}(0, U)$ and $\sqrt{n}(M_n - Z)$ converges stably to $\mathcal{N}(0, U+V)$, with $U+V>0$ almost surely (see Remark 3.10).

Remark 3.14. Regarding the limit random variance U, we note that, by Jensen inequality, we have $(E[N_n|\mathcal{F}_{n-1}])^2 \leq E[N_n^2|\mathcal{F}_{n-1}]$ and $E[N_n|\mathcal{F}_{n-1}]^{-1} \leq E[N_n^{-1}|\mathcal{F}_{n-1}]$ and so we have $N^2 \leq Q$ and $1/N \leq L$. Therefore, we get

$$V \geq Z(1-Z)\frac{N[(1-Z)^2 q_A + Z^2 q_B + 2Z(1-Z)q_{AB}] + Z(1-Z)N^2[q_A + q_B - 2q_{AB}]}{(mN)^2} \quad \text{and}$$

$$L - \frac{2}{N} \geq -\frac{1}{N}\,.$$

Moreover, since $N_n \geq 1$ for each n, we have $N \geq 1$ and so $N^2 \geq N$. It follows the relation $V \geq Z(1-Z)[(1-Z)q_A + Z q_B]/(mN)^2$ and hence

$$U \geq \frac{Z(1-Z)}{N}\left[\frac{(1-Z)q_A + Z q_B}{m^2} - 1\right].$$

Since $q_A \geq m^2$ and $q_B \geq m^2$ and $P(Z=0) = P(Z=1) = 0$, the quantity in the right side of the last inequality is always greater or equal than zero almost surely and it is equal to zero if and only if $q_A = q_B = m^2$. Summing up, the rate of convergence of $(M_n - Z_n)$ to zero is $1/2$ whenever $q_A > m^2$ or $q_B > m^2$ and, otherwise, it could be even greater.

Proof. Thanks to what we have already proven in the previous proof, it suffices to verify that the following condition is satisfied (see Theorem C.2 applied to $Y_n = X'_n$):

c3) $n^{-1}\sum_{j=1}^{n}\left[X'_j - Z_{j-1} + j(Z_{j-1} - Z_j)\right]^2 \xrightarrow{P} U$.

To this purpose, we apply Lemma B.1 with

$$Y_j = \left[X'_j - Z_{j-1} + j(Z_{j-1} - Z_j)\right]^2.$$

Indeed, by the assumptions, we have $\sum_{j\geq 1} j^{-2} E[Y_j^2] < +\infty$. Moreover, from what we have already seen in the previous proof, we can get

$$j^2 E[(Z_{j-1} - Z_j)^2|\mathcal{F}_{j-1}] \xrightarrow{a.s.} V\,.$$

Moreover, leveraging the above computations, we have

$$\begin{aligned} E[(X'_j - Z_{j-1})^2|\mathcal{F}_{j-1}] &= E[(X'_j)^2|\mathcal{F}_{j-1}] - Z_{j-1}^2 \\ &= Z_{j-1}(1-Z_{j-1})(S_{j-1}-1)^{-1}\left(S_{j-1}E[N_j^{-1}|\mathcal{F}_{j-1}] - 1\right) \xrightarrow{a.s.} Z(1-Z)L\,. \end{aligned}$$

Finally, we observe that

$$j(X'_j - Z_{j-1})(Z_{j-1} - Z_j) = -j(X'_j - Z_{j-1})\frac{(1-Z_{j-1})A_j N_j X'_j - Z_{j-1}B_j N_j(1-X'_j)}{S_j} =$$

$$-\frac{j(1-Z_{j-1})A_j N_j (X'_j)^2}{S_j} + \frac{jZ_{j-1}(1-Z_{j-1})A_j N_j X'_j}{S_j} + \frac{jZ_{j-1}B_j N_j X'_j(1-X'_j)}{S_j} - \frac{jZ_{j-1}^2 B_j N_j(1-X'_j)}{S_j} =$$

$$-U_{1,j} + U_{2,j} + U_{3,j} - U_{4,j}\,.$$

With the same techniques adopted in the previous proof, we can get

$$\begin{aligned} T_{1,j-1} &= E[U_{1,j}|\mathcal{F}_{j-1}] \xrightarrow{a.s.} T_1 = Z(1-Z)^2/N + Z^2(1-Z) \\ T_{2,j-1} &= E[U_{2,j}|\mathcal{F}_{j-1}] \xrightarrow{a.s.} T_2 = Z^2(1-Z) \\ T_{3,j-1} &= E[U_{3,j}|\mathcal{F}_{j-1}] \xrightarrow{a.s.} T_3 = Z^2 - Z^2(1-Z)/N - Z^3 = -Z^2(1-Z)/N + Z^2(1-Z) \\ T_{4,j-1} &= E[U_{4,j}|\mathcal{F}_{j-1}] \xrightarrow{a.s.} T_4 = Z^2(1-Z) \end{aligned}$$

Summing up, we obtain the almost sure convergence of $E[Y_j|\mathcal{F}_{j-1}]$ to $U = V + Z(1-Z)L + 2(-T_1 + T_2 + T_3 - T_4) = V + Z(1-Z)(L - 2N^{-1})$. ∎

3.3. **Probability distribution of the limit proportion in the case of equal reinforcement means.** When we are in case 2), the distribution of the limit proportion Z is unknown except in a few particular cases (see [3]). What we are able to prove in the general case is that it is diffuse (see Theorem 3.15 below) and to leverage the above central limit theorems in order to get asymptotic confidence intervals for Z (see Subsection 3.4 below).

Theorem 3.15. *Assume the same assumptions as in Theorem 3.9, then $P(Z = z) = 0$ for all $z \in [0, 1]$.*

Proof. We already know that $P(Z = 0) = P(Z = 1) = 0$ (see Theorem 3.8) In order to prove that $P(Z = z) = 0$ for all $z \in (0, 1)$, we can argue exactly as done in [12, Cor. 4.1] or in Th. 3.2 in [14]. Since the key issue on which the proof is based is the almost sure conditional convergence of $\sqrt{n}(Z_n - Z)$ with respect to $\mathcal{F} = (\mathcal{F}_n)$ to a Gaussian kernel $\mathcal{N}(0, V)$, for some $V > 0$ on $\{Z \in (0, 1)\}$. ∎

3.4. **Asymptotic confidence intervals for the limit proportion in the case of equal reinforcement means.** Suppose to be in case 2). By means of Theorem 3.9 and Theorem 3.13 (together with Theorem C.1), we can construct *asymptotic confidence intervals* for the limit proportion Z. More precisely, assume $A_n \vee B_n \vee N_n \leq C$ for each n and conditions (10), (11), and (12). Then, by Lemma B.1, the random variables

$$\widehat{m}_n = \frac{\sum_{j=1}^n A_j}{n}, \qquad \widehat{q}_{A,n} = \frac{\sum_{j=1}^n A_j^2}{n}, \qquad \widehat{q}_{B,n} = \frac{\sum_{j=1}^n B_j^2}{n}, \qquad \widehat{q}_{AB,n} = \frac{\sum_{j=1}^n A_j B_j}{n} \tag{18}$$

are strongly consistent estimators of the constants m, q_A, q_B and q_{AB} (supposed unknown), respectively. By Lemma B.1 again, the random variables

$$\widehat{\mu}_n = \frac{\sum_{j=1}^n N_j}{n}, \qquad \widehat{q}_{N,n} = \frac{\sum_{j=1}^n N_j^2}{n}, \tag{19}$$

are strongly consistent estimators of the random variables N and Q. Hence, the random variable

$$V_n = Z_n(1 - Z_n)\times$$
$$\frac{(1 - Z_n)\widehat{q}_{A,n}[(1 - Z_n)\widehat{\mu}_n + Z_n\widehat{q}_{N,n}] + Z_n\widehat{q}_{B,n}[Z_n\widehat{\mu}_n + (1 - Z_n)\widehat{q}_{N,n}] - 2Z_n(1 - Z_n)\widehat{q}_{AB,n}(\widehat{q}_{N,n} - \widehat{\mu}_n)}{(\widehat{m}_n\widehat{\mu}_n)^2}$$

results a strongly consistent estimator of the random variable V (defined in Theorem 3.9). Recalling that $V > 0$ almost surely (see Remark 3.10), by Theorem 3.9, together with Theorem C.1, we obtain that a confidence interval for Z is

$$Z_n \pm q_{1-\frac{\alpha}{2}}\sqrt{\frac{V_n}{n}}, \tag{20}$$

where $q_{1-\frac{\alpha}{2}}$ is the quantile of order $1 - \frac{\alpha}{2}$ of the standard normal distribution.

When $N_n = k$, with k a known constant, for V_n we can employ the simpler formula (14) with $\widehat{q}_{A,n}$, $\widehat{q}_{B,n}$ and $\widehat{q}_{AB,n}$ instead of q_A, q_B and q_{AB}.

If condition (17) is also satisfied, then, again by Lemma B.1, $\widehat{\eta}_n = \frac{\sum_{j=1}^n N_j^{-1}}{n}$ is a strongly consistent estimator of the random variable L (defined in Theorem 3.13) and so, setting

$$W_n = 2V_n' + M_n(1 - M_n)[\widehat{\eta}_n - 2(\widehat{\mu}_n)^{-1}],$$

where V_n' is equal to V_n but with M_n instead of Z_n, is a strongly consistent estimator of the random variable $W = U + V$. Since $U + V > 0$ almost surely, by Theorem 3.13, together with Theorem C.1), we get that

$$M_n \pm q_{1-\frac{\alpha}{2}} \sqrt{\frac{W_n}{n}} \tag{21}$$

is a confidence interval for Z. Note that this second interval does not depend on the initial composition of the urn, which could be unknown.

A remark useful for applications follows.

Remark 3.16. The estimators of m, q_A, q_B and q_{AB} defined in (18) presuppose that we can observe both A_j and B_j for each $j = 1, \dots, n$. Actually, in applications, we can observe A_j (respectively, B_j) only when $X_j > 0$ (respectively, $X_j < N_j$). Therefore, it makes more sense to use the following estimators:

$$\begin{aligned} \widehat{m}_n &= \frac{\sum_{j=1}^n \left(A_j I_{\{X_j>0\}} + B_j I_{\{X_j=0\}} \right)}{n}, \\ \widehat{q}_{A,n} &= \frac{\sum_{j=1}^n A_j^2 I_{\{X_j>0\}}}{\sum_{j=1}^n I_{\{X_j>0\}}}, \qquad \widehat{q}_{B,n} = \frac{\sum_{j=1}^n B_j^2 I_{\{X_j<N_j\}}}{\sum_{j=1}^n I_{\{X_j<N_j\}}}, \\ \widehat{q}_{AB,n} &= \frac{\sum_{j=1}^n A_j B_j I_{\{0<X_j<N_j\}}}{\sum_{j=1}^n I_{\{0<X_j<N_j\}}}. \end{aligned} \tag{22}$$

Note that $\widehat{m}_n \xrightarrow{a.s.} m$ by Lemma B.1 (applied with $Y_j = A_j I_{\{X_j>0\}} + B_j I_{\{X_j=0\}} \leq C$ and $\mathcal{F}_j = \mathcal{G}_j$). Indeed, we have

$$\begin{aligned} E[A_j I_{\{X_j>0\}} + B_j I_{\{X_j=0\}} | \mathcal{G}_{j-1}] &= E\left[E[A_j I_{\{X_j>0\}} + B_j I_{\{X_j=0\}} | \mathcal{H}_{j-1}] \, | \mathcal{G}_{j-1} \right] \\ &= E[A_j P(X_j > 0 | \mathcal{H}_{j-1}) + B_j P(X_j = 0 | \mathcal{H}_{j-1}) \, | \mathcal{G}_{j-1}] = m_j \,, \end{aligned}$$

where the last equality is due to the fact that the conditional distribution of X_j given $\mathcal{H}_{j-1}$ depends on N_j, S_{j-1} and H_{j-1} (and so coincides with the one given $\mathcal{G}_{j-1}$) and to relation (3). The convergence $\widehat{q}_{A,n} \xrightarrow{a.s.} q_A$ also follows from by Lemma B.1. Indeed, we have

$$E[A_j^2 I_{\{X_j>0\}} | \mathcal{H}_{j-1}] = A_j^2 \left[1 - \frac{\binom{S_{j-1}-H_{j-1}}{N_j}}{\binom{S_{j-1}}{N_j}} \right]$$

Then, conditioning with respect to $\mathcal{G}_{j-1}$ and using (3), we get $E[A_j^2 I_{\{X_j>0\}} | \mathcal{G}_{j-1}] = q_{A,j} \varphi(N_j, S_{j-1}, H_{j-1})$ with $\varphi(N, S, H) = \left[1 - \frac{\binom{S-H}{N}}{\binom{S}{N}} \right]$. Finally, conditioning with respect to $\mathcal{F}_{j-1}$, we find

$$E[A_j^2 I_{\{X_j>0\}} | \mathcal{F}_{j-1}] = q_{A,j} \sum_{k=1}^{C} \varphi(k, S_{j-1}, H_{j-1}) P(N_j = k | \mathcal{F}_{j-1}) \,.$$

Assuming that $P(N_j = k | \mathcal{F}_{j-1}) \xrightarrow{a.s.} \nu(k)$ (with $\nu(k)$ possibly random), as a consequence of Proposition 3.6 and the above equality, we have

$$E[A_j^2 I_{\{X_j>0\}} | \mathcal{F}_{j-1}] \overset{a.s.}{\sim} q_{A,j} \sum_{k=1}^{C} \left[1 - (1 - \frac{H_{j-1}}{S_{j-1}})^k \right] P(N_j = k | \mathcal{F}_{j-1}) \xrightarrow{a.s.} q_A \sum_{k=1}^{C} \left[1 - (1 - Z)^k \right] \nu(k) \,.$$

Similarly, we have $E[I_{\{X_j>0\}}|\mathcal{F}_{j-1}] \xrightarrow{a.s.} \sum_{k=1}^{C} \left[1-(1-Z)^k\right] \nu(k)$ and so, by Lemma B.1, we obtain

$$\widehat{q}_{A,n} = \frac{\sum_{j=1}^{n} A_j^2 I_{\{X_j>0\}}/n}{\sum_{j=1}^{n} I_{\{X_j>0\}}/n} \xrightarrow{a.s.} \frac{q_A \sum_{k=1}^{C}[1-(1-Z)^k]\nu(k)}{\sum_{k=1}^{C}[1-(1-Z)^k]\nu(k)} = q_A\,.$$

Exactly with the same argument, we get $\widehat{q}_{B,n} \xrightarrow{a.s.} q_B$. For the almost sure convergence of $\widehat{q}_{AB,n}$ to q_{AB}, we can argue in the similar way, but we need $P(\nu(1)<1)=1$ in order to guarantee that $\sum_{k=1}^{C}\left[1-(1-Z)^k-Z^k\right]\nu(k)>0$ almost surely.

4. Examples and numerical illustrations

Before considering special cases as illustration through numerical simulations, let us formulate some general remarks.

Remark 4.1. *($[A_n, B_n]$ identically distributed)* If all the random vectors $[A_n, B_n]$ (that are independent by assumption (A3)) are also identically distributed, then we simply have $m = m_n = E[A_n] = E[B_n]$ and condition (12) is satisfied with $q_A = q_{A,n} = E[A_n^2]$, $q_B = q_{B,n} = E[B_n^2]$ and $q_{AB} = q_{AB,n} = E[A_nB_n]$).

Remark 4.2. *(N_n independent of the past)*
If, for each n, the random variable N_n is independent of $\mathcal{F}_{n-1}$, then we simply have $E[N_n|\mathcal{F}_{n-1}] = E[N_n]$, $E[N_n^2|\mathcal{F}_{n-1}] = E[N_n^2]$ and $E[N_n^{-1}|\mathcal{F}_{n-1}] = E[N_n^{-1}]$. Therefore, conditions (10), (11) and (17) are satisfied whenever the above sequences of mean values converge to suitably constants N, Q and L. For instance, this happens when all the random variables N_n are identically distributed. More precisely, in this last case, assuming $N_n \le a+b$ (so that we are sure that $N_n \le S_{n-1}$ for each n), with mean value μ and variance σ^2, conditions (10), (11) and (17) are satisfied with $N = E[N_n] = \mu$, $Q = E[N_n^2] = q_N = \sigma^2+\mu^2$ and $L = E[N_n^{-1}] = \eta$.

Remark 4.3. *(N_n dependent on Z_{n-1})* When N_n depends on the urn proportion at time $n-1$, i.e. Z_{n-1}, in such a way that, for each n, we have

$$E[N_{n+1}|\mathcal{F}_n] = f(Z_n), \quad E[N_{n+1}^2|\mathcal{F}_n] = g(Z_n), \quad E[N_{n+1}^{-1}|\mathcal{F}_n] = h(Z_n)\,,$$

where f, g and h are continuous functions, then conditions (10), (11) and (17) are satisfied with $N = f(Z)$, $Q = g(Z)$ and $L = h(Z)$. Note that, if the functions f, g and h are known, we can obtain asymptotic confidence intervals for Z replacing $\widehat{\mu}_n$ and $\widehat{q}_{N,n}$ in the expression for V_n by $f(Z_n)$ and $g(Z_n)$, respectively, and replacing $\widehat{\mu}_n$, $\widehat{q}_{N,n}$ and $\widehat{\eta}_n$ in the expression for W_n by $f(M_n)$, $g(M_n)$ and $h(M_n)$, respectively.

Remark 4.4. *(N_n almost surely convergent)*
If (N_n) is a sequence of integer-valued random variables with $1 \le N_n \le C$ and converging almost surely to a random variable N, then (by Lemma B.2) conditions (10), (11) and (17) are satisfied and $Q = N^2$ and $L = N^{-1}$. See, for instance, Example 4.2 in [12], where (N_n) is a symmetric random walk with two absorbing barriers.

The following examples regard the case 2) (that is the case of equal reinforcement means) and they deal with the different situations described in the above general remarks.

Example 1a
Take each N_n independent of $\mathcal{F}_{n-1}$ and uniformly distributed on $\{1,\dots,5\}$. Moreover, take A_n and B_n satisfying assumption (A3), independent and uniformly distributed on $\{1,\dots,5\}$. We set $a=b=5$. See Fig. 1 for samples.

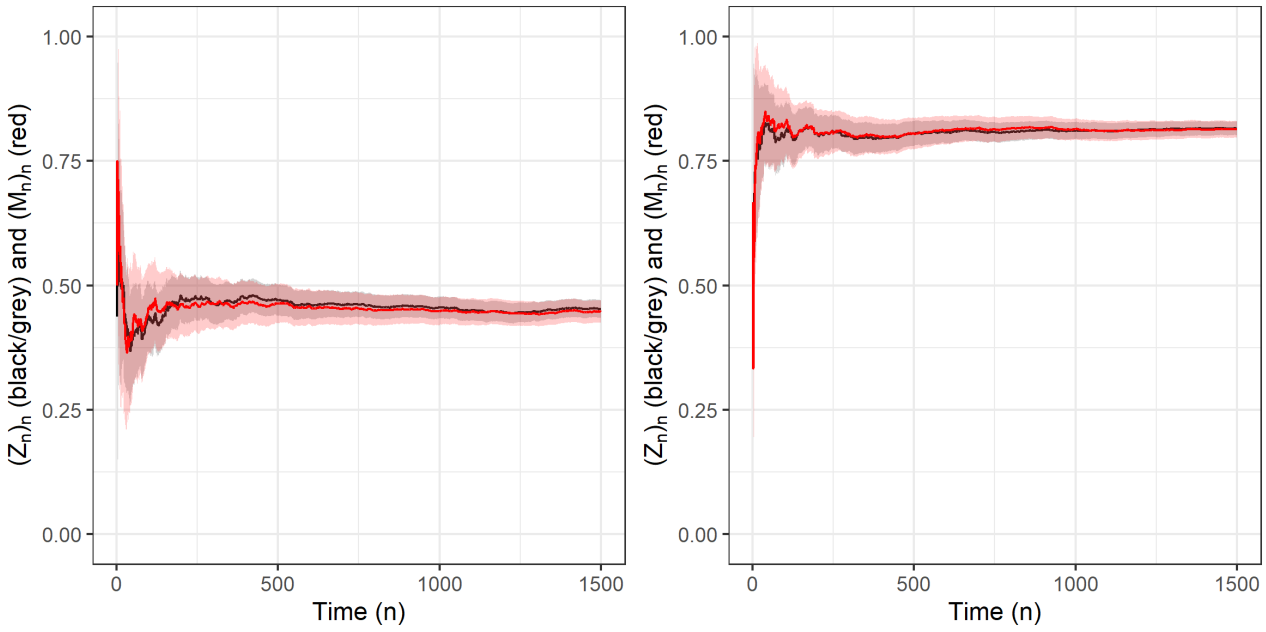

FIGURE 1. Case 1a. Time-horizon 1500. On each picture, one sample plot of $(Z_n)_n$ (black) and $(M_n)_n$ (red) with the corresponding confidence intervals for Z with $\alpha = 0.05$ (resp. grey and red).

Example 1b
Take each N_n independent of $\mathcal{F}_{n-1}$ and uniformly distributed on $\{1, \dots, 5\}$. In particular, assumption (i) in Section 3.4 is satisfied. Moreover, take $[A_n, B_n]$ satisfying assumption (A3) and such that

$$A_n \overset{d}{=} 1 + Y_1 \qquad \text{and} \qquad B_n \overset{d}{=} 1 + Y_2\,,$$

where Y_1 and Y_2 are, respectively, the first and the second component of a multinomial distribution associated to the parameters: size= 12, probabilities= $(4/15, 4/15, 7/15)$. Thus the random variables A_n and B_n are negatively correlated. We set $a = b = 5$. See Fig. 2 for samples.

Example 1c
Set $(N_n)_n$ be a sequence of random variables such that

$$N_n|\mathcal{F}_{n-1} \overset{d}{=} 1 + \mathcal{B}(\kappa, Z_{n-1}).$$

Moreover, take A_n and B_n satisfying assumption (A3), independent and uniformly distributed on $\{1, \dots, 5\}$. In particular, we are in the situation described in Remark 4.3. Indeed, we have:

$$\begin{aligned}
E[N_{n+1}|\mathcal{F}_n] &= 1 + \kappa Z_n \xrightarrow{a.s.} N = 1 + \kappa Z \\
E[N_{n+1}^2|\mathcal{F}_n] &= \kappa Z_n(1 - Z_n) + (1 + \kappa Z_n)^2 \xrightarrow{a.s.} Q = \kappa Z(1 - Z) + (1 + \kappa Z)^2 \\
E[N_{n+1}^{-1}|\mathcal{F}_n] &= \frac{1 - (1 - Z_n)^{\kappa+1}}{(\kappa + 1)Z_n} \xrightarrow{a.s.} \frac{1 - (1 - Z)^{\kappa+1}}{(\kappa + 1)Z}
\end{aligned}$$

(recall that $P(Z = 0) = 0$ by Theorem 3.15). We set $\kappa = 10$ and $a = b = 6$. See Fig. 3 for samples.

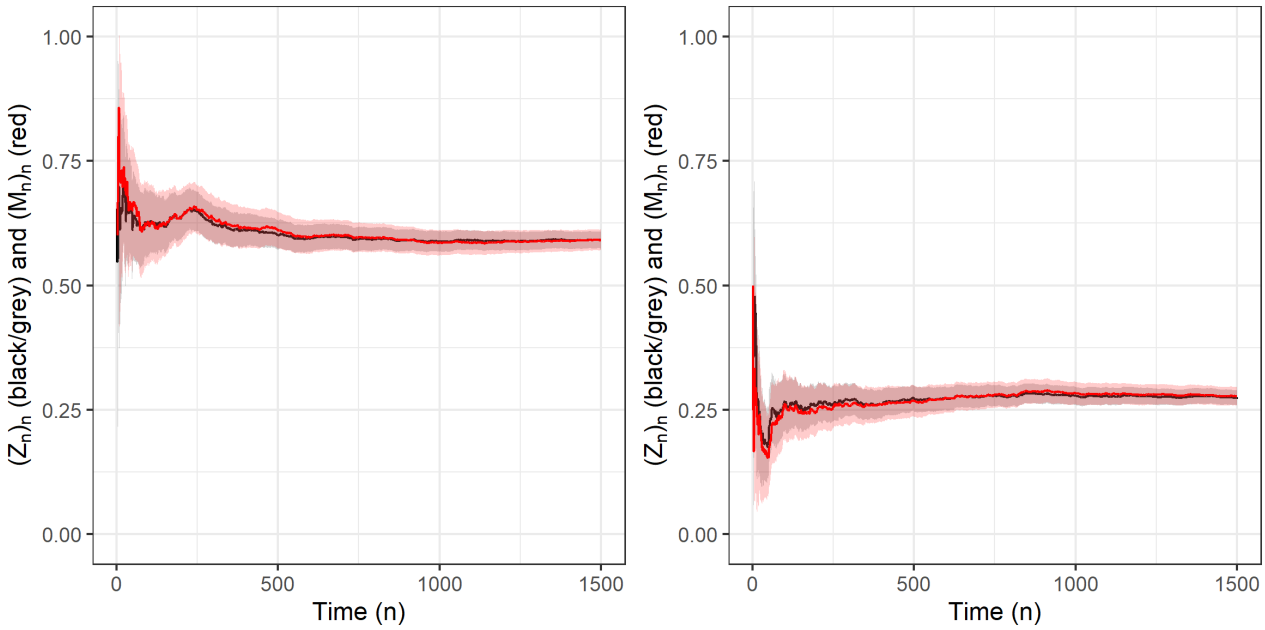

FIGURE 2. Case 1b. Time-horizon 1500. On each picture, one sample plot of $(Z_n)_n$ (black) and $(M_n)_n$ (red) with the corresponding confidence intervals for Z with $\alpha = 0.05$ (resp. grey and red).

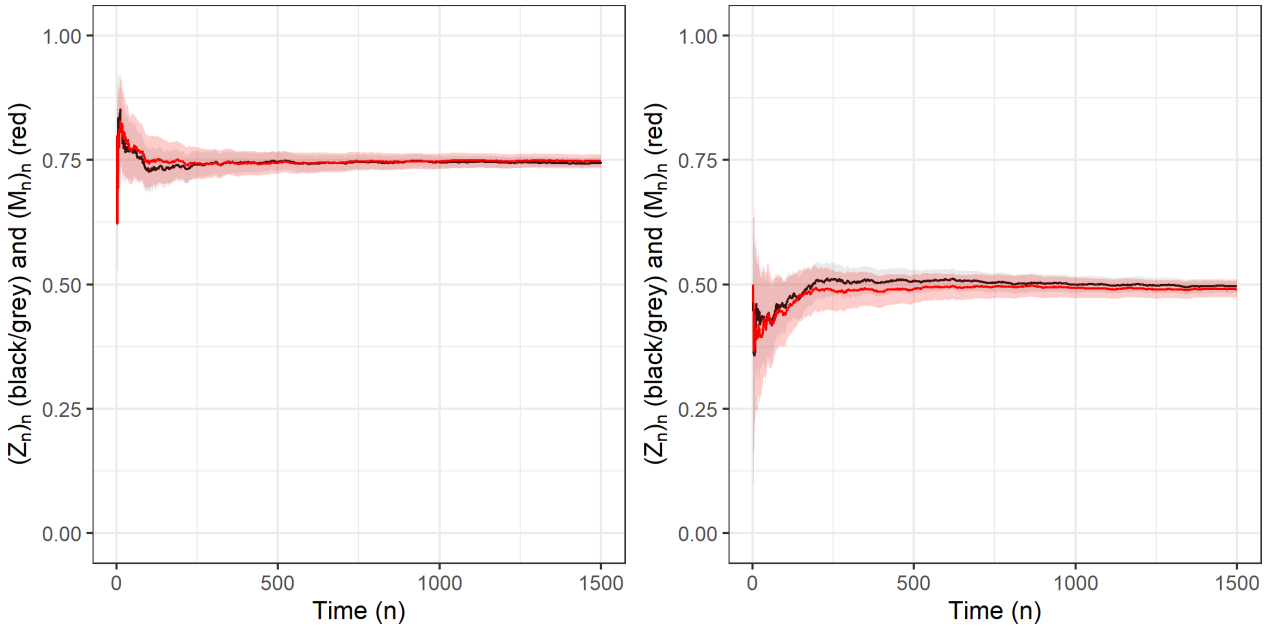

FIGURE 3. Case 1c. Time-horizon 1500. On each picture, one sample plot of $(Z_n)_n$ (black) and $(M_n)_n$ (red) with the corresponding confidence intervals for Z with $\alpha = 0.05$ (resp. grey and red). The confidence intervals are the ones given in Remark 4.3, taking parameter κ known (that is with the functions f, g and h known).

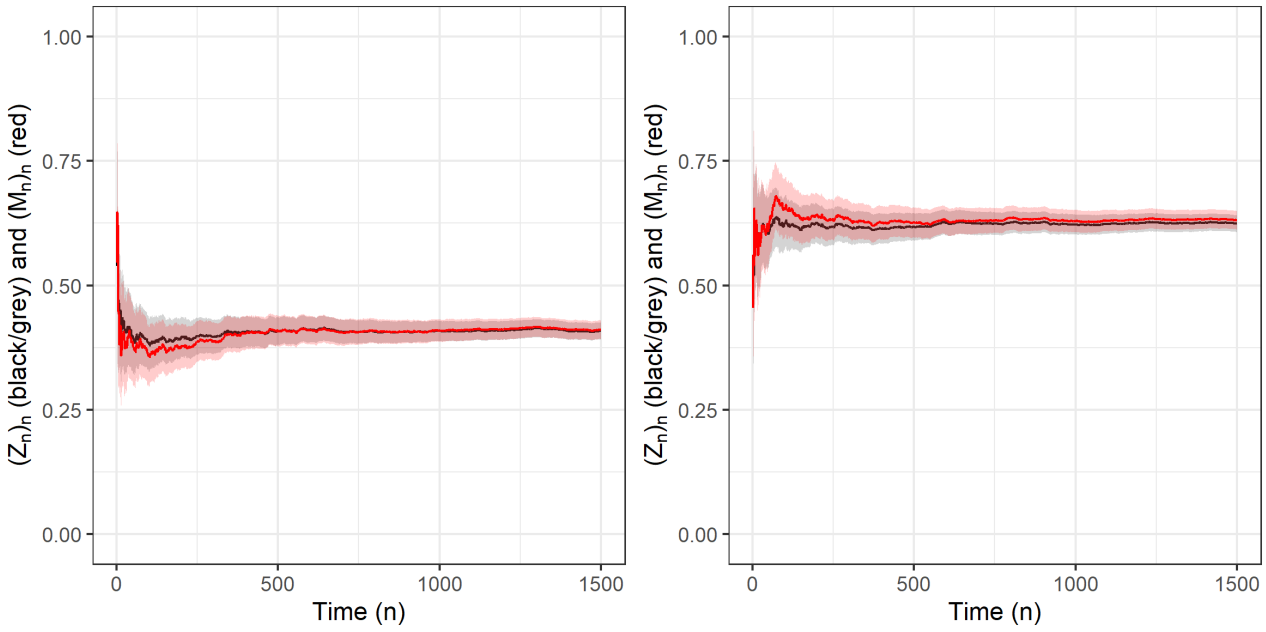

FIGURE 4. Case 1d. Time-horizon 1500. On each picture, one sample plot of $(Z_n)_n$ and $(M_n)_n$ with the corresponding confidence intervals for Z with $\alpha = 0.05$ (resp. grey and red).

Example 1d

Take each N_n independent of $\mathcal{F}_{n-1}$ and such that

$$N_n \overset{d}{=} 2 + \mathcal{B}(\kappa, p_n),$$

with $\kappa = 10$ and $p_n = 1/\sqrt{n}$. Moreover, take A_n and B_n satisfying assumption (A3), independent and such that

$$A_n \overset{d}{=} B_n \overset{d}{=} 1 + \mathcal{B}(\kappa', q_n),$$

with $\kappa' = 5$ and $q_n = \min(1, \frac{1}{2} + \frac{1}{\sqrt{n}})$. We take $a = b = 6$. See Fig. 4 for samples.

Example 1e

This example is associated to Remark 4.4. Following Example 4.2 in [12], take $(N_n)_n$ be a sequence of random variables defined through a symmetric nearest neighbors random walk with absorbing barriers. Given $h \in \mathbb{N}$, with $3 \leq h \leq a + b$, let $\widetilde{N}_1$ be a random variable with distribution concentrated on $\{2, \ldots, h-1\}$ and set

$$\widetilde{N}_n = \widetilde{N}_1 + \sum_{j=2}^{n} Y_j \text{ for } n \geq 2\,,$$

$$T_1 = \inf\{n : \widetilde{N}_n = 1\}, \qquad T_h = \inf\{n : \widetilde{N}_n = h\}$$

and

$$N_n = \widetilde{N}_{T \wedge n} \quad \text{for } n \geq 1, \quad \text{with } T = T_1 \wedge T_h\,,$$

where each Y_j is independent of $[\widetilde{N}_1, X_1, A_1, B_1, Y_1, X_2, A_2, B_2, \ldots, Y_{j-1}, X_{j-1}, A_{j-1}, B_{j-1}]$ and such that $P(Y_j = -1) = P(Y_j = 1) = p \in (0, \frac{1}{2}]$ and $P(Y_j = 0) = 1 - 2p$. Then $N_n \xrightarrow{a.s.} N = \widetilde{N}_T$ where

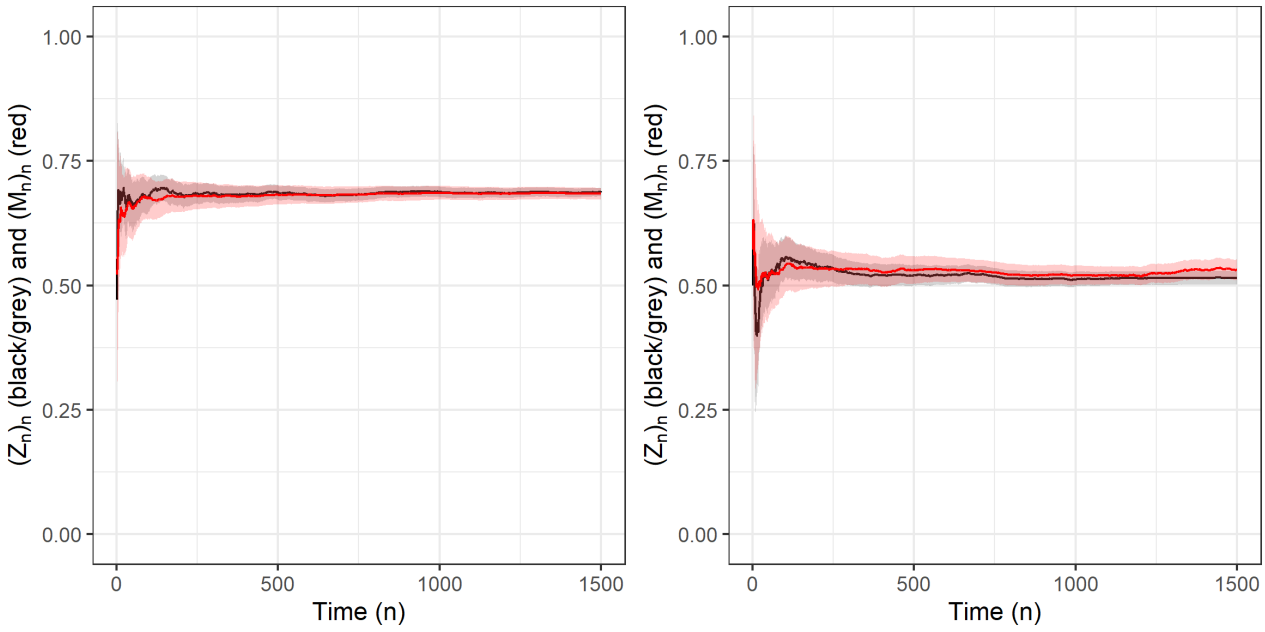

FIGURE 5. Case 1e. Time-horizon 1500. On each picture, one sample plot of $(Z_n)_n$ and $(M_n)_n$ with the corresponding confidence intervals for Z with $\alpha = 0.05$ (resp. grey and red).

$N = \mathbb{1}_{\{T=T_1\}} + h\mathbb{1}_{\{T=T_h\}}$. We take A_n and B_n satisfying assumption (A3), independent and such that

$$A_n \overset{d}{=} B_n \overset{d}{=} 1 + \mathcal{B}(\kappa', q_n).$$

We consider specifically $a = b = 30$, $h = 50$, $\widetilde{N}_1$ uniformly distributed on $\{2, \dots, h-1\}$, $p = 1/4$, $\kappa' = 5$ and $q_n = \min(1, \frac{1}{2} + \frac{1}{\sqrt{n}})$. Note that also in this case it is possible to contruct confidence intervals for Z (see Remark 4.3). See Fig. 5 for samples.

In the following example, the random variables N_n, A_n and B_n are not bounded, but condition (7) is satisfied.

Example 2

For each $n \geq 1$, take $\widetilde{N}_n$ independent of $[\widetilde{N}_1, X_1, A_1, B_1, \dots, \widetilde{N}_{n-1}, X_{n-1}, A_{n-1}, B_{n-1}]$ and such that

$$\widetilde{N}_n \overset{d}{=} 1 + \mathcal{B}(\kappa + \left\lceil n^{\frac{1}{3}} \right\rceil, p)$$

with $\kappa = 3$ and $p = 1/10$. Set $N_n = \widetilde{N}_n \wedge S_{n-1}$ for each $n \geq 1$. Take A_n and B_n satisfying assumption (A3), independent and such that

$$A_n \overset{d}{=} B_n \overset{d}{=} 1 + \mathrm{neg}\mathcal{B}(r, p_n)\,,$$

where $\mathrm{neg}\mathcal{B}(r, p_n)$ means the negative binomial distribution with parameters $r = 3$ and $p_n = 1/\sqrt{n+1}$, that is with mean value equal to $rp_n/(1-p_n)$ and variance equal to $rp_n/(1-p_n)^2$. Condition (7) is satisfied because

$$E[(A_n + B_n)^2] = O(1) \quad \text{and} \qquad E[N_n^2] \leq E[\widetilde{N}_n^2] = O(n^{2/3}).$$

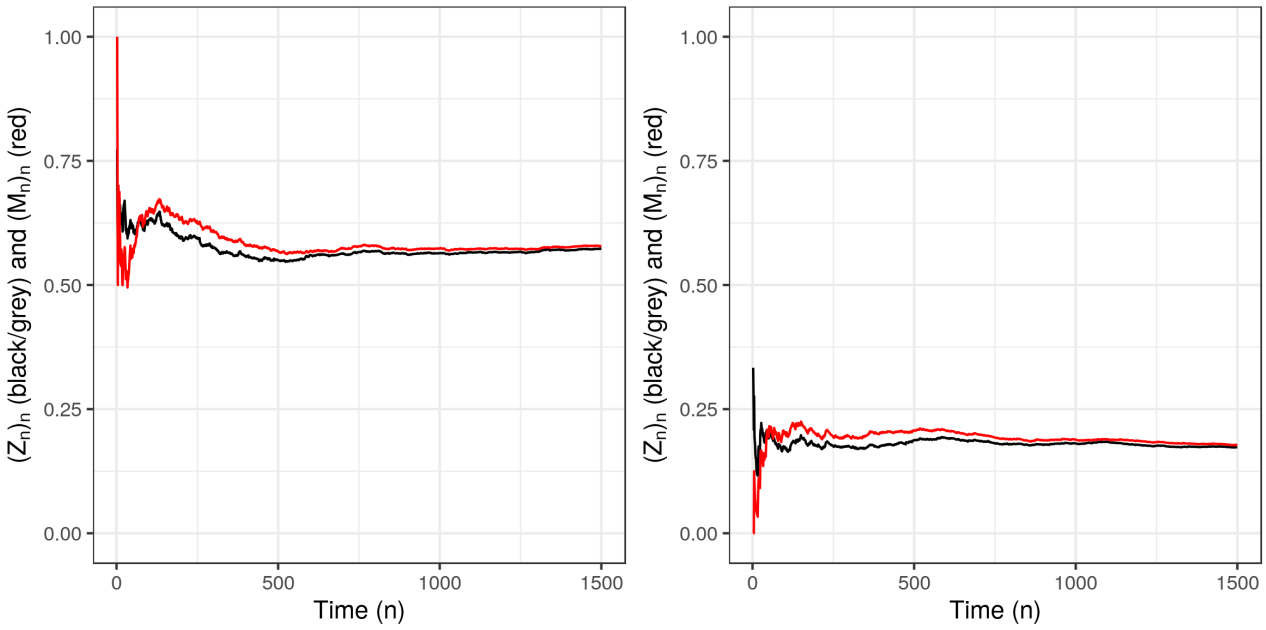

FIGURE 6. Case 2. Time-horizon 1500. On each picture, one sample plot of $(Z_n)_n$ and $(M_n)_n$ (resp. black and red).

We set $a = b = 5$. See Fig. 6 for samples.

The last two examples below are related to the case $m_A > m_B$. Note that the time of the almost sure convergence to 1, proven above, depends on the difference $m_A - m_B$. Thus, when this difference is small, it may be difficult to guess the right asymptotic behavior only through simulations.

Example 3a
Take each N_n independent of $\mathcal{F}_{n-1}$ and uniformly distributed on $\{1, \ldots, 5\}$. Take $[A_n, B_n]$ satisfying assumption (A3) and taking values $(1, 1)$, $(3, 1)$, $(1, 3)$, $(3, 3)$ with respective probabilities $\frac{3}{16}, \frac{1}{4}, \frac{1}{16}, \frac{1}{2}$. It holds $m_A = 2.5$ and $m_B = 2.125$. We set $a = b = 5$. See Fig. 7 for samples.

Example 3b
Take each N_n independent of $\mathcal{F}_{n-1}$ and uniformly distributed on $\{1, \ldots, 5\}$. Take $[A_n, B_n]$ satisfying assumption (A3) and taking values $(1, 1)$, $(10, 1)$, $(1, 3)$, $(10, 3)$ with respective probabilities $\frac{1}{5}, \frac{2}{5}, \frac{1}{5}, \frac{1}{5}$. It holds $m_A = 6.4$ and $m_B = 1.8$. We set $a = b = 5$. See Fig. 8 for samples.

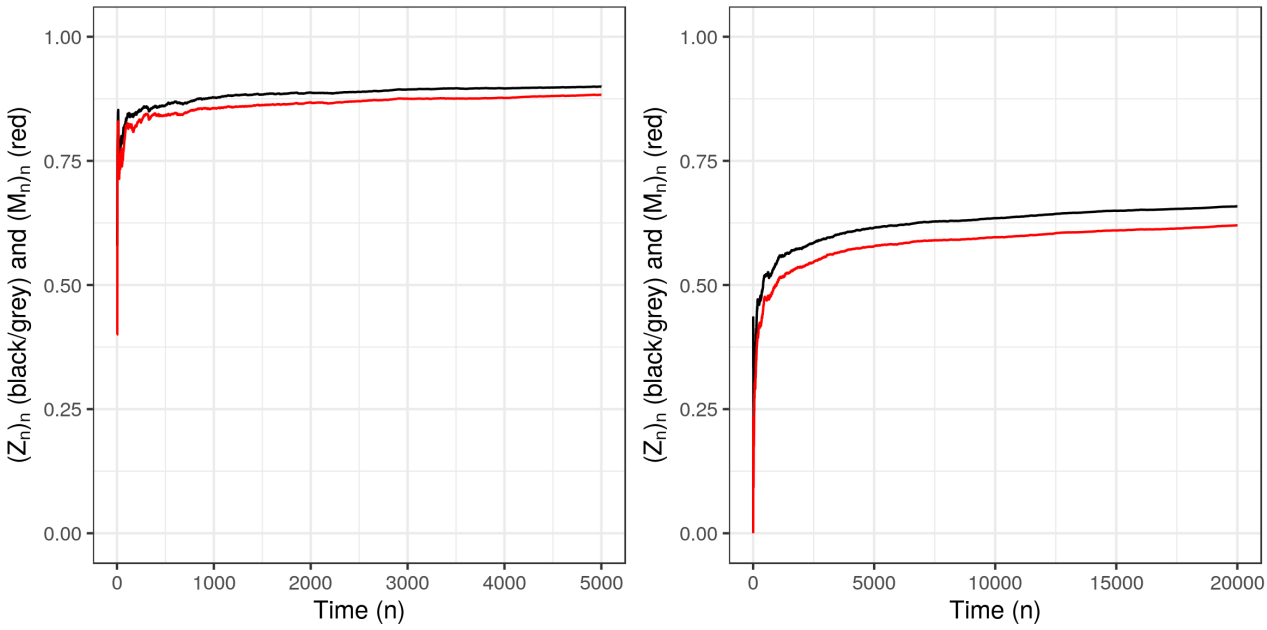

FIGURE 7. Case 3a. Time-horizon 5.000 (left), 20.000 (right). On each picture, one sample plot of $(Z_n)_n$ and $(M_n)_n$ (resp. black and red).

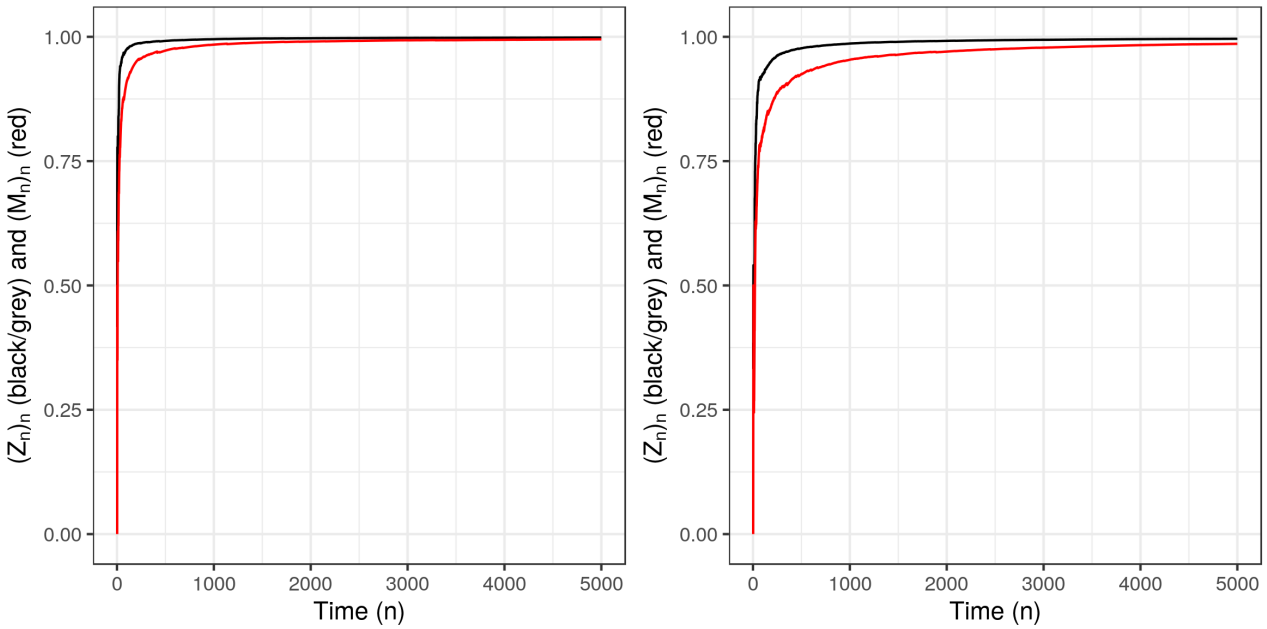

FIGURE 8. Case 3b. Time-horizon 5.000. On each picture, one sample plot of $(Z_n)_n$ and $(M_n)_n$ (resp. black and red).

Acknowledgments
Irene Crimaldi and Ida Minelli are members of the Italian Group "Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni" of the Italian Institute "Istituto Nazionale di Alta Matematica". P.-Y. Louis acknowledges the International Associated Laboratory Ypatia Laboratory of Mathematical Sciences (LYSM) for funding travel expenses.

Funding Sources
Irene Crimaldi is partially supported by the Italian "Programma di Attività Integrata" (PAI), project "TOol for Fighting FakEs" (TOFFE) funded by IMT School for Advanced Studies Lucca.

Declaration
All the authors equally contributed to this work.

Appendix A. Technical results

Consider the model and the assumptions described in Section 2.

Lemma A.1. *Suppose $A_n \vee B_n \vee N_n \leq C$ for some (integer) constant C. Let $p_{n+1,k} = p_k(N_{n+1}, S_n, H_n)$ be the values of the hypergeometric distribution with parameters N_{n+1}, S_n and H_n (see (5)). Then, we have*

$$1 - p_{n+1,N_{n+1}} = \frac{K_n}{S_n}\,(1 + O(1)) = O(K_n/S_n)\,.$$

Proof. If $N_{n+1} = 1$, we simply have $1 - p_{n+1,N_{n+1}} = K_n/S_n$. By Lemma 3.1, we have $H_n \geq C$ for n large enough (and so $H_n \geq N_{n+1}$ for n large enough). Therefore, for n large enough, we have

$$\begin{aligned}
1 - p_{n+1,N_{n+1}} &= 1 - \prod_{j=1}^{N_{n+1}} \frac{H_n - j + 1}{S_n - j + 1} = \frac{H_n + K_n}{S_n} - \prod_{j=1}^{N_{n+1}} \frac{H_n - j + 1}{S_n - j + 1} \\
&= \frac{K_n}{S_n} + \frac{H_n}{S_n}\left[1 - \prod_{j=2}^{N_{n+1}} \frac{H_n - j + 1}{S_n - j + 1}\right] \\
&= \frac{K_n}{S_n} + \frac{H_n}{S_n} \frac{\prod_{j=1}^{N_{n+1}-1}(S_n - j) - \prod_{j=1}^{N_{n+1}-1}(H_n - j)}{\prod_{j=1}^{N_{n+1}-1}(S_n - j)} \\
&= \frac{K_n}{S_n} + \frac{H_n}{S_n} \frac{(S_n - H_n) f(S_n, H_n)}{\prod_{j=1}^{N_{n+1}-1}(S_n - j)} = \frac{K_n}{S_n}\left(1 + \frac{H_n f(S_n, H_n)}{\prod_{j=1}^{N_{n+1}-1}(S_n - j)}\right),
\end{aligned}$$

where $f(x, y) = 1$ when $N_{n+1} = 2$ and $f(x, y) = \sum_{j=1}^{N_{n+1}-2} a_j x^j + b_j y^j + c$ when $N_{n+1} \geq 3$. Therefore, since $H_n \leq S_n$ and $S_n \to +\infty$ almost surely (by Lemma 3.1), we have $H_n f(S_n, H_n) / \prod_{j=1}^{N_{n+1}-1}(S_n - j) = O(1)$. ∎

Lemma A.2. *Suppose to be in case 2). For $e > 1$, H_n/K_n^e and K_n/H_n^e are eventually (positive) supermartingales and so they converge almost surely to a finite random variable.*

Proof. The proof used in order to prove that $Q_n = K_n/H_n^e$ is eventually a positive supermartingale in the proof of Theorem 3.2 does not work now, because we have $e > 1$ and the inequality $(1-x)^e \leq$

$1-ex$ is not true. Therefore we need a different proof. We observe that

$$E\left[\frac{H_{n+1}}{K_{n+1}^e}-\frac{H_n}{K_n^e}\,|\,\mathcal{H}_n\right]=E\left[\frac{H_{n+1}}{K_n^e}-\frac{H_n}{K_n^e}+\frac{H_{n+1}}{K_{n+1}^e}-\frac{H_{n+1}}{K_n^e}\,|\,\mathcal{H}_n\right]=$$
$$\sum_{k\in\mathcal{X}_{n+1}}p_{n+1,k}\left(\frac{H_n+A_{n+1}k}{K_n^e}-\frac{H_n}{K_n^e}\right)+p_{n+1,k}(H_n+A_{n+1}k)\left(\frac{1}{(K_n+B_{n+1}(N_{n+1}-k))^e}-\frac{1}{K_n^e}\right)=$$
$$\sum_{k\in\mathcal{X}_{n+1}}p_{n+1,k}\frac{A_{n+1}k}{K_n^e}+\sum_{k\in\mathcal{X}_{n+1}\setminus\{N_{n+1}\}}p_{n+1,k}(H_n+A_{n+1}k)\left(\frac{1}{(K_n+B_{n+1}(N_{n+1}-k))^e}-\frac{1}{K_n^e}\right).$$

Using the Taylor expansion of the function $f(x)=1/(c+x)^e$ with $c=K_n$ and $x=B_{n+1}(N_{n+1}-k)$, we can choose a constant θ such that eventually

$$\left(\frac{1}{(K_n+B_{n+1}(N_{n+1}-k))}-\frac{1}{K_n^e}\right)\le-\frac{e}{K_n^{e+1}}\left(B_{n+1}(N_{n+1}-k)-\frac{\theta}{K_n}\right).$$

Therefore the last term of the above equalities is eventually smaller or equal than

$$\frac{H_n}{K_n^e}\left\{\sum_{k\in\mathcal{X}_{n+1}}\left(\frac{A_{n+1}k}{H_n}-e\frac{B_{n+1}(N_{n+1}-k)}{K_n}\right)p_{n+1,k}+e\theta\sum_{k\in\mathcal{X}_{n+1}\setminus\{N_{n+1}\}}\frac{(1+A_{n+1}k/H_n)}{K_n^2}p_{n+1,k}\right\}.$$

Now, we observe that

$$E\left[\sum_{k\in\mathcal{X}_{n+1}}\left(\frac{A_{n+1}k}{H_n}-e\frac{B_{n+1}(N_{n+1}-k)}{K_n}\right)p_{n+1,k}\,|\,\mathcal{G}_n\right]=m_{n+1}\frac{N_{n+1}}{S_n}(1-e)$$

and (using $\lim_n m_n=m>0$, $N_{n+1}\ge1$ and Lemma A.1)

$$E\left[\sum_{k\in\mathcal{X}_{n+1}\setminus\{N_{n+1}\}}\frac{(1+A_{n+1}k/H_n)}{K_n^2}p_{n+1,k}\,|\,\mathcal{G}_n\right]\le\frac{(1-p_{n+1,N_{n+1}})}{K_n^2}+\frac{m_{n+1}N_{n+1}}{S_nK_n^2}=O(1/(S_nK_n)).$$

Therefore, we have

$$E\left[\frac{H_{n+1}}{K_{n+1}^e}-\frac{H_n}{K_n^e}\,|\,\mathcal{G}_n\right]\le m_{n+1}\frac{H_n}{K_n^e}\frac{N_{n+1}}{S_n}\left[-(e-1)+O(1/K_n)\right]$$

and so, since $e>1$ and $K_n\uparrow+\infty$ (by Lemma 3.1), we can conclude that the above conditional expectation is definitely negative. ∎

Lemma A.3. *Under the assumptions of Theorem 3.8, we have* $1/K_n=O(1/n^\gamma)$ *and* $1/H_n=O(1/n^\gamma)$ *for some* $\gamma>0$.

Proof. This proof is essentially the same as the one of Lemma A.1(iv) in [32]. However, for the reader's convenience, we here rewrite it with all the details. Since $S_n/n=(H_n+K_n)/n$ converges almost surely to mN, we have that eventually $S_n=(H_n+K_n)>nmN3/4$ almost surely. Let $F_H=\{H_n>nmN/4\ \text{ eventually}\}$ and $F_K=\{K_n>nmN/4\ \text{eventually}\}$. Since (Z_n) converges almost surely to Z with values in $[0,1]$, then $H_n/K_n=Z_n/(1-Z_n)$ converges almost surely to a random variable with values in $[0,+\infty]$. It follows that $P(F_H\cup F_K)=1$. Indeed, on $(F_H\cup F_K)^c=F_H^c\cap F_K^c$, we have $\liminf H_n/n\le mN/4$, $\liminf_n K_n/n\le mN/4$ and $H_n+K_n>nmN3/4$ almost surely and so, since we can write $K_n/H_n=(H_n+K_n)/H_n-1$ and $H_n/K_n=(H_n+K_n)/K_n-1$, we have $\liminf_n H_n/K_n\le1/2<2\le\limsup_n H_n/K_n$. This means that on $(F_H\cup F_K)^c$, H_n/K_n does not converge and hence $P((F_H\cap F_K)^c)=0$. In order to conclude, it is enough to prove that on F_H (resp. F_K), K_n (resp. H_n) is eventually greater than n^γ for $\gamma>0$

(up to a multiplicative constant).

Now, by Lemma A.2, H_n/K_n^e is bounded and we know that $K_n \uparrow +\infty$ (see Lemma 3.1). Therefore, for each $\epsilon > 0$, we have $H_n/K_n^{e+\epsilon} \to 0$ almost surely and so $H_n/K_n^{e+\epsilon} < 1$ eventually. Therefore on F_H, we eventually have $K_n^{e+\epsilon} = (H_n/K_n^{e+\epsilon})^{-1} H_n > nmN/4 \geq nm/4$, *i.e.* $K_n > n^\gamma$ eventually (up to a multiplicative constant) with $\gamma = 1/(e+\epsilon) > 0$. Similarly, on F_K, we have $H_n > n^\gamma$ eventually (up to a multiplicative constant) with $\gamma = 1/(e+\epsilon) > 0$. ∎

Appendix B. Some auxiliary results

For reader's convenience, we state here some general results:

Lemma B.1. *(Lemma 2 in [5])*
Let (Y_n) be a sequence of real random variables, adapted to a filtration $\mathcal{F}$. If $\sum_{j\geq 1} j^{-2} E[Y_j^2] < +\infty$ and $E[Y_j|\mathcal{F}_{j-1}] \xrightarrow{a.s.} Y$ for some real random variable Y, then

$$n \sum_{j \geq n} \frac{Y_j}{j^2} \xrightarrow{a.s.} Y, \qquad \frac{1}{n} \sum_{j=1}^{n} Y_j \xrightarrow{a.s.} Y.$$

Lemma B.2. *(Th. 2 in [7] or a special case of Lemma A.2 in [11])*
Let $\mathcal{F}$ be a filtration and set $\mathcal{F}_\infty = \bigvee_n \mathcal{F}_n$. Then, for each sequence (Y_n) of integrable complex random variables, which is dominated in L^1 and which converges almost surely to a complex random variable Y, the conditional expectation $E[Y_n|\mathcal{F}_n]$ converges almost surely to the conditional expectation $E[Y|\mathcal{F}_\infty]$.

Appendix C. Stable convergence and its variants

This brief appendix contains some basic definitions and results concerning stable convergence and its variants. For more details, we refer the reader to [11, 13, 16, 18] and the references therein.

Let $(\Omega, \mathcal{A}, P)$ be a probability space, and let S be a Polish space, endowed with its Borel σ-field. A *kernel* on S, or a random probability measure on S, is a collection $K = \{K(\omega) : \omega \in \Omega\}$ of probability measures on the Borel σ-field of S such that, for each bounded Borel real function f on S, the map

$$\omega \mapsto Kf(\omega) = \int f(x)\, K(\omega)(dx)$$

is $\mathcal{A}$-measurable. Given a sub-σ-field $\mathcal{H}$ of $\mathcal{A}$, a kernel K is said $\mathcal{H}$-measurable if all the above random variables Kf are $\mathcal{H}$-measurable.

On $(\Omega, \mathcal{A}, P)$, let $(Y_n)_n$ be a sequence of S-valued random variables, let $\mathcal{H}$ be a sub-σ-field of $\mathcal{A}$, and let K be a $\mathcal{H}$-measurable kernel on S. Then we say that Y_n converges *$\mathcal{H}$-stably* to K, and we write $Y_n \longrightarrow K$ $\mathcal{H}$-stably, if

$$P(Y_n \in \cdot \,|\, H) \xrightarrow{weakly} E\left[K(\cdot) \,|\, H\right] \qquad \text{for all } H \in \mathcal{H} \text{ with } P(H) > 0,$$

where $K(\cdot)$ denotes the random variable defined, for each Borel set B of S, as $\omega \mapsto KI_B(\omega) = K(\omega)(B)$. In the case when $\mathcal{H} = \mathcal{A}$, we simply say that Y_n converges *stably* to K and we write $Y_n \longrightarrow K$ stably. Clearly, if $Y_n \longrightarrow K$ $\mathcal{H}$-stably, then Y_n converges in distribution to the probability distribution $E[K(\cdot)]$. Moreover, the $\mathcal{H}$-stable convergence of Y_n to K can be stated in terms of the following convergence of conditional expectations:

$$E[f(Y_n) \,|\, \mathcal{H}] \xrightarrow{\sigma(L^1, L^\infty)} Kf \tag{23}$$

for each bounded continuous real function f on S.

in [16] the notion of $\mathcal{H}$-stable convergence is firstly generalized in a natural way replacing in (23) the single sub-σ-field $\mathcal{H}$ by a collection $\mathcal{G} = (\mathcal{G}_n)_n$ (called conditioning system) of sub-σ-fields of $\mathcal{A}$ and then it is strengthened by substituting the convergence in $\sigma(L^1, L^\infty)$ by the one in probability (*i.e.* in L^1, since f is bounded). Hence, according to [16], we say that Y_n converges to K *stably in the strong sense*, with respect to $\mathcal{G} = (\mathcal{G}_n)_n$, if

$$E\left[f(Y_n) \,|\, \mathcal{G}_n\right] \xrightarrow{P} Kf \tag{24}$$

for each bounded continuous real function f on S.

Finally, a strengthening of the stable convergence in the strong sense can be naturally obtained if in (24) we replace the convergence in probability by the almost sure convergence (see [11]): given a conditioning system $\mathcal{G} = (\mathcal{G}_n)_n$, we say that Y_n converges to K in the sense of the *almost sure conditional convergence*, with respect to $\mathcal{G}$, if

$$E\left[f(Y_n) \,|\, \mathcal{G}_n\right] \xrightarrow{a.s.} Kf$$

for each bounded continuous real function f on S.

We conclude recalling two results. In particular, for the second one, we denote by $\mathcal{N}(\mu, \sigma^2)$ the Gaussian probability distribution with mean μ and variance $\sigma^2 \geq 0$ (where $\mathcal{N}(\mu, 0)$ means the Dirac distribution δ_μ concentrated in μ). Therefore, when U is a positive random variable, the symbol $\mathcal{N}(0, U)$ denotes the Gaussian kernel $\{\mathcal{N}(0, U(\omega)) : \omega \in \Omega\}\}$.

Theorem C.1. *(Lemma 1 in [5])*
Suppose that C_n and D_n are S-valued random variables, that M and N are kernels on S, and that $\mathcal{G} = (\mathcal{G}_n)_n$ is a filtration satisfying $\sigma(C_n) \subseteq \mathcal{G}_n$ and $\sigma(D_n) \subseteq \sigma(\cup_n \mathcal{G}_n)$ for all n. If C_n stably converges to M and D_n converges to N stably in the strong sense, with respect to $\mathcal{G}$, then $[C_n, D_n] \longrightarrow M \otimes N$ stably. (Here, $M \otimes N$ is the kernel on $S \times S$ such that $(M \otimes N)(\omega) = M(\omega) \otimes N(\omega)$ for all ω.)

This last result contains as a special case the fact that stable convergence and convergence in probability combine well: that is, if C_n stably converges to M and D_n converges in probability to a random variable D, then (C_n, D_n) stably converges to $M \otimes \delta_D$, where δ_D denotes the Dirac kernel concentrated in D. In particular, if M is the Gaussian kernel $\mathcal{N}(0, D)$, we have $C_n/\sqrt{D_n} \longrightarrow \mathcal{N}(0, 1)$ stably.

Theorem C.2. *(See Th. 1 together with Prop. 1 in [5] and Th. 10 in [6])*
Let (Y_n) be a bounded sequence of real random variables, adapted to a filtration $\mathcal{G} = (\mathcal{G}_n)$. Set

$$Z_n = E[Y_{n+1}|\mathcal{G}_n] \quad \textit{and} \quad M_n = \frac{1}{n}\sum_{j=1}^{n} Y_j.$$

Suppose that $n^3 E\left[\,(E[Z_{n+1}|\mathcal{G}_n] - Z_n)^2\,\right] \to 0$.

Then, $Z_n \xrightarrow{a.s.} Z$ and $M_n \xrightarrow{a.s.} Z$ for some real random variable Z. Moreover, $\sqrt{n}(Z_n - Z)$ converges in the sense of the almost sure conditional convergence with respect to $\mathcal{G}$ toward the Gaussian kernel $\mathcal{N}(0, V)$ for some real random variable V, provided

c1) $E\left[\sup_{j\geq 1} \sqrt{j}\,|Z_{j-1} - Z_j|\,\right] < +\infty$,

c2) $n\sum_{j\geq n}(Z_{j-1} - Z_j)^2 \xrightarrow{a.s.} V$.

If condition

c3) $n^{-1}\sum_{j=1}^{n}\left[Y_j - Z_{j-1} + j(Z_{j-1} - Z_j)\right]^2 \xrightarrow{P} U$

is also satisfied for some real random variable U, then

$$\left[\sqrt{n}\left(M_n - Z_n\right), \sqrt{n}(Z_n - Z)\right] \overset{stably}{\longrightarrow} \mathcal{N}\big(0,\, U\big) \otimes \mathcal{N}(0, V).$$

In particular, we have $\sqrt{n}\left(M_n - Z_n\right) \longrightarrow \mathcal{N}(0, U)$ stably and $\sqrt{n}\left(M_n - Z\right) \longrightarrow \mathcal{N}(0, U + V)$ stably.

References

[1] R. Aguech, N. Lasmar, and O. Selmi. A generalized urn with multiple drawing and random addition. *Annals of the Institute of Statistical Mathematics*, 71(2):389–408, Apr. 2019.

[2] R. Aguech and O. Selmi. Unbalanced multi-drawing urn with random addition matrix. *Arab Journal of Mathematical Sciences*, 2019.

[3] D. A. Aoudia and F. Perron. A new randomized Pólya urn model. *Appl. Math.*, 3:2118–2122, 2012.

[4] P. Berti, I. Crimaldi, L. Pratelli, and P. Rigo. Central limit theorems for multicolor urns with dominated colors. *Stoch. Process. their Appl.*, 120(8):1473–1491, 2010.

[5] P. Berti, I. Crimaldi, L. Pratelli, and P. Rigo. A central limit theorem and its applications to multicolor randomly reinforced urns. *Journal of Applied Probability*, 48(2):527–546, 2011.

[6] P. Berti, I. Crimaldi, L. Pratelli, and P. Rigo. Central limit theorems for an Indian buffet model with random weights. *The Annals of Applied Probability*, 25(2):523–547, 2015.

[7] D. Blackwell and L. Dubins. Merging of opinions with increasing information. *The Annals of Mathematical Statistics*, 33(3):882–886, 1962.

[8] M.-R. Chen. A time dependent Pólya urn with multiple drawings. *Probab. Eng. Informational Sci.*, 34(4):469–483, 2020.

[9] M.-R. Chen and M. Kuba. On generalized Pólya urn models. *J. Appl. Prob.*, 50:1169–1186, 2013.

[10] M.-R. Chen and C.-Z. Wei. A new urn model. *Journal of Applied Probability*, 42(4):964–976, Dec. 2005.

[11] I. Crimaldi. An almost sure conditional convergence result and an application to a generalized Pólya urn. *Int. Math. Forum*, 4(21-24):1139–1156, 2009.

[12] I. Crimaldi. Central limit theorems for a hypergeometric randomly reinforced urn. *Journal of Applied Probability*, 53(3):899–913, 2016.

[13] I. Crimaldi. *Introduzione alla nozione di convergenza stabile e sue varianti (Introduction to the notion of stable convergence and its variants)*, volume 57. Unione Matematica Italiana, Monograf s.r.l., Bologna, Italy., 2016. Book written in Italian.

[14] I. Crimaldi, P. Dai Pra, and I. G. Minelli. Fluctuation theorems for synchronization of interacting Pólya's urns. *Stochastic Process. Appl.*, 126(3):930–947, 2016.

[15] I. Crimaldi and F. Leisen. Asymptotic Results for a Generalized Pólya Urn with "Multi-Updating" and Applications to Clinical Trials. *Commun. Stat. - Theory Methods*, 37(17):2777–2794, July 2008.

[16] I. Crimaldi, G. Letta, and L. Pratelli. A strong form of stable convergence. In *Séminaire de Probabilités XL*, volume 1899 of *Lecture Notes in Math.*, pages 203–225. Springer, Berlin, 2007.

[17] F. Eggenberger and G. Pólya. Über die Statistik verketteter Vorgänge. *Z. Angewandte Math. Mech.*, 3:279–289, 1923.

[18] P. Hall and C. C. Heyde. *Martingale limit theory and its application*. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1980. Probability and Mathematical Statistics.

[19] I. Higueras, J. Moler, F. Plo, and M. San Miguel. Central limit theorems for generalized Pólya urn models. *Journal of Applied Probability*, 43(4):938–951, 2006.

[20] S. Idriss and N. Lasmar. Limit Theorems for Stochastic Approximations Algorithms With Application to General Urn Models. Hal-01726014, 2018.

[21] N. Johnson, S. Kotz, and H. Mahmoud. Pólya-Type Urn Models with Multiple Drawings. *J. Iran. Stat. Soc.*, 3(2):165–173, 2004.

[22] S. Kotz and N. Balakrishnan. *Advances in Urn Models during the Past Two Decades*, chapter 14, pages 203–257. Statistics for Industry and Technology. Birkhäuser Boston, 1997.

[23] M. Kuba. Classification of urn models with multiple drawings. Preprint Arxiv 1612.04354, 2016.

[24] M. Kuba, H. Mahmoud, and A. Panholzer. Analysis of a generalized Friedman's urn with multiple drawings. *Discrete Appl. Math.*, 161(18):2968–2984, Dec. 2013.

[25] M. Kuba and H. M. Mahmoud. Two-color balanced affine urn models with multiple drawings. *Adv. in Appl. Math.*, 90:1–26, Sept. 2017.

[26] M. Kuba and H. Sulzbach. On martingale tail sums in affine two-color urn models with multiple drawings. *J. Appl. Probab.*, 54(1):96–117, 2017.

[27] B. Laslier and J.-F. Laslier. Reinforcement learning from comparisons: Three alternatives are enough, two are not. *Ann. Appl. Probab.*, 27(5):2907–2925, Oct. 2017.
[28] N. Lasmar, C. Mailler, and O. Selmi. Multiple drawing multi-colour urns by stochastic approximation. *J. Appl. Probab.*, 55(1):254–281, 2018.
[29] M. Launay. Urns with simultaneous drawing. Preprint Arxiv 1201.3495, 2012.
[30] H. M. Mahmoud. *Pólya urn models*. Texts in Statistical Science Series. CRC Press, Boca Raton, FL, 2009.
[31] H. M. Mahmoud. Drawing multisets of balls from tenable balanced linear urns. *Probability in the Engineering and Informational Sciences*, 27(2):147–162, 2013.
[32] C. May and N. Flournoy. Asymptotics in response-adaptive designs generated by a two-color, randomly reinforced urn. *Ann. Statist.*, 37(2):1058–1078, 04 2009.
[33] R. Pastor-Satorras, C. Castellano, P. Van Mieghem, and A. Vespignani. Epidemic processes in complex networks. *Rev. Modern Phys.*, 87(3):925–979, 2015.
[34] R. Pemantle. A survey of random processes with reinforcement. *Probab. Surv.*, 4(1-79):1–79, 2007.
[35] R. Pemantle and S. Volkov. Vertex-reinforced random walk on **Z** has finite range. *Ann. Probab.*, 27(3):1368–1388, July 1999.