



# Elimination ideals and Bezout relations

Zbigniew Jelonek, André Galligo

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# FEW REMARKS ON IDEALS

ANDRE GALIGO & ZBIGNIEW JELONEK

## 1. INTRODUCTION

Let  $I \subset \mathbb{K}[x_1, \dots, x_n]$  be an ideal such that  $\dim V(I) = q$ . Using Hilbert Nullstellensatz we can easily see, that in the set  $I \cap \mathbb{K}[x_1, \dots, x_{q+1}]$  there exist non-zero polynomials. It is interesting to know the degree of the minimal polynomial of this type. Here using ideas from [1] we get a sharp estimate for the degree of such minimal polynomial in terms of degrees of generators of the ideal  $I$ . In fact, in general we solve this problem only for fields of characteristic zero.

## 2. MAIN RESULT

Let us recall (see [1]):

**Theorem 1.** (*Perron Theorem*) Let  $\mathbb{L}$  be a field and let  $Q_1, \dots, Q_{n+1} \in \mathbb{L}[x_1, \dots, x_m]$  be non-constant polynomials with  $\deg Q_i = d_i$ . If the mapping  $Q = (Q_1, \dots, Q_{n+1}) : \mathbb{L}^n \rightarrow \mathbb{L}^{n+1}$  is generically finite, then there exists a non-zero polynomial  $W(T_1, \dots, T_{n+1}) \in \mathbb{L}[T_1, \dots, T_{n+1}]$  such that

- (a)  $W(Q_1, \dots, Q_{n+1}) = 0$ ,
- (b)  $\deg W(T_1^{d_1}, T_2^{d_2}, \dots, T_{n+1}^{d_{n+1}}) \leq \prod_{j=1}^{n+1} d_j$ .

and (see [1]):

**Lemma 2.** Let  $\mathbb{K}$  be an infinite field. Let  $X \subset \mathbb{K}^m$  be an affine algebraic variety of dimension  $n$ . For sufficiently general numbers  $a_{ij} \in \mathbb{K}$  the mapping

$$\pi : X \ni (x_1, \dots, x_m) \rightarrow \left( \sum_{j=1}^m a_{1j}x_j, \sum_{j=2}^m a_{2j}x_j, \dots, \sum_{j=n}^m a_{1j}x_j \right) \in \mathbb{K}^n$$

is finite.  $\square$

**Theorem 3.** Let  $\mathbb{K}$  be an algebraically closed field and let  $f_1, \dots, f_s \in \mathbb{K}[x_1, \dots, x_n]$  be polynomials such  $\deg f_i = d_i$  where  $d_1 \geq d_2 \geq \dots \geq d_s$ . Assume that  $I = (f_1, \dots, f_s) \in \mathbb{K}[x_1, \dots, x_n]$  is an ideal, such that  $V(I)$  has dimension  $q$ . If we take a sufficiently general system of coordinates  $(x_1, \dots, x_n)$ , then there exist polynomials  $g_j \in \mathbb{K}[x_1, \dots, x_n]$  and a non-zero polynomial  $\phi(x) \in \mathbb{K}[x_1, \dots, x_{q+1}]$  such that

- (a)  $\deg g_j f_j \leq d_s \prod_{i=1}^{n-q-1} d_i$ ,
- (b)  $\phi(x) = \sum_{j=1}^k g_j f_j$ .

*Proof.* Take  $F_{n-q} = f_s$  and  $F_i = \sum_{j=i}^s \alpha_{ij} f_j$  for  $i = 1, \dots, n-q-1$ , where  $\alpha_{ij}$  are sufficiently general. Take  $J = (F_1, \dots, F_{n-q})$ . Then  $\deg F_{n-q} = d_s$  and  $\deg F_i = d_i$  for  $i = 1, \dots, n-q-1$ . Moreover,  $V(J)$  has pure dimension  $q$  and  $J \subset I$ . The mapping

$$\Phi : \mathbb{K}^n \times \mathbb{K} \ni (x, z) \rightarrow (F_1(x)z, \dots, F_{n-q}(x)z, x) \in \mathbb{K}^{n-q} \times \mathbb{K}^n$$

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is a (non-closed) embedding outside the set  $V(J) \times \mathbb{K}$ . Take  $\Gamma = \text{cl}(\Phi(\mathbb{K}^n \times \mathbb{K}))$ . Let  $\pi : \Gamma \rightarrow \mathbb{K}^{n+1}$  be a generic projection. Define  $\Psi := \pi \circ \Phi(x, z)$ . By Lemma 2 we can assume that

$$\Psi = \left( \sum_{j=1}^{n-q} \gamma_{1j} F_j z + l_1(x), \dots, \sum_{j=n-q}^{n-q} \gamma_{nj} F_j z + l_n(x), l_{n-q+1}(x), \dots, l_{n+1}(x) \right),$$

where  $l_1, \dots, l_{n+1}$  are generic linear form. In particular we can assume that  $l_{n-q+i}$ ,  $i = 1, \dots, q+1$  is the variable  $x_i$  in a new generic system of coordinates.

Apply Theorem 1 to  $\mathbb{L} = \mathbb{K}(z)$ , the polynomials  $\Psi_1, \dots, \Psi_{n+1} \in \mathbb{L}[x]$ . Thus there exists a non-zero polynomial  $W(T_1, \dots, T_{n+1}) \in \mathbb{L}[T_1, \dots, T_{n+1}]$  such that

$$W(\Psi_1, \dots, \Psi_{n+1}) = 0 \text{ and } \deg W(T_1^{d_1}, T_2^{d_2}, \dots, T_k^{d_k}, T_{k+1}, \dots, T_{n+1}) \leq d_s \prod_{j=1}^{n-q-1} d_j,$$

where  $k = n - q$ . Since the coefficients of  $W$  are in  $\mathbb{K}(z)$ , there is a non-zero polynomial  $\tilde{W} \in \mathbb{K}[T_1, \dots, T_{n+1}, Y]$  such that

- (a)  $\tilde{W}(\Psi_1(x, z), \dots, \Psi_{n+1}(x, z), z) = 0$ ,
- (b)  $\deg_T \tilde{W}(T_1^{d_1}, T_2^{d_2}, \dots, T_k^{d_k}, T_{k+1}, \dots, T_{n+1}, Y) \leq d_s \prod_{j=1}^{n-q-1} d_j$ , where  $\deg_T$  denotes the degree with respect to the variables  $T = (T_1, \dots, T_{n+1})$ .

Note that the mapping  $\Psi = (\Psi_1, \dots, \Psi_{n+1}) : \mathbb{K}^n \times \mathbb{K} \rightarrow \mathbb{K}^{n+1}$  is finite outside the set  $V(J) \times \mathbb{K}$ . Let  $\phi' = 0$  describes the image of the projection

$$\pi : V(J) \ni x \mapsto (x_1, \dots, x_{q+1}) \in \mathbb{K}^{q+1}$$

(recall that we consider generic system of coordinates).

The set of non-properness of the mapping  $\Psi$  is contained in the hypersurface  $S = \{T \in \mathbb{K}^{n+1} : \phi'(T) = 0\}$ . Since the mapping  $\Psi$  is finite outside  $S$ , for every  $H \in \mathbb{K}[x_1, \dots, x_n, z]$  there is a minimal polynomial  $P_H(T, Y) \in \mathbb{K}[T_1, \dots, T_{n+1}][Y]$  such that  $P_H(\Psi_1, \dots, \Psi_{n+1}, H) = \sum_{i=0}^r b_i(\Psi_1, \dots, \Psi_{n+1}) H^{r-i} = 0$  and the coefficient  $b_0$  satisfies  $\{T : b_0(T) = 0\} \subset S$ . Now set  $H = z$ .

We have

$$\deg_T P_z(T_1^{d_1}, T_2^{d_2}, \dots, T_n^{d_n}, T_{n+1}, Y) \leq d_s \prod_{j=1}^{n-q-1} d_j$$

and consequently we obtain the equality  $b_0(x_1, \dots, x_{q+1}) + \sum F_i g_i = 0$ , where  $\deg F_i g_i \leq \prod_{j=1}^{n-q} d_j$ . Set  $\phi = b_0$ . By the construction the polynomial  $\phi$  has zeros only on the image of the projection

$$\pi : V(J) \ni x \mapsto (x_1, \dots, x_{q+1}) \in \mathbb{K}^{q+1}.$$

□

**Remark 4.** Simple application of the Bezout theorem shows that our estimations on the degree of  $\phi$  is sharp.

**Corollary 5.** *Let  $I$  be as above. If  $V(I)$  has pure dimension  $q$  and  $I$  has not embedded components, then there is a polynomial  $\phi_1 \in \mathbb{K}[x_1, \dots, x_{q+1}]$  which describes the image of the projection*

$$\pi : V(I) \ni x \mapsto (x_1, \dots, x_{q+1}) \in \mathbb{K}^{q+1}$$

such that

- (a)  $\phi_1 \in I$ ,

$$(b) \deg \phi_1 \leq d_s \prod_{i=1}^{n-q-1} d_i.$$

*Proof.* Let  $I = \bigcap^r I_k$  be a primary decomposition of  $I$ . Then  $\dim V(I_k) = q$  for every  $k$ . Let  $\phi$  be a polynomial as above. If  $\phi = \phi_1 \phi_2$ , where  $\phi_2$  does not vanish on any component of  $V(I)$  then  $\phi_1 \in I_k$  for every  $k$  (by properties of primary ideals) and consequently  $\phi_1 \in I$ . But  $\phi_1$  describes the image of the projection

$$\pi : V(I) \ni x \mapsto (x_1, \dots, x_{q+1}) \in \mathbb{K}^{q+1}.$$

□

**Theorem 6.** *Let  $f_1, \dots, f_s \in \mathbb{C}[x_1, \dots, x_n]$  be polynomials such  $\deg f_i = d_i$  where  $d_1 \geq d_2 \dots \geq d_s$ . Let  $I = (f_1, \dots, f_s)$  be the ideal in  $\mathbb{C}[x_1, \dots, x_n]$  such that  $\dim V(I) = q$ . Then in  $I$  we can find a non-zero polynomial  $\phi(x) \in \mathbb{C}[x_1, \dots, x_{q+1}]$  such that  $\deg \phi \leq d_s \prod_{i=1}^{n-q-1} d_i$ .*

*Proof.* By theorem 3 for generic  $\alpha = (\alpha_{i,j}; i = 1, \dots, q+1, j \geq i)$  there exist a non-zero polynomial  $\phi_\alpha \in \mathbb{C}[t_1, \dots, t_{q+1}]$  such that

$$a) \deg \phi_\alpha \leq d_s \prod_{i=1}^{n-q-1} d_i,$$

$$b) \phi_\alpha(X_1(\alpha), X_2(\alpha), \dots, X_{q+1}(\alpha)) \in I, \text{ where } X_1(\alpha) = \alpha_{1,1}x_1 + \dots + \alpha_{1,n}x_n, X_2(\alpha) = \alpha_{2,2}x_1 + \dots + \alpha_{2,n}x_n, \dots, X_{q+1}(\alpha) = \alpha_{q+1,q+1}x_{q+1} + \dots + \alpha_{q+1,n}x_n.$$

For a polynomial  $p = \sum_\alpha a_\alpha x^\alpha \in \mathbb{C}[x_1, \dots, x_n]$  we define a norm  $\|p\| = \max_\alpha |a_\alpha|$ . In particular for every generic  $\alpha$  we can assume that  $\|\phi_\alpha\| = 1$  (we consider the polynomial  $\frac{\phi}{\|\phi\|}$  instead of  $\phi$ ). Moreover, we can take generic  $\alpha_m$  in this way that  $X_i(\alpha_m) \rightarrow x_i$  for  $m \rightarrow \infty$  and for  $i = 1, \dots, q+1$ . Polynomial  $\phi_\alpha$  we can treat as an element of a vector space  $B(D)$  of all polynomials from  $\mathbb{C}[x_1, \dots, x_n]$  of degree bounded by  $D = d_s \prod_{i=1}^{n-q-1} d_i$ . Since the norm of every  $\phi_{\alpha_m}$  is bounded by 1, we can assume that this sequence converges to a polynomial  $\phi$ , with norm 1 and of degree bounded by  $D = d_s \prod_{i=1}^{n-q-1} d_i$ . Thus also a sequence  $\phi_{\alpha_m}(X_1(\alpha_m), \dots, X_{q+1}(\alpha_m))$  tends to the same polynomial  $\phi$ . Of course it is non-zero because  $\|\phi\| = 1$ . It is enough to show that  $\phi \in I$ . However the space  $I \cap B(D)$  is a linear subspace of a finitely dimensional complex vector space  $B(D)$ , hence it is closed subset of  $B(D)$ . This finishes the proof. □

**Remark 7.** By Lefschetz Principle Theorem 6 holds for every field of characteristic zero.

**Corollary 8.** *Let  $f_1, \dots, f_s \in \mathbb{C}[x_1, \dots, x_n]$  be polynomials such  $\deg f_i = d_i$  where  $d_1 \geq d_2 \dots \geq d_s$ . Let  $I = (f_1, \dots, f_s)$  be the ideal in  $\mathbb{C}[x_1, \dots, x_n]$  such that  $\dim V(I) = 1$ . Assume that the image  $S$  of the projection  $x \mapsto (x_1, x_2)$  of a one dimensional part of  $V(I)$  into  $\mathbb{C}^2$  is a curve. Then in  $I$  we can find a non-zero polynomial  $\phi(x_1, x_2) = \phi_1(x_1, x_2)\rho(x_1) \in \mathbb{C}[x_1, x_2]$  such that*

$$a) \deg \phi \leq D^2 - 2D + 2, \text{ where } D = d_s \prod_{i=1}^{n-2} d_i,$$

$$b) \phi_1 \text{ describes the image of the projection } x \mapsto (x_1, x_2) \text{ of a one dimensional part of } V(I) \text{ into } \mathbb{C}^2.$$

*Proof.* Consider the ideal  $I_\alpha = (f_s, \sum_{i=1}^{s-1} \alpha_{1i} f_i, \sum_{i=2}^{s-1} \alpha_{2i} f_i, \dots, \sum_{i=n-2}^{s-1} \alpha_{2i} f_i)$ . By Theorem 6 we have that there exists a non-zero polynomial  $\phi_\alpha \in I \cap \mathbb{C}[x_1, x_2]$  of degree bounded by  $D$ . We can write  $\phi_\alpha = \phi_{1,\alpha} \phi_{2,\alpha}$ , where  $\phi_{1,\alpha}$  describes  $S$  and  $\phi_{2,\alpha}$  does not vanish on  $S$ . Since the degree of  $\phi_{1,\alpha}$  is bounded, there is infinitely many  $\alpha_i$  such that  $\phi_{1,\alpha_i}$  up to a multiplicative constant is the same. Moreover, the ideal  $J = (\phi_{2,\alpha_i}, i = 1, 2, \dots)$  describes the zero dimensional part of the image of the projection  $x \mapsto (x_1, x_2)$  of  $V(I)$  into  $\mathbb{C}^2$ . If  $\deg \phi_1 = a$ , then  $\deg \phi_{2,\alpha_i} \leq D - a$ . Consequently we can find a non zero polynomial  $\rho(x_1) \in J$ , such that  $\deg \rho \leq (D - a)^2$ . The ideal  $I$  contains a polynomial  $\phi\rho$  of degree

bounded by  $a + (D - a)^2$ . The expression  $a + (D - a)^2$ , where  $0 < a \leq D$  attains its maximal value for  $a = 1$ . This finishes the proof.  $\square$

**Remark 9.** If the ideal  $I$  has not embedded components, then we can assume that the polynomial  $\rho$  describes the image of the projection  $x \mapsto x_1$  of a zero dimensional part of  $V(I)$  into  $\mathbb{C}$ . In general case it can describe also some extra points.

#### REFERENCES

- [1] Jelonek, Z. *On the Effective Nullstellensatz*. Invent. Math. 162(2005), pp 1-17.