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FEW REMARKS ON IDEALS

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1. INTRODUCTION

Let $I \subset \mathbb{K}[x_1, \dots, x_n]$ be an ideal such that $\dim V(I) = q$. Using Hilbert Nullstellensatz we can easily see, that in the set $I \cap \mathbb{K}[x_1, \dots, x_{q+1}]$ there exist non-zero polynomials. It is interesting to know the degree of the minimal polynomial of this type. Here using ideas from [1] we get a sharp estimate for the degree of such minimal polynomial in terms of degrees of generators of the ideal I . In fact, in general we solve this problem only for fields of characteristic zero.

2. MAIN RESULT

Let us recall (see [1]):

Theorem 1. (*Perron Theorem*) Let \mathbb{L} be a field and let $Q_1, \dots, Q_{n+1} \in \mathbb{L}[x_1, \dots, x_m]$ be non-constant polynomials with $\deg Q_i = d_i$. If the mapping $Q = (Q_1, \dots, Q_{n+1}) : \mathbb{L}^n \rightarrow \mathbb{L}^{n+1}$ is generically finite, then there exists a non-zero polynomial $W(T_1, \dots, T_{n+1}) \in \mathbb{L}[T_1, \dots, T_{n+1}]$ such that

- (a) $W(Q_1, \dots, Q_{n+1}) = 0$,
- (b) $\deg W(T_1^{d_1}, T_2^{d_2}, \dots, T_{n+1}^{d_{n+1}}) \leq \prod_{j=1}^{n+1} d_j$.

and (see [1]):

Lemma 2. Let \mathbb{K} be an infinite field. Let $X \subset \mathbb{K}^m$ be an affine algebraic variety of dimension n . For sufficiently general numbers $a_{ij} \in \mathbb{K}$ the mapping

$$\pi : X \ni (x_1, \dots, x_m) \rightarrow \left(\sum_{j=1}^m a_{1j}x_j, \sum_{j=2}^m a_{2j}x_j, \dots, \sum_{j=n}^m a_{1j}x_j \right) \in \mathbb{K}^n$$

is finite. \square

Theorem 3. Let \mathbb{K} be an algebraically closed field and let $f_1, \dots, f_s \in \mathbb{K}[x_1, \dots, x_n]$ be polynomials such $\deg f_i = d_i$ where $d_1 \geq d_2 \geq \dots \geq d_s$. Assume that $I = (f_1, \dots, f_s) \in \mathbb{K}[x_1, \dots, x_n]$ is an ideal, such that $V(I)$ has dimension q . If we take a sufficiently general system of coordinates (x_1, \dots, x_n) , then there exist polynomials $g_j \in \mathbb{K}[x_1, \dots, x_n]$ and a non-zero polynomial $\phi(x) \in \mathbb{K}[x_1, \dots, x_{q+1}]$ such that

- (a) $\deg g_j f_j \leq d_s \prod_{i=1}^{n-q-1} d_i$,
- (b) $\phi(x) = \sum_{j=1}^k g_j f_j$.

Proof. Take $F_{n-q} = f_s$ and $F_i = \sum_{j=i}^s \alpha_{ij} f_j$ for $i = 1, \dots, n-q-1$, where α_{ij} are sufficiently general. Take $J = (F_1, \dots, F_{n-q})$. Then $\deg F_{n-q} = d_s$ and $\deg F_i = d_i$ for $i = 1, \dots, n-q-1$. Moreover, $V(J)$ has pure dimension q and $J \subset I$. The mapping

$$\Phi : \mathbb{K}^n \times \mathbb{K} \ni (x, z) \rightarrow (F_1(x)z, \dots, F_{n-q}(x)z, x) \in \mathbb{K}^{n-q} \times \mathbb{K}^n$$

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is a (non-closed) embedding outside the set $V(J) \times \mathbb{K}$. Take $\Gamma = \text{cl}(\Phi(\mathbb{K}^n \times \mathbb{K}))$. Let $\pi : \Gamma \rightarrow \mathbb{K}^{n+1}$ be a generic projection. Define $\Psi := \pi \circ \Phi(x, z)$. By Lemma 2 we can assume that

$$\Psi = \left(\sum_{j=1}^{n-q} \gamma_{1j} F_j z + l_1(x), \dots, \sum_{j=n-q}^{n-q} \gamma_{nj} F_j z + l_n(x), l_{n-q+1}(x), \dots, l_{n+1}(x) \right),$$

where l_1, \dots, l_{n+1} are generic linear form. In particular we can assume that l_{n-q+i} , $i = 1, \dots, q+1$ is the variable x_i in a new generic system of coordinates.

Apply Theorem 1 to $\mathbb{L} = \mathbb{K}(z)$, the polynomials $\Psi_1, \dots, \Psi_{n+1} \in \mathbb{L}[x]$. Thus there exists a non-zero polynomial $W(T_1, \dots, T_{n+1}) \in \mathbb{L}[T_1, \dots, T_{n+1}]$ such that

$$W(\Psi_1, \dots, \Psi_{n+1}) = 0 \text{ and } \deg W(T_1^{d_1}, T_2^{d_2}, \dots, T_k^{d_k}, T_{k+1}, \dots, T_{n+1}) \leq d_s \prod_{j=1}^{n-q-1} d_j,$$

where $k = n - q$. Since the coefficients of W are in $\mathbb{K}(z)$, there is a non-zero polynomial $\tilde{W} \in \mathbb{K}[T_1, \dots, T_{n+1}, Y]$ such that

- (a) $\tilde{W}(\Psi_1(x, z), \dots, \Psi_{n+1}(x, z), z) = 0$,
- (b) $\deg_T \tilde{W}(T_1^{d_1}, T_2^{d_2}, \dots, T_k^{d_k}, T_{k+1}, \dots, T_{n+1}, Y) \leq d_s \prod_{j=1}^{n-q-1} d_j$, where \deg_T denotes the degree with respect to the variables $T = (T_1, \dots, T_{n+1})$.

Note that the mapping $\Psi = (\Psi_1, \dots, \Psi_{n+1}) : \mathbb{K}^n \times \mathbb{K} \rightarrow \mathbb{K}^{n+1}$ is finite outside the set $V(J) \times \mathbb{K}$. Let $\phi' = 0$ describes the image of the projection

$$\pi : V(J) \ni x \mapsto (x_1, \dots, x_{q+1}) \in \mathbb{K}^{q+1}$$

(recall that we consider generic system of coordinates).

The set of non-properness of the mapping Ψ is contained in the hypersurface $S = \{T \in \mathbb{K}^{n+1} : \phi'(T) = 0\}$. Since the mapping Ψ is finite outside S , for every $H \in \mathbb{K}[x_1, \dots, x_n, z]$ there is a minimal polynomial $P_H(T, Y) \in \mathbb{K}[T_1, \dots, T_{n+1}][Y]$ such that $P_H(\Psi_1, \dots, \Psi_{n+1}, H) = \sum_{i=0}^r b_i(\Psi_1, \dots, \Psi_{n+1}) H^{r-i} = 0$ and the coefficient b_0 satisfies $\{T : b_0(T) = 0\} \subset S$. Now set $H = z$.

We have

$$\deg_T P_z(T_1^{d_1}, T_2^{d_2}, \dots, T_n^{d_n}, T_{n+1}, Y) \leq d_s \prod_{j=1}^{n-q-1} d_j$$

and consequently we obtain the equality $b_0(x_1, \dots, x_{q+1}) + \sum F_i g_i = 0$, where $\deg F_i g_i \leq \prod_{j=1}^{n-q} d_j$. Set $\phi = b_0$. By the construction the polynomial ϕ has zeros only on the image of the projection

$$\pi : V(J) \ni x \mapsto (x_1, \dots, x_{q+1}) \in \mathbb{K}^{q+1}.$$

□

Remark 4. Simple application of the Bezout theorem shows that our estimations on the degree of ϕ is sharp.

Corollary 5. *Let I be as above. If $V(I)$ has pure dimension q and I has not embedded components, then there is a polynomial $\phi_1 \in \mathbb{K}[x_1, \dots, x_{q+1}]$ which describes the image of the projection*

$$\pi : V(I) \ni x \mapsto (x_1, \dots, x_{q+1}) \in \mathbb{K}^{q+1}$$

such that

- (a) $\phi_1 \in I$,

$$(b) \deg \phi_1 \leq d_s \prod_{i=1}^{n-q-1} d_i.$$

Proof. Let $I = \bigcap^r I_k$ be a primary decomposition of I . Then $\dim V(I_k) = q$ for every k . Let ϕ be a polynomial as above. If $\phi = \phi_1 \phi_2$, where ϕ_2 does not vanish on any component of $V(I)$ then $\phi_1 \in I_k$ for every k (by properties of primary ideals) and consequently $\phi_1 \in I$. But ϕ_1 describes the image of the projection

$$\pi : V(I) \ni x \mapsto (x_1, \dots, x_{q+1}) \in \mathbb{K}^{q+1}.$$

□

Theorem 6. *Let $f_1, \dots, f_s \in \mathbb{C}[x_1, \dots, x_n]$ be polynomials such $\deg f_i = d_i$ where $d_1 \geq d_2 \dots \geq d_s$. Let $I = (f_1, \dots, f_s)$ be the ideal in $\mathbb{C}[x_1, \dots, x_n]$ such that $\dim V(I) = q$. Then in I we can find a non-zero polynomial $\phi(x) \in \mathbb{C}[x_1, \dots, x_{q+1}]$ such that $\deg \phi \leq d_s \prod_{i=1}^{n-q-1} d_i$.*

Proof. By theorem 3 for generic $\alpha = (\alpha_{i,j}; i = 1, \dots, q+1, j \geq i)$ there exist a non-zero polynomial $\phi_\alpha \in \mathbb{C}[t_1, \dots, t_{q+1}]$ such that

$$a) \deg \phi_\alpha \leq d_s \prod_{i=1}^{n-q-1} d_i,$$

$$b) \phi_\alpha(X_1(\alpha), X_2(\alpha), \dots, X_{q+1}(\alpha)) \in I, \text{ where } X_1(\alpha) = \alpha_{1,1}x_1 + \dots + \alpha_{1,n}x_n, X_2(\alpha) = \alpha_{2,2}x_1 + \dots + \alpha_{2,n}x_n, \dots, X_{q+1}(\alpha) = \alpha_{q+1,q+1}x_{q+1} + \dots + \alpha_{q+1,n}x_n.$$

For a polynomial $p = \sum_\alpha a_\alpha x^\alpha \in \mathbb{C}[x_1, \dots, x_n]$ we define a norm $\|p\| = \max_\alpha |a_\alpha|$. In particular for every generic α we can assume that $\|\phi_\alpha\| = 1$ (we consider the polynomial $\frac{\phi}{\|\phi\|}$ instead of ϕ). Moreover, we can take generic α_m in this way that $X_i(\alpha_m) \rightarrow x_i$ for $m \rightarrow \infty$ and for $i = 1, \dots, q+1$. Polynomial ϕ_α we can treat as an element of a vector space $B(D)$ of all polynomials from $\mathbb{C}[x_1, \dots, x_n]$ of degree bounded by $D = d_s \prod_{i=1}^{n-q-1} d_i$. Since the norm of every ϕ_{α_m} is bounded by 1, we can assume that this sequence converges to a polynomial ϕ , with norm 1 and of degree bounded by $D = d_s \prod_{i=1}^{n-q-1} d_i$. Thus also a sequence $\phi_{\alpha_m}(X_1(\alpha_m), \dots, X_{q+1}(\alpha_m))$ tends to the same polynomial ϕ . Of course it is non-zero because $\|\phi\| = 1$. It is enough to show that $\phi \in I$. However the space $I \cap B(D)$ is a linear subspace of a finitely dimensional complex vector space $B(D)$, hence it is closed subset of $B(D)$. This finishes the proof. □

Remark 7. By Lefchetz Principle Theorem 6 holds for every field of characteristic zero.

Corollary 8. *Let $f_1, \dots, f_s \in \mathbb{C}[x_1, \dots, x_n]$ be polynomials such $\deg f_i = d_i$ where $d_1 \geq d_2 \dots \geq d_s$. Let $I = (f_1, \dots, f_s)$ be the ideal in $\mathbb{C}[x_1, \dots, x_n]$ such that $\dim V(I) = 1$. Assume that the image S of the projection $x \mapsto (x_1, x_2)$ of a one dimensional part of $V(I)$ into \mathbb{C}^2 is a curve. Then in I we can find a non-zero polynomial $\phi(x_1, x_2) = \phi_1(x_1, x_2)\rho(x_1) \in \mathbb{C}[x_1, x_2]$ such that*

$$a) \deg \phi \leq D^2 - 2D + 2, \text{ where } D = d_s \prod_{i=1}^{n-2} d_i,$$

b) ϕ_1 describes the image of the projection $x \mapsto (x_1, x_2)$ of a one dimensional part of $V(I)$ into \mathbb{C}^2 .

Proof. Consider the ideal $I_\alpha = (f_s, \sum_{i=1}^{s-1} \alpha_{1i}f_i, \sum_{i=2}^{s-1} \alpha_{2i}f_i, \dots, \sum_{i=n-2}^{s-1} \alpha_{2i}f_i)$. By Theorem 6 we have that there exists a non-zero polynomial $\phi_\alpha \in I \cap \mathbb{C}[x_1, x_2]$ of degree bounded by D . We can write $\phi_\alpha = \phi_{1,\alpha}\phi_{2,\alpha}$, where $\phi_{1,\alpha}$ describes S and $\phi_{2,\alpha}$ does not vanish on S . Since the degree of $\phi_{1,\alpha}$ is bounded, there is infinitely many α_i such that ϕ_{1,α_i} up to a multiplicative constant is the same. Moreover, the ideal $J = (\phi_{2,\alpha_i}, i = 1, 2, \dots)$ describes the zero dimensional part of the image of the projection $x \mapsto (x_1, x_2)$ of $V(I)$ into \mathbb{C}^2 . If $\deg \phi_1 = a$, then $\deg \phi_{2,\alpha_i} \leq D - a$. Consequently we can find a non zero polynomial $\rho(x_1) \in J$, such that $\deg \rho \leq (D - a)^2$. The ideal I contains a polynomial $\phi\rho$ of degree

bounded by $a + (D - a)^2$. The expression $a + (D - a)^2$, where $0 < a \leq D$ attains its maximal value for $a = 1$. This finishes the proof. \square

Remark 9. If the ideal I has not embedded components, then we can assume that the polynomial ρ describes the image of the projection $x \mapsto x_1$ of a zero dimensional part of $V(I)$ into \mathbb{C} . In general case it can describe also some extra points.

REFERENCES

- [1] Jelonek, Z. *On the Effective Nullstellensatz*. Invent. Math. 162(2005), pp 1-17.