

Elimination ideals and Bezout relations

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FEW REMARKS ON IDEALS

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1. INTRODUCTION

Let $I \subset \mathbb{K}[x_1, ..., x_n]$ be an ideal such that dim V(I) = q. Using Hilbert Nullstellensatz we can easily see, that in the set $I \cap \mathbb{K}[x_1, ..., x_{q+1}]$ there exist non-zero polynomials. It is interesting to know the degree of the minimal polynomial of this type. Here using ideas from [1] we get a sharp estimate for the degree of such minimal polynomial in terms of degrees of generators of the ideal I. In fact, in general we solve this problem only for fields of characteristic zero.

2. Main Result

Let us recall (see [1]):

Theorem 1. (Perron Theorem) Let \mathbb{L} be a field and let $Q_1, \ldots, Q_{n+1} \in \mathbb{L}[x_1, \ldots, x_m]$ be non-constant polynomials with deg $Q_i = d_i$. If the mapping $Q = (Q_1, \ldots, Q_{n+1}) :$ $\mathbb{L}^n \to \mathbb{L}^{n+1}$ is generically finite, then there exists a non-zero polynomial $W(T_1, \ldots, T_{n+1}) \in$ $\mathbb{L}[T_1, \ldots, T_{n+1}]$ such that

(a) $W(Q_1, \dots, Q_{n+1}) = 0,$ (b) deg $W(T_1^{d_1}, T_2^{d_2}, \dots, T_{n+1}^{d_{n+1}}) \le \prod_{j=1}^{n+1} d_j.$

Lemma 2. Let \mathbb{K} be an infinite field. Let $X \subset \mathbb{K}^m$ be an affine algebraic variety of dimension n. For sufficiently general numbers $a_{ij} \in \mathbb{K}$ the mapping

$$\pi: X \ni (x_1, \dots, x_m) \to \left(\sum_{j=1}^m a_{1j} x_j, \sum_{j=2}^m a_{2j} x_j, \dots, \sum_{j=n}^m a_{1j} x_j\right) \in \mathbb{K}^n$$

is finite. \Box

Theorem 3. Let \mathbb{K} be an algebraically closed field and let $f_1, \ldots, f_s \in \mathbb{K}[x_1, \ldots, x_n]$ be polynomials such deg $f_i = d_i$ where $d_1 \ge d_2 \ldots \ge d_s$. Assume that $I = (f_1, \ldots, f_s) \in \mathbb{K}[x_1, \ldots, x_n]$ is an ideal, such that V(I) has dimension q. If we take a sufficiently general system of coordinates (x_1, \ldots, x_n) , then there exist polynomials $g_j \in \mathbb{K}[x_1, \ldots, x_n]$ and a non-zero polynomial $\phi(x) \in \mathbb{K}[x_1, \ldots, x_{q+1}]$ such that

- (a) deg $g_j f_j \leq d_s \prod_{i=1}^{n-q-1} d_i$,
- (b) $\phi(x) = \sum_{j=1}^{k} g_j f_j.$

Proof. Take $F_{n-q} = f_s$ and $F_i = \sum_{j=i}^s \alpha_{ij} f_j$ for i = 1, ..., n-q-1, whre α_{ij} are sufficiently general. Take $J = (F_1, ..., F_{n-q})$. Then deg $F_{n-q} = d_s$ and deg $F_i = d_i$ for i = 1, ..., n-q-1. Moreover, V(J) has pure dimension q and $J \subset I$. The mapping

$$\Phi: \mathbb{K}^n \times \mathbb{K} \ni (x, z) \to (F_1(x)z, \dots, F_{n-q}(x)z, x) \in \mathbb{K}^{n-q} \times \mathbb{K}^n$$

and (see [1]):

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is a (non-closed) embedding outside the set $V(J) \times \mathbb{K}$. Take $\Gamma = cl(\Phi(\mathbb{K}^n \times \mathbb{K}))$. Let $\pi : \Gamma \to \mathbb{K}^{n+1}$ be a generic projection. Define $\Psi := \pi \circ \Phi(x, z)$. By Lemma 2 we can assume that

$$\Psi = \left(\sum_{j=1}^{n-q} \gamma_{1j} F_j z + l_1(x), \dots, \sum_{j=n-q}^{n-q} \gamma_{nj} F_j z + l_n(x), l_{n-q+1}(x), \dots, l_{n+1}(x)\right),$$

where l_1, \ldots, l_{n+1} are generic linear form. In particular we can assume that l_{n-q+i} , $i = 1, \ldots, q+1$ is the variable x_i in a new generic system of coordinates.

Apply Theorem 1 to $\mathbb{L} = \mathbb{K}(z)$, the polynomials $\Psi_1, \ldots, \Psi_{n+1} \in \mathbb{L}[x]$. Thus there exists a non-zero polynomial $W(T_1, \ldots, T_{n+1}) \in \mathbb{L}[T_1, \ldots, T_{n+1}]$ such that

$$W(\Psi_1, \dots, \Psi_{n+1}) = 0$$
 and $\deg W(T_1^{d_1}, T_2^{d_2}, \dots, T_k^{d_k}, T_{k+1}, \dots, T_{n+1}) \le d_s \prod_{j=1}^{n-q-1} d_j,$

where k = n - q. Since the coefficients of W are in $\mathbb{K}(z)$, there is a non-zero polynomial $\tilde{W} \in \mathbb{K}[T_1, \ldots, T_{n+1}, Y]$ such that

(a) $\tilde{W}(\Psi_1(x,z),\ldots,\Psi_{n+1}(x,z),z) = 0,$

(b) $\deg_T \tilde{W}(T_1^{d_1}, T_2^{d_2}, \ldots, T_k^{d_k}, T_{k+1}, \ldots, T_{n+1}, Y) \leq d_s \prod_{j=1}^{n-q-1} d_j$, where \deg_T denotes the degree with respect to the variables $T = (T_1, \ldots, T_{n+1})$.

Note that the mapping $\Psi = (\Psi_1, \ldots, \Psi_{n+1}) : \mathbb{K}^n \times \mathbb{K} \to \mathbb{K}^{n+1}$ is finite outside the set $V(J) \times \mathbb{K}$. Let $\phi' = 0$ describes the image of the projection

$$\pi: V(J) \ni x \mapsto (x_1, \dots, x_{q+1}) \in \mathbb{K}^{q+1}$$

(recall that we consider generic system of coordinates).

The set of non-properness of the mapping Ψ is contained in the hypersurface $S = \{T \in \mathbb{K}^{n+1} : \phi'(T) = 0\}$. Since the mapping Ψ is finite outside S, for every $H \in \mathbb{K}[x_1, \ldots, x_n, z]$ there is a minimal polynomial $P_H(T, Y) \in \mathbb{K}[T_1, \ldots, T_{n+1}][Y]$ such that $P_H(\Psi_1, \ldots, \Psi_{n+1}, H) = \sum_{i=0}^r b_i(\Psi_1, \ldots, \Psi_{n+1})H^{r-i} = 0$ and the coefficient b_0 satisfies $\{T : b_0(T) = 0\} \subset S$. Now set H = z.

We have

$$\deg_T P_z(T_1^{d_1}, T_2^{d_2}, \dots, T_n^{d_n}, T_{n+1}, Y) \le d_s \prod_{j=1}^{n-q-1} d_j$$

and consequently we obtain the equality $b_0(x_1, ..., x_{q+1}) + \sum F_i g_i = 0$, where deg $F_i g_i \leq \prod_{j=1}^{n-q} d_j$. Set $\phi = b_0$. By the construction the polynomial ϕ has zeros only on the image of the projection

$$\pi: V(J) \ni x \mapsto (x_1, \dots, x_{q+1}) \in \mathbb{K}^{q+1}.$$

Remark 4. Simple application of the Bezout theorem shows that our estimations on the degree of ϕ is sharp.

Corollary 5. Let I be as above. If V(I) has pure dimension q and I has not embedded components, then there is a polynomial $\phi_1 \in \mathbb{K}[x_1, ..., x_{q+1}]$ which describes the image of the projection

$$\pi: V(I) \ni x \mapsto (x_1, \dots, x_{q+1}) \in \mathbb{K}^{q+1}$$

such that

(a) $\phi_1 \in I$,

(b) $\deg \phi_1 \le d_s \prod_{i=1}^{n-q-1} d_i$.

Proof. Let $I = \bigcap^r I_k$ be a primary decomposition of I. Then dim $V(I_k) = q$ for every k. Let ϕ be a polynomial as above. If $\phi = \phi_1 \phi_2$, where ϕ_2 does not vanish on any component of V(I) then $\phi_1 \in I_k$ for every k (by properties of primary ideals) and consequently $\phi_1 \in I$. But ϕ_1 describes the image of the projection

$$\pi: V(I) \ni x \mapsto (x_1, \dots, x_{q+1}) \in \mathbb{K}^{q+1}.$$

Theorem 6. Let $f_1, \ldots, f_s \in \mathbb{C}[x_1, \ldots, x_n]$ be polynomials such deg $f_i = d_i$ where $d_1 \ge d_2 \ldots \ge d_s$. Let $I = (f_1, \ldots, f_s)$ be the ideal in $\mathbb{C}[x_1, \ldots, x_n]$ such that dim V(I) = q. Then in I we can find a non-zero polynomial $\phi(x) \in \mathbb{C}[x_1, \ldots, x_{q+1}]$ such that deg $\phi \le d_s \prod_{i=1}^{n-q-1} d_i$.

Proof. By theorem 3 for generic $\alpha = (\alpha_{i,j}; i = 1, ..., q + 1, j \ge i)$ there exist a non-zero polynomial $\phi_{\alpha} \in \mathbb{C}[t_1, ..., t_{q+1}]$ such that

a) deg $\phi_{\alpha} \leq d_s \prod_{i=1}^{n-q-1} d_i$,

b) $\phi_{\alpha}(X_1(\alpha), X_2(\alpha), ..., X_{q+1}(\alpha)) \in I$, where $X_1(\alpha) = \alpha_{1,1}x_1 + ... + \alpha_{1,n}x_n$, $X_2(\alpha) = \alpha_{2,2}x_1 + ... + \alpha_{2,n}x_n$, ..., $X_{q+1}(\alpha) = \alpha_{q+1,q+1}x_{q+1} + ... + \alpha_{q+1,n}x_n$.

For a polynomial $p = \sum_{\alpha} a_{\alpha} x^{\alpha} \in \mathbb{C}[x_1, ..., x_n]$ we define a norm $||p|| = max_{\alpha}|a_{\alpha}|$. In particular for every generic α we can assume that $||\phi_{\alpha}|| = 1$ (we consider the polynomial $\frac{\phi}{||\phi||}$ instead of ϕ). Moreover, we can take generic α_m in this way that $X_i(\alpha_m) \to x_i$ for $m \to \infty$ and for i = 1, ..., q + 1. Polynomial ϕ_{α} we can treat as an element of a vector space B(D) of all polynomials from $\mathbb{C}[x_1, ..., x_n]$ of degree bounded by $D = d_s \prod_{i=1}^{n-q-1} d_i$. Since the norm of every ϕ_{α_m} is bounded by 1, we can assume that this sequence converges to a polynomial ϕ , with norm 1 and of degree bounded by $D = d_s \prod_{i=1}^{n-q-1} d_i$. Thus also a sequence $\phi_{\alpha_m}(X_1(\alpha_m), ..., X_{q+1}(\alpha_m))$ tends to the same polynomial ϕ . Of course it is non-zero because $||\phi|| = 1$. It is enough to show that $\phi \in I$. However the space $I \cap B(D)$ is a linear subspace of a finitely dimensional complex vector space B(D), hence it is closed subset of B(D). This finishes the proof.

Remark 7. By Lefchetz Principle Theorem 6 holds for every field of characteristic zero.

Corollary 8. Let $f_1, \ldots, f_s \in \mathbb{C}[x_1, \ldots, x_n]$ be polynomials such deg $f_i = d_i$ where $d_1 \geq d_2 \ldots \geq d_s$. Let $I = (f_1, \ldots, f_s)$ be the ideal in $\mathbb{C}[x_1, \ldots, x_n]$ such that dim V(I) = 1. Assume that the image S of the projection $x \mapsto (x_1, x_2)$ of a one dimensional part of V(I) into \mathbb{C}^2 is a curve. Then in I we can find a non-zero polynomial $\phi(x_1, x_2) = \phi_1(x_1, x_2)\rho(x_1) \in \mathbb{C}[x_1, x_2]$ such that

a) deg $\phi \leq D^2 - 2D + 2$, where $D = d_s \prod_{i=1}^{n-2} d_i$,

b) ϕ_1 describes the image of the projection $x \mapsto (x_1, x_2)$ of a one dimensional part of V(I) into \mathbb{C}^2 .

Proof. Consider the ideal $I_{\alpha} = (f_s, \sum_{i=1}^{s-1} \alpha_{1i} f_i, \sum_{i=2}^{s-1} \alpha_{2i} f_i, \dots, \sum_{i=n-2}^{s-1} \alpha_{2i} f_i)$. By Theorem 6 we have that there exists a non-zero polynomial $\phi_{\alpha} \in I \cap \mathbb{C}[x_1, x_2]$ of degree bounded by D. We can write $\phi_{\alpha} = \phi_{1,\alpha} \phi_{2,\alpha}$, where $\phi_{1,\alpha}$ describes S and $\phi_{2,\alpha}$ does not vanish on S. Since the degree of $\phi_{1,\alpha}$ is bounded, there is infinitely many α_i such that ϕ_{1,α_i} up to a multiplicative constant is the same. Moreover, the ideal $J = (\phi_{2,\alpha_i}, i = 1, 2, \dots)$ describes the zero dimensional part of the image of the projection $x \mapsto (x_1, x_2)$ of V(I) into \mathbb{C}^2 . If deg $\phi_1 = a$, then deg $\phi_{2,\alpha_i} \leq D - a$. Consequently we can find a non zero polynomial $\rho(x_1) \in J$, such that deg $\rho \leq (D - a)^2$. The ideal I contains a polynomial $\phi\rho$ of degree

bounded by $a + (D - a)^2$. The expression $a + (D - a)^2$, where $0 < a \le D$ attains its maximal value for a = 1. This finishes the proof.

Remark 9. If the ideal I has not embedded components, then we can assume that the polynomial ρ describes the image of the projection $x \mapsto x_1$ of a zero dimensional part of V(I) into \mathbb{C} . In general case it can describe also some extra points.

References

[1] Jelonek, Z. On the Effective Nullstellensatz. Invent. Math. 162(2005), pp 1-17.