# Elimination ideals and Bezout relations 

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# FEW REMARKS ON IDEALS 

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## 1. Introduction

Let $I \subset \mathbb{K}\left[x_{1}, \ldots, x_{n}\right]$ be an ideal such that $\operatorname{dim} V(I)=q$. Using Hilbert Nullstellensatz we can easily see, that in the set $I \cap \mathbb{K}\left[x_{1}, \ldots, x_{q+1}\right]$ there exist non-zero polynomials. It is interesting to know the degree of the minimal polynomial of this type. Here using ideas from [1] we get a sharp estimate for the degree of such minimal polynomial in terms of degrees of generators of the ideal $I$. In fact, in general we solve this problem only for fields of characteristic zero.

## 2. Main Result

Let us recall (see [1]):
Theorem 1. (Perron Theorem) Let $\mathbb{L}$ be a field and let $Q_{1}, \ldots, Q_{n+1} \in \mathbb{L}\left[x_{1}, \ldots, x_{m}\right]$ be non-constant polynomials with $\operatorname{deg} Q_{i}=d_{i}$. If the mapping $Q=\left(Q_{1}, \ldots, Q_{n+1}\right)$ : $\mathbb{L}^{n} \rightarrow \mathbb{L}^{n+1}$ is generically finite, then there exists a non-zero polynomial $W\left(T_{1}, \ldots, T_{n+1}\right) \in$ $\mathbb{L}\left[T_{1}, \ldots, T_{n+1}\right]$ such that
(a) $W\left(Q_{1}, \ldots, Q_{n+1}\right)=0$,
(b) $\operatorname{deg} W\left(T_{1}^{d_{1}}, T_{2}^{d_{2}}, \ldots, T_{n+1}^{d_{n+1}}\right) \leq \prod_{j=1}^{n+1} d_{j}$.
and (see [1]):
Lemma 2. Let $\mathbb{K}$ be an infinite field. Let $X \subset \mathbb{K}^{m}$ be an affine algebraic variety of dimension $n$. For sufficiently general numbers $a_{i j} \in \mathbb{K}$ the mapping

$$
\pi: X \ni\left(x_{1}, \ldots, x_{m}\right) \rightarrow\left(\sum_{j=1}^{m} a_{1 j} x_{j}, \sum_{j=2}^{m} a_{2 j} x_{j}, \ldots, \sum_{j=n}^{m} a_{1 j} x_{j}\right) \in \mathbb{K}^{n}
$$

is finite.
Theorem 3. Let $\mathbb{K}$ be an algebraically closed field and let $f_{1}, \ldots, f_{s} \in \mathbb{K}\left[x_{1}, \ldots, x_{n}\right]$ be polynomials such $\operatorname{deg} f_{i}=d_{i}$ where $d_{1} \geq d_{2} \ldots \geq d_{s}$. Assume that $I=\left(f_{1}, \ldots, f_{s}\right) \in$ $\mathbb{K}\left[x_{1}, \ldots, x_{n}\right]$ is an ideal, such that $V(I)$ has dimension $q$. If we take a sufficiently general system of coordinates $\left(x_{1}, \ldots, x_{n}\right)$, then there exist polynomials $g_{j} \in \mathbb{K}\left[x_{1}, \ldots, x_{n}\right]$ and a non-zero polynomial $\phi(x) \in \mathbb{K}\left[x_{1}, \ldots, x_{q+1}\right]$ such that
(a) $\operatorname{deg} g_{j} f_{j} \leq d_{s} \prod_{i=1}^{n-q-1} d_{i}$,
(b) $\phi(x)=\sum_{j=1}^{k} g_{j} f_{j}$.

Proof. Take $F_{n-q}=f_{s}$ and $F_{i}=\sum_{j=i}^{s} \alpha_{i j} f_{j}$ for $i=1, \ldots, n-q-1$, whre $\alpha_{i j}$ are sufficiently general. Take $J=\left(F_{1}, \ldots, F_{n-q}\right)$. Then $\operatorname{deg} F_{n-q}=d_{s}$ and $\operatorname{deg} F_{i}=d_{i}$ for $i=1, \ldots, n-q-1$. Moreover, $V(J)$ has pure dimension $q$ and $J \subset I$. The mapping

$$
\Phi: \mathbb{K}^{n} \times \mathbb{K} \ni(x, z) \rightarrow\left(F_{1}(x) z, \ldots, F_{n-q}(x) z, x\right) \in \mathbb{K}^{n-q} \times \mathbb{K}^{n}
$$

[^0]is a (non-closed) embedding outside the set $V(J) \times \mathbb{K}$. Take $\Gamma=\operatorname{cl}\left(\Phi\left(\mathbb{K}^{n} \times \mathbb{K}\right)\right)$. Let $\pi: \Gamma \rightarrow \mathbb{K}^{n+1}$ be a generic projection. Define $\Psi:=\pi \circ \Phi(x, z)$. By Lemma 2 we can assume that
$$
\Psi=\left(\sum_{j=1}^{n-q} \gamma_{1 j} F_{j} z+l_{1}(x), \ldots, \sum_{j=n-q}^{n-q} \gamma_{n j} F_{j} z+l_{n}(x), l_{n-q+1}(x), \ldots, l_{n+1}(x)\right),
$$
where $l_{1}, \ldots, l_{n+1}$ are generic linear form. In particular we can assume that $l_{n-q+i}, i=$ $1, . ., q+1$ is the variable $x_{i}$ in a new generic system of coordinates.

Apply Theorem 1 to $\mathbb{L}=\mathbb{K}(z)$, the polynomials $\Psi_{1}, \ldots, \Psi_{n+1} \in \mathbb{L}[x]$. Thus there exists a non-zero polynomial $W\left(T_{1}, \ldots, T_{n+1}\right) \in \mathbb{L}\left[T_{1}, \ldots, T_{n+1}\right]$ such that

$$
W\left(\Psi_{1}, \ldots, \Psi_{n+1}\right)=0 \text { and } \operatorname{deg} W\left(T_{1}^{d_{1}}, T_{2}^{d_{2}}, \ldots, T_{k}^{d_{k}}, T_{k+1}, \ldots, T_{n+1}\right) \leq d_{s} \prod_{j=1}^{n-q-1} d_{j}
$$

where $k=n-q$. Since the coefficients of $W$ are in $\mathbb{K}(z)$, there is a non-zero polynomial $\tilde{W} \in \mathbb{K}\left[T_{1}, \ldots, T_{n+1}, Y\right]$ such that
(a) $\tilde{W}\left(\Psi_{1}(x, z), \ldots, \Psi_{n+1}(x, z), z\right)=0$,
(b) $\operatorname{deg}_{T} \tilde{W}\left(T_{1}^{d_{1}}, T_{2}^{d_{2}}, \ldots, T_{k}^{d_{k}}, T_{k+1}, \ldots, T_{n+1}, Y\right) \leq d_{s} \prod_{j=1}^{n-q-1} d_{j}$, where $\operatorname{deg}_{T}$ denotes the degree with respect to the variables $T=\left(T_{1}, \ldots, T_{n+1}\right)$.
Note that the mapping $\Psi=\left(\Psi_{1}, \ldots, \Psi_{n+1}\right): \mathbb{K}^{n} \times \mathbb{K} \rightarrow \mathbb{K}^{n+1}$ is finite outside the set $V(J) \times \mathbb{K}$. Let $\phi^{\prime}=0$ describes the image of the projection

$$
\pi: V(J) \ni x \mapsto\left(x_{1}, \ldots, x_{q+1}\right) \in \mathbb{K}^{q+1}
$$

(recall that we consider generic system of coordinates).
The set of non-properness of the mapping $\Psi$ is contained in the hypersurface $S=$ $\left\{T \in \mathbb{K}^{n+1}: \phi^{\prime}(T)=0\right\}$. Since the mapping $\Psi$ is finite outside $S$, for every $H \in$ $\mathbb{K}\left[x_{1}, \ldots, x_{n}, z\right]$ there is a minimal polynomial $P_{H}(T, Y) \in \mathbb{K}\left[T_{1}, \ldots, T_{n+1}\right][Y]$ such that $P_{H}\left(\Psi_{1}, \ldots, \Psi_{n+1}, H\right)=\sum_{i=0}^{r} b_{i}\left(\Psi_{1}, \ldots, \Psi_{n+1}\right) H^{r-i}=0$ and the coefficient $b_{0}$ satisfies $\left\{T: b_{0}(T)=0\right\} \subset S$. Now set $H=z$.

We have

$$
\operatorname{deg}_{T} P_{z}\left(T_{1}^{d_{1}}, T_{2}^{d_{2}}, \ldots, T_{n}^{d_{n}}, T_{n+1}, Y\right) \leq d_{s} \prod_{j=1}^{n-q-1} d_{j}
$$

and consequently we obtain the equality $b_{0}\left(x_{1}, \ldots, x_{q+1}\right)+\sum F_{i} g_{i}=0$, where deg $F_{i} g_{i} \leq$ $\prod_{j=1}^{n-q} d_{j}$. Set $\phi=b_{0}$. By the construction the polynomial $\phi$ has zeros only on the image of the projection

$$
\pi: V(J) \ni x \mapsto\left(x_{1}, \ldots, x_{q+1}\right) \in \mathbb{K}^{q+1}
$$

Remark 4. Simple application of the Bezout theorem shows that our estimations on the degree of $\phi$ is sharp.
Corollary 5. Let I be as above. If $V(I)$ has pure dimension $q$ and $I$ has not embedded components, then there is a polynomial $\phi_{1} \in \mathbb{K}\left[x_{1}, \ldots, x_{q+1}\right]$ which describes the image of the projection

$$
\pi: V(I) \ni x \mapsto\left(x_{1}, \ldots, x_{q+1}\right) \in \mathbb{K}^{q+1}
$$

such that
(a) $\phi_{1} \in I$,
(b) $\operatorname{deg} \phi_{1} \leq d_{s} \prod_{i=1}^{n-q-1} d_{i}$.

Proof. Let $I=\bigcap^{r} I_{k}$ be a primary decomposition of $I$. Then $\operatorname{dim} V\left(I_{k}\right)=q$ for every $k$. Let $\phi$ be a polynomial as above. If $\phi=\phi_{1} \phi_{2}$, where $\phi_{2}$ does not vanish on any component of $V(I)$ then $\phi_{1} \in I_{k}$ for every $k$ (by properties of primary ideals) and consequently $\phi_{1} \in I$. But $\phi_{1}$ describes the image of the projection

$$
\pi: V(I) \ni x \mapsto\left(x_{1}, \ldots, x_{q+1}\right) \in \mathbb{K}^{q+1}
$$

Theorem 6. Let $f_{1}, \ldots, f_{s} \in \mathbb{C}\left[x_{1}, \ldots, x_{n}\right]$ be polynomials such $\operatorname{deg} f_{i}=d_{i}$ where $d_{1} \geq$ $d_{2} \ldots \geq d_{s}$. Let $I=\left(f_{1}, \ldots, f_{s}\right)$ be the ideal in $\mathbb{C}\left[x_{1}, \ldots, x_{n}\right]$ such that $\operatorname{dim} V(I)=q$. Then in $I$ we can find a non-zero polynomial $\phi(x) \in \mathbb{C}\left[x_{1}, \ldots, x_{q+1}\right]$ such that $\operatorname{deg} \phi \leq d_{s} \prod_{i=1}^{n-q-1} d_{i}$.

Proof. By theorem 3 for generic $\alpha=\left(\alpha_{i, j} ; i=1, \ldots, q+1, j \geq i\right)$ there exist a non-zero polynomial $\phi_{\alpha} \in \mathbb{C}\left[t_{1}, \ldots, t_{q+1}\right]$ such that
a) $\operatorname{deg} \phi_{\alpha} \leq d_{s} \prod_{i=1}^{n-q-1} d_{i}$,
b) $\phi_{\alpha}\left(X_{1}(\alpha), X_{2}(\alpha), \ldots, X_{q+1}(\alpha)\right) \in I$, where $X_{1}(\alpha)=\alpha_{1,1} x_{1}+\ldots+\alpha_{1, n} x_{n}, X_{2}(\alpha)=$ $\alpha_{2,2} x_{1}+\ldots \alpha_{2, n} x_{n}, \ldots, X_{q+1}(\alpha)=\alpha_{q+1, q+1} x_{q+1}+\ldots+\alpha_{q+1, n} x_{n}$.

For a polynomial $p=\sum_{\alpha} a_{\alpha} x^{\alpha} \in \mathbb{C}\left[x_{1}, \ldots, x_{n}\right]$ we define a norm $\|p\|=\max _{\alpha}\left|a_{\alpha}\right|$. In particular for every generic $\alpha$ we can assume that $\left\|\phi_{\alpha}\right\|=1$ (we consider the polynomial $\frac{\phi}{\|\phi\|}$ instead of $\left.\phi\right)$. Moreover, we can take generic $\alpha_{m}$ in this way that $X_{i}\left(\alpha_{m}\right) \rightarrow x_{i}$ for $m \rightarrow \infty$ and for $i=1, \ldots, q+1$. Polynomial $\phi_{\alpha}$ we can treat as an element of a vector space $B(D)$ of all polynomials from $\mathbb{C}\left[x_{1}, \ldots, x_{n}\right]$ of degree bounded by $D=d_{s} \prod_{i=1}^{n-q-1} d_{i}$. Since the norm of every $\phi_{\alpha_{m}}$ is bounded by 1 , we can assume that this sequence converges to a polynomial $\phi$, with norm 1 and of degree bounded by $D=d_{s} \prod_{i=1}^{n-q-1} d_{i}$. Thus also a sequence $\phi_{\alpha_{m}}\left(X_{1}\left(\alpha_{m}\right), \ldots, X_{q+1}\left(\alpha_{m}\right)\right)$ tends to the same polynomial $\phi$. Of course it is non-zero because $\|\phi\|=1$. It is enough to show that $\phi \in I$. However the space $I \cap B(D)$ is a linear subspace of a finitely dimensional complex vector space $B(D)$, hence it is closed subset of $B(D)$. This finishes the proof.
Remark 7. By Lefchetz Principle Theorem 6 holds for every field of characteristic zero.
Corollary 8. Let $f_{1}, \ldots, f_{s} \in \mathbb{C}\left[x_{1}, \ldots, x_{n}\right]$ be polynomials such $\operatorname{deg} f_{i}=d_{i}$ where $d_{1} \geq$ $d_{2} \ldots \geq d_{s}$. Let $I=\left(f_{1}, \ldots, f_{s}\right)$ be the ideal in $\mathbb{C}\left[x_{1}, \ldots, x_{n}\right]$ such that $\operatorname{dim} V(I)=1$. Assume that the image $S$ of the projection $x \mapsto\left(x_{1}, x_{2}\right)$ of a one dimensional part of $V(I)$ into $\mathbb{C}^{2}$ is a curve. Then in I we can find a non-zero polynomial $\phi\left(x_{1}, x_{2}\right)=\phi_{1}\left(x_{1}, x_{2}\right) \rho\left(x_{1}\right) \in$ $\mathbb{C}\left[x_{1}, x_{2}\right]$ such that
a) $\operatorname{deg} \phi \leq D^{2}-2 D+2$, where $D=d_{s} \prod_{i=1}^{n-2} d_{i}$,
b) $\phi_{1}$ describes the image of the projection $x \mapsto\left(x_{1}, x_{2}\right)$ of a one dimensional part of $V(I)$ into $\mathbb{C}^{2}$.

Proof. Consider the ideal $I_{\alpha}=\left(f_{s}, \sum_{i=1}^{s-1} \alpha_{1 i} f_{i}, \sum_{i=2}^{s-1} \alpha_{2 i} f_{i}, \ldots, \sum_{i=n-2}^{s-1} \alpha_{2 i} f_{i}\right)$. By Theorem 6 we have that there exists a non-zero polynomial $\phi_{\alpha} \in I \cap \mathbb{C}\left[x_{1}, x_{2}\right]$ of degree bounded by $D$. We can write $\phi_{\alpha}=\phi_{1, \alpha} \phi_{2, \alpha}$, where $\phi_{1, \alpha}$ describes $S$ and $\phi_{2, \alpha}$ does not vanish on $S$. Since the degree of $\phi_{1, \alpha}$ is bounded, there is infinitely many $\alpha_{i}$ such that $\phi_{1, \alpha_{i}}$ up to a multiplicative constant is the same. Moreover, the ideal $J=\left(\phi_{2, \alpha_{i}}, i=1,2, \ldots\right)$ describes the zero dimensional part of the image of the projection $x \mapsto\left(x_{1}, x_{2}\right)$ of $V(I)$ into $\mathbb{C}^{2}$. If $\operatorname{deg} \phi_{1}=a$, then $\operatorname{deg} \phi_{2, \alpha_{i}} \leq D-a$. Consequently we can find a non zero polynomial $\rho\left(x_{1}\right) \in J$, such that $\operatorname{deg} \rho \leq(D-a)^{2}$. The ideal $I$ contains a polynomial $\phi \rho$ of degree
bounded by $a+(D-a)^{2}$. The expression $a+(D-a)^{2}$, where $0<a \leq D$ attains its maximal value for $a=1$. This finishes the proof.
Remark 9. If the ideal $I$ has not embedded components, then we can assume that the polynomial $\rho$ describes the image of the projection $x \mapsto x_{1}$ of a zero dimensional part of $V(I)$ into $\mathbb{C}$. In general case it can describe also some extra points.

## References

[1] Jelonek, Z. On the Effective Nullstellensatz. Invent. Math. 162(2005), pp 1-17.


[^0]:    1991 Mathematics Subject Classification. 14 D 06, 14 Q 20.

