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A. Elkharrat[‡], *C. Frougny*[§], *J.-P. Gazeau*[‡], and *J.-L. Verger-Gaugry*[†].

[‡] *Laboratoire de Physique Théorique de la Matière Condensée, Boîte 7020, Université Paris 7-Denis Diderot, 75251 Paris Cedex 05, France*
[§] *Laboratoire d'Informatique Algorithmique : Fondements et Applications, UMR 7089 CNRS, Boîte 7014, Université Paris 7-Denis Diderot, 75251 Paris Cedex 05, France*
[†] *Institut Fourier, Université Joseph Fourier Grenoble, CNRS URA 188, BP 74- Domaine Universitaire, 38402- Saint Martin d'Hères, France.*

Abstract.

We present the construction of a symmetry plane-group for a quasiperiodic point-set in the plane named τ-lattice, τ being the golden-ratio. The τ-lattice generalizes the notion of periodic lattice to quasiperiodicity. The algebraic framework is issued from the counting system of τ-integers. The set of τ-integers can be equipped with an abelian group structure and an internal multiplicative law. These arithmetic structures lead to a freely generated symmetry plane-group for the τ-lattice, based on repetitions of discrete "τ-rotations" and "τ-translations" in the plane. Hence the τ-lattice, endowed with these adapted rotations and translations, can be viewed as a lattice with "rotational" symmetries.

1. Introduction

Historically, periodic tilings and lattices merge from Crystallography, and are associated with crystals. A crystallographic group of $\mathbb{R}^d$, is a discrete group of isometries whose maximal translation subgroup is of rank d. A crystal, as it is commonly used in Physics, is the orbit under the action of a crystallographic group of a finite number of points of $\mathbb{R}^d$. Let us recall the example of the square lattice $\Lambda = \mathbb{Z} + \mathbb{Z}e^{\frac{\pi i}{2}}$. The symmetry space-group G of Λ is the semi-direct product of the group of translations of Λ by its group of rotations $G = \Lambda \times \{1, -1, e^{i\frac{\pi}{2}}, e^{-i\frac{\pi}{2}}\}$, its composition law being defined by $(\lambda, R)(\lambda', R') = (\lambda + R\lambda', RR')$, with $\lambda, \lambda' \in \Lambda$ and $R, R' \in \{1, -1, e^{i\frac{\pi}{2}}, e^{-i\frac{\pi}{2}}\}$.

For quasicrystals, as a consequence of their aperiodicity, we generally do not have such an algebraic frame. For quasicrystals determined by some Pisot number $\beta > 1$, we can introduce an underlying structure, generically called *beta-lattice* [3]. Experimentally observed quasicrystals are related to well known Pisot numbers: $\tau = \frac{1+\sqrt{5}}{2}$ with symmetry of order 5 or 10, $\delta = 1 + \sqrt{2}$ with symmetry of order 8, and $\theta = 2 + \sqrt{3}$ with symmetry of order 12. Beta-lattices are Meyer sets, which have the property to be self-similar, with β as self-similarity factor [5]. Beta-lattices are eligible "counting systems" for quasicrystals, that is an algebraic framework in which one could study the properties of quasiperiodic point-sets and tilings.

The objective of the present work consists in generalizing the above context of crystals to beta-lattices, the τ-lattice, $\Gamma(\tau)$, being an illustration (Fig. 1). The algebraic framework

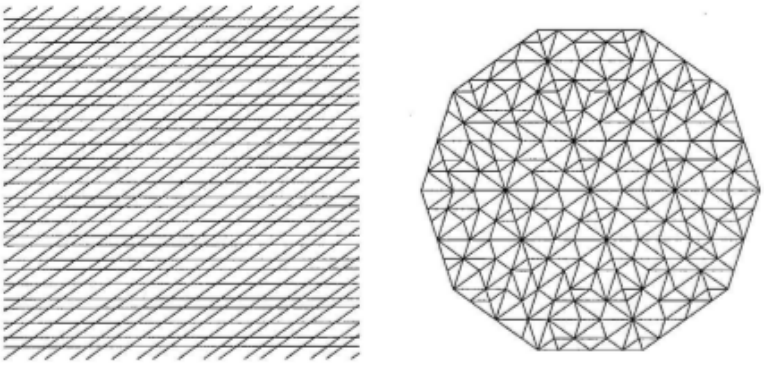

FIGURE 1 Two tilings of the τ-lattice $\Gamma(\tau)$.

is issued from the counting system called τ-integers and denoted by $\mathbb{Z}_\tau$. We show that this τ-lattice is a "crystal" for a "plane-group" that we will explicitly determine. First, we recall the internal laws, called τ-addition and quasi-multiplication, on the set $\mathbb{Z}_\tau$, then we establish the algebraic operations of τ-translations and τ-rotations, leaving the τ-lattice invariant, and finally we construct a plane-group for the τ-lattice which possesses the same algebraic structure as a plane-group in the classical lattice case.

2. The Set of τ-Integers $\mathbb{Z}_\tau$

Any real number $x > 0$ can be developed in series of powers of τ, $x = \sum_{i \leq k} x_i \tau^i$, such that the sequence of digits $(x_i)_{i \leq k}$, called the τ-*expansion*, is computed through the *greedy algorithm* (see [4] and references therein). The set of positive τ-integers is the set of positive real numbers x who's τ-expansions are such that $(x_i)_{0 \leq i \leq k}$, and $x = \sum_{0 \leq i \leq k} x_i \tau^i$. For example $\tau^2 + 1$ and τ^3 are τ-integers, but $1/\tau$ and 2 are not (Table 1). The set of τ-integers $\mathbb{Z}_\tau$ is self-similar and symmetrical with respect to the origin

$$\tau \mathbb{Z}_\tau \subset \mathbb{Z}_\tau, \quad \mathbb{Z}_\tau = -\mathbb{Z}_\tau. \tag{1}$$

The set of τ-integers is associated to a tiling of the real line, with two tiles L and S, the lengths of the tiles being $\ell(L) = 1$, and $\ell(S) = 1/\tau$.

The set of τ-integers verifies the following arithmetic property. Let b_m and b_n be the mth and nth τ-integers respectively. Then $b_m + b_n = b_{m+n} + \{0, \pm 1/\tau\}$, and $b_m \times b_n = b_{mn - \rho_S(m)\rho_S(n)} + \{0, \pm 1/\tau\}$. The function $\rho_S(j)$ counts the number of small tiles between 0 and b_j in the tiling associated to $\mathbb{Z}_\tau$. For instance, $\rho_S(2) = 1$, since there is one long tile and one small tile between 0 and $\tau = b_2$. From these relations, we introduce the following

TABLE 1 Some τ-expansions

	$\cdots$	τ^3	τ^2	τ	1		$\frac{1}{\tau}$	$\frac{1}{\tau^2}$	$\cdots$
τ				1	0	$\cdot$	0	0	
$\tau^2 + 1$			1	0	1	$\cdot$	0	0	
$\tau^3 = \tau^2 + \tau$		1	0	0	0	$\cdot$	0	0	
2				1	0	$\cdot$	0	1	

internal laws [2]

$$b_m \oplus b_n = b_{m+n}\,, \tag{2}$$

$$b_m \otimes b_n = b_{mn-\rho_S(m)\rho_S(n)}\,, \tag{3}$$

respectively called here τ-addition and quasi-multiplication. The τ-addition is consistent with addition on $\mathbb{R}$, and the quasi-multiplication is consistent with multiplication on $\mathbb{R}$, with regards to the additive and multiplicative properties of τ-integers stated above. Note that $(\mathbb{Z}_\tau, \oplus)$ is an abelian group, but since the quasi-multiplication is neither associative, nor is it distributive with respect to the τ-addition, $(\mathbb{Z}_\tau, \oplus, \otimes)$ is not a ring.

3. Tau-Lattices and Plane-Groups

We now focus on the τ-lattices in the plane

$$\Gamma(\tau) = \mathbb{Z}_\tau + \mathbb{Z}_\tau \zeta\,, \quad \zeta = e^{i\frac{\pi}{5}}\,. \tag{4}$$

Note that the latter is not rotationally invariant. The material of this section can be found [1].

A point group for the τ-lattice in the plane—Since the τ-lattice $\Gamma(\tau)$ is not rotationally invariant, we shall enforce invariance by changing the additive and multiplicative laws by τ-additive and quasi-multiplicative laws. Let η_q and ν_q the real numbers such that $\zeta^q = \eta_q + \nu_q \zeta$.

Definition 1. *The following* 10 *operators* r_q, $q = 0, 1, \cdots, 9$, *are defined on* $\Gamma(\tau)$ *by*:

$$r_q \odot (b_m + b_n\zeta) = \eta_q b_m \oplus (-\nu_q b_n) + ((\eta_q + \tau\nu_q)b_n \oplus \nu_q b_m)\zeta. \tag{5}$$

Introducing the identity action Id, *the space inversion* σ *in the complex plane,* $\sigma \odot z = -z$, *and the associative composition rule,* $(rr') \odot z = r \odot (r' \odot z)$, *we trivially check that* $r_0 = Id$ *and* $r_{q+n/2} = \sigma r_q = r_q \sigma$ *for* $q = 0, 1, \cdots, 4$.

Now the computing of composition of any two of such τ-rotations r_q yields the following important result.

Proposition 1. *Let* $\Re_n$ *denote the semi-group freely generated by all* r_q, $q \in \{0, 1, \cdots, n-1\}$. *Among all such extended rotations, only* $r_0, r_1, r_{n/2-1}, r_{n/2+1}, r_{n-1}, \sigma$ *have their inverse in* $\Re_n$.

As an immediate consequence, we can assert the following on the existence of an extended symmetry group for $\Gamma(\tau)$.

Corollary 1. *The group* $\mathcal{R}_{10}$ *freely generated by the four-element set* $\{r_0, \sigma, r_1, r_{n/2-1}\}$ *is a symmetry group for the* τ*-lattice* $\Gamma(\tau)$. *It will be called in the present context the point symmetry group of* $\Gamma(\tau)$.

A plane-group for the τ-lattice—We now introduce into the present formalism the following τ-translations acting on $\Gamma(\tau)$.

Definition 2. *Let* $z_0 = b_{m_0} + b_{n_0}\zeta$ *be an element of* $\Gamma(\tau)$. *There corresponds to it the following internal action* $t_{z_0} : \Gamma(\tau) \mapsto \Gamma(\tau)$

$$t_{z_0}(z) = z \oplus z_0 \stackrel{def}{=} b_m \oplus b_{m_0} + (b_n \oplus b_{n_0})\zeta. \tag{6}$$

The set of such τ*-translations forms an abelian group isomorphic to* $\Gamma(\tau)$, *considered itself as a group for the law* $\oplus$. *For this reason it will be also denoted by* $\Gamma(\tau)$.

A simple computation leads to the following result.

Proposition 2. *The group* $\mathcal{S}_{10}$, *freely generated by the five-element set* $\{r_0, \sigma, r_1, r_4, t_1\}$ *is a symmetry group for the* τ*-lattice* $\Gamma(\tau)$. *This group is the semi-direct product of* $\Gamma(\tau)$ *and* $\mathcal{R}_{10}$

$$\mathcal{S}_{10} = \Gamma(\tau) \times \mathcal{R}_{10} = \{(b, R) \in \Gamma(\tau) \times \mathcal{R}_{10}\} \tag{7}$$

with the composition rule

$$(b, R)(b', R') = (b \oplus R \odot b', RR'). \tag{8}$$

In the present context, $\mathcal{S}_n$ will be called the space symmetry group of $\Gamma(\tau)$. The action of an element of $\mathcal{S}_{10}$ on $\Gamma(\tau)$ is thus defined as

$$(b, R) \cdot z = b \oplus R \odot z = t_b(R \odot z) \in \Gamma(\tau). \tag{9}$$

Cut-and-project sets under group actions—Now that we have constructed a space-group for $\Gamma(\tau)$ we can observe the action on this group on cut-and-project sets. Figure 2 displays a cut-and-project set of decagonal window, and its transforms under the action of t_1 and of $t_2 r_1$.

Tiles under group action—We also would like to illustrate the action of $\mathcal{S}_{10}$ by showing how a tile of $\Gamma(\tau)$ is transformed under the action of an element of $\mathcal{S}_{10}$. Let $z = b_m + b_n\zeta \in \Gamma(\tau)$.

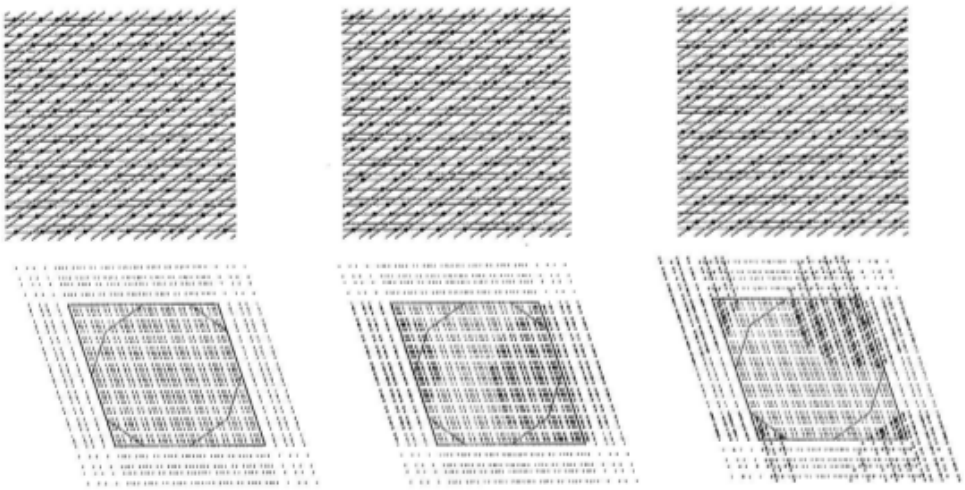

FIGURE 2 Embedding of a decagonal cut-and-project set in $\Gamma(\tau)$ (left), and of its transforms under t_1 (center) and $t_2 r_1$ (right).

FIGURE 3 Rotation operator r_1 applied to elementary tiles of the τ-lattice. Note how the tiles are deformed in order for the vertices to remain in $\Gamma(\tau)$.

An *elementary quadrilateral tile on* z is $\mathbf{T}(z) = \{z, z \oplus 1, z \oplus \zeta, z \oplus (1+\zeta)\}$. Let us apply the rotation operator r_1 on $\mathbf{T}(z)$ and around one of the edges of $\mathbf{T}(z)$. For instance, a rotation around z is defined as

$$t_z(r_1 \odot t_{-z}(\mathbf{T}(z))) = \{z, z \oplus \zeta, z \oplus (-1+\tau\zeta), z \oplus (-1+\tau^2\zeta)\}\,. \qquad (10)$$

Examples of such rotation operations are displayed on Fig. 3. This operation not only rotates, but distorts the tiles.

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