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# PCP-nets and voting rules: some observations

Hénoïk Willot and Sébastien Destercke and Sylvain Lagrue<sup>1</sup>

**Abstract.** PCP-nets can be seen as a compact representation of a probability distribution over CP-nets. Based on this property, we can interpret each CP-net represented in the PCP-net as a voter, with an associated probability/weight in the population, voting on the space of all the outcomes. We can then see various inference problems in CP-nets, and particularly recommendation, as a voting problem. This paper is a reflection on such queries. In particular, it is usual to determine the result of a recommendation query as the top-outcome having highest probability, which can be assimilated as the result of a plurality vote. In this paper, we will see that PCP-nets also suffer from usual problems met in voting rules, and we will make a first discussion on the use of alternative classical voting methods such as the Borda rule.

## 1 Introduction

Preferences are ubiquitous in our everyday life, be it to make light or impactful decisions. Imagine, for instance being in a vacation camp where activities are proposed and coaches are available on demand for you. The vacation company needs you to express your choices over these activities and the available coaches to plan your stay depending on the weather, the time, etc. Note that while this decision and preferences probably have little impact on you, they may have a huge impact for the company in the long run.

Modelling such preferences and making recommendations from such models is an old topic, for which many proposals have been made. Among them are the CP-nets introduced by Boutilier et al. [3]. Due to their graphical nature, CP-nets have the advantage of being compact over the ranked alternatives and easily understandable. Since then, extensions able to cope with uncertainty, such as probabilistic [1, 4] or incomplete CP-nets [6] have been proposed.

In this paper, we focus on the probabilistic extension of PCP-net, and discuss in particular inference rules proposed in the past. More precisely, we link them to voting rules, and criticize them on this basis. It also leads us to a preliminary investigation on how inference mechanisms based on other voting rules, such as Borda's, can be applied to PCP-net. Note that linking PCP-net to voting rules is also quite in-line with their interpretation as models of preference of a population of users.

Sections 2 and 3 give respectively a refresher on both CP-nets and their probabilistic extension. Section 4, the core of the paper, shows that PCP-nets are subject to Condorcet-style paradoxes, and investigate the application of the Borda rule to this model, contrasting it with the classical inference method that can be assimilated to a plurality rule.

## 2 CP-nets: a brief reminder

We consider a set  $\mathcal{V}$  of  $n$  variables. Each variable  $X \in \mathcal{V}$  has its own finite domain  $D_X$ . We call an outcome a full assignment  $o \in \mathcal{O} = \times_{X \in \mathcal{V}} D_X$ . We will denote by  $D_Q = \times_{X \in Q} D_X$  the domain corresponding to a subset  $Q \subseteq \mathcal{V}$  of variables. We will also denote by  $o_Q$  the values of an outcome restricted on  $Q$ .

**Definition 1** (Preference rule). *A preference rule  $\succ^X$  is a total order (reflexive, antisymmetric and transitive binary relation) over the domain  $D_X$  of a variable  $X \in \mathcal{V}$ . The preference rule  $\succ_q^X$  can be conditioned by a value  $q$  of  $D_Q$ , with  $Q \subseteq \mathcal{V} \setminus X$ .*

**Definition 2** (Conditional Preference Network). *A CP-net is a graphical representation of preferences working under the ceteris paribus assumption, all other things being equal. A CP-net  $\mathcal{X}$  is defined by a directed graph  $\mathcal{G} = (\mathcal{V}, E)$  and a conditional preference table (CP-table) for each  $X \in \mathcal{V}$ . An edge from a variable  $X$  to a variable  $Y$  of  $\mathcal{V}$  means that the preferences over  $D_Y$  are conditioned by the values of  $D_X$ . For a variable  $X$ , we denote by  $PA(X) \subseteq \mathcal{V} \setminus X$  its parents, and by  $pa(X)$  a specific instance of these parents. The CP-table of  $X$  is the set of preference rules  $\succ_{pa(X)}^X$  specified for each  $pa(X)$ .*

In this paper, we only consider graphs  $\mathcal{G}$  that are acyclic, as those considerably simplify the model and ensure its consistency. Also, PCP-nets are usually defined over acyclic graphs. We also consider that preference rules are strict preferences, as no indifference in the CP-tables ensures that CP-nets have a unique best outcome (Boutilier [3]).

**Example 1.** *We have the choice between 2 activities, Tennis and Frisbee, and two coaches, Bob and Alice. I prefer unconditionally playing Frisbee over Tennis, and I prefer playing Tennis with Bob and Frisbee with Alice. This preference can be summarised by the following CP-net :*

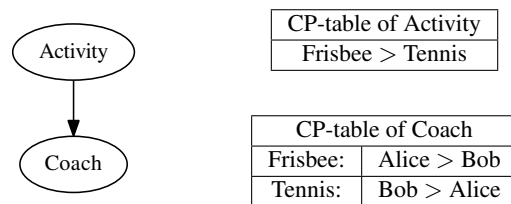


Figure 1: Simple CP-net for vacation activities

**Definition 3** (Order on outcomes). *Given a fixed CP-net  $\mathcal{N}$ , we can use  $\mathcal{N}$  to create a partial order  $\succ_{\mathcal{N}}$  on  $\mathcal{O}$ . CP-nets works under the ceteris paribus assumption. That is, when two outcomes  $o, o'$  differ only on one variable  $X$ ,  $o \succ_{\mathcal{N}} o'$  iff  $o_X \succ_{o_{PA(X)}}^X o'_X$ , with the pair*

<sup>1</sup> Sorbonne Universités, Université de Technologie de Compiègne, Heudiasyc, UMR CNRS 7253, BP 20529, 60205 Compiègne, France

$(o_X, o'_X)$  called a *worsening flip* (swapping value  $o_X$  for  $o'_X$  worsen the outcome).

Two arbitrary outcomes  $o_1, o_2$  are then such that  $o_1 \succ_N o_2$  only if a sequence of worsening flips from  $o_1$  to  $o_2$  exists, i.e. a transitive chain of preference rules  $\succ_{o_{PA(X)}}^X$  starting with  $o_1$  and ending with  $o_2$ . If it is not possible to find such a sequence between  $o_1$  and  $o_2$ , the two outcomes are not comparable, resulting in  $\succ_N$  being a partial order.

**Definition 4** (Optimization query). *The optimization query, or outcome optimization, determine which outcome is the best outcome among the ranking defined by the CP-net  $N$ . We denote the resulting outcome  $o$  by  $opt(N)$ .*

**Example 2.** *If we come back to the CP-net of example 1, we can build a complete ranking outcomes, we obtain:  $(Frisbee, Alice) \succ (Frisbee, Bob) \succ (Tennis, Bob) \succ (Tennis, Alice)$ . The result of the optimization query is  $(Frisbee, Alice)$ .*

In this paper we only consider strict total orders for preference rules, i.e. all the values of the domain are ordered and there is no indifference. We also only consider complete CP-nets, all preference rules are total orders known for every value of the parents.

### 3 PCP-nets

Probabilistic CP-net [1, 4] have emerged in an attempt to take into account ill-known preferences to be able to answer the question "what is the probability of having  $o$  preferred to  $o'$  for an unknown agent?". To handle ill-known preferences, coming from lack or noisy information or multi-agent preferences, each preference statement is now associated with a probability into a Probabilistic CP-table.

**Definition 5.** (Probabilistic CP-net) *A PCP-net  $\mathcal{N}$  is defined by a directed graph  $\mathcal{G} = (\mathcal{V}, E)$  and a conditional (local) probabilistic preference table (PCP-table) for each  $X \in \mathcal{V}$ . A PCP-table on  $X$  is a conditional probability  $p(\cdot | PA(X))$  defined over  $\mathcal{L}(X)$ , the set of linear ordering on  $X$ .*

If we consider a CP-net  $N$  (with specified CP-tables) that has the same graphical structure as a PCP-net  $\mathcal{N}$ , we will say that it is  $\mathcal{N}$ -compatible. We can associate to it a probability such that

$$p_{\mathcal{N}}(N) = \prod_{X \in \mathcal{V}} \prod_{PA(X)} p(\succ_{pa(X)}^X | pa(X))$$

with  $\succ_{pa(X)}^X$  the preference on  $X$  for  $N$ . By abuse of notation, we will also denote by  $N \in \mathcal{N}$  the fact that  $N$  is  $\mathcal{N}$ -compatible.

If we associate to each CP-net  $N$  its partial order,  $p_{\mathcal{N}}$  can then be seen as a probability distribution over partial orders, with  $p_{\mathcal{N}}(N)$  that can be understood as the weight of  $\succ_N$ , e.g., the number of voters/agents giving  $\succ_N$  as preferences, or the probability that  $\succ_N$  represents the true preference of a single agent.

**Example 3.** *Let's get back to our vacation's activities plan. You came with friends and you have to give the preferences for the whole group to the company. After consultation with the group you concluded that Frisbee is preferred over Tennis for 80% of the group. For the Frisbee activity, Alice is preferred to Bob with a probability of 60% and playing Tennis with Bob is preferred in 90% of the time. We obtain the PCP-net in figure 2.*

**Remark** *The CP-net  $N$  of Figure 1 is induced by the PCP-net  $\mathcal{N}$  pictured in Figure 2. Its probability is  $p_{\mathcal{N}}(N) = 0.8 * 0.6 * 0.9 = 0.432$ .*

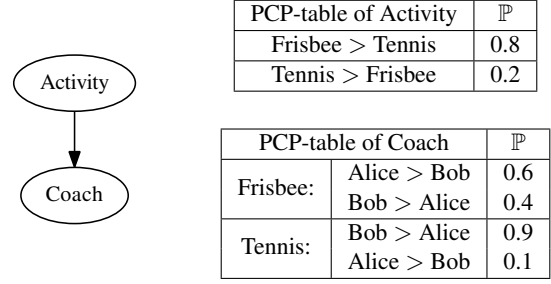


Figure 2: PCP-net of the group for vacation activities

## 4 PCP-nets and voting rules

Seeing PCP-nets as weights over rankings or voters, we discuss some problems of classical PCP-nets inference tools through the lens of voting rules.

### 4.1 PCP-nets and Condorcet paradox

A first remark is that, if one focus on pairwise preferences, PCP-nets are likely to be prone to Condorcet paradox. The dominance probability  $p_{\mathcal{N}}(o \succ o') = \sum_{N \in \mathcal{N}, o \succ_N o'} p_{\mathcal{N}}(N)$  of a pair  $(o, o')$  is indeed such that it can create cycles, in the sense that we can find a sequence  $o_1, \dots, o_c$  such that  $p_{\mathcal{N}}(o_i \succ o_{i+1}) > 0.5$  for  $i \in 1, \dots, c-1$  and  $p_{\mathcal{N}}(o_c \succ o_1) > 0.5$ .

The next trivial example shows that this paradox may occur for more complex CP-nets.

**Example 4.** *Let  $A$  be a variable over  $D_A = \{a_1, a_2, a_3\}$  and consider the trivial PCP-net of Figure 3.*

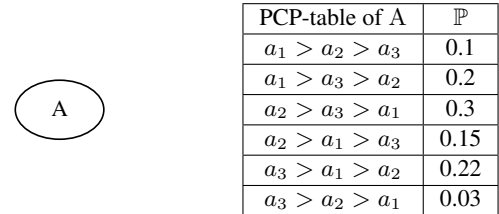


Figure 3: PCP-net of the group for vacation activities

After rather simple computations, we get

$$\begin{aligned} p_{\mathcal{N}}(a_1 \succ a_2) &= 0.1 + 0.2 + 0.22 &= 0.52 > 0.5 \\ p_{\mathcal{N}}(a_2 \succ a_3) &= 0.1 + 0.3 + 0.15 &= 0.55 > 0.5 \\ p_{\mathcal{N}}(a_3 \succ a_1) &= 0.3 + 0.22 + 0.03 &= 0.55 > 0.5 \end{aligned}$$

We obtain a Condorcet cycle between  $a_1, a_2$  and  $a_3$ .

**Remark 1.** *This behaviour could have been expected, as it is well known in the probabilistic literature that statistical preferences (the fact of stating that  $X \succ Y$  if  $P(X > Y) > 0.5$ ) can produce cycles [5].*

Note that other pairwise rule such as unanimity ( $p_{\mathcal{N}}(o \succ o') = 1$ ) have been discussed for incomplete CP-nets [7], yet the one considered here is a natural definition commonly used in probabilistic literature [2].

## 4.2 PCP-nets, plurality and Borda

Usually, as defined in [1], the optimal outcome  $opt_{\mathcal{N}}(\mathcal{O})$  for a PCP-net is defined as

$$opt_{\mathcal{N}}(\mathcal{O}) = \arg \max_{o \in \mathcal{O}} \sum_{N \in \mathcal{N}, opt(N)=o} p_{\mathcal{N}}(N)$$

In practice, this means that the optimal outcome is defined as the one that is preferred by the (weighted) majority of the  $\mathcal{N}$ -compatible CP-nets. This is therefore equivalent to the plurality rule.

The plurality rule is certainly one of the simplest voting rule, but it violates many axioms, in the light of which  $opt_{\mathcal{N}}(\mathcal{O})$  may not be perceived as a good (group) recommendation. Another commonly used rule is the Borda rule, that for a given voter, assigns the score  $r_o$  to an outcome  $o$ , where  $r_o$  is the rank of the outcome (the worst outcome having rank 1). The recommended outcome is then the one whose sum of scores over all voters (in our case, all  $N \in \mathcal{N}$ ) is the highest. A question that ensues is whether the Borda rule applied to CP-net would provide the same answer as the plurality rule, or suggest another outcome.

As Borda rule requires a complete total order over the set of candidates, it cannot be directly applied to PCP-nets, as CP-nets generally induce a partial order. In such cases, we can look for possible winners [10] over all completions of these partial orders in complete ones. Since computing those is NP-complete, we propose next an approximate computation.

To deal with the incomparabilities of a CP-net  $N$ , we will define for each outcome a *Borda* (resp. *Borda*), approximating in a conservative manner the worst (resp. best) rank it can have in the linear extensions of  $\succ_N$ . We will use the following upper-bounding heuristic: we compute the transitive closure of  $\succ_N$ . We then simply define the following bounds

$$\underline{Borda}_N(o) = |\{o' \in \mathcal{O} : o' \prec_N o\}| + 1$$

$$\overline{Borda}_N(o) = |\mathcal{O}| - |\{o' \in \mathcal{O} : o' \succ_N o\}|.$$

This gives us, for each outcome  $o$  and CP-net  $N$ , an interval  $[\underline{Borda}_N(o), \overline{Borda}_N(o)]$  and we can then check for possible winners given this information. Given a PCP-net  $\mathcal{N}$ , we define the associated lower/upper Borda scores as

$$\underline{Borda}_{\mathcal{N}}(o) = \sum_{N \in \mathcal{N}} p_{\mathcal{N}}(N) \underline{Borda}_N(o)$$

$$\overline{Borda}_{\mathcal{N}}(o) = \sum_{N \in \mathcal{N}} p_{\mathcal{N}}(N) \overline{Borda}_N(o).$$

Denoting by  $\mathbf{Borda}_{\mathcal{N}} = \max_o \underline{Borda}_{\mathcal{N}}(o)$  the maximal obtained lower bound, we define the approximate set of possible winners as

$$APoss_{Borda} = \{o \in \mathcal{O} : \overline{Borda}_{\mathcal{N}}(o) \geq \mathbf{Borda}_{\mathcal{N}}\}. \quad (1)$$

This set is a superset of the set of the exact set of possible winners, as  $\underline{Borda}_{\mathcal{N}}, \overline{Borda}_{\mathcal{N}}$  are only lower/upper bounds of the exact lower/upper Borda scores.

**Example 5.** *To help clients with their choice of activities, the company now gives the weather report and set two time periods for the activities. A new activity is also available, Pingpong. The preferences are now evolving with 4 variables: Time: AM and PM, Weather: Fair (F) and Rain (R), Activity: Tennis (T), Frisbee (F) and Pingpong (PP), and Coach: Bob (B) and Alice (A). After consulting with the group, you determined that, globally, according to the Weather, you will*

*prefer either to make the Activity in the morning or the afternoon. The choice of Activity will depend on the Weather and the period of the Day. The choice of Coach, however, is only linked to the choice of Activity. We will not express all the preferences and their associated probabilities as the model start to be complex. This results in the PCP-net in Figure 4.*

*Now that the preferences are completely set for the group, the company can now select the conditions preferred by the group. In Table 1, we compare the results for the 24 outcomes. In red is the winner of the plurality vote, and in blue are the outcomes within the set given by Equation 1 that form a superset of the possible Borda winners.*

Outcome $o$	$p_{\mathcal{N}, Plur}(o)$	$\underline{Borda}_{\mathcal{N}}(o)$	$\overline{Borda}_{\mathcal{N}}(o)$
(F,PM,PP,B)	0.135	<b>10.694</b>	<b>16.394</b>
(F,AM,PP,B)	0.0	7.340	11.244
(R,PM,PP,B)	0.0	6.754	10.766
(R,AM,PP,B)	<b>0.140</b>	10.921	14.745
(F,PM,PP,A)	0.045	<b>10.101</b>	<b>16.182</b>
(F,AM,PP,A)	0.0	6.567	11.610
(R,PM,PP,A)	0.0	5.940	11.302
(R,AM,PP,A)	0.047	10.805	14.338
(F,PM,F,B)	0.0	<b>9.614</b>	<b>15.892</b>
(F,AM,F,B)	0.0	8.761	13.266
(R,PM,F,B)	0.0	8.318	12.518
(R,AM,F,B)	0.0	5.193	10.295
(F,PM,F,A)	0.090	<b>12.857</b>	<b>18.644</b>
(F,AM,F,A)	0.0	<b>10.979</b>	<b>17.199</b>
(R,PM,F,A)	0.0	<b>10.584</b>	<b>16.315</b>
(R,AM,F,A)	0.0	7.676	13.238
(F,PM,T,B)	0.054	<b>12.381</b>	<b>18.371</b>
(F,AM,T,B)	0.045	<b>10.918</b>	<b>18.048</b>
(R,PM,T,B)	0.054	<b>10.545</b>	<b>17.221</b>
(R,AM,T,B)	0.010	8.431	13.545
(F,PM,T,A)	0.126	<b>15.295</b>	<b>19.422</b>
(F,AM,T,A)	0.105	<b>13.109</b>	<b>19.181</b>
(R,PM,T,A)	0.126	<b>12.782</b>	<b>18.246</b>
(R,AM,T,A)	0.023	11.003	14.452

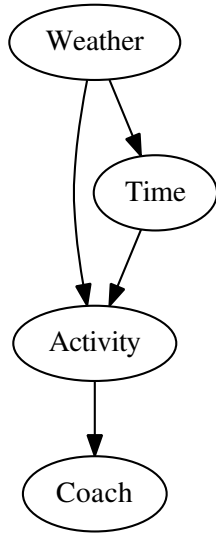
**Table 1:** Results of plurality and Borda vote for PCP-net of Figure 4, with  $opt_{\mathcal{N}}(\mathcal{O})$  in red,  $\underline{Borda}_{\mathcal{N}}$  in bold blue and  $o \in APoss_{Borda}$  in blue

*The first thing we can notice is that the winner of the plurality vote  $opt_{\mathcal{N}}(\mathcal{O}) = (R, AM, PP, B)$  only has a probability equal to 0.140, which is very low to establish a consensus for the group. Once we also notice that  $(R, AM, PP, B)$  does not belong to the potential winners of the Borda vote, despite our approximation. This questions even more the legitimacy of suing plurality to make a recommendation, and shows that the choice of the decision or voting rule is not without consequences.*

This example is sufficient to show that  $opt_{\mathcal{N}}(\mathcal{O})$  may actually not be a very good recommendation in terms of compromise. This shows the interest to investigate alternative rules issued from voting theory, in particular from the computational perspective (left out in this preliminary discussion).

## 5 Discussion and conclusion

In this paper, we have discussed some aspects of classical inference mechanisms of PCP-nets, interpreting them as elements of voting rules. This allowed us to expose some potential problems of these inferences, and to propose a quick (and dirty) approximate Borda procedure that we compared to usual outcome optimisation. We would



PCP-table of Weather		$\mathbb{P}$
Fair (F)	> Rain (R)	0.6
Rain (R)	> Fair (F)	0.4

PCP-table of Time		$\mathbb{P}$
F:	PM > AM	0.75
	AM > PM	0.25
R:	AM > PM	0.55
	PM > AM	0.45

PCP-table of Activity		$\mathbb{P}$
PM,F:	Pingpong (PP) > Tennis (T) > Frisbee (F)	0.4
	Tennis (T) > Frisbee (F) > Pingpong (PP)	0.4
	Frisbee (F) > Tennis (T) > Pingpong (PP)	0.2
PM,R:	Tennis (T) > Frisbee (F) > Pingpong (P)	1
AM,F:	Tennis (T) > Frisbee (F) > Pingpong (P)	1
AM,R:	Pingpong (P) > Tennis (T) > Frisbee (F)	0.85
	Tennis (T) > Frisbee (F) > Pingpong (P)	0.15

PCP-table of Coach		$\mathbb{P}$
PP:	Bob (B) > Alice (A)	0.75
	Alice (A) > Bob (B)	0.25
F:	Alice (A) > Bob (B)	1
T:	Alice (A) > Bob (B)	0.7
	Bob (B) > Alice (A)	0.3

Figure 4: Final PCP-net of the group for vacation activities

like to go further in this direction, and also to link voting rules and inferences with loss or distance minimization procedures [9]. There have been some previous attempts to do so [8], but with voting rules and loss functions that were specific to CP-nets, rather than usual ones used within social choice theory.

This work can also be seen as a will to bring together PCP-nets and voting rules in an attempt to answer the optimisation query in a multi-agent setting represented by the PCP-net itself. The common conception is to use the plurality vote, but our examples show that other voting rules can be used and could provide distinct answers. However, it is known [1] that finding the outcome with the maximal probability can be done in linear time on the variables for Tree-structured PCP-net, whereas our heuristic requires to build the partial order over  $\mathcal{O}$ . We would like to improve this heuristic, as it only provides an outer-approximation of the set of possible winners.

## Acknowledgements

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