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# On the Broadcast Independence Number of Circulant Graphs

Abdelamin LAOUAR \*    Isma BOUCHEMAKH \*    Éric SOPENA †

February 6, 2021

## Abstract

An independent broadcast on a graph  $G$  is a function  $f : V \rightarrow \{0, \dots, \text{diam}(G)\}$  such that (i)  $f(v) \leq e(v)$  for every vertex  $v \in V(G)$ , where  $\text{diam}(G)$  denotes the diameter of  $G$  and  $e(v)$  the eccentricity of vertex  $v$ , and (ii)  $d(u, v) > \max\{f(u), f(v)\}$  for every two distinct vertices  $u$  and  $v$  with  $f(u)f(v) > 0$ . The broadcast independence number  $\beta_b(G)$  of  $G$  is then the maximum value of  $\sum_{v \in V} f(v)$ , taken over all independent broadcasts on  $G$ .

We prove that every circulant graph of the form  $C(n; 1, a)$ ,  $3 \leq a \leq \lfloor \frac{n}{2} \rfloor$ , admits an optimal 2-bounded independent broadcast, that is, an independent broadcast  $f$  satisfying  $f(v) \leq 2$  for every vertex  $v$ , except when  $n = 2a + 1$ , or  $n = 2a$  and  $a$  is even. We then determine the broadcast independence number of various classes of such circulant graphs, and prove that, for most of these classes, the equality  $\beta_b(C(n; 1, a)) = \alpha(C(n; 1, a))$  holds, where  $\alpha(C(n; 1, a))$  denotes the independence number of  $C(n; 1, a)$ .

**Keywords:** Broadcast; Independent broadcast; Circulant graph.

**MSC 2010:** 05C12, 05C69.

## 1 Introduction

All the graphs considered in this paper are undirected and simple. For such a graph  $G$ , we denote by  $V(G)$  and  $E(G)$  its set of vertices and its set of edges, respectively. Let  $G$  be a nontrivial connected graph, that is, a connected graph with at least one edge. The *distance* from a vertex  $u$  to a vertex  $v$  in  $G$ , denoted  $d_G(u, v)$ , or simply  $d(u, v)$  when  $G$  is clear from the context, is the length (number of edges) of a shortest path from  $u$  to  $v$ . The *eccentricity* of a vertex  $v$  in  $G$ , denoted  $e_G(v)$ , is the maximum distance from  $v$  to any other vertex of  $G$ . The minimum eccentricity in  $G$  is the *radius* of  $G$ , denoted  $\text{rad}(G)$ , while the maximum eccentricity in  $G$  is its *diameter*, denoted  $\text{diam}(G)$ . Two vertices  $u$  and  $v$  with  $d_G(u, v) = \text{diam}(G)$  are said to be *antipodal*.

A function  $f : V(G) \rightarrow \{0, \dots, \text{diam}(G)\}$  is a *broadcast on  $G$*  if  $f(v) \leq e_G(v)$  for every vertex  $v \in V$ . For each vertex  $v$ ,  $f(v)$  is the  *$f$ -value* of  $v$ , or the *broadcast value* of  $v$  if  $f$  is clear from the context. Given such a broadcast  $f$ , an  *$f$ -broadcast vertex* is a vertex  $v$  for which  $f(v) > 0$ . The set of all  $f$ -broadcast vertices is denoted  $V_f^+(G)$ . If  $v$  is a broadcast vertex and

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$u$  a vertex such that  $d(u, v) \leq f(u)$ , then the vertex  $v$   $f$ -dominates the vertex  $u$ . The cost of a broadcast  $f$  on  $G$  is the value  $\sigma(f) = \sum_{v \in V_f^+} f(v)$ .

A broadcast  $f$  is *independent* if no broadcast vertex  $f$ -dominates another broadcast vertex, or, equivalently, if  $d(u, v) > \max\{f(u), f(v)\}$  for every two distinct broadcast vertices  $u$  and  $v$ . The maximum cost of an independent broadcast on  $G$  is the *broadcast independence number* of  $G$ , denoted  $\beta_b(G)$ . An independent broadcast with cost  $\beta_b(G)$  is referred to as a  $\beta_b$ -broadcast.

A subset  $S$  of  $V(G)$  is an *independent set* if no two vertices in  $S$  are adjacent in  $G$ . The *independence number* of  $G$ , denoted  $\alpha(G)$ , is then the maximum cardinality of an independent set in  $G$ . Note that the characteristic function  $f_S$  of every maximal independent set  $S$  in a graph  $G$  is an independent broadcast and, therefore,  $\alpha(G) \leq \beta_b(G)$  for every graph  $G$ .

Broadcast independence and broadcast domination were introduced by Erwin [9] in his Ph.D. dissertation, using the terms cost independence and cost domination, respectively. He also discussed several other types of broadcast parameters and gave relationships between them. Most of the corresponding results are published in [8, 10]. Since then, several papers have been devoted to the study of these broadcast parameters, but there were not so many results concerning the broadcast independence number [6, 8] until recently (see [1–5, 7]). In particular, Bessy and Rautenbach discussed the algorithmic complexity of broadcast independence in [3] and the links between girth, minimum degree, independence number and broadcast independence number in [4, 5].

In this paper, we study the broadcast independence number of circulant graphs. Recall that for every integer  $n \geq 3$ , and every sequence of integers  $a_1, \dots, a_k$ ,  $k \geq 1$ , satisfying  $1 \leq a_1 \leq \dots \leq a_k \leq \lfloor \frac{n}{2} \rfloor$ , the *circulant graph*  $G = C(n; a_1, \dots, a_k)$  is the graph defined by

$$V(G) = \{v_0, v_1, \dots, v_{n-1}\} \text{ and } E(G) = \{v_i v_{i+a_j} \mid a_j \in \{a_1, \dots, a_k\}\}$$

(subscripts are taken modulo  $n$ ). Note that, in particular,  $C(n; a_1, \dots, a_k)$  is  $2k$ -regular and vertex-transitive (see [12] for a survey on properties of undirected circulant graphs).

Our paper is organized as follows. In Section 2, we give some preliminary results and determine the broadcast independence number of circulant graphs of the form  $C(2a; 1, a)$  and  $C(3a; 1, a)$ . In Section 3 we prove that almost all circulant graphs of the form  $C(n; 1, a)$  admit an optimal independent broadcast all whose broadcast values are at most 2. General upper and lower bounds on the cost of independent broadcasts on circulant graphs of the form  $C(n; 1, a)$  are proposed in Section 4. We then determine the value of the broadcast independence number of various classes of circulant graphs in Section 5. We finally propose a few concluding remarks in Section 6.

## 2 Preliminary results

Let  $\mu(G)$  denote the maximum cardinality of a set of pairwise antipodal vertices in  $G$ . Dunbar *et al.* proved the following lower bound on the broadcast independence number of a graph.

**Proposition 1** (Dunbar *et al.* [8]). *For every graph  $G$ ,*

$$\beta_b(G) \geq \mu(G)(\text{diam}(G) - 1) \geq 2(\text{diam}(G) - 1).$$

*Moreover, this bound is sharp.*

In addition to grid graphs  $G_{m,n} = P_m \square P_n$  with  $m \in \{2, 3, 4\}$  and  $m \leq n$  [6] and paths [9], the relation  $\beta_b(G) = 2(\text{diam}(G) - 1)$  also holds for cycles of order at least 4 [7].

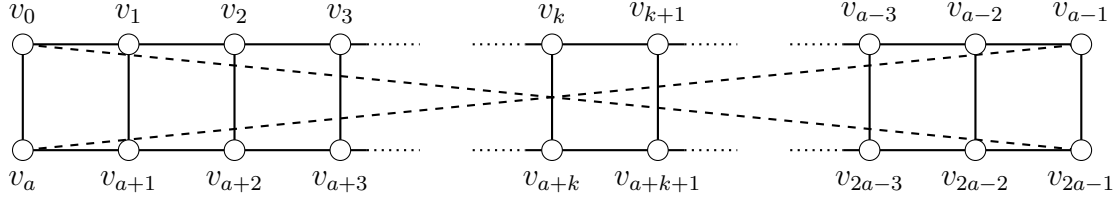


Figure 1: The circulant graph  $C(2a;1,a)$ .

It can also be observed that the value  $2(\text{diam}(G) - 1)$  is an upper bound on the cost of some independent broadcasts.

In order to compare the values of the independence number and of the broadcast independence number of the graphs we will consider in Section 5, the following observation will be useful.

**Observation 2.** *For every graph  $G$ ,  $\beta_b(G) \geq \alpha(G)$ . Moreover,  $\beta_b(G) = \alpha(G)$  if and only if there exists a  $\beta_b$ -broadcast  $f$  on  $G$  such that  $f(v) = 1$  for every broadcast vertex  $v \in V_f^+$ .*

Indeed, the fact that the characteristic function  $f_S$  of every maximal independent set  $S$  in a graph  $G$  is an independent broadcast on  $G$ , as noticed in the previous section, gives the inequality and the necessity of the condition for the second part of the statement, while the sufficiency follows from the fact that  $V_f^+$  is always an independent set.

Before considering general cases in the next sections, we will determine in the following the independence number and the broadcast independence number of circulant graphs of the form  $C(n; 1, a)$  for two particular cases, namely when  $n = 2a$  or  $n = 3a$ .

**Lemma 3.** *For every integer  $a \geq 2$ ,*

$$\alpha(C(2a; 1, a)) = \begin{cases} a, & \text{if } a \text{ is odd,} \\ a - 1, & \text{if } a \text{ is even.} \end{cases}$$

**Proof.** Since  $C(4; 1, 2) = K_4$  and  $\alpha(K_4) = 1$ , we can assume  $a \geq 3$ .

If  $a$  is odd, then the set  $S = \{v_i \mid i \text{ is even}\}$  is an independent set of  $C(2a; 1, a)$ , and thus  $\alpha(C(2a; 1, a)) \geq |S| = a$ . Since  $C_{2a}$  is a subgraph of  $C(2a; 1, a)$ , we get  $\alpha(C(2a; 1, a)) \leq \alpha(C_{2a}) = a$  and the result follows.

If  $a$  is even, then the set  $S' = \{v_i \mid 0 \leq i \leq a-2, i \text{ is even}\} \cup \{v_i \mid a+1 \leq i \leq 2a-3, i \text{ is odd}\}$  is an independent set of  $C(2a; 1, a)$ , and thus  $\alpha(C(2a; 1, a)) \geq |S'| = a-1$ . Note that the odd cycle  $C = v_0 v_1 \dots v_a v_0$ , with  $\alpha(C) = \frac{a}{2}$ , is a subgraph of  $C(2a; 1, a)$ . Therefore, for every independent set  $I$  of  $C(2a; 1, a)$ , there are at least two consecutive vertices  $v_i, v_{i+1}$  with  $v_i, v_{i+1} \notin I$ . This implies  $\alpha(C(2a; 1, a)) \leq \frac{2a-1}{2}$ , which gives  $\alpha(C(2a; 1, a)) = a-1$ .  $\square$

**Theorem 4.** *For every integer  $a \geq 2$ ,*

$$\beta_b(C(2a; 1, a)) = \begin{cases} \alpha(C(2a; 1, a)) = a, & \text{if } a \text{ is odd,} \\ \alpha(C(2a; 1, a)) = a - 1, & \text{if } a = 2^p \text{ for some integer } p \geq 1, \\ a & \text{otherwise.} \end{cases}$$

**Proof.** The case  $a = 2$  directly follows from Lemma 3. We can thus assume  $a \geq 3$ . The graph  $C(2a; 1, a)$  can be viewed as the Cartesian product graph  $P_a \square K_2$  with two additional edges (see

Figure 1, where the two additional edges are drawn as dashed lines). Recall that  $\alpha(C(2a; 1, a))$  is given by Lemma 3, and let  $f$  be any independent  $\beta_b$ -broadcast on  $C(2a; 1, a)$ . If  $|V_f^+| = 1$ , then

$$\sigma(f) = \text{diam}(C(2a; 1, a)) = \left\lfloor \frac{a+1}{2} \right\rfloor < \alpha(C(2a; 1, a)),$$

which gives  $\sigma(f) < \beta_b(C(2a; 1, a))$  by Observation 2, a contradiction. Therefore,  $|V_f^+| \geq 2$ . Since each vertex  $v \in V_f^+$   $f$ -dominates exactly  $4f(v)$  vertices, each  $f$ -broadcast vertex  $v$  is  $f$ -dominated exactly once, each non-broadcast vertex is  $f$ -dominated at most three times, and at most  $|V_f^+|$  vertices can be dominated three times (namely the vertices  $v_{i+a}$  when  $v_i \in V_f^+$ ), we get

$$4f(V_f^+) \leq 3|V_f^+| + |V_f^+| + 2(2a - 2|V_f^+|) = 4a, \quad (1)$$

and thus

$$\beta_b(C(2a; 1, a)) = \sigma(f) = \sum_{v \in V_f^+} f(v) = f(V_f^+) \leq a. \quad (2)$$

We now consider the three cases of the statement of the theorem separately.

1.  $a$  is odd.

Let  $g$  be the mapping from  $V(C(2a; 1, a))$  to  $\{0, 1\}$  defined by  $g(v_i) = 1$  if and only if  $i$  is even. Since  $a$  is odd,  $g$  is an independent broadcast on  $C(2a; 1, a)$ . This gives  $\beta_b(C(2a; 1, a)) \geq \sigma(g) = a$  and, since  $g$  satisfies (2),  $\beta_b(C(2a; 1, a)) = a$ . By Observation 2, we then get  $\beta_b(C(2a; 1, a)) = \alpha(C(2a; 1, a)) = a$ .

2.  $a = 2^p$  for some integer  $p \geq 1$ .

By Observation 2 and Lemma 3, we have  $\beta_b(C(2a; 1, a)) \geq \alpha(C(2a; 1, a)) = a - 1$ , since  $a$  is even. Let  $g$  be any independent  $\beta_b$ -broadcast on  $C(2a; 1, a)$ .

Suppose first that not all  $g$ -broadcast vertices have the same  $g$ -value, and let  $v_i$  and  $v_j$  be any two vertices with  $g(v_i) < g(v_j)$  such that the distance  $d(v_i, v_j)$  is minimum among all  $g$ -broadcast vertices with distinct  $g$ -values. Without loss of generality, we can assume  $i < j$ . We consider two subcases, depending on whether  $v_i$  and  $v_j$  are on the same side of the ‘‘ladder’’ (refer to Figure 1) or not.

(a)  $j - i > a$  ( $v_i$  and  $v_j$  are not on the same side of the ladder).

Since no  $g$ -broadcast vertex lies on a shortest path linking  $v_i$  and  $v_j$ ,  $v_{j+a}$  is not  $g$ -dominated by  $v_i$  and is thus  $g$ -dominated at most twice. Therefore, the inequality (1) becomes

$$4g(V_g^+) \leq 3(|V_g^+| - 1) + |V_g^+| + 2(2a - 2|V_g^+| + 1) = 4a - 1,$$

which gives

$$\beta_b(C(2a; 1, a)) = \sigma(g) = \sum_{v \in V_g^+} g(v) = g(V_g^+) \leq \left\lfloor \frac{4a - 1}{4} \right\rfloor = a - 1.$$

(b)  $j - i < a$  ( $v_i$  and  $v_j$  are on the same side of the ladder).

If there exists a  $g$ -broadcast vertex  $v_k$  with  $i + a < k < j + a$  then, since no  $g$ -broadcast vertex lies on a shortest path linking  $v_i$  and  $v_j$ , we necessarily have  $v_k \in \{v_{i+a+1}, v_{j+a-1}\}$ . By considering either  $v_i$  and  $v_k$ , or  $v_j$  and  $v_k$ , instead of  $v_i$  and  $v_j$ , we are back to the previous subcase.

If no such vertex exists, then both  $v_{i+a}$  and  $v_{j+a}$  are  $g$ -dominated at most twice and thus, using the same argument as before, we get  $\beta_b(C(2a; 1, a)) \leq a - 1$ .

We thus get  $\beta_b(C(2a; 1, a)) = a - 1$  in both subcases, as required. Suppose now that  $g(v) = k$  for every vertex  $v \in V_g^+$  and let  $v_i$  be any such vertex.

If  $v_{i+a+k} \notin V_g^+$ , then the vertex  $v_{i+a}$  is  $g$ -dominated at most twice and thus, as previously, we get  $\beta_b(C(2a; 1, a)) = \sigma(g) = a - 1$ . The same conclusion arises if  $v_{i-a-k} \notin V_g^+$ .

Suppose finally  $v_{i+a+k}, v_{i-a-k} \in V_g^+$  for every vertex  $v_i \in V_g^+$  and assume, without loss of generality,  $v_0 \in V_g^+$ . We thus have (recall that indices are taken modulo  $2a$ )

$$\begin{aligned} V_g^+ &= \{v_0, v_{a+k}, v_{2k}, v_{a+3k}, \dots\}, \\ &= \{v_0, v_{2k}, v_{4k}, \dots, v_{a+k}, v_{a+3k}, v_{a+5k}, \dots\}. \end{aligned}$$

Hence, every  $g$ -broadcast vertex  $v_i$  with  $0 \leq i < a$  satisfies  $i = 2kt$  for some  $t$ ,  $0 \leq t < \frac{a}{2k}$ . Since  $v_{-a-k} = v_{a-k}$  is a  $g$ -broadcast vertex and  $0 < a - k < a$ , we get  $a - k = 2kt'$ , for some  $t'$ ,  $0 < t' < \frac{a}{2k}$ . This gives  $a = (2t' + 1)k$ , contradicting the assumption  $a = 2^p$ , so that this last case cannot appear.

3.  $a$  is even and  $a \neq 2^p$  for every  $p > 0$ .

This implies  $a = (2\ell + 1)2^k$  for some positive integers  $k$  and  $\ell$ . Let  $g$  be the mapping from  $V(C(2a; 1, a))$  to  $\{0, 2^k\}$  defined by  $g(v_i) = 2^k$  if and only if  $i \equiv 0 \pmod{2^{k+1}}$ , which gives  $V_g^+ = \{v_{p2^{k+1}} \mid 0 \leq p \leq 2\ell\}$ .

For any two  $g$ -broadcast vertices  $v_i = v_{p2^{k+1}}$  and  $v_j = v_{q2^{k+1}}$ ,  $0 \leq p < q \leq 2\ell$ , we have

$$d(v_i, v_j) = \min\{(q - p)2^{k+1}, |(p - q)2^{k+1} + a| + 1, (p - q)2^{k+1} + 2a\},$$

which gives, since  $a = (2\ell + 1)2^k$ ,

$$d(v_i, v_j) = \min\{(q - p)2^{k+1}, |2(p - q + \ell) + 1|2^k + 1, (p - q + 2\ell + 1)2^{k+1}\} \geq 2^k + 1.$$

Therefore,  $g$  is an independent broadcast on  $C(2a; 1, a)$ , with cost  $\sigma(g) = 2^k \left(\frac{2a}{2^{k+1}}\right) = a$ , which gives  $\beta_b(C(2a; 1, a)) \geq \sigma(g) = a$  and thus, since  $g$  satisfies (2),  $\beta_b(C(2a; 1, a)) = a$ .

This completes the proof.  $\square$

We finally consider the case  $n = 3a$ .

**Theorem 5.** *For every integer  $a \geq 3$ ,*

$$\beta_b(C(3a; 1, a)) = \alpha(C(3a; 1, a)) = a.$$

**Proof.** Let  $f$  be an independent  $\beta_b$ -broadcast on  $C(3a; 1, a)$ . For each vertex  $v_i \in V_f^+$ , we let

$$C_f^i = \{v_i, \dots, v_{i+f(v_i)-1}\} \cup \{v_{i+a}, \dots, v_{i+a+f(v_i)-1}\} \cup \{v_{i+2a}, \dots, v_{i+2a+f(v_i)-1}\}.$$

We clearly have  $|C_f^i| = 3f(v_i)$  for every  $v_i \in V_f^+$ , and  $C_f^i \cap C_f^{i'} = \emptyset$  for every two distinct vertices  $v_i$  and  $v_{i'}$  in  $V_f^+$ , since otherwise we would have  $d(v_i, v_{i'}) \leq \max\{f(v_i), f(v_{i'})\}$ , contradicting the fact that  $f$  is an independent broadcast. This gives

$$3f(V_f^+) = \sum_{v_i \in V_f^+} |C_f^i| \leq 3a,$$

and thus

$$\beta_b(C(3a; 1, a)) = \sigma(f) = \sum_{v \in V_f^+} f(v) = f(V_f^+) \leq \frac{3a}{3} = a.$$

Consider now the mapping  $f$  from  $V(C(3a; 1, a))$  to  $\{0, 1\}$  defined as follows, depending on the parity of  $a$ .

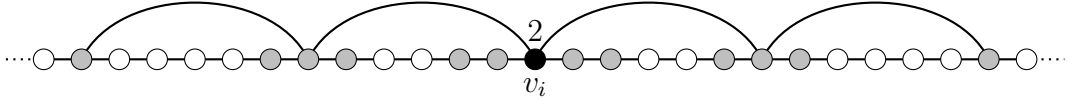


Figure 2: The set  $D_f(v_i)$  (black vertex and grey vertices), with  $a = 6$  and  $f(v_i) = 2$ .

1. If  $a$  is odd, then  $f(v_i) = 1$  if and only if  $i$  is even and  $i < 2a$ .
2. If  $a$  is even, then  $f(v_i) = 1$  if and only if  $(i \bmod a + 1)$  is odd and  $i \leq 2a$ .

In both cases,  $f$  is clearly an independent broadcast on  $V(C(3a; 1, a))$  with  $\sigma(f) = a$ . This implies  $\beta_b(C(3a; 1, a)) \geq a$  and thus, thanks to Observation 2,  $\beta_b(C(3a; 1, a)) = \alpha(C(3a; 1, a)) = a$ .  $\square$

### 3 2-bounded optimal independent broadcasts

Recall that we denote by  $v_0, v_1, \dots, v_{n-1}$  the vertices of  $C(n; 1, a)$  and that subscripts are always considered modulo  $n$ . We will say that an edge  $v_i v_j$  is a  $k$ -edge for some integer  $k$ ,  $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$ , if  $j = i + k$  or  $i = j + k$ . Therefore, every edge in  $C(n; 1, a)$  is either a 1-edge or an  $a$ -edge.

Let  $f$  be an independent broadcast on  $C(n; 1, a)$ . For every  $f$ -broadcast vertex  $v_i \in V_f^+$ , we denote by  $D_f(v_i)$  the set of vertices that are  $f$ -dominated by  $v_i$ , that is

$$D_f(v_i) = \bigcup_{k=0}^{f(v_i)} \{v_j \mid i - (f(v_i) - k)a - k \leq j \leq i - (f(v_i) - k)a + k\} \\ \cup \bigcup_{k=0}^{f(v_i)} \{v_j \mid i + (f(v_i) - k)a - k \leq j \leq i + (f(v_i) - k)a + k\}.$$

Figure 2 illustrates this definition on a circulant graph of the form  $C(n; 1, 6)$  (with  $n \geq 26$ ) for a vertex  $v_i$  with  $f(v_i) = 2$ .

Let us say that an independent broadcast  $f$  is  $\ell$ -bounded, for some integer  $\ell \geq 1$ , if  $f(v) \leq \ell$  for every vertex  $v$ . In particular, a 1-bounded independent broadcast is the characteristic function of an independent set. This implies that such a broadcast always exists for every graph, and thus that every graph admits an  $\ell$ -bounded independent broadcast for every  $\ell \geq 1$ .

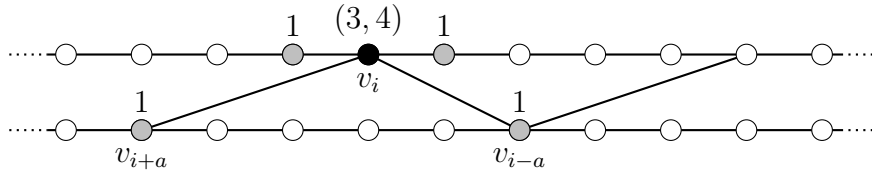
Our goal in this section is to prove that almost all circulant graphs of the form  $C(n; 1, a)$ ,  $2 \leq a \leq \frac{n}{2}$ , admit a 2-bounded optimal independent broadcast. Considering the  $\beta_b$ -broadcasts used in the proofs of Theorems 4 and 5, we already have the following result.

**Proposition 6.** *For every integer  $a \geq 2$ , the following holds.*

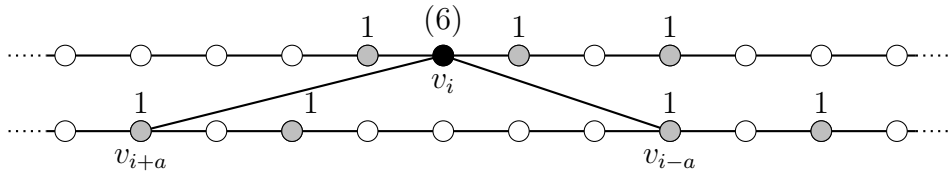
1.  $C(2a; 1, a)$  admits a 2-bounded  $\beta_b$ -broadcast if  $a$  is odd or  $a = 2^p$  for some  $p \geq 1$ .
2.  $C(3a; 1, a)$  admits a 2-bounded  $\beta_b$ -broadcast.

We will now prove that every circulant graph of the form  $C(n; 1, a)$ ,  $a \geq 3$  and  $n \geq 2a + 2$ , admits a 2-bounded  $\beta_b$ -broadcast. We first consider the case when  $2a + 2 \leq n < 3a$ .

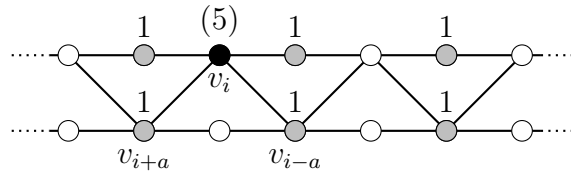
**Lemma 7.** *If  $n$ ,  $a$  and  $r$  are three integers such that  $n = 2a + r$ ,  $3 \leq a \leq \lfloor \frac{n}{2} \rfloor$  and  $2 \leq r < a$ , then  $C(n; 1, a)$  admits a 2-bounded  $\beta_b$ -broadcast.*



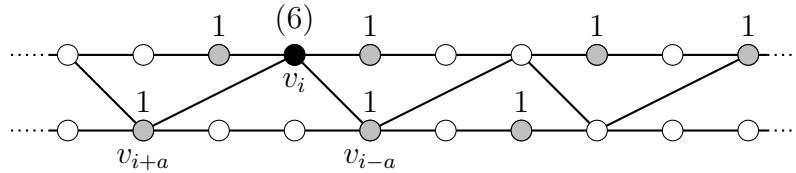
(a) Item 1:  $f(v_i) \in \{3, 4\}$ ,  $a = 10$  and  $r = 5$  (so that  $p = 0$  and  $n = 25$ )



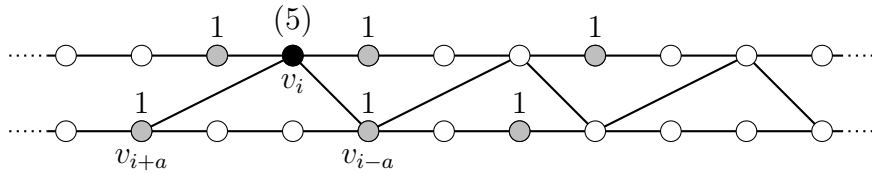
(b) Item 1:  $f(v_i) = 6$ ,  $a = 8$  and  $r = 7$  (so that  $p = 2$  and  $n = 23$ )



(c) Item 2:  $f(v_i) = 5$ ,  $a = 12$  and  $r = 2$  (so that  $p = 2$  and  $n = 26$ )



(d) Item 3:  $f(v_i) = 6$ ,  $a = 12$  and  $r = 3$  (so that  $d = 2$  and  $n = 27$ )



(e) Item 3:  $f(v_i) = 5$ ,  $a = 12$  and  $r = 3$  (so that  $d = 1$  and  $n = 27$ )

Figure 3: Construction of the mapping  $g$  in the proof of Lemma 7.



**Proof.** Note that it is enough to prove that for every independent broadcast  $f$  on  $C(n; 1, a)$ , there exists an independent broadcast  $g$  on  $C(n; 1, a)$  such that  $\sigma(g) \geq \sigma(f)$  and  $g(v) \leq 2$  for every vertex  $v \in V_g^+$ .

Let  $f$  be any independent broadcast on  $C(n; 1, a)$ , and  $g$  be the mapping from  $V(C(n; 1, a))$  to  $\{0, 1, 2\}$  defined as follows (the construction of the mapping  $g$  is illustrated in Figure 3, where the value of  $f(v_i)$  is indicated in brackets; not all  $a$ -edges are drawn, but the missing  $a$ -edges are parallel to the drawn ones; note also that  $v_{i-a} = v_{i+a+r}$ , and thus  $v_{i+a}$  and  $v_{i-a}$  are separated by  $r - 1$  vertices).

1. If  $v_i$  is an  $f$ -broadcast vertex such that  $2 < f(v_i) \leq r$ , then we let

$$g(v_j) = \begin{cases} 0 & \text{if } j = i, \\ 1 & \text{if } j = i - a, \\ & \text{or } i - 1 \leq j \leq i + p + 1 \text{ and } j - i + 1 \text{ is even,} \\ & \text{or } i + a \leq j \leq i + a + p \text{ and } j - i - a \text{ is even,} \end{cases}$$

where  $p = f(v_i) - 3$  if  $f(v_i)$  is odd, and  $p = f(v_i) - 4$  if  $f(v_i)$  is even (see Figure 3(a,b)).

2. If  $v_i$  is an  $f$ -broadcast vertex such that  $\left\lfloor \frac{a}{2} \right\rfloor > f(v_i) \geq r + 1$  and  $r$  is even, then we let

$$g(v_j) = \begin{cases} 0 & \text{if } j = i, \\ 1 & \text{if } i - 1 \leq j \leq i + p + 1 \text{ and } j - i + 1 \text{ is even,} \\ & \text{or } i + a \leq j \leq i + a + r + p \text{ and } j - i - a \text{ is even,} \end{cases}$$

where  $p = f(v_i) - 3$  if  $f(v_i)$  is odd, and  $p = f(v_i) - 4$  if  $f(v_i)$  is even (see Figure 3(c)).

3. If  $v_i$  is an  $f$ -broadcast vertex such that  $f(v_i) \geq r + 1$  and  $r$  is odd, then we let

$$g(v_j) = \begin{cases} 0 & \text{if } j = i, \\ 1 & \text{if } i - 2 \leq j \leq i + \left\lfloor \frac{d+2}{2} \right\rfloor r + (d \bmod 2) \\ & \text{and } (j - i + 2) \bmod (r + 2) \text{ is odd,} \\ & \text{or } i + a \leq j \leq i + a + \left\lceil \frac{d+2}{2} \right\rceil r + 1 - (d \bmod 2) \\ & \text{and } (j - i - a + 3) \bmod (r + 2) \text{ is odd,} \end{cases}$$

where  $d = f(v_i) - (r + 1)$  (see Figure 3(d,e)).

4. For every other vertex  $v_k$ , we let  $g(v_k) = f(v_k)$ .

Note that, in particular,  $g(v_i) = f(v_i)$  for every  $f$ -broadcast vertex  $v_i$  with  $f(v_i) \leq 2$ . Moreover, all vertices set to 1 in the above items are distinct from  $v_i$  and at distance not greater than  $f(v_i)$  from  $v_i$ , which means that their  $f$ -value was 0.

We now prove that  $g$  is an independent broadcast on  $C(n; 1, a)$ . For that, we first prove the following claim.

**Claim A.** *For every vertex  $v_j$  whose  $g$ -value is set to 1 in Item 1, 2 or 3 above, we have  $d(v_i, v_j) \leq f(v_i) - 2$ .*

**Proof.** In Item 1, every vertex whose  $g$ -value is set to 1 is at distance at most  $p+1 \leq f(v_i) - 2$  from  $v_i$ . In Item 2, every vertex whose  $g$ -value is set to 1 is at distance at most  $\max\{p+1, r-1\} \leq f(v_i) - 2$  from  $v_i$ .

Consider now a vertex  $v_j$  whose  $g$ -value is set to 1 in Item 3 and whose distance to  $v_i$  is maximal (see Figure 3(d,e)), and suppose first that  $i-2 \leq j \leq i + \lfloor \frac{d+2}{2} \rfloor r + (d \bmod 2)$ . Since  $r \geq 3$  and  $r$  is odd,  $v_j$  is at distance  $d(v_i, v_j) = 2 \lfloor \frac{d+2}{2} \rfloor + \lfloor \frac{r}{2} \rfloor - 1$  from  $v_i$  (going to  $v_{i+\lfloor \frac{d+2}{2} \rfloor r}$  using  $(2 \lfloor \frac{d+2}{2} \rfloor)$   $a$ -edges, and then back to  $v_j$  using  $(\lfloor \frac{r}{2} \rfloor - 1)$  1-edges). Since  $d = f(v_i) - r - 1$  and  $r \geq 3$ , this gives

$$d(v_i, v_j) \leq f(v_i) - r + 1 + \frac{r-1}{2} - 1 = f(v_i) - \frac{r+1}{2} \leq f(v_i) - 2.$$

Suppose finally that  $i+a \leq j \leq i+a + \lceil \frac{d+2}{2} \rceil r + 1 - (d \bmod 2)$ . In that case,  $v_j$  is at distance  $d(v_i, v_j) = 1 + 2(\lfloor \frac{d+2}{2} \rfloor - 1) + \lfloor \frac{r}{2} \rfloor - 1$  from  $v_i$  (going to  $v_{i+a+\lceil \frac{d+2}{2} \rceil r}$  using  $(1 + 2(\lfloor \frac{d+2}{2} \rfloor - 1))$   $a$ -edges, and then back to  $v_j$  using  $(\lfloor \frac{r}{2} \rfloor - 1)$  1-edges). As before, since  $d = f(v_i) - r - 1$  and  $r \geq 3$ , this gives

$$\begin{aligned} d(v_i, v_j) &\leq 1 + 2 \left( \left\lfloor \frac{d+2}{2} \right\rfloor - 1 \right) + \frac{r-1}{2} - 1 = 2 \left( \left\lfloor \frac{f(v_i) - r + 1}{2} \right\rfloor - 1 \right) + \frac{r-1}{2} \\ &\leq 2 \left( \frac{f(v_i) - r + 1}{2} \right) + \frac{r-1}{2} - 2 = f(v_i) - \frac{r-1}{2} - 2 \\ &\leq f(v_i) - 2, \end{aligned}$$

which concludes the proof of the claim.  $\square$

Thanks to this claim, and since  $f$  was an independent broadcast on  $C(n; 1, a)$ , no  $g$ -broadcast vertex  $v_i$  with  $g(v_i) = f(v_i) \in \{1, 2\}$   $g$ -dominates a vertex whose  $g$ -value has been set to 1. Therefore, in order to prove that  $g$  is indeed an independent broadcast on  $C(n; 1, a)$ , it remains to prove that the set of vertices whose  $g$ -value has been set to 1 is an independent set. Moreover, thanks to Claim A, Items 1, 2 and 3 can be considered separately. This is readily the case for vertices whose  $g$ -value has been set to 1 in Item 1 and 2. In Item 3, thanks to the parity of their subscript, no two such vertices are linked by a 1-edge. Moreover, any two such vertices cannot be linked by an  $a$ -edge since  $(j - i + 2) - (j - i - a + 3) = a - 1$ .

In order to finish the proof, we only need to show that we have  $\sigma(g) \geq \sigma(f)$ . Indeed, in Item 1, the number of vertices set to 1 is  $n_1 = 1 + \frac{p+4}{2} + \frac{p+2}{2} = p+4$ , which gives  $n_1 = f(v_i)$  if  $f(v_i)$  is even, or  $n_1 = f(v_i) + 1 > f(v_i)$  if  $f(v_i)$  is odd.

In Item 2, the number of vertices set to 1 is  $n_2 = \frac{p+4}{2} + \frac{p+2}{2} + \frac{r}{2} = \frac{2p+6+r}{2}$ , which gives  $n_2 = \frac{2f(v_i)-2+r}{2} \geq f(v_i)$  (recall that  $r \geq 2$ ) if  $f(v_i)$  is even, or  $n_2 = \frac{2f(v_i)+r}{2} > f(v_i)$  if  $f(v_i)$  is odd.

Finally consider Item 3. Observe that, since  $r+2$  is odd, in every sequence of  $r+2$  consecutive vertices lying between  $v_{i-2}$  and  $v_{i+\lfloor \frac{d+2}{2} \rfloor r + (d \bmod 2)}$ , or between  $v_{i+a}$  and  $v_{i+a+\lceil \frac{d+2}{2} \rceil r + 1 - (d \bmod 2)}$ , exactly  $\frac{r+1}{2}$  vertices are set to 1. Therefore, the number of vertices set to 1 in Item 3 is either

$$n_3 = \left\lceil \frac{r+1}{2(r+2)} \left( \frac{d+2}{2} r + 3 \right) \right\rceil + \left\lceil \frac{r+1}{2(r+2)} \left( \frac{d+2}{2} r + 2 \right) \right\rceil,$$

if  $d$  is even, or

$$n_3 = \left\lceil \frac{r+1}{2(r+2)} \left( \frac{d+1}{2} r + 4 \right) \right\rceil + \left\lceil \frac{r+1}{2(r+2)} \left( \frac{d+3}{2} r + 1 \right) \right\rceil,$$



2. If  $v_i$  is an  $f$ -broadcast vertex such that  $f(v_i) \geq a + 1$  and  $a$  is odd, then we let

$$g(v_j) = \begin{cases} 0 & \text{if } j = i, \\ 1 & \text{if } i - a \leq j \leq i + (1 + d)a \text{ and } j - i + a \text{ is even,} \end{cases}$$

where  $d = f(v_i) - (a + 1)$  (see Figure 4(b)).

3. If  $v_i$  is an  $f$ -broadcast vertex such that  $f(v_i) \geq a + 1$  and  $a$  is even, then we let

$$g(v_j) = \begin{cases} 0 & \text{if } j = i \\ 1 & \text{if } i - a - 1 \leq j \leq i + (2 + d)a \text{ and } (j - i + a + 1) \pmod{a + 1} \text{ is odd,} \end{cases}$$

where  $d = f(v_i) - (a + 1)$  (see Figure 4(c)).

4. For every other vertex  $v_k$ , we let  $g(v_k) = f(v_k)$ .

Note that, as in the proof of the previous lemma,  $g(v_i) = f(v_i)$  for every  $f$ -broadcast vertex  $v_i$  with  $f(v_i) \leq 2$ . Moreover, all vertices set to 1 in the above items are also distinct from  $v_i$  and at distance not greater than  $f(v_i)$  from  $v_i$ , which means that their  $f$ -value was 0.

We now prove that  $g$  is an independent broadcast on  $C(n; 1, a)$ . For that, we first prove the following claim.

**Claim B.** *For every vertex  $v_j$  whose  $g$ -value is set to 1 in Item 1, 2 or 3 above, we have  $d(v_i, v_j) \leq f(v_i) - 2$ .*

**Proof.** In Item 1, every vertex whose  $g$ -value is set to 1 is at distance at most  $p + 1 \leq f(v_i) - 2$  from  $v_i$ .

Among the vertices whose  $g$ -value might be set to 1 in Item 2, the vertex whose distance to  $v_i$  is maximal is, since  $a$  is odd, the vertex  $v_j$  with  $j = da + \frac{a+1}{2}$ , which gives

$$d(v_i, v_j) = d + \frac{a+1}{2} = f(v_i) - \frac{a+1}{2} \leq f(v_i) - 2$$

(recall that, in that case, we have  $a \geq 3$ ).

Similarly, among the vertices whose  $g$ -value might be set to 1 in Item 3, the vertex whose distance to  $v_i$  is maximal is, since  $a$  is even, the vertex  $v_j$  with  $j = (d + 1)a + \frac{a}{2}$ , which gives

$$d(v_i, v_j) = d + 1 + \frac{a}{2} = f(v_i) - \frac{a}{2} \leq f(v_i) - 2$$

(recall that, in that case, we have  $a \geq 4$ ).

This concludes the proof of the claim. □

Thanks to this claim, and since  $f$  was an independent broadcast on  $C(n; 1, a)$ , no  $g$ -broadcast vertex  $v_i$  with  $g(v_i) = f(v_i) \in \{1, 2\}$   $g$ -dominates a vertex whose  $g$ -value has been set to 1. Therefore, in order to prove that  $g$  is indeed an independent broadcast on  $C(n; 1, a)$ , it remains to prove that the set of vertices whose  $g$ -value has been set to 1 is an independent set. Moreover, thanks to Claim B, Items 1, 2 and 3 can be considered separately. This is readily the case for vertices whose  $g$ -value has been set to 1 in Item 1. It follows from the parity of their subscript in Item 2 (neither a 1-edge nor an  $a$ -edge, since  $a$  is odd, can link any two such vertices), and from the value modulo  $(a + 1)$  of their subscript in Items 3 (which, again, implies that neither a 1-edge nor an  $a$ -edge, since  $a$  is even, can link any two such vertices).

In order to finish the proof, we only need to show that we have  $\sigma(g) \geq \sigma(f)$ . Indeed, in Item 1, the number of vertices set to 1 is  $n_1 = \frac{p+4}{2} + \frac{p+2}{2} + \frac{p+2}{2} = \frac{3p+8}{2}$ , which gives  $n_1 = \frac{3f(v_i)-4}{2} \geq f(v_i)$  if  $f(v_i)$  is even (in that case,  $f(v_i) \geq 4$ , and equality holds only when  $f(v_i) = 4$ ), or  $n_1 = \frac{3f(v_i)-1}{2} > f(v_i)$  if  $f(v_i)$  is odd (in that case,  $f(v_i) \geq 3$ ).

In Item 2, the number of vertices set to 1 is  $n_2 = \left\lceil \frac{(2+d)a+1}{2} \right\rceil = \left\lceil \frac{2a+1+da}{2} \right\rceil$ . If  $d = 0$ , since  $a$  is odd, we get  $n_2 = \left\lceil \frac{2a+1}{2} \right\rceil = a + 1 = f(v_i)$ . Otherwise, that is, if  $d \geq 1$ , since  $a \geq 3$ , we get  $n_2 = \left\lceil \frac{2a+1+da}{2} \right\rceil \geq a + \frac{1}{2} + \frac{ad}{2} \geq a + d + 1 = f(v_i)$ .

Finally, in Item 3, note that, for every sequence of  $a + 1$  consecutive vertices,  $\frac{a}{2}$  of them are set to 1. Therefore, the total number of vertices set to 1 is

$$n_3 = \left\lceil \frac{a}{2(a+1)} ((d+3)a+2) \right\rceil \geq \frac{3a^2 + da^2 + 2a}{2(a+1)} = a + \frac{da^2}{2(a+1)} + \frac{a^2}{2(a+1)}.$$

Since  $a \geq 4$ , we have  $\frac{a^2}{2(a+1)} > 1$ , which gives  $n_3 > a + d + 1 = f(v_i)$ .

We thus have  $\sigma(g) \geq \sigma(f)$ , as required. This completes the proof.  $\square$

From Proposition 6 and Lemmas 7 and 8, we directly get the following theorem.

**Theorem 9.** *Every circulant graph of the form  $C(n; 1, a)$ ,  $3 \leq a \leq \lfloor \frac{n}{2} \rfloor$ , admits a 2-bounded  $\beta_b$ -broadcast if none of the following conditions is satisfied: (i)  $n = 2a$  and  $a$  is even, or (ii)  $n = 2a + 1$ .*

The following example will show that when  $a = 2$  or  $n = 2a + 1$ , not all circulant graphs of the form  $C(n; 1, a)$  admit a 2-bounded  $\beta_b$ -broadcast. Consider the circulant graph  $C(21; 1, 2)$ , and let  $f$  be the mapping from  $V(C(21; 1, 2))$  to  $\{0, 3\}$  defined by  $f(v_0) = f(v_7) = f(v_{14}) = 3$  and  $f(v_i) = 0$  otherwise. Since 2 is even,  $f$  is clearly an independent broadcast on  $C(21; 1, 2)$ , with cost  $\sigma(f) = 9$ , and thus  $9 \leq \beta_b(C(21; 1, 2))$ . Now, suppose that there exists a 2-bounded  $\beta_b$ -broadcast  $g$  on  $C(21; 1, 2)$ . If  $|V_g^+| \leq 4$ , we immediately get  $\sigma(g) \leq 8$ , since  $g$  is 2-bounded. Suppose now  $|V_g^+| > 4$ . Each vertex  $v_i \in V_g^+$  dominates at most three vertices among  $\{v_i, v_{i+1}, v_{i+2}\}$  (subscripts are taken modulo 21), and none of these vertices is dominated more than once. Therefore, since  $g$  is an independent broadcast, we get

$$\sum_{v_i \in V_g^+} (1 + 2g(v_i)) = |V_g^+| + 2g(V_g^+) \leq 21,$$

which gives (recall that  $|V_g^+| > 4$ )

$$\sigma(g) = g(V_g^+) \leq \frac{21 - |V_g^+|}{2} \leq 8.$$

In both cases, we get a contradiction to the optimality of  $g$ . Finally, since  $C(21; 1, 10)$  is isomorphic to  $C(21; 1, 2)$ , we get that there also exist circulant graphs of the form  $C(2a + 1; 1, a)$  that do not admit any 2-bounded  $\beta_b$ -broadcast.

## 4 General bounds on the independence broadcast number of $C(n; 1, a)$

In this section, we will provide some general upper and lower bounds on the cost of independent broadcasts on circulant graphs of the form  $C(n; 1, a)$ ,  $2 \leq a \leq \lfloor \frac{n}{2} \rfloor$ , that will be useful in the next section.



Similarly, we necessarily have  $B_f^j \cap B_f^{j'} = \emptyset$  for every two distinct vertices  $v_j$  and  $v_{j'}$  in  $V_f^2$ , since otherwise we would have  $d(v_j, v_{j'}) \leq \max\{f(v_j), f(v_{j'})\} = 2$ , contradicting the fact that  $f$  is an independent broadcast. The same argument gives  $A_f^i \cap B_f^j = \emptyset$  for every two vertices  $v_i \in V_f^1$  and  $v_j \in V_f^2$ . All together, these three properties imply that Item 3 also holds.  $\square$

The next result provides a general upper bound on the broadcast independence number of circulant graphs of the form  $C(n; 1, a)$ , with  $3 \leq a \leq \lfloor \frac{n}{2} \rfloor$  and  $3a \leq n$ .

**Proposition 11.** *If  $n$  and  $a$  are two integers such that  $3 \leq a \leq \lfloor \frac{n}{2} \rfloor$  and  $3a \leq n$ , then, for every 2-bounded independent broadcast  $f$  on  $C(n; 1, a)$ , we have*

$$\sigma(f) \leq \left\lfloor \frac{n - |V_f^2|}{2} \right\rfloor.$$

**Proof.** Let  $f$  be any 2-bounded independent broadcast on  $C(n; 1, a)$  (it follows from Theorem 9 that such broadcasts exist). From Lemma 10, we get

$$\sum_{v_i \in V_f^1} |A_f^i| + \sum_{v_j \in V_f^2} |B_f^j| = 2f(V_f^1) + |V_f^1| + 3f(V_f^2) - |V_f^2| \leq n,$$

which gives

$$2f(V_f^+) = 2f(V_f^1) + 2f(V_f^2) \leq n - |V_f^1| + |V_f^2| - f(V_f^2) \leq n + |V_f^2| - f(V_f^2).$$

Now, since  $f(v_j) = 2$  for every  $v_j \in V_f^2$ , we have  $f(V_f^2) = 2|V_f^2|$ , and thus

$$\sigma(f) = f(V_f^+) \leq \left\lfloor \frac{n - |V_f^2|}{2} \right\rfloor.$$

This completes the proof.  $\square$

When  $a$  is even, the upper bound given in Proposition 11 can be improved as follows.

**Proposition 12.** *If  $n$  and  $a$  are two integers such that  $2 \leq a \leq \lfloor \frac{n}{2} \rfloor$  and  $a$  is even, then, for every 2-bounded independent broadcast  $f$  on  $C(n; 1, a)$ , we have*

$$\sigma(f) \leq \left\lfloor \frac{a}{2(a+1)} \left( n - \frac{a-4}{a} |V_f^2| \right) \right\rfloor.$$

**Proof.** Let  $f$  be any 2-bounded independent broadcast on  $C(n; 1, a)$ . Observe first that we necessarily have  $|A_f^i| \leq a+1$  for every vertex  $v_i \in V_f^1$ , since otherwise this would give  $f(v_i) = f(v_{i+a}) = 1$ , contradicting the fact that  $f$  is an independent broadcast. This implies  $f(A_f^i) \leq \frac{a}{2}$ . Using item 1 of Lemma 10, we then get

$$\frac{|A_f^i|}{f(A_f^i)} = \frac{2f(A_f^i) + 1}{f(A_f^i)} = 2 + \frac{1}{f(A_f^i)} \geq 2 + \frac{2}{a} = \frac{2(a+1)}{a},$$

and thus

$$|A_f^i| \geq \frac{2(a+1)}{a} f(A_f^i).$$

From Lemma 10, we then get

$$n \geq \sum_{v_i \in V_f^1} |A_f^i| + \sum_{v_j \in V_f^2} |B_f^j| \geq \frac{2(a+1)}{a} f(V_f^1) + 3f(V_f^2) - |V_f^2|,$$

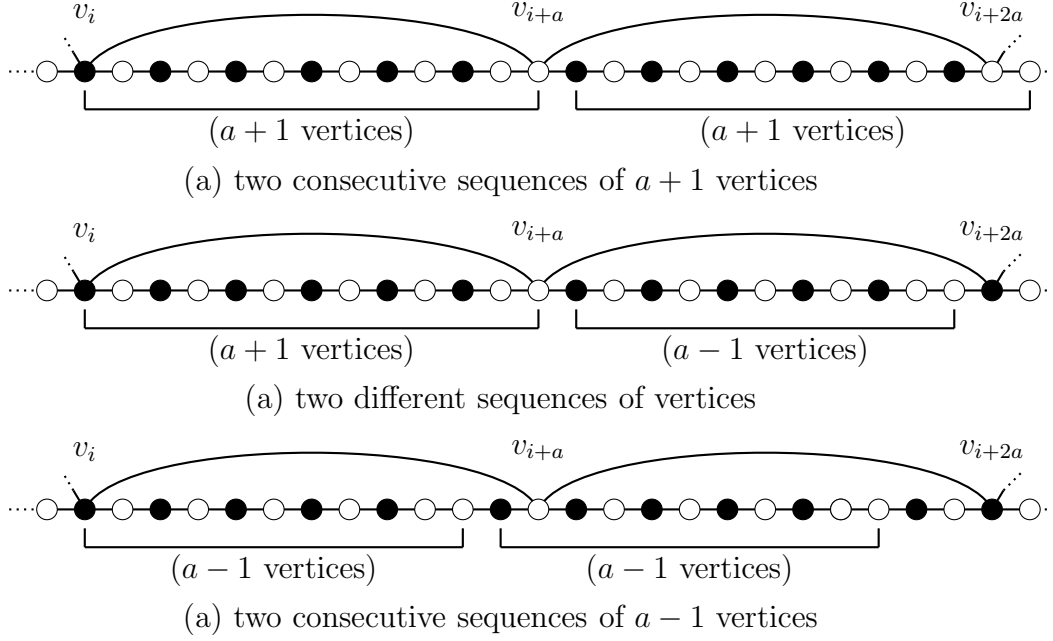


Figure 6: Construction of the mapping  $f$  in the proof of Proposition 13 ( $a = 12$ ).

which gives

$$n \geq \frac{2(a+1)}{a}f(V) + \frac{a-2}{a}f(V_f^2) - |V_f^2|.$$

Finally, since  $f(V_f^2) = 2|V_f^2|$ , we get

$$\frac{2(a+1)}{a}f(V) \leq n + |V_f^2| - \frac{2(a-2)}{a}|V_f^2| = n - \frac{a-4}{a}|V_f^2|,$$

and thus

$$\sigma(f) = f(V) \leq \left\lfloor \frac{a}{2(a+1)} \left( n - \frac{a-4}{a}|V_f^2| \right) \right\rfloor.$$

This completes the proof.  $\square$

**Proposition 13.** *If  $n$ ,  $a$ ,  $k_1$  and  $k_2$  are four integers such that  $n = k_1(a+1) + k_2(a-1)$ ,  $6 \leq a \leq \lfloor \frac{n}{2} \rfloor$ , and  $a$  is even, then, for every independent broadcast  $f$  on  $C(n; 1, a)$ , we have*

$$\sigma(f) \geq k_1 \binom{a}{2} + k_2 \left( \frac{a}{2} - 1 \right).$$

**Proof.** Let  $n = k_1(a+1) + k_2(a-1)$ . The circulant graph  $C(n; 1, a)$  consists of  $k_1$  sequences of  $a+1$  vertices and  $k_2$  sequences of  $a-1$  vertices. Let  $f$  be a mapping from  $V(C(n; 1, a))$  to  $\{0, 1\}$ , defined as follows (see Figure 6 for the case  $a = 12$ ). For every sequence of  $a+1$  or  $a-1$  vertices, we let the broadcast values of the form 1010...10100. Since  $a$  is even, for every two consecutive sequences, the  $f$ -broadcast vertices are pairwise non adjacent and then,  $f$  is an independent broadcast on  $C(n; 1, a)$ , with cost  $\sigma(f) = k_1 \binom{a}{2} + k_2 \left( \frac{a}{2} - 1 \right)$ . Hence,

$$\sigma(f) \geq k_1 \binom{a}{2} + k_2 \left( \frac{a}{2} - 1 \right).$$

This completes the proof.  $\square$



## 5 Some exact values

We determine in this section the broadcast independence number of circulant graphs of the form  $C(n; 1, a)$ , for various values of  $n$  and  $a$ . In several cases, we prove, thanks to Observation 2, that the independence number and the broadcast independence number of these graphs coincide.

In [11], Liancheng, Zunquan and Yuansheng determined the exact value of the independence number of some circulant graphs of the form  $C(n; 1, a)$ .

**Proposition 14** (Liancheng *et al.* [11]). *For every two integers  $n$  and  $a$  with  $2 \leq a \leq \lfloor \frac{n}{2} \rfloor$ , we have*

1.  $\alpha(C(n; 1, a)) = \frac{n}{2}$ , for even  $n$  and odd  $a$ ,
2.  $\alpha(C(n; 1, a)) = \frac{n-k}{2}$ , for odd  $n$  and  $a \in \{3, 5\}$ ,
3.  $\alpha(C(n; 1, 2)) = \lfloor \frac{n}{3} \rfloor$ ,
4.  $\alpha(C(n; 1, 4)) = \lfloor \frac{2n}{5} \rfloor$ .

Several of our results in this section will thus extend the results of Proposition 14.

We first consider the case of circulant graphs of the form  $C(n; 1, 2)$ ,  $n \geq 4$ . It is not difficult to check that, for every  $n \geq 4$ , antipodal vertices in  $C(n; 1, 2)$  are at distance  $\lceil \frac{n-1}{4} \rceil$  apart from each other. We thus have the following.

**Observation 15.** *For every integer  $n$ ,  $n \geq 4$ ,  $\text{diam}(C(n; 1, 2)) = \lceil \frac{n-1}{4} \rceil$ .*

The broadcast independence number of circulant graphs of the form  $C(n; 1, 2)$  is given by the following result.

**Theorem 16.** *For every integer  $n \geq 4$ ,*

$$\beta_b(C(n; 1, 2)) = \begin{cases} \alpha(C(n; 1, 2)) = 1, & \text{if } n \in \{4, 5\}, \\ \frac{n-3}{2}, & \text{if } n \equiv 9 \pmod{12}, \\ 2(\text{diam}(C(n; 1, 2)) - 1) = 2\left(\lceil \frac{n-1}{4} \rceil - 1\right), & \text{otherwise.} \end{cases}$$

**Proof.** Since  $C(4; 1, 2)$  and  $C(5; 1, 2)$  are both complete graphs, the result obviously holds for  $n \in \{4, 5\}$ .

Suppose now  $n \geq 6$ . By Proposition 1,  $\beta_b(C(n; 1, 2)) \geq 2(\text{diam}(C(n; 1, 2)) - 1)$  holds for every  $n$ . We will prove that we have  $\beta_b(C(n; 1, 2)) \leq 2(\text{diam}(C(n; 1, 2)) - 1)$  if  $n \not\equiv 9 \pmod{12}$ , and  $\beta_b(C(n; 1, 2)) = \frac{n-3}{2}$  otherwise.

Let  $f$  be an independent  $\beta_b$ -broadcast on  $C(n; 1, 2)$ . Each vertex  $v \in V_f^+$   $f$ -dominates  $4f(v) + 1$  vertices. Moreover, each  $f$ -broadcast vertex is  $f$ -dominated exactly once, and each non-broadcast vertex is  $f$ -dominated at most twice. This gives

$$4f(V_f^+) + |V_f^+| \leq 2(n - |V_f^+|) + |V_f^+|,$$

and thus

$$\beta_b(C(n; 1, 2)) = \sigma(f) = \sum_{v \in V_f^+} f(v) = f(V_f^+) \leq \frac{n - |V_f^+|}{2}.$$

We now consider three cases, depending on the value of  $|V_f^+|$ .

1.  $|V_f^+| \leq 2$ .

If  $|V_f^+| = 1$ , then  $V_f^+ = \{v_i\}$  for some vertex  $v_i$ , and thus

$$\sigma(f) = f(v_i) \leq e(v_i) = \text{diam}(C(n; 1, 2)) \leq 2(\text{diam}(C(n; 1, 2)) - 1).$$

If  $|V_f^+| = 2$ , then  $V_f^+ = \{v_i, v_j\}$  for some distinct vertices  $v_i$  and  $v_j$ , and thus

$$\sigma(f) = f(v_i) + f(v_j) \leq 2(\text{diam}(C(n; 1, 2)) - 1).$$

2.  $|V_f^+| \geq 4$ .

In that case, we get

$$\sigma(f) \leq \left\lfloor \frac{n - |V_f^+|}{2} \right\rfloor \leq \left\lfloor \frac{n - 4}{2} \right\rfloor \leq 2 \left( \left\lceil \frac{n - 1}{4} \right\rceil - 1 \right)$$

and thus  $\sigma(f) \leq 2 \left( \left\lceil \frac{n-1}{4} \right\rceil - 1 \right) = 2(\text{diam}(C(n; 1, 2)) - 1)$  by Observation 15.

3.  $|V_f^+| = 3$ .

Let  $V_f^+ = \{v_{i_0}, v_{i_1}, v_{i_2}\}$ , with  $0 \leq i_0 < i_1 < i_2 < n - 1$ . We consider two subcases, depending in the parity of  $n$ .

(a)  $n$  is even.

Since  $f$  is a  $\beta_b$ -broadcast, we have

$$f(v_{i_j}) = \min \{d(v_{i_j}, v_{i_{j-1}}) - 1, d(v_{i_j}, v_{i_{j+1}}) - 1\}$$

for every  $j$ ,  $0 \leq j \leq 2$  (subscripts are taken modulo 3).

Moreover, since  $\min\{x, y\} \leq \frac{x+y}{2}$  for every two integers  $x$  and  $y$ , we get

$$\sigma(f) = f(v_{i_0}) + f(v_{i_1}) + f(v_{i_2}) \leq d(v_{i_0}, v_{i_1}) + d(v_{i_1}, v_{i_2}) + d(v_{i_2}, v_{i_0}) - 3.$$

Now, since

$$d(v_{i_j}, v_{i_{j'}}) = \left\lceil \frac{|i_j - i_{j'}|}{2} \right\rceil \leq \frac{|i_j - i_{j'}| + 1}{2}$$

for every two distinct vertices  $v_{i_j}$  and  $v_{i_{j'}}$ , we get

$$\sigma(f) \leq \left\lfloor \frac{|i_0 - i_1| + |i_1 - i_2| + |i_2 - i_0| + 3}{2} \right\rfloor - 3 = \left\lfloor \frac{n - 3}{2} \right\rfloor.$$

Finally, since  $n$  is even, we get

$$\sigma(f) \leq \left\lfloor \frac{n - 3}{2} \right\rfloor = \frac{n - 4}{2} \leq 2 \left( \left\lceil \frac{n - 1}{4} \right\rceil - 1 \right) = 2(\text{diam}(C(n; 1, 2)) - 1).$$

(b)  $n$  is odd.

If every non-broadcast vertex is  $f$ -dominated exactly twice, then we necessarily have  $f(v_{i_0}) = f(v_{i_1}) = f(v_{i_2}) = \ell$  for some value  $\ell$ . Moreover, since each vertex  $v_{i_j}$ ,  $0 \leq j \leq 2$ ,  $f$ -dominates  $4f(v_{i_j}) + 1 = 4\ell + 1$  vertices, we get  $12\ell + 3 = 2(n - 3) + 3$  (each vertex in  $V_f^+$  is  $f$ -dominated only once), and thus  $\ell = \frac{n-3}{6}$ . This implies  $n \equiv 3 \pmod{6}$  and  $\sigma(f) = \frac{n-3}{2}$ .

Now, we have

$$\sigma(f) = \frac{n-3}{2} = 3\ell = 2 \left( \left\lceil \frac{n-1}{4} \right\rceil - 1 \right) = 2(\text{diam}(C(n; 1, 2)) - 1)$$

if  $\ell$  is even, that is  $n \equiv 3 \pmod{12}$ , while we have

$$\sigma(f) = \frac{n-3}{2} = 3\ell > 3\ell - 1 = 2 \left( \left\lceil \frac{n-1}{4} \right\rceil - 1 \right) = 2(\text{diam}(C(n; 1, 2)) - 1)$$

if  $\ell$  is odd, that is  $n \equiv 9 \pmod{12}$ .

Suppose now that at least one non-broadcast vertex is  $f$ -dominated only once, which implies  $4f(V_f^+) + 3 \leq 2(n-4) + 4 = 2n-4$  and thus

$$\sigma(f) = f(V_f^+) \leq \left\lfloor \frac{2n-7}{4} \right\rfloor.$$

Since  $n$  is odd, we get

$$\sigma(f) \leq \left\lfloor \frac{2n-7}{4} \right\rfloor = \left\lfloor \frac{2n-8}{4} \right\rfloor \leq 2 \left( \left\lceil \frac{n-1}{4} \right\rceil - 1 \right) = 2(\text{diam}(C(n; 1, 2)) - 1).$$

In all cases, we thus get  $\beta_b(C(n; 1, 2)) = \sigma(f) \leq f(V_f^+) \leq 2(\text{diam}(C(n; 1, 2)) - 1)$  if  $n \not\equiv 9 \pmod{12}$ , and  $\beta_b(C(n; 1, 2)) = \sigma(f) = \frac{n-3}{2}$  if  $n \equiv 9 \pmod{12}$ , which completes the proof.  $\square$

Comparing the value of  $\alpha(C(n; 1, 2))$  given in [11] with  $\beta_b(C(n; 1, 2))$ , it is clearly seen that  $\alpha(C(n; 1, 2)) < \beta_b(C(n; 1, 2))$  is almost always true.

Since the circulant graphs  $C(2a+1; 1, a)$  and  $C(2a+1; 1, 2)$  are isomorphic for every integer  $a$ ,  $a \geq 2$ , Theorem 16 admits the following corollary.

**Corollary 17.** *For every integer  $a \geq 2$ ,*

$$\beta_b(C(2a+1; 1, a)) = \begin{cases} a-1, & \text{if } a=2, \text{ or } a \equiv 4 \pmod{6}, \\ 2 \left( \left\lceil \frac{a}{2} \right\rceil - 1 \right), & \text{otherwise.} \end{cases}$$

We now determine the broadcast independence number of circulant graphs of the form  $C(n; 1, a)$  when  $n$  is even and  $a$  is odd.

**Theorem 18.** *If  $n$  and  $a$  are two integers such that  $n$  is even,  $n \geq 6$ ,  $a$  is odd and  $3 \leq a \leq \lfloor \frac{n}{2} \rfloor$ , then*

$$\beta_b(C(n; 1, a)) = \alpha(C(n; 1, a)) = \frac{n}{2}.$$

**Proof.** From Proposition 11, we get that  $\sigma(g) \leq \left\lfloor \frac{n-|V_g^2|}{2} \right\rfloor$  for every 2-bounded independent broadcast  $g$  on  $C(n; 1, a)$ , which implies  $\beta_b(C(n; 1, a)) \leq \frac{n}{2}$ . Consider now the mapping  $f$  from  $V(C(n; 1, a))$  to  $\{0, 1\}$  defined by  $f(v_i) = 1$  if and only if  $i$  is even. Since  $a$  is odd,  $f$  is clearly an independent broadcast on  $C(n; 1, a)$ . This implies  $\beta_b(C(n; 1, a)) \geq \sigma(f) = \frac{n}{2}$  and thus, thanks to Observation 2,  $\beta_b(C(n; 1, a)) = \alpha(C(n; 1, a)) = \frac{n}{2}$ . This completes the proof.  $\square$

We are now able to determine the broadcast independence number of circulant graphs of the form  $C(n; 1, 3)$ .

**Theorem 19.** For every integer  $n \geq 6$ ,

$$\beta_b(C(n; 1, 3)) = \alpha(C(n; 1, 3)) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even,} \\ \frac{n-3}{2}, & \text{otherwise.} \end{cases}$$

**Proof.** If  $n$  is even, the result directly follows from Theorem 18.

Suppose now that  $n$  is odd and consider the mapping  $f$  from  $V(C(n; 1, 3))$  to  $\{0, 1\}$  defined by  $f(v_i) = 1$  if and only if  $i$  is even and  $i \leq n - 5$ . Since all broadcast vertices have an even index not greater than  $n - 5$  and 3 is odd,  $f$  is clearly a 1-bounded independent broadcast on  $C(n; 1, 3)$  with  $\sigma(f) = \frac{n-3}{2}$  and  $V_f^2 = \emptyset$ . We thus get  $\beta_b(C(n; 1, 3)) \geq \frac{n-3}{2}$  and, thanks to Observation 2,  $\beta_b(C(n; 1, 3)) = \alpha(C(n; 1, 3))$ .

From Proposition 11, we get that  $\sigma(g) \leq \lfloor \frac{n-|V_g^2|}{2} \rfloor$  for every 2-bounded independent broadcast  $g$  on  $C(n; 1, 3)$ . If  $|V_g^2| \geq 2$ , then  $\sigma(g) \leq \lfloor \frac{n-2}{2} \rfloor = \frac{n-3}{2} = \sigma(f)$ . If  $|V_g^2| = 1$ , say  $V_g^2 = \{v_j\}$ , then we necessarily have  $g(v_{j-1}) = g(v_{j-2}) = g(v_{j-3}) = g(v_{j-4}) = 0$ , and thus  $v_{j-1}$  and  $v_{j-2}$  do not belong to any set  $A_f^i$  for any  $v_i \in V_f^1$ . Using this remark together with Lemma 10, we then get

$$\sum_{v_i \in V_g^1} |A_g^i| + \sum_{v_j \in V_g^2} |B_g^j| = 2f(V_g^1) + |V_g^1| + 3f(v_j) - 1 \leq n - 2,$$

which gives, since  $f(v_j) \geq 2$ ,

$$\sigma(g) = f(V_g^1) + f(v_j) \leq \frac{n-2-|V_g^1|-f(v_j)+1}{2} \leq \frac{n-3}{2} = \sigma(f).$$

Finally, if  $|V_g^2| = 0$  then, since  $n$  is odd, there necessarily exists a vertex  $v_i \in V_g^1$  such that  $g(v_{i+2}) = 0$ , which implies  $g(v_{i+1}) = g(v_{i+3}) = 0$ . This implies that  $v_{i+3}$  does not belong to any set  $A_g^{i'}$  for any  $v_{i'} \in V_g^1$ . Using this remark together with Lemma 10, we then have  $2f(V_g^1) + |V_g^1| \leq n - 1$ , and thus

$$\sigma(g) = f(V_g^1) \leq \frac{n-1-|V_g^1|}{2} \leq \lfloor \frac{n-2}{2} \rfloor = \frac{n-3}{2} = \sigma(f).$$

Hence, in all the previous cases, we have  $\sigma(g) \leq \sigma(f) = \frac{n-3}{2}$ , which completes the proof.  $\square$

We now determine the broadcast independence number of circulant graphs of the form  $C(n; 1, 4)$ .

**Theorem 20.** For every integer  $n \geq 8$ ,

$$\beta_b(C(n; 1, 4)) = \alpha(C(n; 1, 4)) = \lfloor \frac{2n}{5} \rfloor.$$

**Proof.** From Proposition 12, we get that

$$\sigma(g) \leq \left\lfloor \frac{a}{2(a+1)} \left( n - \frac{a-4}{a} |V_g^2| \right) \right\rfloor$$

for every 2-bounded independent broadcast  $g$  on  $C(n; 1, a)$ , which gives  $\sigma(g) \leq \lfloor \frac{2n}{5} \rfloor$ , and thus  $\beta_b(C(n; 1, 4)) \leq \lfloor \frac{2n}{5} \rfloor$ .

We now construct a mapping  $f$  from  $V(C(n; 1, 4))$  to  $\{0, 1\}$ . Let  $n = 5k + r$  with  $0 \leq r \leq 4$ . We consider five cases, depending on the value of  $r$ .

1.  $r = 0$ .

We let  $f(v_i) = 1$  if  $(i \bmod 5)$  is odd, and  $f(v_i) = 0$  otherwise.

2.  $r = 1$ .

We let  $f(v_i) = 1$  if  $(i \bmod 5)$  is odd and  $i \leq n - 7$ ,  $f(v_{n-2}) = f(v_{n-5}) = 1$ , and  $f(v_i) = 0$  otherwise.

3.  $r = 2$ .

We let  $f(v_i) = 1$  if  $(i \bmod 5)$  is odd and  $i \leq n - 3$ , and  $f(v_i) = 0$  otherwise.

4.  $r = 3$ .

We let  $f(v_i) = 1$  if  $(i \bmod 5)$  is odd, and  $f(v_i) = 0$  otherwise.

5.  $r = 4$ .

We let  $f(v_i) = 1$  if  $(i \bmod 5)$  is odd and  $i \leq n - 7$ ,  $f(v_{n-2}) = f(v_{n-5}) = f(v_{n-5}) = 1$ , and  $f(v_i) = 0$  otherwise.

Clearly, in each of the previous cases,  $f$  is a 1-bounded independent broadcast on  $C(n; 1, 4)$  such that  $\sigma(f) = \lfloor \frac{2n}{5} \rfloor$  and  $V_f^2 = \emptyset$ . Hence,  $\beta_b(C(n; 1, 4)) = \lfloor \frac{2n}{5} \rfloor$  and, thanks to Observation 2,  $\beta_b(C(n; 1, 4)) = \alpha(C(n; 1, 4))$ . This completes the proof.  $\square$

Thanks to Proposition 12, we are now able to determine the broadcast independence number of circulant graphs of the form  $C((a+1)k; 1, a)$  with  $a \geq 5$  and  $k \geq 2$  (the cases  $a = 2, 3$  and  $4$  are already covered by Theorems 16, 19 and 20, respectively).

**Theorem 21.** *If  $a$  and  $k$  are two integers such that  $a \geq 5$  and  $k \geq 2$ , then we have*

$$\beta_b(C((a+1)k; 1, a)) = \alpha(C((a+1)k; 1, a)) = \begin{cases} \frac{ak}{2}, & \text{if } a \text{ is even,} \\ \frac{(a+1)k}{2}, & \text{otherwise.} \end{cases}$$

**Proof.** If  $a$  is odd, then  $(a+1)k$  is even and the result directly follows from Theorem 18.

Suppose now that  $a$  is even, which implies  $a \geq 6$ . From Proposition 12, we get that

$$\sigma(g) \leq \left\lfloor \frac{a}{2(a+1)} \left( (a+1)k - \frac{a-4}{a} |V_g^2| \right) \right\rfloor \leq \left\lfloor \frac{a}{2(a+1)} (a+1)k \right\rfloor = \frac{ak}{2}$$

for every 2-bounded independent broadcast  $g$  on  $C((a+1)k; 1, a)$ , which implies  $\beta_b(C((a+1)k; 1, a)) \leq \frac{ak}{2}$ .

Consider now the mapping  $f$  from  $V(C((a+1)k; 1, a))$  to  $\{0, 1\}$  defined by  $f(v_i) = 1$  if and only if  $(i \bmod a+1)$  is odd. Since  $a$  is even,  $f$  is clearly a 1-bounded independent broadcast on  $C((a+1)k; 1, a)$  with  $\sigma(f) = \frac{ak}{2}$  and  $V_f^2 = \emptyset$ . This implies  $\beta_b(C((a+1)k; 1, a)) \geq \frac{ak}{2}$  and thus, thanks to Observation 2,  $\beta_b(C((a+1)k; 1, a)) = \alpha(C((a+1)k; 1, a)) = \frac{ak}{2}$ . This completes the proof.  $\square$

We now consider the case of circulant graphs  $C(n; 1, a)$  when  $a$  divides  $n$ . We first introduce two new sets of vertices, slightly modifying the definition of the sets  $A_f^i$  and  $B_f^j$  defined in Section 2, using  $a$ -edges instead of 1-edges. Let  $f$  be any 2-bounded independent broadcast on  $C(n; 1, a)$ . Now consider any vertex  $v_i \in V_f^1$  such that  $f(v_{i-a}) = f(v_{i-2a}) = 0$ . Since  $f$  is an independent broadcast, we necessarily have  $f(v_{i+a}) = 0$ . Moreover, we then have either  $f(v_{i+2a}) = 0$  or  $f(v_{i+2a}) = 1$ . Therefore, the broadcast values of the sequence of vertices  $v_i v_{i+a} v_{i+2a} \dots$  is of the form either 100, 10100 or 1010...100.

For each vertex  $v_i \in V_f^1$  such that  $f(v_{i-a}) = f(v_{i-2a}) = 0$ , we then let

$$A_f^i = \{v_{i+\ell a}, 0 \leq \ell \leq 2p+2\}$$

be the set of vertices satisfying (i)  $f(v_{i+2ka}) = 1$  and  $f(v_{i+(2k+1)a}) = 0$  for every  $k$ ,  $0 \leq k \leq p$ , and (ii)  $f(v_{i+(2p+2)a}) = 0$ .

Now, for each vertex  $v_j \in V_f^2$ , we let

$$B_f^j = \{v_j\} \cup \{v_{j+a-1}, v_{j+a}, v_{j+a+1}\} \cup \{v_{j+2a}\}.$$

These sets satisfy the same properties as those of the sets  $A_f^i$  and  $B_f^j$  given in Lemma 10. The proof is similar to the proof of Lemma 10 and is omitted.

**Lemma 22.** *For every 2-bounded independent broadcast  $f$  on  $C(n; 1, a)$ , if any, the following holds.*

1. For every vertex  $v_i \in V_f^1$ ,  $|A_f^i| = 2f(A_f^i) + 1$ .
2. For every vertex  $v_j \in V_f^2$ ,  $|B_f^j| = 5$ .
3.  $\sum_{v_i \in V_f^1} |A_f^i| + \sum_{v_j \in V_f^2} |B_f^j| \leq n$ .

We are now ready to determine the independent broadcast number of circulant graphs of the form  $C(qa; 1, a)$ ,  $a \geq 5$  and  $q \geq 4$ . Recall that the cases  $a = 2, 3$  and  $4$  are already covered by Theorems 16, 19 and 20, respectively, while the cases  $q = 2$  and  $q = 3$  are covered by Theorems 4 and 5, respectively.

**Theorem 23.** *If  $a$  and  $q$  are two integers such that  $a \geq 5$  and  $q \geq 4$ , then we have*

$$\begin{aligned} \beta_b(C(qa; 1, a)) &= \alpha(C(qa; 1, a)) \\ &= \begin{cases} \frac{qa}{2}, & \text{if } a \text{ is odd, and } q \text{ is even,} \\ \frac{(q-1)a}{2}, & \text{if } a \text{ and } q \text{ are odd,} \\ \left\lfloor \frac{qa^2}{2(a+1)} \right\rfloor, & \text{if } a \text{ and } q \text{ are even,} \\ \min \left\{ \left\lfloor \frac{qa^2}{2(a+1)} \right\rfloor, \frac{(q-1)a}{2} \right\}, & \text{otherwise.} \end{cases} \end{aligned}$$

**Proof.** We consider the four cases separately.

1.  $a$  is odd and  $q$  is even.

In that case,  $qa$  is even and the result directly follows from Theorem 18.

2.  $a$  and  $q$  are odd.

In that case,  $q \geq 5$  and we know by Lemma 8 that  $C(qa; 1, a)$  admits a 2-bounded  $\beta_b$ -broadcast. Let  $f$  be any 2-bounded independent broadcast on  $C(qa; 1, a)$ . Observe first that, since  $q$  is odd, we necessarily have  $|A_f^i| \leq q$  for every vertex  $v_i \in V_f^1$ , since otherwise this would give  $f(v_i) = f(v_{i+(q-1)a}) = f(v_{i-a}) = 1$ , contradicting the fact that  $f$  is an

independent broadcast. Therefore  $f(A_f^i) \leq \frac{q-1}{2}$ , and thanks to Item 1 of Lemma 22, we get

$$\frac{|A_f^i|}{f(A_f^i)} = \frac{2f(A_f^i) + 1}{f(A_f^i)} = 2 + \frac{1}{f(A_f^i)} \geq 2 + \frac{2}{q-1} = \frac{2q}{q-1},$$

which gives

$$|A_f^i| \geq \frac{2q}{q-1} f(A_f^i).$$

Now, using Item 2 and Item 3 of Lemma 22, we get

$$qa \geq \sum_{v_i \in V_f^1} |A_f^i| + \sum_{v_j \in V_f^2} |B_f^j| \geq \frac{2q}{q-1} f(V_f^1) + \frac{5}{2} f(V_f^2),$$

which gives

$$qa \geq \frac{2q}{q-1} \sigma(f) + \frac{q-5}{2(q-1)} f(V_f^2) \geq \frac{2q}{q-1} \sigma(f),$$

and then

$$\sigma(f) \leq \frac{(q-1)a}{2}.$$

Consider now the mapping  $g$  from  $V(C(qa; 1, a))$  to  $\{0, 1\}$  defined by  $g(v_i) = 1$  if  $i$  is even and  $i \leq (q-1)a - 1$ , and  $g(v_i) = 0$  otherwise. Clearly,  $g$  is a 1-bounded independent broadcast on  $C(qa; 1, a)$ . Moreover,

$$\beta_b(C(qa; 1, a)) \geq \sigma(g) = \frac{(q-1)a}{2},$$

and thus, thanks to Observation 2,

$$\beta_b(C(qa; 1, a)) = \alpha(C(qa; 1, a)) = \frac{(q-1)a}{2}.$$

3.  $a$  and  $q$  are even.

Note first that if  $a+1$  divides  $q$ , say  $q = \ell(a+1)$  for some integer  $\ell \geq 1$ , which gives  $qa = \ell a(a+1)$ , the result directly follows from Theorem 21 for  $k = \ell a$ , since

$$\frac{ak}{2} = \frac{\ell a^2}{2} = \frac{qa^2}{2(a+1)} = \left\lfloor \frac{qa^2}{2(a+1)} \right\rfloor.$$

Assume now that this is not the case, so that  $k(a+1) < q < (k+2)(a+1)$  for some even integer  $k \geq 2$ . Let  $q = k(a+1) + 2\ell$  (recall that  $k$  and  $q$  are even) for some integer  $\ell$ ,  $1 \leq \ell \leq a$ . From Proposition 12, we get that

$$\sigma(f) \leq \left\lfloor \frac{a}{2(a+1)} \left( qa - \frac{a-4}{a} |V_f^2| \right) \right\rfloor \leq \left\lfloor \frac{qa^2}{2(a+1)} \right\rfloor$$

for every 2-bounded independent broadcast  $f$  on  $C(qa; 1, a)$ . Moreover, we have

$$\left\lfloor \frac{qa^2}{2(a+1)} \right\rfloor = \frac{(a-1)q}{2} + \left\lfloor \frac{q}{2(a+1)} \right\rfloor = \frac{(a-1)q}{2} + \frac{k}{2},$$

which implies

$$\beta_b(C(qa; 1, a)) \leq \frac{(a-1)q}{2} + \frac{k}{2}.$$

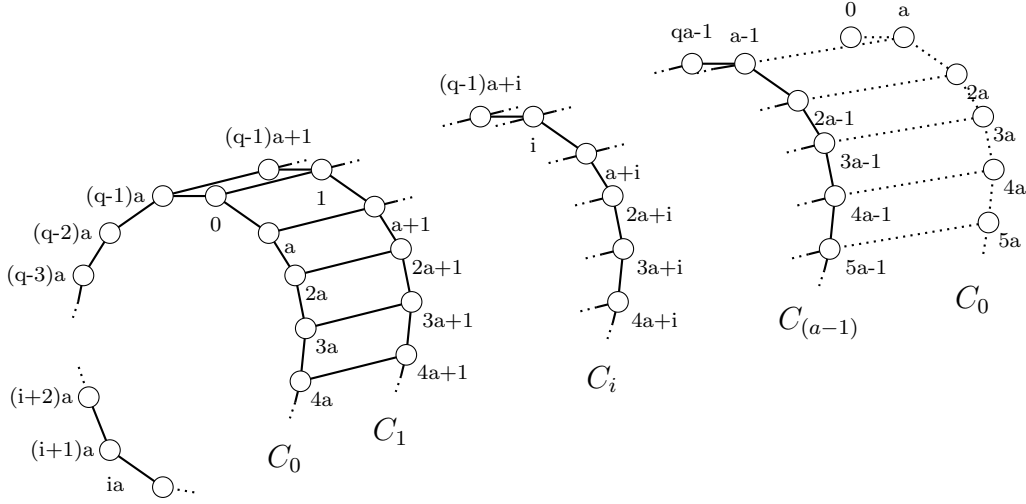


Figure 7: The circulant graph  $C(qa; 1, a)$  (only subscripts of vertices are indicated)

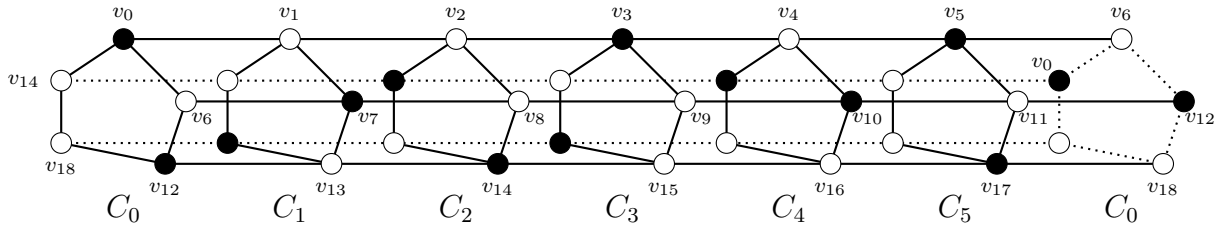


Figure 8: Construction of the sets  $S_i$  in the proof of Theorem 23 ( $a = 6, q = 5$ )

Since  $qa = (ka + \ell)(a + 1) + \ell(a - 1)$ , and thanks to Proposition 13, we get

$$\beta_b(C(qa; 1, a)) \geq \frac{(ka + \ell)a}{2} + \frac{\ell(a - 2)}{2} = \frac{ka^2}{2} + \ell(a - 1) = \frac{ka^2}{2} + \frac{q - k(a + 1)}{2}(a - 1),$$

which gives

$$\beta_b(C(qa; 1, a)) \geq \frac{(a - 1)q}{2} + \frac{k}{2}.$$

Finally, by Observation 2, we get

$$\beta_b(C(qa; 1, a)) = \alpha(C(qa; 1, a)) = \left\lfloor \frac{qa^2}{2(a + 1)} \right\rfloor.$$

4.  $a$  is even and  $q$  is odd.

The graph  $C(qa; 1, a)$  can be seen as  $a$  copies  $C_0, \dots, C_{a-1}$  of a  $q$ -cycle, with  $C_k = \{v_k, v_{k+a}, v_{k+2a}, \dots, v_{k+(q-1)a}\}$ , for every  $k, 0 \leq k \leq a - 1$ , cyclically connected as depicted in Figure 7.

We consider three subcases.

(a)  $q = a - 1$ .

Since  $q$  is odd, similarly to Case 2 ( $q$  and  $a$  odd), we have

$$\sigma(f) \leq \frac{(q - 1)a}{2}$$



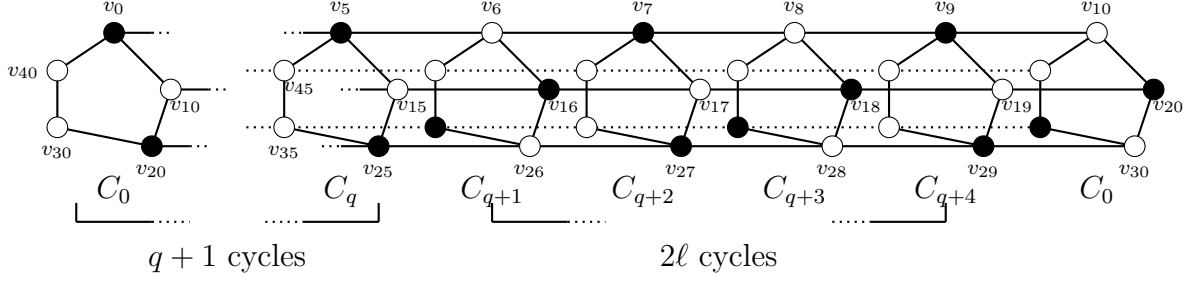


Figure 9: Construction of the sets  $S_i$  in the proof of Theorem 23 ( $a = 10$ ,  $q = 5$ ,  $\ell = 2$ )

for every independent broadcast  $f$  on  $C(qa; 1, a)$ . Consider the sets  $S_k$ ,  $0 \leq k \leq a - 1$  defined as follows (see Figure 8 for the case  $a = 6$  and  $q = 5$ ).

$$\begin{aligned}
S_0 &= \{v_0, v_{2a}, v_{4a}, \dots, v_{(q-5)a}, v_{(q-3)a}\}, \\
S_1 &= \{v_{1+a}, v_{1+3a}, v_{1+5a}, \dots, v_{1+(q-4)a}, v_{1+(q-2)a}\}, \\
S_2 &= \{v_{2+2a}, v_{2+4a}, v_{2+6a}, \dots, v_{2+(q-3)a}, v_{2+(q-1)a}\}, \\
&\vdots \\
S_{a-1} &= \{v_{a-1}, v_{3a-1}, v_{5a-1}, \dots, v_{n-4a-1}, v_{n-2a-1}\}.
\end{aligned}$$

From this definition, we clearly get that  $\bigcup_{k=0}^{a-1} S_k$  is a independent set in  $V(G)$ . This gives

$$\left| \bigcup_{k=0}^{a-1} S_k \right| = a \frac{q-1}{2} \leq \alpha(C(qa; 1, a)) \leq \beta_b(C(qa; 1, a)).$$

Thanks to Observation 2, we then get

$$\beta_b(C(qa; 1, a)) = \alpha(C(qa; 1, a)) = \frac{(q-1)a}{2}.$$

(b)  $q < a - 1$ .

Let  $a = q + 1 + 2\ell$  (recall that  $a$  and  $q$  have different parity) for some integer  $\ell$ ,  $\ell \geq 1$ . For every cycle  $C_k$ ,  $k = 0, 1, \dots, q$ , let  $S_k$  be the set defined as in the previous subcase and, for every cycle  $C_k$ ,  $k = q + 1, \dots, q + 2\ell$ , let (see Figure 9 for the case  $a = 10$ ,  $q = 5$  and  $\ell = 2$ )

$$\begin{aligned}
S_k &= \{v_k, v_{k+2a}, v_{k+4a}, \dots, v_{k+(q-5)a}, v_{k+(q-3)a}\}, \text{ if } k \text{ is even,} \\
S_k &= \{v_{k+a}, v_{k+3a}, v_{k+5a}, \dots, v_{k+(q-4)a}, v_{k+(q-2)a}\}, \text{ if } k \text{ is odd.}
\end{aligned}$$

From this definition, we clearly get that  $\bigcup_{k=0}^{a-1} S_k$  is an independent set of  $C(qa; 1, a)$ . We then have

$$\left| \bigcup_{k=0}^{a-1} S_k \right| = a \frac{q-1}{2} \leq \alpha(C(qa; 1, a)) \leq \beta_b(C(qa; 1, a))$$

and, since  $q$  is odd, we get

$$\beta_b(C(qa; 1, a)) \leq a \left( \frac{q-1}{2} \right).$$

Thanks to Observation 2, we finally get

$$\beta_b(C(qa; 1, a)) = \alpha(C(qa; 1, a)) = \frac{(q-1)a}{2}.$$

(c)  $q > a + 1$ .

Note first that if  $a + 1$  divides  $q$ , say  $q = \ell(a + 1)$  for some integer  $\ell \geq 1$ , which gives  $qa = \ell a(a + 1)$ , the result directly follows from Theorem 21 for  $k = \ell a$ , since

$$\frac{ak}{2} = \frac{\ell a^2}{2} = \frac{qa^2}{2(a+1)} = \left\lfloor \frac{qa^2}{2(a+1)} \right\rfloor.$$

Suppose now that this is not the case, so that  $k(a + 1) < q < (k + 2)(a + 1)$ , for some odd integer  $k \geq 1$ . Let  $q = k(a + 1) + 2\ell$  (recall that  $k$  and  $q$  are odd) for some integer  $\ell$ ,  $1 \leq \ell \leq a$ . From Proposition 12 we get

$$\begin{aligned} \beta_b(C(qa; 1, a)) &\leq \left\lfloor \frac{qa^2}{2(a+1)} \right\rfloor = \left\lfloor \frac{q(a-1)(a+1) + q}{2(a+1)} \right\rfloor \\ &= \left\lfloor \frac{q(a-1)(a+1) + (a+1)}{2(a+1)} + \frac{q - (a+1)}{2(a+1)} \right\rfloor \\ &= \left\lfloor \frac{q(a-1)(a+1) + (a+1)}{2(a+1)} + \frac{(k-1)(a+1) + 2\ell}{2(a+1)} \right\rfloor \\ &= \frac{(a-1)q + 1}{2} + \frac{k-1}{2}. \end{aligned}$$

Moreover, we have  $qa = (ka + \ell)(a + 1) + \ell(a - 1)$  and, thanks to Proposition 13, we get

$$\beta_b(C(qa; 1, a)) \geq (ka + \ell)\frac{a}{2} + \ell\left(\frac{a}{2} - 1\right) = \frac{ka^2}{2} + \ell(a - 1) = \frac{ka^2}{2} + \frac{q - k(a + 1)}{2}(a - 1),$$

which gives

$$\beta_b(C(qa; 1, a)) \geq \frac{(a-1)q + 1}{2} + \frac{k-1}{2}.$$

Therefore, by Observation 2, we finally get

$$\beta_b(C(qa; 1, a)) = \alpha(C(qa; 1, a)) = \frac{(a-1)q + 1}{2} + \frac{k-1}{2} = \left\lfloor \frac{qa^2}{2(a+1)} \right\rfloor.$$

This completes the proof. □

Our last general result is the following.

**Theorem 24.** *Let  $n$ ,  $a$ ,  $q$  and  $r$  be integers such that  $n = qa + r$ ,  $a$  is even,  $a \geq 6$  and  $q \geq \max\{2, r\}$ . If either  $q$  and  $r$  have the same parity, or  $q$  and  $r$  have different parity and  $q + r \geq a - 1$ , then we have*

$$\beta_b(C(n; 1, a)) = \alpha(C(n; 1, a)) = \left\lfloor \frac{a}{2(a+1)}n \right\rfloor$$

**Proof.** From Proposition 12, we get that

$$\sigma(f) \leq \left\lfloor \frac{a}{2(a+1)} \left( n - \frac{a-4}{a} |V_f^2| \right) \right\rfloor \leq \left\lfloor \frac{a}{2(a+1)} n \right\rfloor$$

for every 2-bounded independent broadcast  $f$  on  $C(n; 1, a)$ . We consider two cases.

1.  $q$  and  $r$  have the same parity.

Note first that if  $a+1$  divides  $q-r$ , say  $q-r = \ell(a+1)$  for some integer  $\ell$ , which gives  $qa+r = (q-\ell)(a+1)$ , the result directly follows from Theorem 21 for  $k = q-\ell$ , since

$$\frac{ak}{2} = \frac{a(q-\ell)}{2} = \frac{(qa+r)a}{2(a+1)} = \left\lfloor \frac{(qa+r)a}{2(a+1)} \right\rfloor.$$

Suppose now that this is not the case, so that  $k(a+1) < q-r < (k+2)(a+1)$ , for some even integer  $k$ . Let  $q-r = k(a+1) + 2\ell$  (recall that  $k$  and  $q-r$  are even) for some integer  $\ell$ ,  $1 \leq \ell \leq a$ . We have

$$\left\lfloor \frac{a}{2(a+1)}(qa+r) \right\rfloor = \left\lfloor \frac{qa(a+1) - (q-r)a}{2(a+1)} \right\rfloor = \frac{aq}{2} + \left\lfloor -\frac{(q-r)a}{2(a+1)} \right\rfloor.$$

Since  $-\ell < -\frac{\ell a}{a+1} < -\ell + 1$ , we get

$$\left\lfloor -\frac{(q-r)a}{2(a+1)} \right\rfloor = \left\lfloor -\frac{(k(a+1) + 2\ell)a}{2(a+1)} \right\rfloor = -\frac{ka}{2} + \left\lfloor -\frac{\ell a}{a+1} \right\rfloor = -\frac{ka}{2} - \ell,$$

which gives

$$\beta_b(C(qa+r; 1, a)) \leq \left\lfloor \frac{a}{2(a+1)}(qa+r) \right\rfloor = \frac{aq}{2} - \frac{ak}{2} - \ell = \frac{(q-k)a}{2} - \ell.$$

Furthermore, since

$$qa+r = \frac{q+r}{2}(a+1) + \frac{q-r}{2}(a-1) = \frac{q+r+k(a-1)}{2}(a+1) + \ell(a-1),$$

and thanks to Proposition 13, we get

$$\beta_b(C(qa+r; 1, a)) \geq \frac{q+r+k(a-1)}{2} \binom{a}{2} + \ell \binom{a}{2} - 1 = \frac{q+r+k(a-1)+2\ell}{2} \binom{a}{2} - \ell.$$

Again, since  $q-r = k(a+1) + 2\ell$ , we have

$$\beta_b(C(qa; 1, a)) \geq \frac{q+r+q-r-2k}{2} \binom{a}{2} - \ell = \frac{(q-k)a}{2} - \ell,$$

and thus, thanks to Observation 2, we finally get

$$\beta_b(C(qa+r; 1, a)) = \alpha(C(qa+r; 1, a)) = \frac{(a-1)q+1}{2} + \frac{k-1}{2} = \left\lfloor \frac{qa^2}{2(a+1)} \right\rfloor.$$

2.  $q$  and  $r$  have different parity.

Similarly to the previous case, if  $a + 1$  divides  $q - r$ , then

$$\frac{ak}{2} = \left\lfloor \frac{(qa + r)a}{2(a + 1)} \right\rfloor.$$

Suppose now that this is not the case. We consider two subcases, depending on whether  $q - r$  is greater than  $a + 1$  or not.

(a)  $q - r < a + 1$ .

In that case, we have

$$\begin{aligned} \beta_b(C(qa + r; 1, a)) &\leq \left\lfloor \frac{a}{2(a + 1)}(qa + r) \right\rfloor = \left\lfloor \frac{qa(a + 1) - (q - r)a}{2(a + 1)} \right\rfloor \\ &= \frac{aq}{2} + \left\lfloor -\frac{(q - r)a}{2(a + 1)} \right\rfloor. \end{aligned}$$

Since  $q - r < a + 1$ , we get

$$-\frac{q - r + 1}{2} < -\frac{(q - r)a}{2(a + 1)} < -\frac{q - r + 1}{2} + 1,$$

and thus

$$\beta_b(C(qa + r; 1, a)) \leq \frac{aq}{2} - \frac{q - r + 1}{2}.$$

Since  $qa + r = \frac{q + r + 1 - a}{2}(a + 1) + \frac{q - r + 1 + a}{2}(a - 1)$ , and thanks to Proposition 13, we have

$$\beta_b(C(qa + r; 1, a)) \geq \left( \frac{q + r + 1 - a}{2} \right) \frac{a}{2} + \left( \frac{q - r + 1 + a}{2} \right) \left( \frac{a}{2} - 1 \right),$$

which gives

$$\beta_b(C(qa; 1, a)) \geq \frac{aq}{2} - \frac{q - r + 1}{2}.$$

Finally, thanks to Observation 2, we get

$$\beta_b(C(qa; 1, a)) = \alpha(C(qa; 1, a)) = \frac{aq}{2} - \frac{q - r + 1}{2}.$$

(b)  $q - r > a + 1$ .

In that case, we have  $k(a + 1) < q - r < (k + 2)(a + 1)$ , for some odd integer  $k \geq 1$ . Let  $q - r = k(a + 1) + 2\ell$  (recall that  $k$  and  $q - r$  are odd) for some integer  $\ell$ ,  $1 \leq \ell \leq a$ . Since

$$\left\lfloor -\frac{(q - r)a}{2(a + 1)} \right\rfloor = \left\lfloor -\frac{(k(a + 1) + 2\ell)a}{2(a + 1)} \right\rfloor = -\frac{ka}{2} + \left\lfloor -\frac{\ell a}{(a + 1)} \right\rfloor$$

and  $-\ell < -\frac{\ell a}{(a + 1)} < -\ell + 1$ , we get

$$\left\lfloor \frac{a(qa + r)}{2(a + 1)} \right\rfloor = \left\lfloor \frac{qa(a + 1) - (q - r)a}{2(a + 1)} \right\rfloor = \frac{aq}{2} + \left\lfloor -\frac{(q - r)a}{2(a + 1)} \right\rfloor = \frac{aq}{2} - \frac{ka}{2} - \ell.$$

This implies

$$\beta_b(C(qa + r; 1, a)) \leq \left\lfloor \frac{a(qa + r)}{2(a + 1)} \right\rfloor = \frac{aq}{2} - \frac{ak}{2} - \ell = \frac{(q - k)a}{2} - \ell.$$

Moreover, since  $qa + r = \frac{q + r + 1 - a}{2}(a + 1) + \frac{q - r + a + 1}{2}(a - 1)$  and  $q - r = k(a + 1) + 2\ell$ , we get

$$qa + r = \frac{q + r + k(a - 1)}{2}(a + 1) + \ell(a - 1).$$

Hence, we have

$$\beta_b(C(qa + r; 1, a)) \geq \frac{q + r + k(a - 1)}{2} \cdot \frac{a}{2} + \ell\left(\frac{a}{2} - 1\right) = \frac{q + r + k(a - 1) + 2\ell}{2} \cdot \frac{a}{2} - \ell,$$

which gives

$$\beta_b(C(qa + r; 1, a)) \geq \frac{q + r + q - r - 2k}{2} \cdot \frac{a}{2} - \ell = \frac{(q - k)a}{2} - \ell.$$

Finally, thanks to Observation 2, we get

$$\beta_b(C(qa + r; 1, a)) = \alpha(C(qa + r; 1, a)) = \left\lfloor \frac{qa^2}{2(a + 1)} \right\rfloor.$$

This completes the proof. □

## 6 Discussion

We proved that every circulant graph of the form  $C(n; 1, a)$ ,  $3 \leq a \leq \lfloor \frac{n}{2} \rfloor$ , admits a 2-bounded  $\beta_b$ -broadcast, except when  $n = 2a + 1$ , or  $n = 2a$  and  $a$  is even. Using this property, we determined the exact value of the broadcast independence number of several classes of circulant graphs of the form  $\beta_b(C(n; 1, a))$ ,  $2 \leq a \leq \lfloor \frac{n}{2} \rfloor$ . In several cases, we showed that  $\beta_b(C(n; 1, a))$  reaches one of its lower bounds, namely  $\alpha(C(n; 1, a))$  or  $2(\text{diam}(C(n; 1, a)) - 1)$ . In particular, whenever  $\beta_b(C(n; 1, a)) = \alpha(C(n; 1, a))$ , we get that  $C(n; 1, a)$  admits a 1-bounded  $\beta_b$ -broadcast.

We finally mention a few open problems that seem worth to be investigated.

1. Determine the value of  $\beta_b(C(n; 1, a))$  for the remaining unsolved cases namely:
  - (a) For odd integers  $a$  and  $n$ , with  $a \geq 5$  and  $a \nmid n$ .
  - (b) For integers  $n$ ,  $a$ ,  $q$  and  $r$ , with  $n = qa + r$ ,  $a \geq 6$  is even,  $n$  is divisible neither by  $a$  nor by  $a + 1$ , and either
    - i.  $q < r$ , or
    - ii.  $q > r$ ,  $q$  and  $r$  have different parity,  $q + r < a - 1$ .
2. Determine the broadcast independent number of other classes of circulant graphs.
3. Characterize the classes of graphs  $G$  for which  $\beta_b(G) = \alpha(G)$  or  $\beta_b(G) = 2(\text{diam}(G) - 1)$ , respectively.

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