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LARGE DEVIATIONS FOR GENERALIZED GIBBS ENSEMBLES OF THE CLASSICAL TODA CHAIN

ALICE GUIONNET AND RONAN MEMIN

Abstract We derive large deviations principles for the distribution of the empirical measure of the equilibrium measure for the Generalized Gibbs ensembles of the classical Toda chain introduced in [11]. We deduce its almost sure convergence and characterize its limit in terms of the limiting measure of Beta-ensembles. Our results apply to general smooth potentials.

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1. INTRODUCTION

In a breakthrough paper [11], Herbert Spohn introduced the generalized Gibbs ensembles of the classical Toda chain as invariant measures of the dynamics of the classical Toda lattice and showed how to analyze it thanks to a beautiful comparison with Dumitriu-Edelman tri-diagonal representations of β -ensembles. Thanks to this comparison, it is shown in [11] that the empirical measure of the eigenvalues of Toda Lax matrices for these Generalized Gibbs ensembles converges towards a probability measure related with the equilibrium measure for β ensembles. One of the key tool of Herbert Spohn analysis

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is the uses of transfer matrices which restricts him to polynomial potentials. We refer the interested reader to subsequent developments in [9, 10, 8].

The main goal of this article is to generalize some of the results of Spohn [11] by using large deviations theory, which will allow us to consider more general potentials. More precisely, we will show the convergence of the free energy and of the empirical measure of the eigenvalues of Toda Lax matrices for these Generalized Gibbs towards limits related to β -ensembles as in [11] but for general continuous potentials instead of polynomial functions. Moreover, we will derive large deviation principle allowing to study a priori tri-diagonal matrices with more general parameters.

More precisely, the Hamiltonian of the Toda chain on sites $j = 1, \dots, N$ is given by

$$H = \sum_{j=1}^N \left(\frac{1}{2} p_j^2 + e^{-r_j} \right), \quad r_j = q_{j+1} - q_j$$

with the periodic conditions $q_{N+j} = q_j$. The equations of motion are then given by

$$\frac{d}{dt} q_j = p_j, \quad \frac{d}{dt} p_j = e^{-r_{j-1}} - e^{-r_j}. \quad (1)$$

Let L_N be the Lax matrix given by the tri-diagonal with entries

$$(L_N)_{j,j} = p_j \text{ and } (L_N)_{j,j+1} = (L_N)_{j+1,j} = e^{-r_j/2} \quad (2)$$

with periodic boundary conditions $(L_N)_{1,N} = (L_N)_{N+1,N}$ and $(L_N)_{N,1} = (L_N)_{N,N+1}$, then for all integer number n ,

$$Q_N^n = \text{Tr}(L_N^n)$$

is conserved by the dynamics (1) as well as $\sum_{i=1}^N r_i$. It is therefore natural to consider that the finite N Toda chain is distributed according to the Gibbs measure with density $e^{-\text{Tr}(W(L_N))}$ with respect to $\prod_{i=1}^N e^{-Pr_i} dr_i dp_i$, where $P > 0$ controls the pressure of the chain. W is a potential to be chosen later, which can be a polynomial or a general measurable function from \mathbb{R} into \mathbb{R} . We will assume it goes to infinity faster than x^2 : namely there exists $a > 0$ and a finite constant C such that for

$$W(x) \geq ax^2 + C. \quad (3)$$

This assumption is used to compare our distribution to the case where $W(x) = x^2$ in which case the entries of the Lax matrix L_N are independent. We can without loss of generality assume $a = \frac{1}{2}$ up to rescaling and therefore put $W(x) = \frac{1}{2}x^2 + V$. In the following we will denote

$$d\mathbb{T}_N^{V,P}(p, r) = \frac{1}{Z_{N,\mathbb{T}}^{V,P}} \exp\{-\text{Tr}(V(L_N)) - \frac{1}{2}\text{Tr}(L_N^2)\} \prod_{i=1}^N e^{-Pr_i} dr_i dp_i \quad (4)$$

We denote in short \mathbb{T}_N^P for $\mathbb{T}_N^{0,P}$. Our goal in this article is to study the empirical measure of the eigenvalues $\lambda_1 \leq \dots \leq \lambda_N$ of L_N , called hereafter the spectral measure of L_N and denoted by

$$\hat{\mu}_{L_N} = \frac{1}{N} \sum_{i=1}^N \delta_{\lambda_i}.$$

Our main result is a large deviation principle for the distribution of $\hat{\mu}_{L_N}$ under $d\mathbb{T}_N^{V,P}$, from which we deduce the almost sure convergence of $\hat{\mu}_{L_N}$ under $d\mathbb{T}_N^{V,P}$.

Theorem 1.1. *Let $P > 0$ and assume that V is continuous. Assume that there exists $k \in \mathbb{N}$, $a \geq 0$, such that*

$$\lim_{|x| \rightarrow \infty} \frac{V(x)}{x^{2k}} = a.$$

Then,

- (1) *the law of $\hat{\mu}_{L_N}$ under $\mathbb{T}_N^{V,P}$ satisfies a large deviation principle in the scale N with good rate function T_P^V ,*
- (2) *T_P^V achieves its minimal value at a unique probability measure ν_P^V ,*
- (3) *As a consequence $\hat{\mu}_{L_N}$ converges almost surely and in L^1 towards ν_P^V .*

Moreover, following [11], we can identify the equilibrium measure ν_P^V using the equilibrium measure for Coulomb gases in dimension one at temperature of order of the dimension. More precisely, for a probability measure μ on the real line, define the function

$$f_P^V(\mu) = \frac{1}{2} \int (W(x) + W(y) - 2P \ln |x - y|) d\mu(x) d\mu(y) + \int \ln \frac{d\mu}{dx} d\mu(x)$$

if $\mu \ll dx$, whereas f_P^V is infinite otherwise. f_P^V achieves its minimal value at a unique probability measure $\mu_P^V \ll dx$ which satisfies the non-linear equation

$$W(x) - 2P \int \ln |x - y| d\mu_P^V(y) + \ln \frac{d\mu_P^V}{dx} = \lambda_P^V \quad a.s \quad (5)$$

where λ_P^V is a finite constant. We show in section 3 that μ_P^V is absolutely continuous with respect to Lebesgue measure with density which is almost surely differentiable with respect to P . We then show that

Theorem 1.2. *For any bounded continuous function f*

$$\int f(x) d\nu_P^V(x) = \partial_P (P \int f(x) d\mu_P^V(x))$$

This result was already shown in [11] when V is a polynomial. Moreover, our result allows to derive large deviation principle for a general variance profile. Namely let L_N^σ be a tri-diagonal symmetric matrix with independent Gaussian variables on the diagonal and independent chi distributed variables above the diagonal with parameter $\sigma(\frac{i}{N})$, $1 \leq i \leq N$. Let $\mathbb{T}_N^{V,\sigma}$ be the distribution with density $e^{-\text{Tr}(V(L_N^\sigma))}/Z$ with respect to the distribution of L_N^σ .

Theorem 1.3. *Assume that V is continuous and such that there exists $k \in \mathbb{N}$, $a \geq 0$, such that*

$$\lim_{|x| \rightarrow \infty} \frac{V(x)}{x^{2k}} = a.$$

Then, if σ is bounded continuous,

- (1) *the law of $\hat{\mu}_{L_N^\sigma}$ under $\mathbb{T}_N^{V,\sigma}$ satisfies a large deviation principle in the scale N with good rate function T_σ^V ,*
- (2) *T_σ^V achieves its minimal value at a unique probability measure $\nu_\sigma^V = \int_0^1 \nu_{\sigma(P)}^V dP$,*
- (3) *As a consequence, $\hat{\mu}_{L_N^\sigma}$ converges almost surely and in L^1 towards ν_σ^V .*

Our strategy is to prove first a large deviation principles in the case when W is quadratic: L_N has then independent entries (modulo the symmetry constraint) under \mathbb{T}_N^P and then obtain large deviations for more general potentials by using Varadhan's Lemma for potentials V which are bounded continuous, see section 2.

Indeed, in the case where V vanishes, the random variables $(p_j, r_j)_{1 \leq j \leq N}$ are independent, $(L_N)_{j,j}$ are standard Gaussian $N(0, 1)$ variables and $\sqrt{2}(L_N)_{j,j+1}$ follows a χ_{2P} distribution with density with respect to Lebesgue measure given by

$$\chi_{2P}(x) = \frac{2^{1-P} x^{2P-1} e^{-x^2/2}}{\Gamma(P)} \mathbf{1}_{x>0}. \quad (6)$$

The central observation is that the eigenvalues of such a matrix, if the parameter P is replaced by $2P_i/N$, would follow a $2P/N$ Beta ensembles by Dumitriu-Edelman [3]. We then relate the free energy, the rate function of the equilibrium measure of the Toda chain with Coulomb gases in section 3. In section 4, we study the case of general potential. The proof is nearly independent from the quadratic case, but requires additional arguments in particular because the eigenvalues of the Toda matrix are not simple functions of the empirical measure of the entries.

2. LARGE DEVIATION PRINCIPLES FOR TRI-DIAGONAL MATRICES

In this section, we consider a tri-diagonal matrix M_N with entries

$$(M_N)_{j,j} = a_j \text{ and } (M_N)_{j,j+1} = (M_N)_{j+1,j} = b_j \quad (7)$$

with periodic boundary conditions, the random variables $(a_i, b_i)_{1 \leq i \leq N}$ being iid, with (a_1, b_1) with law $Q_a \otimes Q_b$ on \mathbb{R}^2 . We denote by $\hat{\mu}_{M_N}$ the spectral measure of M_N and prove the existence of a large deviation principle for the distribution of $\hat{\mu}_{M_N}$. In [12, Theorem 4.2], the author proves a large deviation principle for moments $\hat{\mu}_{M_N}(x^k)$ by noticing that

$$\hat{\mu}_{M_N}(x^k) = \frac{1}{N} \sum_{i=1}^N f_k(a_j, b_j, |i-j| \leq k)$$

and using the large deviation principle for Markov chains, see e.g [2, Theorem 3.1.2], as well as the contraction principle. Here $f_k(a_j, b_j, |i-j| \leq k) = (M_N^k)_{ii}$ is an homogeneous polynomial of degree k . This could be used to deduce the existence of a large deviation principle for $\hat{\mu}_{M_N}(x^k), k \geq p$ for the weak topology after approximations, but the rate function would not be particularly explicit. We prefer to develop a more straightforward sub-additivity argument and prove separately the existence of a weak large deviation principle and exponential tightness, see e.g [2, Lemma 1.2.18].

2.1. Exponential tightness. In this section we assume that

Assumption 2.1. *There exists $\gamma > 0$ such that*

$$D_\gamma := \int e^{\gamma x^2} dQ_a(x) \times \int e^{\gamma y^2} dQ_b(y) < \infty.$$

We equip the set of probability measures on the real line $\mathcal{P}(\mathbb{R})$ with the weak topology. We then show that

Lemma 2.2. *Under Assumption 2.1, the sequence $(\hat{\mu}_M^N)_{N \geq 0}$ is exponentially tight, namely for each $L \geq 0$ there exists a compact set K_L , $K_L = \{\mu \in \mathcal{P}(\mathbb{R}) : \int x^{2d\mu(x)} \leq L\}$, such that*

$$\limsup_N \frac{1}{N} \ln \mathbb{P}(\hat{\mu}_{M_N} \in K_L^c) < -L. \quad (8)$$

Proof. For $N \geq 1$, notice that

$$\begin{aligned} \int x^2 d\hat{\mu}_M^N(x) &= \frac{1}{N} \text{Tr}(M_N^2) \\ &= \frac{1}{N} \sum_{k=1}^N ((M_N)_{j,j})^2 + \frac{2}{N} \sum_{k=1}^N ((M_N)_{j,j+1})^2 \\ &= \frac{1}{N} \sum_{k=1}^N ((M_N)_{j,j})^2 + \frac{1}{N} \sum_{k=1}^N (\sqrt{2}(M_N)_{j,j+1})^2. \end{aligned} \quad (9)$$

As a consequence, Tchebychev's inequality implies

$$\begin{aligned} \mathbb{P}\left(\int_{\mathbb{R}} x^2 d\hat{\mu}_{M_N}(x) > L\right) &\leq e^{-\frac{1}{2}\gamma NL} \mathbb{E}[e^{\frac{1}{2}N\gamma \int_{\mathbb{R}} x^2 d\hat{\mu}_{M_N}(x)}] \\ &\leq e^{-\frac{1}{2}\gamma NL} D_\gamma^N \end{aligned}$$

Since

$$K_L = \left\{ \mu \in \mathcal{P}(\mathbb{R}) \mid \int_{\mathbb{R}} x^2 d\mu(x) \leq \frac{2}{\gamma}(L + \ln D_\gamma) \right\}$$

is a compact subset of $\mathcal{P}(\mathbb{R})$, the conclusion follows. \square

2.2. Weak large deviation principle. We next establish a weak large deviation principle, based on the general ideas developed in [2], see Lemma 6.1.7. To this end, we will use the following distance on $\mathcal{P}(\mathbb{R})$:

$$d(\mu, \nu) = \sup_{\|f\|_{\text{BV}} \leq 1, |f|_{\text{Lip}} \leq 1} \left\{ \left| \int_{\mathbb{R}} f d\mu - \int_{\mathbb{R}} f d\nu \right| \right\}, \quad (10)$$

where $\|f\|_{\text{BV}}$ is the total variation of f given by

$$\|f\|_{\text{BV}} = \sup \sum_{k \in \mathbb{Z}} |f(x_{k+1}) - f(x_k)|,$$

where the supremum holds over all increasing sequences $(x_k)_{k \in \mathbb{Z}}$. $\|f\|_L$ is the Lipschitz norm of f . If f is C^1 and we put without loss of generality $f(0) = 0$, $\|f\|_{\text{BV}} = \int_{-\infty}^{+\infty} |f'(y)| dy$ and $\|f\|_L = \|f'\|_\infty$. The distance d is smaller than the Wasserstein distance where one takes the supremum over all functions whose sum of their L^∞ and Lipschitz norm are bounded by one and is easily seen to be as well compatible with the weak topology. Then, we shall prove that

Lemma 2.3. *For any μ in $\mathcal{P}(\mathbb{R})$, there exists a limit*

$$\lim_{\delta \rightarrow 0} \liminf_N \frac{1}{N} \ln \mathbb{P}(\hat{\mu}_{M_N} \in B_\mu(\delta)) = \lim_{\delta \rightarrow 0} \limsup_N \frac{1}{N} \ln \mathbb{P}(\hat{\mu}_{M_N} \in B_\mu(\delta)). \quad (11)$$

We denote this limit by $-J_M(\mu)$.

Proof. The advantage of the distance d is the following control: For any symmetric $N \times N$ matrices A and B with spectral measures $\hat{\mu}_A$ and $\hat{\mu}_B$, we have:

$$d(\hat{\mu}_A, \hat{\mu}_B) \leq \min \left\{ \frac{\text{rank}(A - B)}{N}, \frac{1}{N} \sum_{i,j} |A(i, j) - B(i, j)| \right\}. \quad (12)$$

Indeed, for any function f with bounded variation we have thanks to Weyl interlacing property, see e.g. [7, (1.17)],

$$\left| \int f d\hat{\mu}_A^N - \int f d\hat{\mu}_B^N \right| \leq \frac{1}{N} \text{rank}(A - B). \quad (13)$$

Moreover, one can check that if f is C^1

$$\begin{aligned} \int f d\hat{\mu}_A - \int f d\hat{\mu}_B &= \int_0^1 \frac{1}{N} \text{Tr}((A - B)f'(\alpha A + (1 - \alpha)B)) d\alpha \\ &= \int_0^1 \left(\frac{1}{N} \sum_{i,j=1}^N (A - B)_{ij} f'(\alpha A + (1 - \alpha)B)_{ji} \right) d\alpha \end{aligned}$$

which implies since for all indices i, j , $|f'(\alpha A + (1 - \alpha)B)_{ji}| \leq \|f'\|_\infty$ that

$$\left| \int f d\hat{\mu}_A - \int f d\hat{\mu}_B \right| \leq \|f'\|_\infty \frac{1}{N} \sum_{i,j=1}^N |(A - B)_{ij}| \quad (14)$$

Since C^1 functions with bounded L^∞ norm are dense in Lipschitz functions, we deduce (12) from (13) and (14). We are now ready to prove Lemma 2.3. To this end we shall approximate our matrix M_N by a diagonal block matrix with independent blocks. Let $q \geq 1$. For $N \geq 1$ we decompose $N = k_N q + r_N$ with $r_N \in \{0, \dots, q - 1\}$ and set $M_N = M_N^q + R_N^q$, where M_N^q is the diagonal block matrix

$$M_N^q = \begin{bmatrix} M_q^1 & & & \\ & \ddots & & \\ & & M_q^{k_N} & \\ & & & B \end{bmatrix}, \quad (15)$$

where for all $i \in \{1, \dots, k_N\}$ M_q^i has the same distribution than M_q and B the same distribution than M_{r_N} . The M_q^i are independent, and independent from B . R_N^q is the self-adjoint matrix with null entries except $R_N^q(1, N) = R_N^q(N, 1) = b_N$, $R_N^q(k_N q + 1, N) = R_N^q(N, k_N q + 1) = -b_N$, and those given, for $k \in \{1, \dots, k_N\}$, $R_N^q(kq + 1, kq) = R_N^q(kq, kq + 1) = b_{kq}$, $R_N^q((k - 1)q + 1, kq) = R_N^q(kq, (k - 1)q + 1) = -b_{kq}$. Therefore $\text{rank}(R_N^q) \leq 2k_N + 2 \leq 4k_N$. By (12), we deduce that

$$d(\hat{\mu}_{M_N}, \hat{\mu}_{M_N^q}) \leq \frac{4}{q}. \quad (16)$$

Moreover, we can write $\hat{\mu}_{M_N^q}$ as the sum

$$\hat{\mu}_{M_N^q} = \sum_{i=1}^{k_N} \frac{q}{N} \hat{\mu}_{M_q^i} + \frac{r_N}{N} \hat{\mu}_B.$$

Therefore, for any $\mu \in \mathcal{P}(\mathbb{R})$ and $\delta > 0$

$$\begin{aligned} \mathbb{P}\left(\hat{\mu}_{M_q^1} \in B_\mu(\delta)\right)^{k_N} \mathbb{P}\left(\hat{\mu}_{M_{r_N}} \in B_\mu(\delta)\right) &= \mathbb{P}\left(\forall i \in \{1, \dots, k_N\}, \hat{\mu}_{M_q^i} \in B_\mu(\delta), \hat{\mu}_B \in B_\mu(\delta)\right) \\ &\leq \mathbb{P}\left(\hat{\mu}_{M_N^q} \in B_\mu(\delta)\right) \\ &\leq \mathbb{P}\left(\hat{\mu}_{M_N} \in B_\mu\left(\delta + \frac{4}{q}\right)\right), \end{aligned}$$

where we used the convexity of balls and (16). As a consequence,

$$u_N(\delta) = -\ln \mathbb{P}\left(\hat{\mu}_{M_N} \in B_\mu(\delta)\right)$$

satisfies

$$u_N(\delta + 4/q) \leq k_N u_q(\delta) + u_{r_N}(\delta).$$

It is easy (and classical) to deduce the convergence of u_N/N when N and then δ goes to infinity. Indeed let $\delta > 0$ be given and choose q large enough so that $\frac{4}{q} < \delta$. Then, since $\delta \rightarrow u_N(\delta)$ is decreasing,

$$\frac{u_N(2\delta)}{N} \leq \frac{u_N(\delta + 4/q)}{N} \leq \frac{u_q(\delta)}{q} + \frac{u_{r_N}(\delta)}{N}. \quad (17)$$

Since $\frac{u_{r_N}(\delta)}{N} \leq \frac{\max_{1 \leq i \leq q-1} u_i(\delta)}{N}$ goes to zero when $N \rightarrow \infty$, we conclude that

$$\limsup_N \frac{u_N(2\delta)}{N} \leq \frac{u_q(\delta)}{q}.$$

Since this is true for all q large enough, we get

$$\limsup_N \frac{u_N(2\delta)}{N} \leq \liminf_N \frac{u_N(\delta)}{N}.$$

Since the left and right hand sides decrease as δ goes to zero, we conclude that

$$\lim_{\delta \rightarrow 0} \limsup_{N \rightarrow \infty} -\frac{1}{N} \ln \mathbb{P}\left(\hat{\mu}_{M_N} \in B_\mu(\delta)\right) \leq \lim_{\delta \rightarrow 0} \liminf_{N \rightarrow \infty} -\frac{1}{N} \ln \mathbb{P}\left(\hat{\mu}_{M_N} \in B_\mu(\delta)\right),$$

and the conclusion follows. \square

2.3. Full large deviation principle. As a consequence of Lemmas 2.2 and 2.3, we have by [2, Theorem 1.2.18] the following large deviation theorem:

Theorem 2.4. *Under Assumption 2.1, the law of $\hat{\mu}_M$ satisfies a large deviation principle in the scale N with a good rate function J_M . Moreover, J_M is convex. In other words,*

- $J_M : \mathcal{P}(\mathbb{R}) \rightarrow [0, +\infty]$ has compact level sets $\{\mu : J_M(\mu) \leq L\}$ for all $L \geq 0$. Moreover, J_M is convex.
- For any closed set $F \subset \mathcal{P}(\mathbb{R})$,

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \ln \mathbb{P}(\hat{\mu}_{M_N} \in F) \leq -\inf_F J_M,$$

whereas for any open set $O \subset \mathcal{P}(\mathbb{R})$

$$\liminf_{N \rightarrow \infty} \frac{1}{N} \ln \mathbb{P}(\hat{\mu}_{M_N} \in O) \geq -\inf_O J_M.$$

Proof. J_M exists and is defined by Lemma 2.3. The lower semi-continuity of J_M follows from [2, Theorem 4.1.11]. We then deduce that the level sets of J_M are compact by the exponential tightness, see [2, Lemma 1.2.18 (b)].

In the spirit of [2, Lemma 4.1.21], we show that J_M is convex. Let $\mu_1, \mu_2 \in \mathcal{P}(\mathbb{R})$. Since $\hat{\mu}_{M_{2N}}$ can be decomposed as the independent sum of $\hat{\mu}_{M_N}$ divided by 2 plus an error term of order $2/N$ by (13), we have for all $\delta_1, \delta_2 > 0$

$$\frac{1}{2} \left(\frac{1}{N} \ln \mathbb{P}(d(\hat{\mu}_{M_N}, \mu_1) < \delta_1) + \frac{1}{N} \ln \mathbb{P}(d(\hat{\mu}_{M_N}, \mu_2) < \delta_2) \right) \leq \frac{1}{2N} \ln \mathbb{P}(d(\hat{\mu}_{M_{2N}}, \delta_3)). \quad (18)$$

for any $\delta_3 \geq \frac{1}{2}(\delta_1 + \delta_2) + \frac{2}{N}$. We then let N going to infinity, δ_1, δ_2 and then δ_3 to zero to conclude that

$$J_M \left(\frac{\mu_1 + \mu_2}{2} \right) \leq \frac{1}{2} \left(J_M(\mu_1) + J_M(\mu_2) \right), \quad (19)$$

The second point, namely that a weak large deviation principle and exponential tightness implies a full large deviation principle, is classical, see [2, Lemma 1.2.18]. \square

2.4. Large deviation principle for the Toda-Chain with quadratic potential. In the case of the Toda chain with Gaussian potential $W(x) = \frac{1}{2}x^2$, that is $V = 0$, with entries following \mathbb{T}_N^P , we take Q_a to be the standard Gaussian law and Q_b to be the chi distribution $\sqrt{2}^{-1} \chi_{2P}$ given in (6). These entries clearly satisfy Assumption 2.1 and therefore we have

Corollary 2.5. *For any $P > 0$, the law of $\hat{\mu}_{L_N(P)}$ with $L_N(P)$ the tridiagonal matrix whose entries follow \mathbb{T}_N^P satisfies a large deviation principle in the scale N with convex good rate function T_P .*

For further use, we show that

Lemma 2.6. *For each $\mu \in P(\mathbb{R})$, the map $s \in (0, +\infty) \mapsto T_s(\mu)$ is lower semi-continuous.*

Proof. We first show that we can couple the matrices $(L_N(s), L_N(s+h))_N$, where $L_N(s)$ follows \mathbb{T}_N^s , so that there exists a finite constant c and a function $A(h)$ going to infinity as h goes to zero so that

$$\mathbb{P} \left(d(\hat{\mu}_{L_N(s)}, \hat{\mu}_{L_N(s+h)}) > \delta \right) \leq e^{N(c-A(h)\delta/2)}, \quad (20)$$

This coupling is done as follows

- The diagonal coefficients are the same set of standard independent Gaussian variables
- The coefficient below and above the diagonal X_u^i , follow a $\sqrt{2}^{-1} \chi_{2u}$ for $u = s$, $u = h$ and $s+h$. By definition of the χ distribution we can construct it so that almost surely

$$X_{s+h}^i = \sqrt{(X_s^i)^2 + (X_h^i)^2}.$$

This coupling allows by (12) to write

$$d(\hat{\mu}_{L_N(s)}, \hat{\mu}_{L_N(s+h)})(s) \leq \frac{2}{N} \sum_{i=1}^N |X_{s+h}^i - X_s^i| = \frac{2}{N} \sum_{i=1}^N (X_{s+h}^i - X_s^i) \leq \frac{2}{N} \sum_{i=1}^N X_h^i.$$

Equation (20) follows by Tchebychev inequality with $A(h) = \sqrt{-\ln(h)}$ since $\mathbb{E}[\exp\{A(h)X_h^i\}]$ is finite, see (35). (20) implies that $(\hat{\mu}_{L_N(s+h)})_{N \geq 0}$ is an exponential approximation of $(\hat{\mu}_{L_N(s)})_{N \geq 0}$ when h goes to zero. By [2, Theorem 4.2.16] we deduce that $\mu \in \mathcal{P}(\mathbb{R})$,

$$T_s(\mu) = \lim_{\delta \rightarrow 0} \liminf_{h \rightarrow 0} \inf_{B_\mu(\delta)} T_{s+h}.$$

By monotonicity of the right hand side and the lower semi-continuity of T_s we deduce (see [2, (4.1.2)]),

$$\lim_{\delta \rightarrow 0} \inf_{B_\mu(\delta)} T_{s+h} = T_{s+h}(\mu),$$

and therefore

$$T_s(\mu) = \lim_{\delta \rightarrow 0} \liminf_{h \rightarrow 0} \inf_{B_\mu(\delta)} T_{s+h} \leq \liminf_{h \rightarrow 0} T_{s+h}(\mu),$$

and so $s \mapsto T_s(\mu)$ is lower semi-continuous. \square

We shall also use later that Corollary 2.5 gives large deviation principle for the empirical measure of the Toda chain with general bounded continuous potential.

Corollary 2.7. *Let V be a bounded continuous function and $P > 0$. Let L_N be the tridiagonal matrix whose entries follow $\mathbb{T}_N^{V,P}$*

- *The law of $\hat{\mu}_{L_N}$ satisfies a large deviation principle in the scale N with convex good rate function*

$$T_P^V(\mu) = T_P(\mu) + \int V d\mu - \inf_{\nu} \{T_P(\nu) + \int V d\nu\}.$$

- *The set M_P^V where T_P^V achieves its minimum value is a compact convex subset of $\mathcal{P}(\mathbb{R})$. It is continuous in the sense that for any $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that for all $\delta < \delta_\varepsilon$, any $(t, s) \in \mathbb{R}^{+*}$ such that for $|t - s| \leq \delta$*

$$M_s^V \subset (M_t^V)^\varepsilon$$

where $A^\varepsilon = \{\mu : d(\mu, A) \leq \varepsilon\}$.

Proof. The first point is a direct consequence of Varadhan's lemma. We hence prove the second point, that is the continuity of $s \mapsto M_s^V$. We let \mathbb{T}_N be the coupling of $L_N(s)$ and $L_N(t)$ introduced in the previous section. By definition for $P = s$ and t

$$\mathbb{T}_N^{V,P}(\hat{\mu}_{L_N} \in \dots) = \frac{1}{Z_{N,\mathbb{T}}^{V,P}} \int \mathbf{1}_{\{\hat{\mu}_{L_N(P)} \in \dots\}} e^{-N \int V(x) d\hat{\mu}_{L_N(P)}(x)} d\mathbb{T}_N$$

Therefore, since $((M_t^V)^\varepsilon)^c$ is open, we can use the previous large deviation principle to state that for any $\kappa > 0$

$$\begin{aligned} - \inf_{((M_t^V)^\varepsilon)^c} T_s^V &\leq \limsup_{N \rightarrow \infty} \frac{1}{N} \ln \frac{1}{Z_{N,\mathbb{T}}^{V,s}} \int_{\{d(\hat{\mu}_{L_N(s)}, M_t^V) > \varepsilon\}} e^{-N \int V(x) d\hat{\mu}_{L_N(s)}(x)} d\mathbb{T}_N \\ &= \max \left\{ \limsup_{N \rightarrow \infty} \frac{1}{N} \ln \frac{1}{Z_{N,\mathbb{T}}^{V,s}} \int_{\{d(\hat{\mu}_{L_N(s)}, M_t^V) > \varepsilon\} \cap \{d(\hat{\mu}_{L_N(s)}, \hat{\mu}_{L_N(t)}) \leq \kappa\}} e^{-N \int V(x) d\hat{\mu}_{L_N(s)}(x)} d\mathbb{T}_N, \right. \\ &\quad \left. 2\|V\|_\infty + c - \sqrt{-\ln|s-t|\kappa} \right\} \end{aligned}$$

where we used (20). For the first term we notice that $\int V(d\mu - d\nu)$ is bounded by $\varepsilon_V(\kappa)$ going to zero as κ does uniformly on $d(\mu, \nu) \leq \kappa$, since V is bounded continuous. Hence

$$\begin{aligned} & \frac{1}{Z_{N,\mathbb{T}}^{V,s}} \int \mathbf{1}_{\{d(\hat{\mu}_{L_N(s)}, M_t^V) > \varepsilon\}} \mathbf{1}_{\{d(\hat{\mu}_{L_N(s)}, \hat{\mu}_{L_N(t)}) \leq \kappa\}} e^{-N \int V(x) d\hat{\mu}_{L_N(s)}(x)} d\mathbb{T}_N \\ & \leq e^{N\varepsilon(\kappa)} \frac{Z_{N,\mathbb{T}}^{V,t}}{Z_{N,\mathbb{T}}^{V,s}} \frac{1}{Z_{N,\mathbb{T}}^{V,t}} \int \mathbf{1}_{\{d(\hat{\mu}_{L_N(t)}, M_t^V) \geq \varepsilon - \kappa\}} \mathbf{1}_{\{d(\hat{\mu}_{L_N(s)}, \hat{\mu}_{L_N(t)}) \leq \kappa\}} e^{-N \int V(x) d\hat{\mu}_{L_N(t)}(x)} d\mathbb{T}_N. \end{aligned}$$

Similarly

$$\begin{aligned} Z_{N,\mathbb{T}}^{V,t} &= \int e^{-N \int V dL_N(t)} \mathbf{1}_{\{d((\hat{\mu}_{L_N(s)}, \hat{\mu}_{L_N(t)}) \leq \kappa\}} d\mathbb{P} + \int e^{-N \int V dL_N(t)} \mathbf{1}_{\{d((\hat{\mu}_{L_N(s)}, \hat{\mu}_{L_N(t)}) > \kappa\}} d\mathbb{T}_N \\ &\leq Z_{N,\mathbb{T}}^{V,s} e^{N\varepsilon(\kappa)} + e^{(2\|V\|_\infty + c - \sqrt{-\ln|s-t|\kappa})N} \leq Z_{N,\mathbb{T}}^{V,s} (e^{N\varepsilon(\kappa)} + e^{(3\|V\|_\infty + c - \sqrt{-\ln|s-t|\kappa})N}) \end{aligned}$$

where we used that the partition function is lower bounded by $e^{-\|V\|_\infty N}$. Moreover the previous large deviation principle implies if $\kappa \leq \varepsilon/2$

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \ln \frac{1}{Z_{N,\mathbb{T}}^{V,t}} \int_{\{d(\hat{\mu}_{L_N(s)}, M_t^V) \geq \varepsilon/2\} \cap \{d(\hat{\mu}_{L_N(s)}, \hat{\mu}_{L_N(t)}) \leq \kappa\}} e^{-N \int V(x) d\hat{\mu}_{L_N(t)}(x)} d\mathbb{T}_N^P \leq - \inf_{d(\mu, M_t^V) \geq \varepsilon/2} \{T_t^V\}.$$

Hence, we conclude that if we choose $\sqrt{-\ln|s-t|\kappa} > 3\|V\|_\infty + c$,

$$- \inf_{((M_t^V)^\varepsilon)^c} T_s^V \leq 2\varepsilon(\kappa) - \inf_{d(\mu, M_t^V) \geq \varepsilon/2} \{T_t^V\}$$

We finally choose $\kappa = (-\ln|s-t|)^{-1/4}$ with $s-t$ small enough so that $2\varepsilon(\kappa) - \inf_{d(\mu, M_t^V) \geq \varepsilon/2} \{T_t^V\} < 0$ and $\kappa \leq \varepsilon/2$. We then conclude that $\inf_{((M_t^V)^\varepsilon)^c} T_s^V > 0$ so that $((M_t^V)^\varepsilon)^c \subset (M_s^V)^c$ and hence the conclusion. \square

3. β -ENSEMBLES

3.1. Large deviation principles for β -ensembles. In this section we consider the β ensembles and collect already known results about their large deviations theorems. We then relate these large deviation principles with the previous ones thanks to Dumitriu-Edelman tri-diagonal representation, as pioneered in [11]. Coulomb gases on the real line are given by

$$d\mathbb{P}_N^{V,\beta}(x_1, \dots, x_N) = \frac{1}{Z_{N,C}^{V,\beta}} \prod_{i < j} |x_i - x_j|^\beta e^{-\sum_{i=1}^N (\frac{1}{2}x_i^2 + V(x_i))} dx_1 \cdots dx_N. \quad (21)$$

V will be a continuous potential. When $V = 0$ and $\beta = 1$, it is well known [1, Section 2.5.2] that $d\mathbb{P}_N^{0,1}$ is the law of the eigenvalues of the Gaussian orthogonal ensemble of random matrices with standard Gaussian entries. In this article we keep the potential to be under the form of a quadratic potential plus a general potential only to have simpler notations later on. In this article we are however interested in the scaling where $\beta = \frac{2P}{N}$. The large deviation principles for the empirical measure $\hat{\mu}_N = \frac{1}{N} \sum_{i=1}^N \delta_{x_i}$ have been derived in [5] and yields the following result.

Theorem 3.1. [5] *Let $W(x) = \frac{1}{2}x^2 + V(x)$ be a continuous function such that for some $P' > P$ there exists a finite constant C_V such that for all x*

$$W(x) \geq P' \ln(|x|^2 + 1) + C_V \quad (22)$$

Then the law of $\hat{\mu}_N$ under $\mathbb{P}_N^{V, \frac{2P}{N}}$ satisfies a large deviation principle in the scale N and with good rate function $I_P^V(\mu) = f_P^V(\mu) - \inf f_P^V$ where

$$f_P^V(\mu) = \frac{1}{2} \int (W(x) + W(y) - 2P \ln|x - y|) d\mu(x) d\mu(y) + \int \ln \frac{d\mu}{dx} d\mu(x)$$

if $\mu \ll dx$, whereas f_P^V is infinite otherwise.

This result can be seen as a consequence of [5], as detailed by his author in a private communication [6], see also [4] for similar idea. In fact, neglecting the singularity of the logarithm, this result would be a direct consequence of Sanov's theorem and Varadhan's lemma. It is not hard to see that

Lemma 3.2. *For any C^1 function W such that (22) holds, any $P > 0$ such that $(P' - P) > 1$*

- $\mu \mapsto I_P^V(\mu)$ is strictly convex,
- I_P^V achieves its minimal value at a unique probability measure $\mu_P^V \ll dx$ which satisfies the non-linear equation

$$W(x) - 2P \int \ln|x - y| d\mu_P^V(y) + \ln \frac{d\mu_P^V}{dx} = \lambda_P^V \quad \text{a.s.} \quad (23)$$

where λ_P^V is a finite constant. Furthermore the support of μ_P^V is the whole real line and the density of $\frac{d\mu_P^V}{dx}$ is bounded by $C_P(|x| + 1)^{2(P - P')}$ where C_P is uniformly bounded on compacts contained in $(0, P' - 1)$. As a consequence, $P \mapsto \inf f_P^V$ is Lipschitz.

- Let D be the distance on $\mathcal{P}(\mathbb{R})$ given by

$$\begin{aligned} D(\mu, \mu') &= \left(- \int \ln|x - y| d(\mu - \mu')(x) d(\mu - \mu')(y) \right)^{1/2} \\ &= \left(\int_0^\infty \frac{1}{t} \left| \int e^{itx} d(\mu - \mu')(x) \right|^2 dt \right)^{1/2} \end{aligned} \quad (24)$$

Then for any R there exists a finite constant C_R such that for all $P, P' \leq R$

$$D(\mu_P^V, \mu_{P'}^V) \leq \frac{C_R}{\min\{P, P'\}} |P - P'|$$

Observe that if f is in L^2 with derivative in L^2 , we can set $\|f\|_{\frac{1}{2}} = (\int_0^\infty t |\hat{f}_t|^2 dt)^{1/2}$. Then, for any measure ν with zero mass,

$$\int f(x) d\nu(x) = \int_{-\infty}^\infty \hat{f}_t \hat{\nu}_t dt = \int_{-\infty}^\infty \sqrt{t} \hat{f}_t \frac{1}{\sqrt{t}} \hat{\nu}_t dt$$

so that by Cauchy-Schwartz inequality

$$\left| \int f(x) d\nu(x) \right|^2 \leq \int_{-\infty}^\infty |t \hat{f}_t|^2 dt \int_{-\infty}^\infty \frac{1}{|t|} |\hat{\nu}_t|^2 dt = 4 \|f\|_{1/2}^2 D(\nu, 0) \quad (25)$$

In particular, the last point in the theorem shows that for any f with finite $\|f\|_{1/2}$, $P \rightarrow \int f d\mu_P^V$ is Lipschitz with uniform Lipschitz norm on $P \geq \varepsilon > 0$.

Proof. For $P'' > 1$, we rewrite f_P^V (up to a constant $\ln Z_{P''}$) as

$$f_P^V(\mu) = \frac{1}{2} \int (\bar{W}(x) + \bar{W}(y) - 2P \ln|x-y|) d\mu(x) d\mu(y) + \int \ln \frac{d\mu}{Z_{P''}^{-1}(|x^2|+1)^{-\frac{P''}{2}}} d\mu(x)$$

where $\bar{W}(y) := W(y) - \frac{1}{2}P'' \ln(|y|^2 + 1)$ and P'' is fixed so that $Z_{P''} = \int (|x^2| + 1)^{-P''/2} dx$ is finite. The advantage is that $\lambda_{P''}(dx) := Z_{P''}^{-1}(|x^2| + 1)^{-P''/2} dx$ is a probability measure so that for all probability measure μ

$$\int \ln \frac{d\mu}{d\lambda_{P''}}(x) d\mu(x) \geq 0.$$

Note that this amounts to change $V(x)$ into $\bar{V}(x) = V(x) - \frac{1}{2}P'' \ln(|x|^2 + 1)$, and therefore to change P' into $P' - \frac{1}{2}P''$ in the hypothesis.

The first point of the lemma is clear as $\mu \mapsto N_V^P(\mu) = \int (W(x) + W(y) - 2P \ln|x-y|) d\mu(x) d\mu(y)$ is strictly convex [1, Lemma 2.6.2] whereas the relative entropy $\mu \mapsto \int \ln \frac{d\mu}{d\lambda_{P''}}(y) d\mu(y)$ is well known to be convex. Since it is a good rate function it achieves its minimal value at a unique probability measure μ_P^V . Writing that for any measure ν with mass zero such that $\mu_P^V + \varepsilon\nu$ is a probability measure for small enough ε , $I_P^V(\mu_P^V + \varepsilon\nu) \geq I_P^V(\mu_P^V)$, we get that (23) holds μ_P^V almost surely and that the left hand side in (23) is greater or equal than the right hand side outside of the support of μ_P^V . Since the left hand side equals $-\infty$ when the density vanishes, we conclude that the support is the whole real line. We finally show the boundedness of the density. Note that (23) implies that

$$\frac{d\mu_P^V}{dx}(x) = e^{\lambda_P^V} e^{-W(x) + 2P \int \ln|x-y| d\mu_P^V(y)} \quad (26)$$

We get from (22), and the fact that $\ln|x-y| \leq \frac{1}{2} \ln(|x|^2 + 1) + \frac{1}{2} \ln(|y|^2 + 1)$ the bound

$$-W(x) + 2P \int \ln|x-y| d\mu_P^V(y) \leq -(P' - P) \ln(|x|^2 + 1) + C_V + P \int \ln(|x|^2 + 1) d\mu_P^V.$$

We thus only need to bound $\int \ln(|x|^2 + 1) d\mu_P^V$ and λ_P^V from above. We first notice that $P \mapsto \inf f_P^V$ is convex since it is the limit of the free energy $N^{-1} \ln Z_N^{V, \frac{2P}{N}}$. This is enough to guarantee that this quantity is uniformly bounded on compact sets (as it is at any given point) :we denote by C such a bound for a fixed compact set. As in [1, Lemma 2.6.2 (b)], since the relative entropy is non-negative we find that

$$\int (\bar{W}(x) - P \ln(|x|^2 + 1)) d\mu_P^V(x) \leq f_P^V(\mu_P^V) \leq C.$$

This implies by our hypothesis (22) that

$$(P' - P'' - P) \int \ln(|x|^2 + 1) d\mu_P^V(x) \leq C - C_V$$

and therefore plugging this estimate in the infimum of f_P^V gives if $P' - P - P'' > 0$ (which is always possible as we assumed $P' - P > 1$)

$$\int W(x) d\mu_P^V(x) \geq C + \frac{C - C_V}{2(P' - P - P'')}$$

Moreover, again because the relative entropy is non-negative,

$$\begin{aligned} -P\Sigma(\mu_P^V) &:= -P \int \ln|x - y| d\mu_P^V(x) d\mu_P^V(y) \\ &\leq C - \int \bar{W}(x) d\mu_P^V(x) \leq C - 2(P' - P'') \int \ln(|x|^2 + 1) d\mu_P^V(x) - C_V \end{aligned}$$

is as well uniformly bounded. Finally, from (23) we have after integration under μ_P^V

$$\lambda_P^V = \inf f_P^V - P \int \ln|x - y| d\mu_P^V(x) d\mu_P^V(y) \quad (27)$$

is thus uniformly bounded from above. This completes the proof of the upper bound of the density: $\frac{d\mu_V}{dx}$ is bounded by $C_P(|x| + 1)^{2(P-P')}$ where C_P is uniformly bounded on compacts so that $P' - P - 1 \geq \varepsilon > 0$ for some fixed ε .

We next study the regularity of the equilibrium measure μ_P^V in the parameter P and let R be a real number in a neighborhood of P . If $\Delta\mu = \mu_P^V - \mu_R^V$, since μ_P^V minimizes f_P^V , we have

$$\begin{aligned} 0 &\geq f_P^V(\mu_P^V) - f_P^V(\mu_R^V) \\ &= \int W(x) d\Delta\mu(x) - 2P \int \ln|x - y| d\mu_R^V(x) d\Delta\mu(y) - P \int \ln|x - y| d\Delta\mu(x) d\Delta\mu(y) \\ &\quad + \int \ln \frac{d\mu_P^V}{dx} d\mu_P^V - \int \ln \frac{d\mu_R^V}{dx} d\mu_R^V \\ &= \int (2R \int \ln|x - y| d\mu_R^V(y) - \ln \frac{d\mu_R^V}{dx})(x) d\Delta\mu(x) - 2P \int \ln|x - y| d\mu_R^V(x) d\Delta\mu(y) \\ &\quad - P \int \ln|x - y| d\Delta\mu(x) d\Delta\mu(y) + \int \ln \frac{d\mu_P^V}{dx} d\mu_P^V - \int \ln \frac{d\mu_R^V}{dx} d\mu_R^V \\ &= 2(R - P) \int \ln|x - y| d\mu_R^V(x) d\Delta\mu(y) - P \int \int \ln|x - y| d\Delta\mu(x) d\Delta\mu(y) + \int \ln \frac{d\mu_P^V}{d\mu_R^V} d\mu_P^V \end{aligned}$$

where in the second line we used (23) and the fact that $\Delta\mu(1) = 0$. By using the Fourier transform of the logarithm, the centering of $\Delta\mu$ and the definition (24) we deduce

$$\int \int \ln \frac{d\mu_P^V}{d\mu_R^V} d\mu_P^V + PD(\mu_P^V, \mu_R^V)^2 \leq 2(P - R) \int \int \ln|x - y| d\mu_R^V(x) d\Delta\mu(y) \quad (28)$$

where the right hand side is uniformly bounded by boundedness of the density of μ_P^V and μ_R^V . Since $\int \ln \frac{d\mu_P^V}{d\mu_R^V} d\mu_P^V \geq 0$ by Jensen's inequality, and $\int \ln|x - y| d\mu_R^V(x)$ is bounded uniformly by the previous estimate, this already gives the existence of a finite constant such that

$$D(\mu_P^V, \mu_R^V) \leq D\sqrt{|P - R|}.$$

We next improve this bound to show the Lipschitz property. If $y \mapsto \int \ln|x-y|d\mu_R^V(x)$ has finite $\|\cdot\|_{1/2}$ norm, we are done by using (25). Because this is unclear, we introduce another probability measure ν in (28) so that

$$\begin{aligned} PD(\mu_P^V, \mu_R^V)^2 &\leq 2(P-R) \int \int \ln|x-y|d(\mu_R^V - \nu)(x)d\Delta\mu(y) \\ &\quad + 2(P-R) \int \int \ln|x-y|d\nu(x)d\Delta\mu(y) \end{aligned} \quad (29)$$

We choose ν such that $\phi_\nu(x) = \int \ln|x-y|d\nu(y)$ is bounded and differentiable with bounded derivative, for instance the Cauchy transform with coefficient 1 for which

$$\phi'_\nu(x) = PV \int \frac{1}{x-y} \frac{1}{\pi(y^2+1)} dy = \frac{1}{x^2+1}$$

Because

$$\|f\|_{1/2}^2 = \int t|\hat{f}_t|^2 dt \leq \frac{1}{2} \left(\int t^2|\hat{f}_t|^2 dt + \int |\hat{f}_t|^2 dt \right) = \frac{1}{2} (\|f\|_\infty^2 + \|f'\|_\infty^2)$$

we deduce that $\|\phi_\nu\|_{1/2}$ is bounded and hence there exists a finite constant C such that

$$\left| \int \int \ln|x-y|d\nu(x)d\Delta\mu(y) \right| \leq CD(\mu_P^V, \mu_R^V).$$

Moreover we notice that by taking the Fourier transform of the logarithm and using Cauchy-Schwartz inequality

$$\left| \int \int \ln|x-y|d(\mu_R^V - \nu)(x)d\Delta\mu(y) \right| = \left| \int \frac{1}{t} (\widehat{\mu_R^V - \nu})_t \overline{\widehat{\Delta\mu}_t} dt \right| \leq D(\mu_R^V, \nu) D(\mu_R^V, \mu_\nu^V).$$

Now

$$D(\mu_R^V, \nu)^2 = - \int \ln|x-y|d(\mu_R^V - \nu)(x)d(\mu_R^V - \nu)(y)$$

is bounded uniformly because μ_R^V and ν have bounded density going to zero sufficiently fast at infinity. Hence, we conclude from (29) that there exists a finite constant C such that

$$PD(\mu_P^V, \mu_R^V)^2 \leq C(P-R)D(\mu_P^V, \mu_R^V)$$

from which the conclusion follows. \square

3.2. Relation with the large deviation principle for Toda matrices with quadratic potential.

When $V = 0$, for any $\beta > 0$, Dumitriu and Edelman [3, Theorem 2.12] have shown that $\mathbb{P}_N^{0,\beta}$ is the law of the eigenvalues of a $N \times N$ tri-diagonal matrix C_N^β such that $((C_N^\beta)_{j,j})_{1 \leq j \leq N}$ are independent standard normal variables, independent from the off diagonal entries $(C_N^\beta)_{j,j+1} = (C_N^\beta)_{j+1,j}$ which are independent and such that $\sqrt{2}C_N^\beta(j, j+1)$ follows a $\chi_{(N-j)\beta}$ distribution. As in the case of Toda measure we hereafter identify $\mathbb{P}_N^{0,\beta}$ with \mathbb{P}_N^β . We are now going to give an alternate large deviation principle for the empirical measure under $\mathbb{P}_N^{2P/N}$ based on this representation, this will allow to relate the rate function $I_P = I_P^0$ of the Coulomb Gas in terms of the large deviation rate function $T_s, s \leq P$ for Toda matrices.

corner entries, put to zero :

$$C_i = \begin{pmatrix} g_{(i-1)k_N+1} & \cdots & 0 \\ \cdots & \cdots & \frac{1}{\sqrt{2}}c_j^i \\ \cdots & \frac{1}{\sqrt{2}}c_j^i & \cdots \\ 0 & \cdots & g_{ik_N} \end{pmatrix}.$$

The $(c_j^i)_{1 \leq j \leq k_N-1}$ are distributed according to $\chi_{2P \frac{N-ik_N}{N}}$.

For $1 \leq i \leq M$ and $1 \leq j \leq k_N - 1$, let $b_j^i = \sqrt{(c_j^i)^2 + \chi_{i,j}^2}$, where $(\chi_{i,j})_{1 \leq i \leq M, 1 \leq j \leq k_N}$ is an independent family of χ variables with parameter $2P \frac{k_N-j}{N}$, independent from U_M^N .

We set, for $1 \leq i \leq M$, B_i to be the matrix

$$B_i = \begin{pmatrix} g_{(i-1)k_N+1} & \cdots & 0 \\ \cdots & \cdots & \frac{1}{\sqrt{2}}b_j^i \\ \cdots & \frac{1}{\sqrt{2}}b_j^i & \cdots \\ 0 & \cdots & g_{ik_N} \end{pmatrix}.$$

The matrix

$$C_N^{2P/N} = \begin{pmatrix} \boxed{B_1} & & & & & \\ & * & & & & \\ & & * & & & \\ & & & \ddots & & \\ & & & & * & \\ & & & & & * \\ & & & & & & \boxed{B_M} \\ & & & & & & & * \\ & & & & & & & & \boxed{R_N^M} \end{pmatrix}$$

is distributed according to $\mathbb{P}_N^{2P/N}$, where the symbols $*$ denote the same coefficients as those of U_N^M . Because the rank of $S_N^M - U_N^M$ is bounded by $2M + r_N \leq 3M$, by (12) we have

$$d(\hat{\mu}_{U_N^M}, \hat{\mu}_{S_N^M}) \leq \frac{3M}{N} = \frac{3}{k_N}. \quad (32)$$

Let $\delta > 0$. Then for N large enough so that k_N verifies $\frac{3}{k_N} \leq \delta/2$,

$$\begin{aligned} \mathbb{P}\left(d(\hat{\mu}_{C_N^{2P/N}}, \hat{\mu}_{S_N^M}) > \delta\right) &\leq \mathbb{P}\left(d(\hat{\mu}_{C_N^{2P/N}}, \hat{\mu}_{U_N^M}) + d(\hat{\mu}_{U_N^M}, \hat{\mu}_{S_N^M}) > \delta\right) \\ &\leq \mathbb{P}\left(d(\hat{\mu}_{C_N^{2P/N}}, \hat{\mu}_{U_N^M}) > \delta/2\right). \end{aligned}$$

Moreover (12) yields

$$d(\hat{\mu}_{U_N^M}, \hat{\mu}_{C_N^{2P/N}}) \leq \frac{2}{N} \sum_{i=1}^N |Y_i|, \quad (33)$$

where $|Y_i|$ is the i th coefficient above or below the (i, i) the coefficient of $C_N^{2P/N} - U_N^M$. Applying the inequality $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ for $a, b \geq 0$ and $a = c_j^i$ and $b = \chi_{i,j}$, we deduce

$$d(\hat{\mu}_{U_N^M}, \hat{\mu}_{C_N^{2P/N}}) \leq \frac{\sqrt{2}}{k_N M} \sum_{i=1}^{k_N M} \chi_{2P/M}^i, \quad (34)$$

where the last sum denotes the sum of iid variables with law $\chi_{2P/M}$ (and we used that there exists a coupling between a $\chi_{2P \frac{k_N - j}{N}}$ and a $\chi_{2P/M}$ variable such that the first is always bounded above by the second.)

Thus for all $\delta > 0$, for any integer numbers N such that $\frac{3}{k_N} \leq \delta/2$ (i.e for N larger than some N_0 depending on M) and for any non-negative function $A : M \mapsto A(M)$

$$\begin{aligned} \mathbb{P} \left(d(\hat{\mu}_{S_N^M}, \hat{\mu}_{C_N^{2P/N}}) > \delta \right) &\leq \mathbb{P} \left(\sum_{i=1}^{k_N M} \chi_{2P/M} > \frac{k_N M \delta}{2\sqrt{2}} \right) \\ &\leq e^{-A(M)k_N M \delta / (2\sqrt{2})} \mathbb{E} \left[e^{A(M)\chi_{2P/M}} \right]^{k_N M}. \end{aligned}$$

It is not hard to see that with $A(M) = \sqrt{\ln(M)}$, there exists a finite constant K such that

$$\sup_{M \geq 0} \mathbb{E} \int e^{A(M)x} d\chi_{1/M}(x) \leq K \quad (35)$$

insuring that

$$\frac{1}{N} \ln \mathbb{P}(d(\hat{\mu}_{C_N^{2P/N}}, \hat{\mu}_{S_N^M}) > \delta) \leq -A(M) \frac{\delta}{2\sqrt{2}} + K,$$

which yields the result. □

We shall use the previous lemma to study the case with a non trivial potential. Indeed, as a direct consequence, Varadhan's lemma yields

Theorem 3.4. *For any continuous function V such that*

$$\limsup_{|x| \rightarrow \infty} \frac{|V(x)|}{x^2} = 0, \quad (36)$$

the law of the empirical measure $\hat{\mu}_N$ under $\mathbb{P}_N^{V, 2P/N}$ satisfies a large deviation principle in the scale N and with good rate function $I_P^V(\mu) = f_P^V(\mu) - \inf f_P^V$ where

$$f_P^V(\mu) = \lim_{\delta \rightarrow 0} \liminf_M \inf_{\substack{\nu_{P/M}, \dots, \nu_{P s.t.} \\ \frac{1}{M} \sum_i \nu_{iP/M} \in B_\mu(\delta)}} \left\{ \frac{1}{M} \sum_{i=1}^M (T_{iP/M}(\nu_{iP/M}) + \int V d\nu_{iP/M}) \right\}. \quad (37)$$

Remark 3.5. *Varadhan's lemma gives the result for bounded continuous function V . However, we can approximate V by $V(x)(1 + \varepsilon x^2)^{-1}$ with overwhelming probability thanks to Lemma 2.2, which allows to conclude for any potential V satisfying (36)*

We shall use this relation to give a better description of the rate function T_s . In fact we first consider the free energy

$$F_{\mathbb{T}}^{V,P} = \lim_{N \rightarrow \infty} \frac{1}{N} \ln Z_{N,\mathbb{T}}^{V,P}, \quad F_C^{V,P} = \lim_{N \rightarrow \infty} \frac{1}{N} \ln Z_{N,C}^{V,P} = \inf f_P^V$$

Recall from Lemma 3.2 that $P \mapsto \inf f_P^V = F_C^{V,P}$ is Lipschitz on \mathbb{R}^{+*} and hence almost surely differentiable. Similarly $P \mapsto \mu_P^V$ is almost surely differentiable in the sense that for f with finite 1/2-norm, $\int f d\mu_P^V$ is differentiable. Then, we claim

Lemma 3.6. *For any continuous function V satisfying (36),*

- $P \mapsto F_C^{V,P} = \inf f_P^V$ is continuously differentiable on \mathbb{R}^{+*} . Moreover, for any $P > 0$

$$F_{\mathbb{T}}^{V,P} = \partial_P(PF_C^{V,P})$$

- Moreover, for almost all P there exists a unique minimizer ν_P^V of $\mu \mapsto T_P(\mu) + \int V d\mu(x)$ and it is given by

$$\nu_P^V = \partial_P(P\mu_P^V). \quad (38)$$

- For any probability measure μ ,

$$T_P(\mu) = - \inf_{V \in \mathcal{C}_0^q} \left\{ \int_{\mathbb{R}} V d\mu + F_{\mathbb{T}}^{V,P} \right\}. \quad (39)$$

Proof. First notice that for any probability measure μ , Lemma 3.3 implies

$$\begin{aligned} f_P^V(\mu) &= I_P(\mu) + \int_{\mathbb{R}} V d\mu \geq \liminf_M \frac{1}{M} \sum_{i=1}^M \inf_{\nu} \left\{ T_{iP/M}(\nu) + \int_{\mathbb{R}} V d\nu \right\} \\ &= \int_0^1 \inf_{\nu} \left\{ T_{sP}(\nu) + \int_{\mathbb{R}} V d\nu \right\} ds = \int_0^1 F_{\mathbb{T}}^{V,Ps} ds. \end{aligned}$$

We claim that this lower bound is achieved. For $s \in [0, 1]$, let ν_{sP}^* be a minimizer of $\mu \mapsto T_{sP}(\mu) + \int V d\mu$. By Corollary 2.7 we can choose ν_{sP}^* such that $s \mapsto \nu_{sP}^*$ is continuous. Hence, $\mu_P^* = \int_0^1 \nu_{sP}^* ds$ makes sense and is a probability measure on \mathbb{R} . We

claim it minimizes f_P^V . Indeed, by Lemma 3.3, we have

$$I_P(\mu_P^*) + \int_{\mathbb{R}} V d\mu_P^* = \lim_{\delta \rightarrow 0} \liminf_M \inf_{\frac{1}{M} \sum_{i=1}^M \nu_{iP/M} \in B_{\mu_P^*}(\delta)} \left\{ \frac{1}{M} \sum_{i=1}^M T_{iP/M}(\nu_{iP/M}) + \int_{\mathbb{R}} V d\nu_{iP/M} \right\} \quad (40)$$

$$= \liminf_M \frac{1}{M} \sum_{i=1}^M \left\{ T_{iP/M}(\nu_{iP/M}^*) + \int_{\mathbb{R}} V d\nu_{iP/M}^* \right\} \quad (41)$$

$$= \liminf_M \frac{1}{M} \sum_{i=1}^M \inf_{\nu} \left\{ T_{iP/M}(\nu) + \int_{\mathbb{R}} V d\nu \right\}$$

$$= \int_0^1 \inf_{\nu} \left\{ T_{sP}(\nu) + \int_{\mathbb{R}} V d\nu \right\} ds.$$

Hence μ_P^* achieves the minimal value of f_P^V and therefore

$$-F_C^{V,P} = \inf f_P^V = I_P(\mu_P^V) + \int_{\mathbb{R}} V d\mu_P^V = - \int_0^1 F_{\mathbb{T}}^{V,P,s} ds.$$

By a change of variable we deduce

$$PF_C^{V,P} = \int_0^P F_{\mathbb{T}}^{V,s} ds.$$

Moreover, $P \mapsto F_C^{V,P}$ is convex and hence almost surely differentiable. As a consequence, for almost all $P > 0$,

$$F_{\mathbb{T}}^{V,P} = \partial_P(PF_C^{V,P}).$$

As $F_{\mathbb{T}}^{V,\cdot}$ is convex, this defines $F_{\mathbb{T}}^{V,\cdot}$ everywhere. Moreover we have seen that I_P^V achieves its minimal value at a unique probability measure μ_P^V and $\mu_P^V = \int_0^1 \nu_{sP}^V ds$ for any continuous minimizing path ν_P^V . This implies that ν_P^V is unique and given by $\partial_P(P\mu_P^V)$. The last point is a direct consequence of [2, Theorem 4.5.10] since T_P^V is convex for all bounded continuous function V . □

By Lemma 3.2, ν_P^V is a probability measure which satisfies almost surely

$$d\nu_P^V(x) = (C_P^V + 2P \int \ln|x-y| d\nu_P^V(y)) d\mu_P^V(x)$$

with C_P^V a constant such that

$$C_P^V + 2P \int \ln|x-y| d\nu_P^V(y) d\mu_P^V(x) = 1$$

Furthermore we must have $C_P^V + 2P \int \ln|x-y| d\nu_P^V(y) \geq 0$ for all x .

4. LARGE DEVIATIONS FOR TODA GIBBS MEASURE WITH GENERAL POTENTIALS

We now consider the measures $\mathbb{T}_N^{V,P}$ given by (4), with polynomial of even degree potential $V : x \in \mathbb{R} \mapsto ax^{2k} + W(x)$, $k \geq 2$, with $W(x)/x^{2k}$ going to zero at infinity. We show that under these laws, the empirical measures $(\hat{\mu}_{L_N})_{N \geq 1}$ still fulfills a large deviations principle, by extending the subadditivity argument previously used. We then identify the

rate function as before. By Varadhan's Lemma, it is enough to consider the case where $W = 0$.

4.1. Exponential tightness. We use the notations of equation (4). In this section we prove that if $V(x) = ax^{2k}$ with $k \geq 2$ and $a > 0$, then the law of the empirical measure of the eigenvalues is exponentially tight under $\mathbb{T}_N^{V,P}$. More precisely, we let $\mathcal{K}_L = \{\mu \in \mathcal{P}(\mathbb{R}) \mid \int_{\mathbb{R}} V d\mu \leq L\}$ which is a compact of $\mathcal{P}(\mathbb{R})$. Then we shall prove

Lemma 4.1. *There exists a finite constant c_V such that*

$$\mathbb{T}_N^{V,P}(\mathcal{K}_L^c) \leq e^{-(M-c_V)N}.$$

Proof. We first bound from below the free energy by Jensen's inequality

$$Z_{N,\mathbb{T}}^{V,P} = \int_{\mathbb{R}^{2N}} e^{-N \int_{\mathbb{R}} V d\hat{\mu}_N} d\mathbb{T}_N^P \geq \exp\{-N \int_{\mathbb{R}^{2N}} V d\hat{\mu}_N d\mathbb{T}_N^P\} \geq e^{-c_V N}. \quad (42)$$

From here we deduce exponential tightness for $(\hat{\mu}_N)_N$ under $\mathbb{T}_N^{V,P}$: for $M > 0$,

$$\begin{aligned} \mathbb{T}_N^{V,P} \left(\int_{\mathbb{R}} V d\hat{\mu}_N \geq M \right) &= \frac{1}{Z_{N,\mathbb{T}}^{V,P}} \int_{\mathbb{R}^{2N}} \mathbf{1}_{\{\int_{\mathbb{R}} V d\hat{\mu}_N \geq M\}} e^{-N \int_{\mathbb{R}} V d\hat{\mu}_N} d\mathbb{T}_N^P \\ &\leq e^{N(c_V - M)}. \end{aligned} \quad (43)$$

□

For later purpose we prove the following result showing that the off diagonal terms do not become too small :

Lemma 4.2. *For any $P > 0$*

$$\limsup_L \limsup_N \frac{1}{N} \ln \mathbb{T}_N^{V,P} \left(\frac{1}{N} \sum_{i=1}^N \ln b_i \leq -L \right) = -\infty.$$

Proof. Since V is bounded from below and we have bounded from below the partition function (42), it is enough to prove this estimate when $V = 0$. But, in this case the entries are independent and so we only need to prove it for independent chi distributed variables. But then, for any $\delta > 0$

$$\mathbb{T}_{N,1}^P \left(\frac{1}{N} \sum_{i=1}^N \ln b_i \leq -L \right) \leq e^{-\delta LN} \frac{Z_{N,\mathbb{T}}^{P-\delta/2}}{Z_{N,\mathbb{T}}^P} = e^{-\delta LN} \left(\frac{\Gamma(P - \delta/2)}{2^{\delta/2} \Gamma(P)} \right)^N$$

from which the result follows by taking $\delta = P/2$ and $P > 0$.

□

4.2. Weak LDP. In this section, we prove that $\hat{\mu}_N$ satisfies a weak large deviation principle, namely Lemma 2.3 in this more general setup, following again a subadditivity argument. We will restrict ourselves to the case where $V(x) = ax^{2k}$, $a > 0$. We first show that the large deviations principle is the same if we remove the entries in the corners $(N, 1)$ and $(1, N)$ in the Toda matrix. Namely, let \tilde{L}_N be the tridiagonal matrix with

entries equal to those of L_N except for the entries $(1, N)$ and $(N, 1)$ which vanish and consider the following modification of $\mathbb{T}_N^{V,P}$ given by

$$d\tilde{\mathbb{T}}_N^{V,P} = \frac{1}{\tilde{Z}_N^{V,P}} e^{-\text{Tr}V(\tilde{L}_N)} d\mathbb{T}_N^P.$$

Lemma 4.3. *Let μ be a probability measure. Then*

$$\lim_{\delta \rightarrow 0} \liminf_{N \rightarrow \infty} \frac{1}{N} \ln \int 1_{d(\hat{\mu}_{L_N}, \mu) < \delta} e^{-\text{Tr}V(L_N)} d\mathbb{T}_N^P = \lim_{\delta \rightarrow 0} \liminf_{N \rightarrow \infty} \frac{1}{N} \ln \int 1_{d(\hat{\mu}_{\tilde{L}_N}, \mu) < \delta} e^{-\text{Tr}V(\tilde{L}_N)} d\tilde{\mathbb{T}}_N^P$$

Moreover,

$$\liminf_{N \rightarrow \infty} \frac{1}{N} \ln \int e^{-\text{Tr}V(L_N)} d\mathbb{T}_N^P = \liminf_{N \rightarrow \infty} \frac{1}{N} \ln \int e^{-\text{Tr}V(\tilde{L}_N)} d\tilde{\mathbb{T}}_N^P.$$

The same results hold if we replace all the *liminf* by *limsup*.

Proof. First notice that $V(L_N) - V(\tilde{L}_N)$ is an homogeneous polynomial of degree $2k$ in L_N and $\Delta L_N = L_N - \tilde{L}_N$, with degree at least one in the latter. Therefore, there exists a finite constant C_k such that on $B_N^{K,M} := \{b_N \leq K\} \cap \{\frac{1}{N} \text{Tr}(L_N)^{2k} \leq M\}$ (or $\tilde{B}_N^{M,K} := \{b_N \leq K\} \cap \{\frac{1}{N} \text{Tr}(\tilde{L}_N)^{2k} \leq M\}$), Hölder's inequality implies

$$\begin{aligned} \left| \frac{1}{N} \text{Tr}(V(L_N) - V(\tilde{L}_N)) \right| &\leq C_k \left(\frac{1}{N} \text{Tr}((\Delta L_N)^{2k}) + \left(\frac{1}{N} \text{Tr}((\Delta L_N)^{2k}) \right)^{1/2k} \left(\frac{1}{N} \text{Tr}(L_N^{2k}) \right)^{\frac{2k-1}{2k}} \right) \\ &\leq C(M, K) N^{-\frac{1}{2k}} \end{aligned}$$

where $C(M, K) \leq 2C_k(K^{2k} + M^{\frac{2k-1}{2k}})$ is a finite constant depending only on M, K, k . Note above that $\text{Tr}(L_N^{2k})$ can be replaced by $\text{Tr}(\tilde{L}_N^{2k})$. Moreover, by (13), $d(\hat{\mu}_{L_N}, \hat{\mu}_{\tilde{L}_N}) \leq 2/N$. Hence for a given probability measure μ

$$\begin{aligned} \int 1_{d(\hat{\mu}_{L_N}, \mu) < \delta} e^{-\text{Tr}V(L_N)} d\mathbb{T}_N^P &\geq e^{-C(M,K)N^{\frac{2k-1}{2k}}} \int 1_{\tilde{B}_N^{M,K} \cap \{d(\hat{\mu}_{\tilde{L}_N}, \mu) < \delta - \frac{2}{N}\}} e^{-\text{Tr}V(\tilde{L}_N)} d\tilde{\mathbb{T}}_N^P \\ &\geq C' e^{-C(M,K)N^{\frac{2k-1}{2k}}} \int 1_{\{\text{Tr}V(\tilde{L}_N) \leq NM\} \cap \{d(\hat{\mu}_{\tilde{L}_N}, \mu) < \delta - \frac{2}{N}\}} e^{-\text{Tr}V(\tilde{L}_N)} d\tilde{\mathbb{T}}_N^P \\ &\geq C'' e^{-C(M,K)N^{\frac{2k-1}{2k}}} \int 1_{\{d(\hat{\mu}_{\tilde{L}_N}, \mu) < \delta - \frac{2}{N}\}} e^{-\text{Tr}V(\tilde{L}_N)} d\tilde{\mathbb{T}}_N^P \end{aligned}$$

where in the second line we integrated over $b_N \leq K$ and in the last line we used that under $\tilde{\mathbb{T}}_N^{P,V}$, for M large enough, $\text{Tr}V(\tilde{L}_N) \leq NM$ with overwhelming probability following the same arguments than in the proof of Lemma 4.1. We deduce that

$$\lim_{\delta \rightarrow 0} \liminf_{N \rightarrow \infty} \frac{1}{N} \ln \int 1_{d(\hat{\mu}_{L_N}, \mu) < \delta} e^{-\text{Tr}V(L_N)} d\mathbb{T}_N^P \geq \lim_{\delta \rightarrow 0} \liminf_{N \rightarrow \infty} \frac{1}{N} \ln \int 1_{d(\hat{\mu}_{\tilde{L}_N}, \mu) < \delta} e^{-\text{Tr}V(\tilde{L}_N)} d\tilde{\mathbb{T}}_N^P.$$

To prove the converse inequality, we notice that there exists one b_i bounded by K with probability greater than $1 - e^{-a(K)N}$ under \mathbb{T}_N^P , with $a(K) = -\ln P(b \leq K) > 0$ which goes to $+$ infinity when K does. By symmetry with respect to the order of the indices,

we may assume it is b_N . Therefore, using that $V \geq 0$ and Lemma 4.1,

$$\begin{aligned} \int \mathbf{1}_{d(\hat{\mu}_{L_N}, \mu) < \delta} e^{-\text{Tr}V(L_N)} d\mathbb{T}_N^P &\leq e^{-Na(K)} + N \int \mathbf{1}_{\{b_N \leq K\} \cap \{d(\hat{\mu}_{L_N}, \mu) < \delta\}} e^{-\text{Tr}V(L_N)} d\mathbb{T}_N^P \\ &\leq Ne^{-Na(K)} + Ne^{-N(M-cv)} + Ne^{C(M,K)N^{\frac{2k-1}{2k}}} \int \mathbf{1}_{B_N^{M,K} \cap \{d(\hat{\mu}_{L_N}, \mu) < \delta + \frac{2}{N}\}} e^{-\text{Tr}V(\tilde{L}_N)} d\mathbb{T}_N^P \\ &\leq e^{-Na(K)} + 2e^{-N(M-cv)} + Ne^{C(M,K)N^{\frac{2k-1}{2k}}} \int \mathbf{1}_{\{d(\hat{\mu}_{L_N}, \mu) < \delta + \frac{2}{N}\}} e^{-\text{Tr}V(\tilde{L}_N)} d\mathbb{T}_N^P \end{aligned}$$

which gives the converse bound, letting N going to infinity, provided K and M are large enough. The same arguments also hold when there is no indicator function, providing the same estimates for the free energy. \square

Lemma 4.4. *Let $V(x) = ax^{2k}$ and $P > 0$. For any μ in $\mathcal{P}(\mathbb{R})$, there exists a limit*

$$\lim_{\delta \rightarrow 0} \liminf_N \frac{1}{N} \ln \mathbb{T}_N^{V,P}(\hat{\mu}_{L_N} \in B_\mu(\delta)) = \lim_{\delta \rightarrow 0} \limsup_N \frac{1}{N} \ln \mathbb{T}_N^{V,P}(\hat{\mu}_{L_N} \in B_\mu(\delta)). \quad (44)$$

We denote this limit by $-T_P^V(\mu)$.

Proof. We use the notations of Lemma 2.3. Let $q \geq 1$ be fixed. For $N \geq 1$ we write $N = k_N q + r_N$, $0 \leq r_N \leq q - 1$, and define L_N^q by removing the off diagonal terms $b_{\ell q} = L_{\ell q, \ell q+1}, L_{\ell q+1, \ell q}$, $1 \leq \ell \leq k_N$ as well as the entries $L_{1,N}, L_{N,1}$ of L_N . We set $R_N^q = L_N - L_N^q$. Let $Z_N^V = Z_{N,\mathbb{T}}^{V,P}$ denote in short the partition function for the Toda Gibbs measure with potential V and set

$$Z_{N,q}^V = \mathbb{E}_{\mathbb{T}_N^P} \left[e^{-\text{Tr}V(L_N^q)} \right] = \int e^{-\text{Tr}V(L_N^q)} d\mathbb{T}_N^P.$$

We first show that there is some constant C_k (independent of N) such that for all $N \geq 1$,

$$\frac{1}{N} \ln \frac{Z_{N,q}^V}{Z_N^V} \geq \frac{C_k}{q^{1/2k}}. \quad (45)$$

By Jensen's inequality we have

$$\frac{1}{N} \ln \frac{Z_{N,q}^V}{Z_N^V} = \frac{1}{N} \ln \mathbb{E}_{\mathbb{T}_N^{V,P}} \left[e^{\text{Tr}(V(L_N) - V(L_N^q))} \right] \geq \frac{1}{N} \mathbb{E}_{\mathbb{T}_N^{V,P}} \left[\text{Tr}(V(L_N) - V(L_N^q)) \right]. \quad (46)$$

To bound the right hand side we first notice that $V(L_N) - V(L_N^q)$ is an homogeneous polynomial of degree $2k$ in L_N and $L_N - L_N^q$, with degree at least one in the later. Therefore, Hölder's inequality implies that there exists a finite constant C depending only on k such that

$$\begin{aligned} \left| \frac{1}{N} \mathbb{E}_{\mathbb{T}_N^{V,P}} \left[\text{Tr}(V(L_N) - V(L_N^q)) \right] \right| &\leq C \mathbb{E}_{\mathbb{T}_N^{V,P}} \left[\frac{1}{N} \text{Tr} \left((L_N - L_N^q)^{2k} \right) \right] \\ &+ C \mathbb{E}_{\mathbb{T}_N^{V,P}} \left[\frac{1}{N} \text{Tr} \left((L_N - L_N^q)^{2k} \right) \right]^{1/2k} \mathbb{E}_{\mathbb{T}_N^{V,P}} \left[\frac{1}{N} \text{Tr}(L_N^{2k}) \right]^{\frac{2k-1}{2k}} \end{aligned}$$

Now, $R_N^q = L_N - L_N^q$ has non zero entries only at the sites $(i, i+1)$ and $(i+1, i)$, $i \in J = \{\ell q, 1 \leq \ell \leq k_N\}$, as well as $(N, 1)$ and $(1, N)$. We take $2k \leq q$ so that there exists a finite constant C_k which only depends on k such that

$$\mathrm{Tr} \left((R_N^q)^{2k} \right) \leq C_k \sum_{i \in J} (L_{i,i+1})^{2k} + C_k (L_{N,1})^{2k}.$$

We next remark that since the application $A \rightarrow A_{ii} = \langle e_i, A e_i \rangle$ is non negative, we can apply Hölder's inequality so that

$$L_{i,i+1}^{2k} \leq ((L_N^2)_{i,i})^k \leq (L_N^{2k})_{i,i}.$$

Because L_N has periodic boundary conditions, the distribution of the entries of L_N are invariant under the shift $\theta : i \rightarrow i+1$, so that under $\mathbb{T}_N^{V,P}$, $L_{i,i+1}$ has the same law than $L_{i+1,i+2}$. Moreover, $\tilde{b}_i = L_{i+1,i+q}^q = L_{i+q,i+1}^q$ is independent of L

$$\mathbb{E}_{\mathbb{T}_N^{V,P}} \left[\frac{1}{N} \mathrm{Tr} \left((L_N - L_N^q)^{2k} \right) \right] \leq \frac{1}{N} C_k \sum_{i \in J} \mathbb{E}_{\mathbb{T}_N^{V,P}} \left[(L)_{i,i}^{2k} \right] = C_k \frac{k_N}{N} \mathbb{E}_{\mathbb{T}_N^{V,P}} \left[\frac{1}{N} \mathrm{Tr} (L_N)^{2k} \right].$$

But (43) implies that $\mathbb{E}_{\mathbb{T}_N^{V,P}} \left[\frac{1}{N} \mathrm{Tr} (L_N^{2k}) \right]$ is bounded by some finite constant independent of N . We therefore deduce (45) from (46).

We next prove the subadditivity property. Let $\delta > 0$ and $L > 0$ be given. Let $\mathcal{K}_L = \{\hat{\mu}_{L_N}(V) \leq L\}$. As in equation (16), we have for q big enough,

$$\mathbb{T}_N^{V,P} (\{\hat{\mu}_{L_N} \in B_\mu(\delta)\} \cap \mathcal{K}_L) \geq \frac{Z_{N,q}^V}{Z_N^V} \frac{1}{Z_{N,q}^V} \int_{\mathcal{K}_L \cap K_A} \mathbf{1}_{\hat{\mu}_{L_N^q} \in B_\mu(\delta-4/q)} e^{-\mathrm{Tr}(V(L_N))} d\mathbb{T}_N^P, \quad (47)$$

where we set $K_A = K_{A,N} = \cap_{i \in J} \{b_i^{2k} \leq A\} \cap \{b_N \leq A\}$. As before, noticing that $V(\tilde{L}_N) - V(L_N^q)$ is a polynomial in L_N^q and $L_N - L_N^q$, we find a finite constant C such that, on $\mathcal{K}_L \cap K_A$,

$$\frac{1}{N} |\mathrm{Tr}(V(L_N) - V(L_N^q))| \leq C \left(\left(\frac{k_N}{N} C_k A \right)^{1/2k} L^{\frac{2k-1}{2k}} + \frac{k_N}{N} C_k A \right).$$

Therefore if we set $\mathcal{K}_L^q = \{\hat{\mu}_{L_N^q}(V) \leq L\}$, we deduce that $K_A \cap \mathcal{K}_L$ contains $K_A \cap \mathcal{K}_{L-\varepsilon(q)}^q$ for some $\varepsilon(q)$ going to zero as q goes to infinity. We deduce from (45) and (47) that

$$\mathbb{T}_N^{V,P} (\{\hat{\mu}_{L_N} \in B_\mu(\delta)\} \cap \mathcal{K}_L) \geq \frac{e^{-\varepsilon(q)N}}{Z_{N,q}^V} \int_{K_A \cap \mathcal{K}_{L-\varepsilon(q)}^q} \mathbf{1}_{\hat{\mu}_{L_N^q} \in B_\mu(\delta-4/q)} e^{-\mathrm{Tr}(V(L_N^q))} d\mathbb{T}_N^P, \quad (48)$$

Since L_N^q is independent of the entries $b_i, i \in J$ and therefore of K_A , we see that we can integrate the indicator function of K_A yielding a contribution $C_A^{k_N}$ for some positive constant C_A depending only on A . We observe as well that L_N^q is a block diagonal matrix $\mathrm{diag}(L_q^1, \dots, L_q^{k_N}, B)$ where L_q^i are independent $\tilde{\mathbb{T}}_q^P$ independent from B with law $\tilde{\mathbb{T}}_{r_N}^P$. Finally, we notice that $\mathcal{K}_{L-\varepsilon(q)}^q$ contains $\cap_{i \in J} \{\frac{1}{q} \mathrm{Tr}((L_q^i)^{2k}) \leq L - \varepsilon(q)\} \cap \{\frac{1}{N-k_N q} \mathrm{Tr}(B^{2k}) \leq L - \varepsilon(q)\}$ since the trace of $(L_N^q)^{2k}$ is a linear combination of the latter traces. Thus by independence of the matrices $L_q^1, \dots, L_q^{k_N}$ under $\frac{1}{Z_{N,q}^V} e^{-\mathrm{Tr}V(L_N^q)} d\mathbb{T}_N^P$ and convexity of balls, we deduce by taking the logarithm that if we set $u_N(\delta, L) = -\ln \mathbb{T}_N^{V,P}(\{\hat{\mu}_{M_N} \in$

$B_\mu(\delta)\} \cap \mathcal{K}_L)$ and $v_N(\delta, L) = -\ln \tilde{\mathbb{T}}_N^{V,P}(\{\hat{\mu}_{\tilde{L}_N} \in B_\mu(\delta)\} \cap \{\text{Tr}(\tilde{L}_N)^{2k} \leq LN\})$, then we have

$$u_N(\delta + 4/q, L + \varepsilon(q)) \leq N\varepsilon(q) + k_N v_q(\delta, L) + v_{r_N}(\delta, L). \quad (49)$$

We conclude as in Lemma 2.3 that

$$\limsup_N \frac{u_N(\delta + 4/q, L + \varepsilon(q))}{N} \leq \frac{v_q(\delta, L)}{q} + \varepsilon(q) \quad (50)$$

We then notice that for all N, δ , $u_N(\delta, L) \geq u_N(\delta, \infty)$ and $v_N(\delta, L) \leq v_N(\delta, \infty) + \ln 2$ for L large enough by Lemma 2.2 (for \tilde{L}_N). If therefore we choose a subsequence q going to infinity along which the liminf is taken, we deduce by Lemma 4.3 that

$$\limsup_N \frac{u_N(2\delta, \infty)}{N} \leq \liminf_{q \rightarrow \infty} \frac{v_q(\delta, \infty)}{q} = \liminf_{q \rightarrow \infty} \frac{u_q(\delta, \infty)}{q}$$

If there is no such subsequence then both sides go to infinity and there is nothing to say. Otherwise we conclude as in Lemma 2.3. \square

4.3. Convergence of the free energy and large deviation principle. In the case still where $V(x) = ax^{2k}$, $a > 0$, the previous two sections showed that a large deviation principle holds for the empirical measure of the eigenvalues of L_N under $\mathbb{T}_N^{V,P}$ with good rate function

$$T_P^V(\mu) = -\inf_{W \in C_b^o} \left\{ \int W d\mu + F_{\mathbb{T}}^{V+W,P} - F_{\mathbb{T}}^{V,P} \right\} \quad (51)$$

where

$$F_{\mathbb{T}}^{V,P} = \lim_{N \rightarrow \infty} \frac{1}{N} \ln \int e^{-\text{Tr}V(L_N)} d\mathbb{T}_N^P.$$

To identify T_P^V and its minimal value our goal is to show that

Lemma 4.5. *For $V(x) = x^{2k} + W(x)$ with $W \in C_0^b(\mathbb{R})$, for all $P > 0$*

$$\int_0^1 F_{\mathbb{T}}^{V,sP} ds = F_C^{V,P}. \quad (52)$$

As a consequence, the unique minimizer of T_P^V is given by $\nu_P^V = \partial_P(P\mu_P^V)$ with μ_P^V the equilibrium measure for the Coulomb gas.

Proof. We first prove (52). Clearly, for all bounded continuous functions W, W' , uniformly in P ,

$$|F_{\mathbb{T}}^{V,x^{2k}+W} - F_{\mathbb{T}}^{V,x^{2k}+W'}| \leq \|W - W'\|_\infty \text{ and } |F_C^{V,x^{2k}+W} - F_C^{V,x^{2k}+W'}| \leq \|W - W'\|_\infty.$$

Therefore it is enough to prove (52) for $W \in C_b^1(\mathbb{R})$. We prove that for $W \in C_b^1(\mathbb{R})$,

$$F_{\mathbb{T}}^{V,P} = \partial_P(PF_C^{V,P}). \quad (53)$$

Let us consider the tridiagonal matrix C_P^N of the Coulomb model with distribution $\mathbb{P}_N^{\frac{2P}{N}}$ we decompose, for $\varepsilon > 0$ this matrix as

$$C_P^N = \begin{pmatrix} M_P^{[\varepsilon N]} & R_N \\ R_N^T & C_{(1-\varepsilon)P}^{(1-\varepsilon)N} \end{pmatrix}$$

where $M_P^{[\varepsilon N]}$ is a $[N\varepsilon] \times [N\varepsilon]$ tri-diagonal symmetric matrix with standard independent Gaussian on the diagonal and chi-square distributed variables above the diagonal with parameters $2\frac{i}{N}P, N - [\varepsilon N] \leq i \leq N - 1$, $C_{(1-\varepsilon)P}^{(1-\varepsilon)N}$ is a $N - [\varepsilon N]$ square tridiagonal Coulomb matrix with parameter $2P(1 - [\varepsilon N]/N - 1/N)$ and R_N has only one non-zero entry r at position $([\varepsilon N], [\varepsilon N] + 1)$. Our first goal is to show that

$$\lim_{N \rightarrow \infty} \frac{1}{\varepsilon N} \ln \mathbb{E}[e^{-\text{Tr}V(M_P^{[\varepsilon N]})}] = \frac{1}{\varepsilon} (F_C^{V,P} - F_C^{V,P-\varepsilon}) + F_C^{V,P-\varepsilon}. \quad (54)$$

We will then complete the argument by showing that

$$\lim_{\varepsilon \downarrow 0} \lim_{N \rightarrow \infty} \frac{1}{\varepsilon N} \ln \mathbb{E}[e^{-\text{Tr}V(M_P^{[\varepsilon N]})}] = F_{\mathbb{T}}^{V,P} \quad (55)$$

We next turn to the proof of (54). Let us denote

$$\tilde{C}_P^N = \begin{pmatrix} M_P^{[\varepsilon N]} & 0 \\ 0 & C_{(1-\varepsilon)P}^{(1-\varepsilon)N} \end{pmatrix}.$$

We next show that

$$\text{Tr}((C_P^N)^{2k}) \geq \text{Tr}((\tilde{C}_P^N)^{2k}). \quad (56)$$

Indeed, by Klein's lemma [1], $B \mapsto \text{Tr}(B)^{2k}$ is convex on the set of symmetric matrices. Moreover $\nabla \text{Tr}(B)^{2k} = (2k(B)^{2k-1})_{ij}$. As a consequence, for any symmetric matrices A, B

$$\text{Tr}((A+B)^{2k}) - \text{Tr}(B)^{2k} \geq \text{Tr}(2k(A)^{2k-1}B).$$

We apply the above inequality with $A = \tilde{C}_P^N$ and $B = C_P^N - \tilde{C}_P^N$ and notice that the entry $[\varepsilon N], [\varepsilon N] + 1$ of any polynomial in \tilde{C}_P^N vanishes so that $\text{Tr}((\tilde{C}_P^N)^{2k-1}(C_P^N - \tilde{C}_P^N)) = 0$. Moreover, if W is C_b^1 ,

$$|\text{Tr}(W(C_P^N)) - \text{Tr}(W(\tilde{C}_P^N))| \leq \int_0^1 |\text{Tr}(W'(\alpha C_P^N + (1-\alpha)\tilde{C}_P^N)(C_P^N - \tilde{C}_P^N))| d\alpha \leq \|W'\|_{\infty} |r|$$

Consequently, using the independence of r and \tilde{C}_P^N and the fact that $C_W = \mathbb{E}[e^{+\|W'\|_{\infty}|r|}]$ is finite since r has sub-Gaussian distribution. We deduce from (56) that

$$\mathbb{E}[e^{-\text{Tr}(V(C_P^N))}] \leq \mathbb{E}[e^{-\text{Tr}(V(\tilde{C}_P^N)) + \|W'\|_{\infty}|r|}] \leq C_W \mathbb{E}[e^{-\text{Tr}(V(\tilde{C}_P^N))}]. \quad (57)$$

As a consequence

$$\mathbb{E}[e^{-\text{Tr}(V(C_P^N))}] \leq C_W \mathbb{E}[e^{-\text{Tr}V(M_P^{[\varepsilon N]})}] \mathbb{E}[e^{-\text{Tr}(V(C_{(1-\varepsilon)P}^{(1-\varepsilon)N}))}]$$

which gives the desired lower bound:

$$\liminf_{N \rightarrow \infty} \frac{1}{N} \ln \mathbb{E}[e^{-\text{Tr}V(M_P^{[\varepsilon N]})}] \geq F_C^{P,V} - (1-\varepsilon)F_C^{P(1-\varepsilon),V} \quad (58)$$

To get the complementary lower bound we restrict ourselves to

$$\{|r| \leq \frac{1}{N}\} \cap \{\|\tilde{C}_P^N\|_{\infty} \leq (NM)^{1/2k}\}$$

On this set $\text{Tr}(V(C_P^N)) - \text{Tr}(V(\tilde{C}_P^N))$ goes to zero uniformly for all M . On the other hand the probability of the set $\{|r| \leq \frac{1}{N}\}$ is of order $1/N$. Again by independence we deduce

that there exists a function $o(N)$ such that $o(N)/N$ goes to zero (and eventually changing from line to line) such that

$$\begin{aligned} \mathbb{E}[e^{-\text{Tr}(V(C_P^N))}] &\geq e^{o(N)} \mathbb{E}[1_{\{\|\tilde{C}_P^N\|_\infty \leq (NM)^{1/2k}\} \cap \{|r| \leq \frac{1}{N}\}} e^{-\text{Tr}(V(\tilde{C}_P^N))}] \\ &\geq e^{o(N)} \left(\mathbb{E}[e^{-\text{Tr}(V(\tilde{C}_P^N))}] - \mathbb{E}[1_{\{\|\tilde{C}_P^N\|_\infty \geq (NM)^{1/2k}\}} e^{-\text{Tr}(V(\tilde{C}_P^N))}] \right) \\ &\geq e^{o(N)} \left(\mathbb{E}[e^{-\text{Tr}(V(\tilde{C}_P^N))}] - \mathbb{E}[1_{\{\text{Tr}((\tilde{C}_P^N)^{2k}) \geq MN\}} e^{-\text{Tr}(V(\tilde{C}_P^N))}] \right). \end{aligned} \quad (59)$$

But we can prove exactly as in the proof of Lemma 4.1 that for M large enough

$$\limsup_{N \rightarrow \infty} \frac{\mathbb{E}[1_{\{\text{Tr}((\tilde{C}_P^N)^{2k}) \geq MN\}} e^{-\text{Tr}(V(\tilde{C}_P^N))}]}{\mathbb{E}[e^{-\text{Tr}(V(\tilde{C}_P^N))}]} \leq \frac{1}{2},$$

yielding the desired lower bound and therefore (54).

To prove (55), we proceed by approximation. We notice that if we denote by f_T^ε the density of the distribution of $M_P^{[\varepsilon N]}$ with respect distribution of a Toda matrix $\tilde{L}_{[\varepsilon N]}$ with parameter P to which we removed the extreme entries at $(1, [\varepsilon N])$ and $([\varepsilon N], 1)$, then we get

$$f_T^\varepsilon = \prod_{i=1}^{N\varepsilon} b_i^{-2P(\frac{i}{N})}$$

Therefore

$$\begin{aligned} \mathbb{E}[e^{-\text{Tr}V(M_P^{[\varepsilon N]})}] &\geq e^{-\varepsilon^2 NM} \mathbb{E}[e^{-\text{Tr}V(\tilde{L}_{[\varepsilon N]})} 1_{-2P \sum_{i=1}^{\varepsilon N} \frac{i}{N} \ln b_i \geq -\varepsilon^2 NM}] \\ &= e^{-\varepsilon^2 NM} \mathbb{E}[e^{-\text{Tr}V(\tilde{L}_{[\varepsilon N]})}] (1 - \tilde{\mathbb{T}}_{[\varepsilon N]}^{V,P}(-2P \sum_{i=1}^{\varepsilon N} \frac{i}{N} \ln b_i \leq -\varepsilon^2 NM)) \end{aligned}$$

On the other hand

$$\{2P \sum_{i=1}^{\varepsilon N} \frac{i}{N} \ln b_i \geq \varepsilon^2 NM\} \subset \{P \frac{1}{N\varepsilon} \sum_{i=1}^{\varepsilon N} b_i^2 \geq M\} \subset \{\frac{1}{N\varepsilon} \text{Tr}((\tilde{L}_{[\varepsilon N]})^2) \geq M/P\}$$

has exponentially small probability under $\tilde{\mathbb{T}}_{[\varepsilon N]}^{V,P}$ for M large enough. This shows that there exists a finite constant M such that

$$\liminf_{N \rightarrow \infty} \frac{1}{N\varepsilon} \ln \mathbb{E}[e^{-\text{Tr}V(M_P^{[\varepsilon N]})}] \geq F_{\mathbb{T}}^{V,P} + M\varepsilon$$

Similarly, we can see that the density $\tilde{f}_T^\varepsilon = \prod_{i=1}^{N\varepsilon} b_i^{2P(\frac{i}{N}-\varepsilon)}$ of the law a Toda matrix $\tilde{L}_{[\varepsilon N]}$ with respect to $M_P^{[\varepsilon N]}$ is bounded below by $-\varepsilon^2 NM$ on $\{\sum_{i=1}^{\varepsilon N} (\varepsilon - \frac{i}{N}) \ln b_i \leq \varepsilon^2 NM\}$ so that we get similarly a finite constant M' such that

$$\limsup_{N \rightarrow \infty} \frac{1}{N\varepsilon} \ln \mathbb{E}[e^{-\text{Tr}V(M_P^{[\varepsilon N]})}] \leq F_{\mathbb{T}}^{V,P(1-\varepsilon)} + M'\varepsilon \quad (60)$$

We hence only need to show the continuity of $\varepsilon \rightarrow F_T^{P(1-\varepsilon),R}$. We already proved above that there exists a finite constant M such that for $\varepsilon > 0$

$$F_{\mathbb{T}}^{V,P} \leq F_{\mathbb{T}}^{V,P(1-\varepsilon)} + M\varepsilon$$

We can get the reverse bound by using Lemma 4.2. Indeed, it insures that for all $P > 0$,

$$\begin{aligned} \int e^{-\text{Tr}V(\tilde{L}_N)} d\mathbb{T}_N^P &\geq \int 1_{\sum \ln b_i \geq -LN} e^{-\text{Tr}V(\tilde{L}_N)} d\mathbb{T}_N^P \\ &\geq \frac{Z_{\mathbb{T}}^{P-\delta}}{Z_{\mathbb{T}}^P} e^{-L\delta N} \int 1_{\sum \ln b_i \geq -L\delta N} e^{-\text{Tr}V(\tilde{L}_N)} d\mathbb{T}_N^{P-\delta} \\ &\geq e^{-2L\delta N} \int 1_{\sum \ln b_i \geq -LN} e^{-\text{Tr}V(\tilde{L}_N)} d\mathbb{T}_N^{P-\delta} \end{aligned}$$

from which we deduce by Lemmas 4.2 and 4.3 that there exists a finite constant M such that

$$F_{\mathbb{T}}^{V,P} \geq F_{\mathbb{T}}^{V,P-\delta} + M\delta.$$

Equality (53) follows then from (60) and the continuity of $F_{\mathbb{T}}^V$.

We finally show that (52) implies that T_P^V achieves its minimum value at $\partial_P(P\mu_P^V)$. Indeed, by (51), for any bounded continuous W , any probability measure ν , we have

$$T_P^V(\nu) \geq - \left(\int W d\nu + F_{\mathbb{T}}^{V+W,P} - F_{\mathbb{T}}^{V,P} \right)$$

We integrate this inequality at $\nu = \nu_{sP}$ a measurable probability measure valued process such that $\mu = \int_0^1 \nu_{sP} ds$ to deduce from (52) that

$$\int_0^1 T_P^V(\nu_{sP}) ds \geq - \left(\int W d\mu + F_C^{V+W,P} - F_C^{V,P} \right).$$

We finally optimize over W to conclude that

$$\int_0^1 T_P^V(\nu_{sP}) ds \geq - \inf_W \left(\int W d\mu + F_C^{V+W,P} - F_C^{V,P} \right) = I_P^V(\mu).$$

Since I_P^V vanishes only at μ_P^V we deduce that any measurable minimizing path $(\nu_{sP})_{0 \leq s \leq 1}$ must satisfy $\int_0^1 \nu_{sP} ds = \mu_P^V$. The last issue we have to address is the existence of non-measurable minimizing paths. But we can follow arguments similar to those of Corollary 2.7 to show that the set M_P^V where T_P^V achieves its minimum value is a compact convex subset of $\mathcal{P}(\mathbb{R})$ and is continuous in the sense that for any $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that for all $\delta < \delta_\varepsilon$, any $(t, s) \in \mathbb{R}^{+*}$ such that for $|t - s| \leq \delta$

$$M_s^V \subset (M_t^V)^\varepsilon.$$

Indeed, even if we do not have the coupling of Corollary 2.7 we have seen just above that the density of $\mathbb{T}_N^{V,s}$ with respect to $\mathbb{T}_N^{V,t}$ is bounded by $e^{MN|t-s|}$ with probability greater than $1 - e^{-c(M)N}$ with $c(M)$ going to infinity when M goes to infinity. This implies that

$$- \inf_{((M_t^V)^\varepsilon)^c} T_s^V \leq \max\{M|s-t| - \inf_{((M_t^V)^\varepsilon)^c} T_t^V, -c(M)N\}$$

which implies that for any $\varepsilon > 0$, for M large enough and $|s-t|$ small enough $\inf_{((M_t^V)^\varepsilon)^c} T_s^V > 0$, from which the continuity follows. \square

5. PROOF OF THEOREM 1.1 AND 1.3

Lemma 44 proved a large deviation principle for the potential $V(x) = ax^{2k}$. If now we consider the case where $V(x)/x^{2k}$ goes to $a > 0$ at infinity, we can always write $V(x) = ax^{2k} + W(x)$ where $W(x)/x^{2k}$ goes to zero at infinity. We have seen that under $\mathbb{T}_N^{P,V}$ the probability that $\{\text{Tr}(L_N^{2k}) \leq M\}$ has exponentially large probability. Let for $\varepsilon > 0$, $V_\varepsilon(x) = ax^{2k} + (1 + \varepsilon x^{2k})^{-1}W(x)$. Then, the large deviation principle for the distribution of $\hat{\mu}_{L_N}$ under $\mathbb{T}_N^{V_\varepsilon,P}$ follows from Varadhan's lemma. Moreover, on $\{\text{Tr}(L_N^{2k}) \leq MN\}$, if $|W(x)| \leq \delta x^{2k}$ on $|x| \geq L$,

$$\begin{aligned} \left| \frac{1}{N} \text{Tr} V(L_N) - \frac{1}{N} \text{Tr} V_\varepsilon(L_N) \right| &\leq \frac{\varepsilon L^{2k}}{1 + \varepsilon L^{2k}} \max_{|x| \leq L} W(x) + \delta \varepsilon \frac{1}{N} \text{Tr} \left(\frac{L_N^{4k}}{1 + \varepsilon L_N^{2k}} \right) \\ &\leq \frac{\varepsilon L^{2k}}{1 + \varepsilon L^{2k}} \max_{|x| \leq L} W(x) + M\delta \end{aligned}$$

which is as small as wished if M is fixed, L taken large so that δ is small, provided ε is taken small enough. This shows that we can approximate $\mathbb{T}_N^{V,P}$ by $\mathbb{T}_N^{V_\varepsilon,P}$ in the exponential scale from which the result follows.

The proof of Theorem 1.3 follows the same arguments than those developed in the last section: we approximated the general variance profile by a stepwise constant profile, remove a neglectable number of off diagonal entries and then use the large deviation principle for the Toda matrices. We leave the details to the reader.

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