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# Computational characterization of surface entropies for $\mathbb{Z}^2$ subshifts of finite type

Antonin Callard  

Université Paris-Saclay, ENS Paris-Saclay, Département Informatique, 91190 Gif-sur-Yvette, France

Pascal Vanier   

Normandie Univ, UNICAEN, ENSICAEN, CNRS, GREYC, 14000 Caen, France

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## Abstract

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Subshifts of finite type (SFTs) are colorings of the plane that avoid a finite family of forbidden patterns. In this article, we are interested in the behavior of the growth of the number of valid patterns in SFTs. While entropy  $h$  corresponds to growths that are squared exponential  $2^{hn^2}$ , surface entropy (introduced in Pace's thesis in 2018) corresponds to the eventual linear term in exponential growths. We give here a characterization of the possible surface entropies of SFTs as the  $\Pi_3$  real numbers of  $[0, +\infty]$ .

**2012 ACM Subject Classification** Mathematics of computing  $\rightarrow$  Discrete mathematics; Theory of computation  $\rightarrow$  Models of computation

**Keywords and phrases** surface entropy, arithmetical hierarchy of real numbers, 2D subshifts, symbolic dynamics

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## 1 Introduction

For a finite alphabet of colors, a two dimensional subshift is a set of all colorings of the plane  $\mathbb{Z}^2$  that respect some local constraints. These constraints are usually given as a family of forbidden patterns. The most studied class of subshift are *Subshifts of Finite Type* (SFTs), subshifts that can be defined by a finite family of forbidden patterns. Wang tiles (unit squares with colored edges that may only be placed side by side when the colors on the edges match) are a famous special case of SFTs: the set of all tilings by some tileset is always an SFT.

Subshifts were introduced in order to discretize continuous dynamical systems and are themselves dynamical systems. While it has long been known that most problems concerning Wang tiles (and thus subshifts) are undecidable [2, 5, 6], computability has since seen its role shift from an obstacle to a tool and has played a major role in the study of SFTs and other related classes of subshifts. An aperiodic subshift has for instance been constructed [4] based on Kleene's *fixed-point theorem* [11], a classical theorem of computability theory. Many conjugacy invariants have been characterized thanks to computability or complexity classes. The first such characterization was for *topological entropy*, which measures the exponential growth of the number of valid colorings of finite patterns. For a subshift  $X$ , its (topological) entropy is defined by:

$$h(X) = \lim \frac{\log N_X(n)}{n^2}$$

where  $N_X(n)$  denotes the number of  $n \times n$  patterns appearing in the subshift  $X$ . Having entropy  $h$  corresponds to having  $N_X = 2^{hn^2 + o(n^2)}$ . It turns out that the possible entropies for  $\mathbb{Z}^2$  are exactly the numbers which are upper semi-computable [8]. Another invariant related

to growth, called entropy dimension, was then characterized using the arithmetical hierarchy of real numbers [13]. Many other invariants have since been linked to computability: for instance periodic data [10] is related to computational complexity classes, subactions [7, 1, 3] can be characterized through recursively enumerable forbidden patterns and many more.

We focus here on the notion of *surface entropy*, a notion which was introduced in Dennis Pace's thesis [14] in order to quantify the linear term inside exponential growth functions. Indeed, topological entropy cannot distinguish between the two following behaviors of the complexity function:

$$N_X(n) \approx 2^{hn^2} \quad \text{and} \quad N_X(n) \approx 2^{hn^2+2sn}$$

Surface entropy corresponds roughly to the  $s$  term in the second behavior and will be introduced more formally in Subsection 2.2. In his thesis, Pace realizes  $\Pi_1$  and  $\Sigma_1$  numbers of the arithmetical hierarchy of real numbers and conjectures that surface entropies are exactly the  $\Pi_3$  numbers. This is exactly what we prove here:

► **Theorem 1.** *The class of surface entropies of  $\mathbb{Z}^2$  SFTs is  $[0, +\infty] \cap \Pi_3$ .*

As a corollary of Theorem 1 and using Corollary 4, we obtain the same result for sofic subshifts, which are the letter-by-letter projections of SFTs, and effective subshifts, which are the subshifts that can be defined with a recursively enumerable family of forbidden patterns :

► **Corollary 2.** *The class of surface entropies of  $\mathbb{Z}^2$  sofic and effective subshifts is  $[0, +\infty] \cap \Pi_3$ .*

The paper is organized as follows. The next section recalls some background and useful definitions. Section 3 and Section 4 focuses on the proof of Theorem 1, and provides in particular a construction which creates  $\mathbb{Z}^2$  SFTs with arbitrary  $\Pi_3$  surface entropies. Some open questions are then discussed in Section 5. Appendix A contains the proofs for the upper and lower bounds on the surface entropy of the construction in Section 4.

## 2 Preliminaries

### 2.1 Subshifts

These paragraphs introduce some standard definitions and facts about subshifts. One may consult [12] for more details.

Let  $\Sigma$  be a finite alphabet of colors. A  $\mathbb{Z}^d$  *configuration* (in this paper,  $d = 1$  or  $d = 2$ ) is a coloring  $x : \mathbb{Z}^d \mapsto \Sigma$ , and the value of  $x$  at position  $z$  is noted  $x_z$ . A (*d-dimensional*) *pattern* is a coloring  $p : D \mapsto \Sigma$ , with  $D \subseteq \mathbb{Z}^d$  a finite domain. For a configuration  $x$ , we say that a pattern  $p$  appears in  $x$  (noted  $p \sqsubseteq x$ ) if there exists some position  $t \in \mathbb{Z}^d$  such that for all  $z \in D$ ,  $p_z = x_{t+z}$ . A *subshift* is a set of colorings/configurations defined by some family of forbidden patterns. Each family of forbidden patterns  $\mathcal{F}$  defines a subshift, possibly empty:

$$X_{\mathcal{F}} = \{(x : \mathbb{Z}^d \mapsto \Sigma) : \forall p \in \mathcal{F}, p \not\sqsubseteq x\}$$

A subshift is *effective* if it can be defined by a recursively enumerable family of forbidden patterns. A subshift is *of finite type* (SFT) if it can be defined by a finite family of forbidden patterns. A subshift is *sofic* if is the projection of an SFT letter-by-letter by a mapping between two alphabets.

## 2.2 Complexity function of $\mathbb{Z}^2$ subshifts

Given a  $\mathbb{Z}^2$  subshift  $X$ , its *complexity function*  $N_X(m, n)$  (for  $m, n \in \mathbb{N}$ ) is the number of different patterns that appear in a rectangle of size  $m \times n$  in the configurations of  $X$  :

$$N_X(m, n) = |\{p \in \Sigma^{m \times n} : \exists x \in X, p \sqsubseteq x\}|$$

This complexity function can be used to define the (topological) entropy  $h(X)$ :

$$h(X) = \lim_{n \rightarrow +\infty} \frac{\log N_X(n, n)}{n^2}$$

This led Dennis Pace to introduce in [14] the notion of *surface entropy*, which corresponds to the “linear term” of the complexity function. Here, we define *surface entropy with eccentricity*  $\alpha$  as :

$$h_s(X, \alpha) = \limsup_{n \rightarrow +\infty} \frac{\log N_X(pn, qn) - pqn^2 h(X)}{(p+q)n} \quad \text{with } \alpha = \frac{p}{q} \in \mathbb{Q}^+$$

and  $p, q$  are relatively prime. This definition differs slightly from the one of [14]; this will be further discussed in Section 5.

Note that in the definition of the topological entropy, only square patterns are used. In fact, any shape of rectangular patterns would generate the same value. Interestingly, this is no longer the case with surface entropy: the *eccentricity* (ratio of the patterns’ widths to heights) affects the value of the calculations. This explains why  $h_s$  is a function of both a subshift  $X$  and a rational parameter  $\alpha = p/q$ . The study in [14] focuses on both the realizability of specific surface entropies, and the behavior of surface entropies as functions of their eccentricities.

While surface entropy is not a conjugacy invariant, it was proved in [14] that its finiteness is one. For more details about surface entropies, one may refer to [14].

## 2.3 Arithmetical hierarchy of real numbers

In order to state our main result, this section recalls from [15] the arithmetical hierarchy of real numbers, which classifies elements of the real line according to their computational properties. Denote by  $\Gamma_{\mathbb{Q}}$  the set of total computable functions  $f : \mathbb{N}^k \mapsto \mathbb{Q}$ . For  $n \geq 1$ , the classes of real numbers  $\Sigma_n, \Pi_n$  and  $\Delta_n$  are defined as follows:

$$\Sigma_n = \{x \in \mathbb{R} : \exists f \in \Gamma_{\mathbb{Q}}, \sup_{i_1} \inf_{i_2} \sup_{i_3} \dots f(i_1, \dots, i_n)\}$$

$$\Pi_n = \{x \in \mathbb{R} : \exists f \in \Gamma_{\mathbb{Q}}, \inf_{i_1} \sup_{i_2} \inf_{i_3} \dots f(i_1, \dots, i_n)\}$$

$$\Delta_n = \Sigma_n \cap \Pi_n$$

It is known that for any  $n \geq 1$ , the inclusions  $\Pi_n \subset \Sigma_{n+1}$  and  $\Sigma_n \subset \Pi_{n+1}$  are proper. One may refer to [15] for more details.

In this paper, we will be interested in the third level of the hierarchy, and one of its equivalent characterization proved in [15]:

$$x \in \Pi_3 \text{ iff there exists } f \in \Gamma_{\mathbb{Q}} \text{ such that } x = \limsup_i \inf_j f(i, j)$$

### 3 Arithmetical restrictions of surface entropies

In this section, we prove the first and easiest direction of Theorem 1:

► **Theorem 3.** *For any  $\mathbb{Z}^2$  SFT  $X$  and  $\alpha \in \mathbb{Q}^+$ :*

$$h_s(X, \alpha) \in [0, +\infty] \cap \Pi_3$$

**Proof.** Let  $X$  be a  $\mathbb{Z}^2$  SFT. First, one should note that  $N_X(m, n)$  is not computable in  $m, n$ . Indeed,  $N_X(m, n) = 0$  if and only if  $X = \emptyset$ ; the latter is well-known for being undecidable.

For  $j \geq m, n$ , define  $N_X^{(j)}(m, n)$  as the number of  $m \times n$  patterns that appear in admissible patterns (patterns in which no forbidden pattern of  $X$  appears) of size  $j \times j$ . These functions are uniformly computable. Additionally,  $N_X(m, n) = \inf_{j \geq m, n} N_X^{(j)}(m, n)$ . In particular, this implies that  $\log N_X(m, n)$  is a  $\Pi_1$  real number.

As proved in [8],  $h(X)$  is a  $\Pi_1$  real number. According to the closure properties proved in [15], this implies that for any  $\alpha = p/q \in \mathbb{Q}^+$ , and for every  $n \in \mathbb{N}$ , the following is a  $\Delta_2$  real number:

$$\frac{\log N_X(pn, qn) - pqn^2 h(X)}{(p+q)n}$$

which then leads to  $h_s(X, \alpha) \in \Pi_3$ . Finally, Pace [14] proved that  $h_s(X, \alpha) \geq 0$ . ◀

► **Corollary 4.** *For any  $\mathbb{Z}^2$  sofic or effective subshift  $X$ ,  $h_s(X) \in [0, +\infty] \cap \Pi_3$ .*

**Proof.** For a sofic or effective subshift  $X$ ,  $\log N_X(m, n)$  is still a  $\Pi_1$  real number. Indeed:

- A sofic subshift is a projection of an SFT. As a projection is a map between two finite sets of colors, it is computable, and  $j, m, n \mapsto N_X^{(j)}(m, n)$  is still a computable function.
- An effective subshift is defined by a computably enumerable family of forbidden patterns. In this context, define  $N_X^{(k)}(m, n)$  (for  $k \geq m, n$ ) as the number of patterns of size  $m \times n$  that appear in squares of size  $k \times k$  in which none of the first  $k$  forbidden patterns enumerated appear. These still define a computable function, and we still have  $N_X(m, n) = \inf_k N_X^{(k)}(m, n)$ .

After obtaining  $\log N_X(m, n) \in \Pi_1$ , the rest of the proof is the same as before. ◀

### 4 Realization of $\Pi_3$ real numbers as surface entropies

In this section, we prove the other (and harder) direction of Theorem 1. Because [14] provides examples of SFTs with infinite surface entropy, we now prove:

► **Theorem 5.** *For any  $x \in [0, +\infty) \cap \Pi_3$ , there exists an SFT  $X$  with surface entropy:*

$$\forall \alpha \in \mathbb{Q}^+, \quad h_s(X, \alpha) = \frac{\min(\alpha, 1)}{1 + \alpha} x$$

Theorem 1 is a consequence of this statement.

**Proof.** Let  $x \in [0, +\infty) \cap \Pi_3$  be a  $\Pi_3$  real number.

Let  $e \in \mathbb{N}$  and  $x' \in [0, 1)$  be such that  $x = e + x'$ . Because  $x'$  is also a  $\Pi_3$  real number, there exists a computable function  $f : \mathbb{N}^2 \mapsto \mathbb{Q}$  such that  $x' = \limsup_k \inf_l f(k, l)$  (see the characterization in Subsection 2.3). We can assume that  $f$  only takes values in  $[0, 1)$ .

In the following subsections, we create an SFT  $X$  which verifies the property of Theorem 5. The proof is organized as follows:

1. Subsection 4.1 introduces our “sparse squares” construction, which aims at creating a set of colorings of the plane with controlled surface entropy. All the sections that follow it focus on implementing this geometrical construction into an actual SFT  $X$ .
2. Subsection 4.2 recalls the Toeplitz sequences, which are sequences of uniform densities. They will be used in the sparse square layout to control the density of each square.
3. In Subsection 4.3, we create a  $\mathbb{Z}$  effective subshift  $X_1$ . Its effectiveness gives us a lot of room to control its patterns, and we will use  $X_1$  as the foundation of  $X$ .
4. We use the “Fixed-Point” construction of [3] to create a  $\mathbb{Z}^2$  SFT which simulates it. Subsection 4.4 provides an intermediary lemma about the entropy and the surface entropy of this construction: in our case, it proves that the intermediary construction has surface entropy zero.
5. We then create the desired SFT  $X$ , which arranges the sparse squares on the plane (with the help of the previous points). It is done in Subsection 4.5.
6. Finally, in Subsection 4.6 we compute the surface entropy of  $X$ . This proves that  $X$  is a valid example for Theorem 5.

#### 4.1 The sparse squares and the sparse square layout

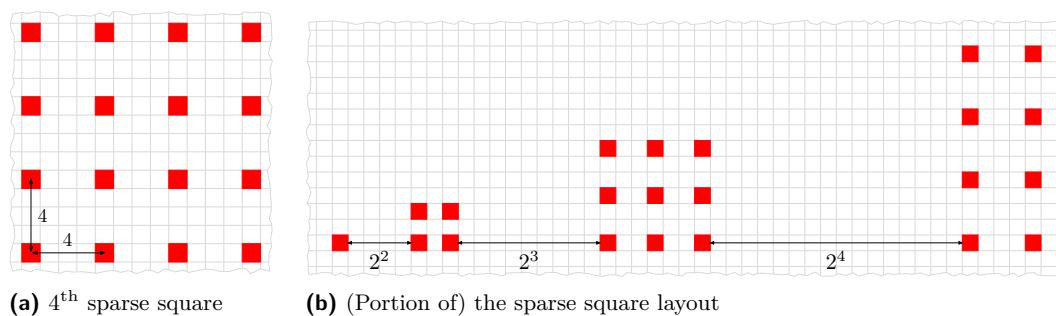
To understand the idea behind this construction, consider the *full shift* over  $\mathbb{Z}^2$  with the alphabet  $\{0, 1\}$ . Configurations are full grids of *free bits*, ie. bits that are allowed to vary freely in  $\{0, 1\}$ . It is not difficult to see that for the full shift,  $\log N_{\text{full}}(n, n) = n^2$ . In particular, its complexity function is quadratic in  $n$ , and its entropy is 1.

To realize specific surface entropies, we first need to figure out a way to contribute *linearly* to  $\log N_X$ , instead of quadratically. To do this, we create a sequence of *sparse squares*. A sparse square is, roughly, a finite piece taken from the full shift, but whose points are moved apart from one another: the square is *sparsified*.

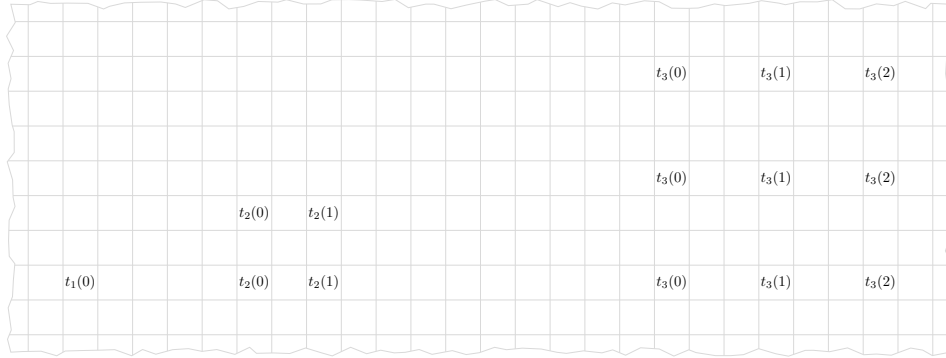
More precisely, the *sparse square of index  $k$*  (see Figure 1a) is a set of positions that form a grid. Any position not in this set of points is blank. In the grid, there are  $k$  columns (the distance between two columns is also  $k$ ), and in each column there are  $k$  points (the distance between two points in a column is also  $k$ ).

The *sparse square layout* (see Figure 1b) makes the sparse squares sit next to one another on a single line, according to their indices. We set the distance between the square of index  $k - 1$  and the square of index  $k$  to  $2^k$ .

The key feature of this geometrical layout lies in its linearity: the  $k^{\text{th}}$  square has edges of size  $k(k - 1) + 1$ : thus it has an area which is roughly  $(k^2)^2$ , while it also contains  $k^2$  points.



■ **Figure 1** Presentation of the sparse square layout



■ **Figure 2** A sequence of words  $(t_k)$  written in (a finite piece of) the sparse square layout

In addition, as the distance between two points increases as one considers squares of greater indices, compactness will only lead to degenerate configurations that contain at most one point. As such, they will not contribute significantly to the complexity function.

For now, the set of positions in the sparse square layout is not very interesting. In order to increase the complexity function, we will allow free bits to vary at each position inside a sparse square. Additionally, to create a surface entropy related to  $x$ , the density of free bits in each square will be related to  $x$ .

More precisely, define  $x'_k = \inf_l f(k, l)$  (recall that  $x = e + x'$ , and  $x' = \limsup_k \inf_l f(k, l)$ ). For each  $k$ , let  $t_k$  be a word of size  $k$  over the alphabet  $\{\text{ON}, \text{OFF}\}$ , whose density of ON is  $x'_k$ . Then, define the subshift  $X'$  as (the closure of) the following configurations (see Figure 2):

- These configurations follow the sparse square layout.
- Each position not marked in the sparse square layout is blank (ie. marked with  $\square$ ).
- The word  $t_k$  is written in each row of the  $k^{\text{th}}$  square.

On the positions marked by ON, we then allow free bits to vary. On every position of the sparse square, we also allow free letters to vary in  $\{1, \dots, 2^e\}$ . With this, the square of index  $k$  contributes to  $\log N_{X'}$  with a term  $k(x'_k + e)$ . If the squares do not interfere too much, one should expect the surface entropy to “converge” towards  $\limsup_k x'_k + e = x' + e = x$ .

The previous paragraphs were a draft of a geometrical construction. Below, we create an actual  $\mathbb{Z}^2$  SFT  $X$  which implements this subshift (with some additional construction lines). Then, we formally prove that  $X$  has the desired surface entropy.

## 4.2 Toeplitz sequences are $\mathbb{Z}$ effective subshifts

In order to create specific densities of letters in a subshift, we recall the useful Toeplitz sequences from [9]. Let  $0 \leq y = \sum_{i=1}^{+\infty} y_i 2^{-i} \leq 1$  be a real number. A *Toeplitz sequence* associated to  $y$  is a bi-infinite sequence  $b \in \{0, 1\}^{\mathbb{Z}}$  such that:

$$b_n = \begin{cases} y_1 & \text{if } n \text{ is even} \\ y_2 & \text{if } n = 4k + 1 \\ y_3 & \text{if } n = 8k + 3 \\ y_4 & \text{if } n = 16k + 5 \\ \text{etc...} & \end{cases}$$

ie. one bit in two is  $y_1$ ; on the remaining bits, one bit in two is  $y_2$ ; etc...

For any  $y \in [0, 1)$ , consider the  $\mathbb{Z}$  subshift  $\mathcal{T}(y)$  defined by:

$$\mathcal{T}(y) = \{(b) \in \{0, 1\}^{\mathbb{Z}} : \exists 0 \leq y' \leq y, (b) \text{ is a Toeplitz sequence associated to } y'\}$$

► **Lemma 6.** *If  $y \in \Pi_1 \cap [0, 1)$ , then  $\mathcal{T}(y)$  is an effective  $\mathbb{Z}$  subshift.*

**Proof.** If  $y \in \Pi_1$ , there exists a computable total function  $f : \mathbb{N} \mapsto \mathbb{Q}$  such that  $y = \inf_n f(n)$ . It is possible to recursively enumerate the different values of  $f$ , to computably forbid patterns of  $\mathcal{T}(y)$  which do not respect the structure of Toeplitz sequences, and to computably forbid patterns of  $\mathcal{T}(y)$  that respect the structure of Toeplitz sequences but whose density is too high. ◀

In the following subsection, we will use subwords of Toeplitz sequences on the alphabet  $\{\text{ON}, \text{OFF}\}$  (rather than  $\{1, 0\}$ ). They have high regularity and tightly controlled densities. Indeed, assume that  $t_n$  is a factor of length  $n$  which appears in  $\mathcal{T}(y)$ . Then the number of letters ON in  $t_n$  is bounded by:

$$0 \leq |t_n|_{\text{ON}} \leq ny + O(1)$$

### 4.3 Building the base line

By definition, any  $x'_k = \inf_l f(k, l)$  is a  $\Pi_1$  real number. As we recalled in the previous section, for any  $k \in \mathbb{N}$ , the subshift  $\mathcal{T}(x'_k)$  composed of all Toeplitz sequences of density lesser or equal to  $x'_k$  is a  $\mathbb{Z}$  effective subshift.

Then it is no surprise that the base line (ie. the line in the sparse square layout on which all the sparse squares sit) forms an effective  $\mathbb{Z}$  subshift. We denote it by  $X_1$  for the rest of the paper. More precisely, for each subword of length  $k$  in

$$\mathcal{T}_k(x'_k) = \{t_k \in \{\text{ON}, \text{OFF}\}^k : t_k \text{ is a subword of } \mathcal{T}(x'_k)\}$$

we create a base  $w_{k, t_k}$  of the  $k^{\text{th}}$  sparse square (ie. one rows of the  $k^{\text{th}}$  sparse square)

$$w_{k, t_k} = t_k(1) B^{k-1} t_k(2) B^{k-1} \dots B^{k-1} t_k(k)$$

where letters  $B$  are blanks, and each  $t_k(j)$  is a letter in  $\{\text{ON}, \text{OFF}\}$  (these letters will later control the density of free bits in the squares). For each sequence  $(t_k)_{k \in \mathbb{N}}$  of words such that  $t_k \in \mathcal{T}_k$ , we can create the base line on which the whole squares sit:

$$B_{(t_k)} = \{\#^\infty s_1 w_{1, t_1} \#^{2^2-2} s_2 w_{2, t_2} \dots s_{k-1} w_{k-1, t_{k-1}} \#^{2^k-k} s_k w_{k, t_k} \dots\}$$

(we highlighted the base of each square in red), where the  $\#$  are another sort of blanks, and the words  $s_k$  are prefixes which we will later use for the construction of the whole squares:

$$s_1 = S \quad \text{and for } k \geq 2, \quad s_k = SB^{k-2}E$$

Finally,  $X_1$  is defined as the closure of all the previous configurations  $B_{(t_k)}$  for all the sequences  $(t_k)$ .  $X_1$  is a  $\mathbb{Z}$  subshift over the alphabet  $\Sigma = \{\#, S, E, B, \text{ON}, \text{OFF}\}$ .

It is possible to computably enumerate all the patterns in which at least two letters of the set  $\{\text{ON}, \text{OFF}\}$  appear, and to forbid each of these patterns that do not respects the structure of the configurations above. Furthermore, each  $x'_k$  is a  $\Pi_1$  real number, and their Toeplitz sequences are effective. These considerations imply that  $X_1$  is an effective  $\mathbb{Z}$  subshift.



#### 4.4 Entropy of the fixed-point realization of effective $\mathbb{Z}$ subshifts as subactions of $\mathbb{Z}^2$ SFTs

We now have a  $\mathbb{Z}$  subshift  $X_1$  which can be used as a foundation in order to implement the whole sparse square layout into an actual SFT. In this section, we recall a particular method (from [3]) which transforms  $\mathbb{Z}$  effective subshifts into  $\mathbb{Z}^2$  SFTs. Additionally, we compute the influence of this method on the surface entropy.

The construction of *fixpoint-based tile sets* was originally introduced in [4]. One particular application of this construction, explained in [3], is the following theorem: for any  $\mathbb{Z}$  effective subshift  $X_1$ , there exists a  $\mathbb{Z}^2$  sofic subshift  $Y_2$  whose configurations are the configurations of  $X_1$  repeated vertically:

$$Y_2 = \{(y^{(2)} : \mathbb{Z}^2 \mapsto \Sigma) : \exists x^{(1)} \in X_1, \forall i, j \in \mathbb{Z}, y_{(i,j)}^{(2)} = x_i^{(1)}\}$$

The principle of the theorem was introduced in [7], which realized  $\mathbb{Z}$  effective subshifts as  $\mathbb{Z}^3$  sofic subshifts. It was also proved in [1] with a different method. In these paragraphs, we focus on the construction of [3] in order to prove:

► **Lemma 7.** *Let  $X_1$  be a  $\mathbb{Z}$  effective subshift. There exists an  $\mathbb{Z}^2$  SFT  $X_2$  composed of two superimposed layers of tilings such that:*

1. *The projection of  $X_2$  on its first layer is a sofic subshift  $Y_2$  whose configurations are the configurations of  $X_1$  repeated vertically.*
2. *The second layer of  $X_2$  is composed of tilings of a fixpoint based tile set.*
3. (New) *For any  $p, q$  relatively prime,*

$$h(X_2) = h(Y_2) = 0 \quad \text{and} \quad h_s(X_2, \alpha) = h_s(Y_2, \alpha) = \frac{\alpha}{1 + \alpha} h(X_1)$$

Points 1 and 2 of this lemma come from [3]. Point 3 is proved in Appendix A.1.

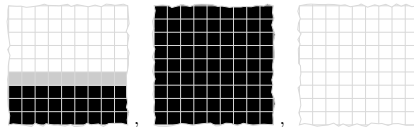
#### 4.5 Building the sparse square layout

In this section, we use the  $\mathbb{Z}$  subshift  $X_1$  to build the whole sparse square layout.

First, apply Lemma 7 from the previous section: there exists a  $\mathbb{Z}^2$  SFT  $X_2$  with two superimposed layers, such that its first layer contains all the vertical replications of configurations of  $X_1$ , and the second layer contains some embedded computations. We then create a 2D SFT  $X_3$  by superimposing a third layer to  $X_2$ , which we describe below.

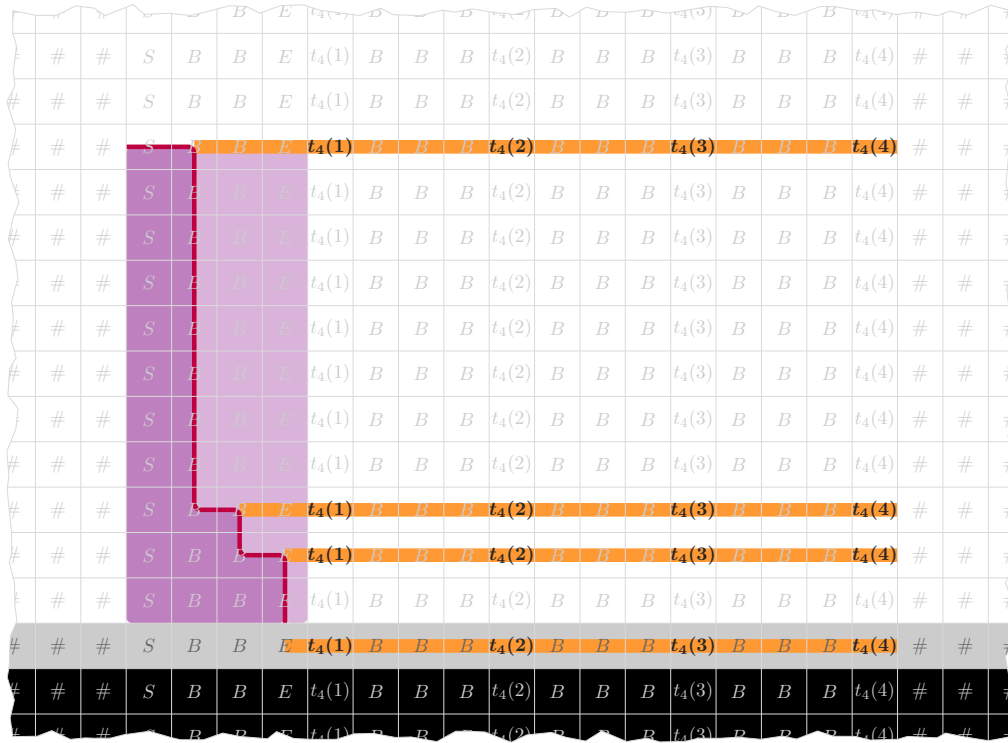
This *Layer 3* is itself a superimposition of several sub-layers:

1. First, one must choose a line on Layer 1 to be the base line (the first layer is composed of the same line repeated vertically: we choose one). To do so, we add a [Layer 3a] with three colors (black, white and gray) whose only type of configurations are the following three:

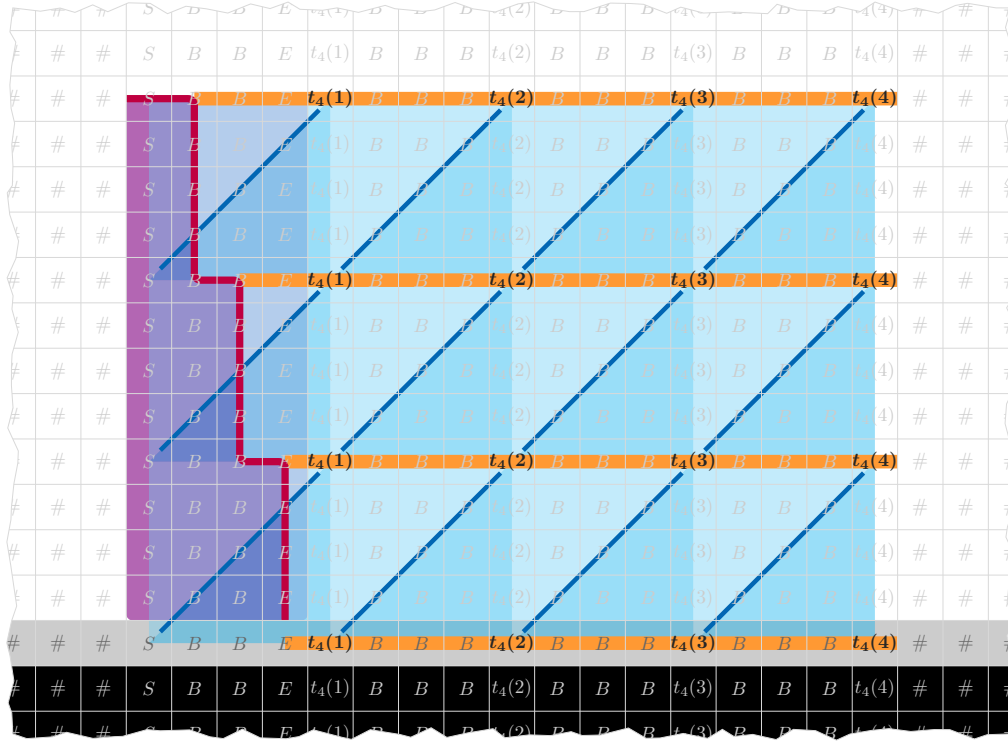


The base line will appear in gray (if it exists).

► **Important.** *The other markings of Layer 3 (Layers 3b to 3d) will only be applied on white and gray areas. Additionally, they are not applied on areas marked by # on Layer 1.*

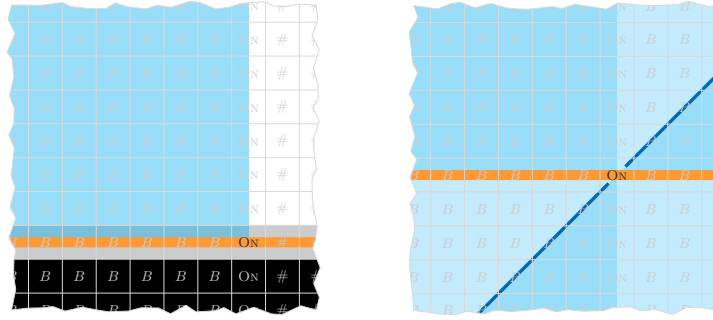


(a) Layers 1 and 3a-3c. Layer 3b ensures there are at most 4 orange lines on Layer 3c.



(b) Layers 1 and 3a-3d. Layer 3d enforces the positions of the orange lines.

■ **Figure 3** Behavior of the construction for a pattern which contains the 4<sup>th</sup> sparse square. These figures reveal the effect of each layer. (Positions of the sparse square are highlighted for the convenience of the reader)



■ **Figure 4** Two examples of degenerate configurations

2. Add [Layer 3b] with purple construction lines (see Figure 3a). Any  $E$  marked in gray (on Layer 3a) starts a line at its top, and this line goes up. Purple lines have the ability to “start” an orange line (on Layer 3c). Each time they start an orange line, they move to their left (it ensures that there are at most  $k$  orange lines in the  $k^{\text{th}}$  sparse square). Purple lines can only end on an  $S$ . Because compactness might lead to surprising results (like infinitely many purple lines behaving erratically), we add colored areas below and above these lines. In these areas of color, we forbid any other purple line to exist: this ensures that there are exactly one purple line per square.
3. Add [Layer 3c] with orange construction lines (see Figure 3a). *These lines will highlight the rows of the sparse squares.* They can only be started by the purple line at their left, and end just before a  $\#$  on the right. Additionally, an  $E$  colored in gray (in [Layer 3a]) must start an orange line. (In the figure, we highlight in bold the  $t_k(i)$  marked by an orange line. These lines mark the future rows of the sparse squares, and these letters ON, OFF marked by an orange line will be at the exact non-blank positions of the sparse squares layout.)
4. Add [Layer 3d] with blue construction lines (see Figure 3b). These lines are diagonals. Each  $t_k(i)$  colored in orange (on Layer 3c) and not in gray (on Layer 3a) should start a line that goes diagonally down and left. Additionally, these lines should end either on a letter  $S$  (colored in purple on Layer 3b), or on the next column marked with  $t_k(i')$ , on another letter that is colored in orange (on Layer 3c). We also impose with colored areas that there can be only one blue line between two orange lines. These blue lines ensure the structure of the sparse squares, and constrain the behavior of degenerate configurations (see Figure 4 for examples of these degenerate configurations).

#### 4.6 Contributing to the entropy

The subshift  $X_3$  reflects the sparse square layout in the following way: by considering letters  $t_k(i)$  marked in orange on Layer 3c, we obtain a set of positions that respects the layout.

The final step of the construction consists in adding a fourth layer to  $X_3$  with *free letters* (ie. letters in the alphabet  $1, \dots, 2^e$ ) and *free bits* (ie. bits in  $\{0, 1\}$ ) to obtain an SFT  $X$  with the right surface entropy. More precisely:

- *Free letters:* At each emplacement in the sparse square layout (ie. for each position marked by ON or OFF on [Layer 1], and which is marked in orange on Layer 3c, we add

a free letter which varies in  $\{1, \dots, 2^e\}$ .

- *Free bits:* At each activated position in the sparse square layout (ie. for each position marked by ON on [Layer 1], and which is marked in orange on Layer 3c), we add a free bit which varies in  $\{0, 1\}$ .

Let  $X$  be the SFT composed of these four superimposed layers. We prove in Appendix A.2:

► **Lemma 8.** *For any  $\alpha = p/q \in \mathbb{Q}^+$  (for  $p$  and  $q$  relatively prime), and for any  $n \in \mathbb{N}$ , if  $k$  is the integer such that  $k(k-1) + 1 \leq \min(p, q)n < (k+1)k - 1$ , then the complexity function of the SFT  $X$  (defined in Section 4) behaves as follows:*

$$\min(p, q)n(e + x'_k) + o(n) \leq \log N_X(pn, qn) \leq \min(p, q)n \left( e + \sup_{\log n \leq i} x'_i \right) + o(n)$$

In particular, this first implies that  $h(X) = 0$ , then that:

$$h_s(X, \alpha) = \limsup_{n \rightarrow +\infty} \frac{\log N_X(pn, qn)}{(p+q)n} = \frac{\min(p, q)}{p+q} \left( e + \limsup_{n \rightarrow +\infty} x'_n \right) = \frac{\min(p, q)}{p+q} x$$

which concludes the proof of Theorem 5. ◀

## 5 Conclusive remarks and open questions

Many questions remain about the notion of surface entropy.

### Computational behavior of the definition in [14]

The definition of *surface entropy* used in this paper differs from the original notion of surface entropy in Pace's thesis [14], which was: for any eccentricity  $\alpha \in \mathbb{R}^+$ ,

$$h_s(X, \alpha) = \sup_{\substack{(x_n, y_n) \in (\mathbb{N}^2)^{\mathbb{N}}: \\ x_n, y_n \rightarrow +\infty, \frac{x_n}{y_n} \rightarrow \alpha}} \limsup_{n \rightarrow +\infty} \frac{\log N_X(x_n, y_n) - x_n y_n h(X)}{x_n + y_n}$$

This definition was chosen in [14] because it provides a unified approach for rational (ie.  $\alpha \in \mathbb{Q}^+$ ) and irrational eccentricities. However, we are currently unsure of how the supremum over all sequences impacts the computational characterization of surface entropies. Our construction still realizes any  $\Pi_3$  surface entropy with the definition of [14], but surface entropies may not be  $\Pi_3$  real numbers anymore. For all we know, they may not be at any level of the arithmetical hierarchy.

### Equivalence between the two definitions

Furthermore, as we modified [14]'s definition of surface entropy, a natural question is whether our new definition coincides with it in the case of rational eccentricities. In other words, can the supremum over all sequences be removed when the eccentricity is a rational number. Our investigations in these directions were not fruitful at the time of writing these lines.

### Arbitrary topological entropy with an arbitrary surface entropy

Finally, in the main section of this paper, we created SFTs with zero topological entropy and any  $\Pi_3$  surface entropy. It was proved in [8] that the class of entropies of  $\mathbb{Z}^2$  SFTs is exactly the class of  $\Pi_1$  real numbers. This led us to wonder whether we could create a family of  $\mathbb{Z}^2$  SFTs with arbitrary  $\Pi_1$  entropy and arbitrary  $\Pi_3$  surface entropy.

As the main constructions of  $\Pi_1$  entropies have non-zero surface entropies, the straightforward construction (ie. a Cartesian product of our construction with one for  $\Pi_1$  entropies) is not an answer to this problem.

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## A

 Proofs in Section 4

### A.1 Proof of Lemma 7

In this appendix, we prove Lemma 7, which we recall below:

► **Lemma 7.** *Let  $X_1$  be a  $\mathbb{Z}$  effective subshift. There exists an  $\mathbb{Z}^2$  SFT  $X_2$  composed of two superimposed layers of tilings such that:*

1. *The projection of  $X_2$  on its first layer is a sofic subshift  $Y_2$  whose configurations are the configurations of  $X_1$  repeated vertically.*
2. *The second layer of  $X_2$  is composed of tilings of a fixpoint based tile set.*
3. (New) *For any  $p, q$  relatively prime,*

$$h(X_2) = h(Y_2) = 0 \quad \text{and} \quad h_s(X_2, \alpha) = h_s(Y_2, \alpha) = \frac{\alpha}{1 + \alpha} h(X_1)$$

**Proof.** Points 1 and 2 of this lemma come directly from the construction of Theorem 1 in [3]. This proof focuses on point 3, and uses notations from [3].

Let  $X_1$  be some  $\mathbb{Z}$  effective subshift, and  $X_2$  (resp.  $Y_2$ ) be the SFT (resp. the sofic subshift) given by the first two points of Lemma 7. Below, we compute the entropies and the surface entropies of  $X_2$  and  $Y_2$ .

First, we prove that  $\log N_{X_2}(pn, qn) = pn h(X_1) + o(n)$ .

By definition of  $\mathbb{Z}$  entropy,  $\log N_{X_1}(n) = n h(X_1) + o(n)$ . As any configuration of  $Y_2$  is entirely determined by a single line, one has  $\log N_{Y_2}(pn, qn) = \log N_{X_1}(pn) = pn h(X_1) + o(n)$ . The complexity function of  $X_2$  is greater than the contribution of its first layer, which leads to  $\log N_{X_2}(pn, qn) \geq \log N_{Y_2}(pn, qn) = pn h(X_1) + o(n)$ .

On the other hand, one can find an upper bound of  $\log N_{X_2}(pn, qn)$  by considering the contributions of the two layers independently. As the contribution of the first layer is the contribution of  $Y_2$ , we now focus on the contribution of all the tilings obtained from the fixpoint-based tile set. Here, we use notions and notations of [3].

In the basic construction of a self-simulating tile set, each macro-tile of level  $i$  (ie. of size  $N_i$ ) is entirely determined by its four macro-colors, which fit in  $O(\log N_i)$  bits. In the construction used in [3] to transform  $\mathbb{Z}$  effective subshifts into  $\mathbb{Z}^2$  SFTs, these macro-colors contain additional data: the level of the macro-tile ( $\log N_i$  bits), one segment of  $l_i$  and three segments of  $l_{i+1}$  letters from configurations of  $X_1$ , and the position in the grand-father macro-tile ( $\log N_{i+2}$  bits). By taking (as in [3])  $N_i = 2^{C2^i}$  with  $C$  being a constant,  $L_i = \prod_{j=0}^{i-1} N_j$  and  $l_i = \log \log L_i$ , we obtain that these macro-colors still fit in  $O(\log N_i)$  bits.

For any  $n$  big enough, there exists  $i$  verifying  $L_i \leq pn \leq L_{i+1}$  and  $qn \leq L_{i+2}$  (indeed,  $\lim_{i \rightarrow +\infty} L_{i+2}/L_{i+1} = +\infty$ ). In this context, a pattern of size  $pn \times qn$  can partially cover at most four macro-tiles of level  $i + 2$ . These macro-tiles are entirely determined by their four macro-colors; each macro-tile of level  $i + 1$  entirely determines the macro-tiles of inferior levels that compose it; and by the previous paragraph each macro-color fits on  $O(\log N_{i+2})$  bits (and the constant in the  $O$  does not depend on  $i$ ): all these considerations imply that the number of patterns of size  $pn \times qn$  on the second layer is at most polynomial in  $N_{i+2}$ .

Considering now the contribution of the two layers independently, one obtains that  $\log N_{X_2}(pn, qn) \leq \log N_{Y_2} + \log \text{poly}(N_{i+2})$ . We have just proved:

$$\log N_{X_2}(pn, qn) = pn h(X_1) + o(n) \quad \text{and} \quad \log N_{Y_2}(pn, qn) = pn h(X_1) + o(n)$$

This immediately leads to:

$$h(X_2) = \lim_{n \rightarrow +\infty} \frac{nh(X_1) + o(n)}{n^2} = 0 = h(Y_2)$$

$$h_s(X_2, p/q) = \limsup_{n \rightarrow +\infty} \frac{pnh(X_1) + o(n)}{(p+q)n} = \frac{p}{p+q}h(X_1) = h_s(Y_2, p/q) \quad \blacktriangleleft$$

## A.2 Computation of the complexity function

Let  $\alpha = p/q$  (for  $p$  and  $q$  relatively prime) be a positive rational number. In this appendix, we compute the complexity function  $N_X(pn, qn)$  of the SFT  $X$  introduced in Section 4. We recall that  $x = e + x'$  is a  $\Pi_3$  real number with  $x' \in \Pi_3 \cap [0, 1)$ , given by  $x' = \limsup_i \inf_l f(i, l)$ , and that we defined  $x'_i = \inf_l f(i, l)$ .

We now recall and prove Lemma 8:

► **Lemma 8.** *For any  $\alpha = p/q \in \mathbb{Q}^+$  (for  $p$  and  $q$  relatively prime), and for any  $n \in \mathbb{N}$ , if  $k$  is the integer such that  $k(k-1) + 1 \leq \min(p, q)n < (k+1)k - 1$ , then the complexity function of the SFT  $X$  (defined in Section 4) behaves as follows:*

$$\min(p, q)n(e + x'_k) + o(n) \leq \log N_X(pn, qn) \leq \min(p, q)n \left( e + \sup_{\log n \leq i} x'_i \right) + o(n)$$

**Proof.** Here is the structure of the proof:

1. First, we provide a lower bound of  $\log N_X(pn, qn)$  (subsection A.2.1).
2. We then consider the contribution of “degenerate configurations” (ie. configurations that do not respect the structure of the sparse square layout: these are obtained when defining  $X_1$  as a closure) to  $\log N_X(pn, qn)$ , and prove that they contribute only as  $o(n)$  (subsection A.2.2).
3. Finally, we consider how many sparse squares can appear simultaneously in a pattern of size  $pn \times qn$  (subsection A.2.3), and:
  - a. We provide a lower bound for the sparse square that appear simultaneously in a pattern of size  $pn \times qn$  (subsection A.2.4). As these squares have very low indices, they only contribute as  $o(n)$ .
  - b. We provide an upper bound for the sparse squares that appear alone (subsection A.2.5; they contribute with the significant term of the upper bound). To obtain it, we bound the number of positions that can appear simultaneously (in a pattern of size  $pn \times qn$ ) of such a sparse square.

Before we begin, in the whole proof we denote by  $k$  the integer such that  $k(k-1) + 1 \leq \min(p, q)n < (k+1)k + 1$ . (Please note that  $k(k-1) + 1$  is the size of the edges of the  $k^{\text{th}}$  sparse square).

### A.2.1 Lower bound of the complexity function

▷ **Claim 9.**

$$\log N_X(pn, qn) \geq \min(p, q)n(e + x'_k) + o(n)$$

**Proof.** As  $k(k-1) + 1 \leq \min(p, q)n$ , the  $k^{\text{th}}$  sparse square can fit entirely in a pattern of size  $pn \times qn$ . To find a lower bound of the complexity function, one can ignore the contribution of Layers 1, 2 and 3, consider the position of the  $k^{\text{th}}$  sparse square fixed, and only look at the contribution induced by its “free bits” and “free letters” on Layer 4.

*Free bits* (For now, we assume the “free letters” are fixed). In the  $k^{\text{th}}$  square appear all the Toeplitz subwords of length  $k$ , associated to a density  $y \leq x'_k$ . To obtain a lower bound of  $\log N_X(pn, qn)$ , we can consider a single Toeplitz subword  $t_k$  associated to the density  $x'_k$ .

Consider a configuration in which the  $k^{\text{th}}$  square is marked with  $t_k$ . Then, the number of free bits in the  $k^{\text{th}}$  sparse square (which is the number of positions in the sparse squares marked with ON) is  $k|t_k|_{\text{ON}}$ . Indeed, there are  $k$  identical lines, and in each line there are  $|t_k|_{\text{ON}}$  positions marked with ON.

Additionally, one has  $|t_k|_{\text{ON}} = kx'_k + O(1)$  (because its density is  $x'_k$ ). This implies that all the free bits of the  $k^{\text{th}}$  sparse square (fixed at this position), when marked with  $t_k$ , contribute to the complexity function with a term

$$\exp_2(k|t_k|_{\text{ON}}) = \exp_2(k^2 x'_k + O(k))$$

*Free letters* Recall that free letters vary in the alphabet  $\{1, \dots, 2^e\}$  at each point in the  $k^{\text{th}}$  sparse squares. There are  $k^2$  points in the  $k^{\text{th}}$  sparse squares, so the previous contribution is multiplied by  $\exp_2(k^2 e)$ . As  $k^2 = \min(p, q)n + o(n)$ , this concludes the proof.  $\triangleleft$

In the rest of the proof, we look for an upper bound of  $\log N_X(pn, qn)$ .

### A.2.2 Contribution of degenerate configurations

We call “degenerate” the configurations that do not respect the structure of the sparse square layout. There are two possible sources for these configurations: some were obtained when  $X_1$  was defined as a closure; and some are obtained if Layer 1 respects the structure of a base line, and Layer 3a is full white or full black (ie. there is no gray base line for the squares to sit on).

$\triangleright$  Claim 10. The contribution of these configurations to  $\log N_X(pn, qn)$  is  $o(n)$ .

Proof. First, we consider the case of a degenerate configuration in the case Layer 1 is not a base line. One should note that there can be at most one letter ON or OFF colored in orange in such a degenerate configuration: indeed, there can be at most one orange line (on Layer 3c) because of the colored areas of the blue line on Layer 3d.

With this in mind, a pattern of a degenerate configuration depends on the position of the gray line on Layer 3a (if it exists at all), of the position of the letter ON or OFF in the pattern, etc... The corresponding varying bit (if the letter is ON) only multiplies the number of patterns by two.

All these depend on finitely many parameters that range from 0 to  $\max(p, q)n$ , so the contribution to  $N_X(pn, qn)$  of these configurations is polynomial in  $n$ , ie. logarithmic to  $\log N_X(pn, qn)$ .

Consider then the case of degenerate configurations in the case Layer 1 is a base line, and Layer 3a does not have a gray line. If Layer 3a is a full black configuration, there are no markings at all on Layer 3 or 4, and the number of patterns depends only on Layer 1. If Layer 3a is a full white configuration, then red lines on Layer 3b can only go up, and never go left: if they did, there would be an orange line on Layer 3c, which is impossible because of the blue areas of color on Layer 3d. This implies that the number of patterns again only depends on Layer 1 (and by Lemma 7, Layer 1 contributes as  $o(n)$ ).

All in all, degenerate configurations contribute to  $\log N_X(pn, qn)$  as  $o(n)$ .  $\triangleleft$

In the rest of the proof, we assume that we consider non-degenerate configurations, ie. configurations that respect the sparse square layout.



We can also assume that the gray line of Layer 3a is fixed at the bottom of the pattern of size  $pn \times qn$  which we consider: this maximizes the number of free bits/free letters in the pattern, which are the main contributors to the complexity function. Furthermore, we will happily forget to count the different horizontal positions of the squares in the patterns.

Indeed, all these other patterns can be taken into account by translating the figure/varying some parameters which range between 0 and  $\max(p, q)n$ : these considerations only multiply the complexity function by a polynomial in  $n$ , or in other words only add a  $o(n)$  to  $\log N_X(pn, qn)$ .

We also define the following set of words for any  $l \in \mathbb{N}$  and  $y \in [0, 1)$  (usually  $\mathcal{T}(y)$  is defined on the alphabet  $\{0, 1\}$  rather than  $\{\text{ON}, \text{OFF}\}$ ; otherwise, there is no difference):

$$\mathcal{T}_l(y) = \{t_l \in \{\text{ON}, \text{OFF}\}^l : t_l \text{ is a subword of } \mathcal{T}(y)\}$$

It is interesting to say that, because we are looking at non-degenerate configurations, Layers 3b-3d (construction layers) are fixed by the line chosen by Layer 3a. This means that we can now focus the different Toeplitz words written on Layer 1, and on the contribution from the free bits and free letters that appear on Layer 4. To count these patterns, we mainly have to compute how many free bits/letters can fit in a pattern at the same time.

### A.2.3 Which sparse squares can only appear alone in a pattern?

To find an upper bound of the complexity function, we ask the following question: how many sparse squares can fit in a pattern of size  $pn \times qn$ ?

▷ **Claim 11.** For  $n \geq p$ , if at least two different sparse squares appear (maybe partially) in a pattern of size  $pn \times qn$ , then their indices are below  $2 \log n$ .

*Proof.* Assume that a range of squares from  $i$  to  $j$ , with  $i < j$ , appear (maybe partially) in a pattern of size  $pn \times qn$ . Then the horizontal space before the square of index  $j$  is entirely contained in the pattern, ie  $2^j < pn$ . Then for any  $n \geq p$ , one has  $j \leq 2 \log n$ . ◁

Reciprocally,

▷ **Claim 12.** If a sparse square can only appear alone in a pattern of size  $pn \times qn$ , then its index is greater than  $\log n$ .

*Proof.* Assume that a square of index  $j$  can “only” appear alone in a pattern of size  $pn \times qn$ . This means that the space before the sparse square, and the space after the sparse square, are bigger than the horizontal size of the pattern  $pn$ . In other words,  $2^j \geq pn$ , which becomes  $j \geq \log n + \log p \geq \log n$ . ◁

### A.2.4 Contribution of simultaneously appearing sparse squares

▷ **Claim 13.** The sparse squares that can appear grouped with others contribute as  $M_1 = o(n)$  to  $\log N_X(pn, qn)$ .

*Proof.* Assume that a range (between  $i$  and  $j$ ,  $i < j$ ) of sparse squares appear (partially) in a pattern of size  $pn \times qn$ . For  $n$  big enough, one has  $j \leq 2 \log n$  by Claim 11. Additionally, because we are interested in an upper bound of  $\log N_X(pn, qn)$ , we can freely assume that all the free bits of the sparse squares of index  $i$  and  $j$  appear in this pattern.

If  $C_{i,j}$  denotes the contribution to  $N_X(pn, qn)$  of this slice of squares between  $i$  and  $j$ , then an upper bound on  $C_{i,j}$  is (we count all the Toeplitz subwords written in the squares on Layer 1, and then their free bits and free letters on Layer 4):

$$\begin{aligned}
C_{i,j} &\leq \sum_{t_i \in \mathcal{T}_i(x'_i), \dots, t_j \in \mathcal{T}_j(x'_j)} \exp_2 \left( \sum_{r=i}^j r |t_r|_{\text{ON}} + e \sum_{t=i}^j r^2 \right) \\
&\leq \sum_{t_i \in \mathcal{T}_i(x'_i), \dots, t_j \in \mathcal{T}_j(x'_j)} \exp_2 \left( \sum_{r=1}^j (r |t_r|_{\text{ON}} + e r^2) \right) \\
&\leq \sum_{t_i \in \mathcal{T}_i(x'_i), \dots, t_j \in \mathcal{T}_j(x'_j)} \exp_2(j^3(1+e)) \\
&\leq \sum_{t_1, \dots, t_j \in \{\text{ON}, \text{OFF}\}^{1+\dots+j}} \exp_2((2 \log n)^3(1+e)) \\
&\leq \exp_2(j^2 + o(j^2)) \exp_2((2 \log n)^3(1+e)) \\
&\leq \exp_2(O((\log n)^3))
\end{aligned}$$

As there are less than  $(2 \log n)^2$  different tuples of  $i, j \leq 2 \log n$ , the contribution  $M_1$  of these sparse squares to  $\log N_X(pn, qn)$  verifies:

$$\begin{aligned}
M_1 &\leq \log \left( \sum_{i < j \leq 2 \log n} C_{i,j} \right) \leq \log \left[ (4(\log n)^2) \exp_2(O((\log n)^3)) \right] \\
&\leq \log \exp_2(O((\log n)^3)) \\
&\leq o(n) \quad \triangleleft
\end{aligned}$$

### A.2.5 Contribution of the other sparse squares

The other sparse squares can only appear alone (maybe partially) in a pattern of size  $pn \times qn$ . We distinguish two cases for them:

- The sparse squares of indices  $i \leq k$ . As they can fit entirely in a pattern of size  $pn \times qn$  (recall that  $k(k+1) - 1 \leq \min(p, q)n \leq (k+1)k + 1$ ), we assume they do (this maximizes the number of free bits that appear simultaneously).
- The sparse squares of indices  $i > k$ . They can only fit partially in a pattern of size  $pn \times qn$ , and we need to “count” the number of their free bits/letters that can appear simultaneously.

#### Contribution of the sparse squares of index $i \leq k$

▷ **Claim 14.** The sparse squares that can only appear alone in a pattern of size  $pn \times qn$ , and of indices  $i \leq k$ , contribute to  $\log N_X(pn, qn)$  with a term:

$$M_2 \leq \min(p, q)n \left( e + \max_{\log n \leq i \leq k} x'_i \right) + o(n)$$

Proof. As  $\#\mathcal{T}_i(x'_i) \leq 2^i$ , each of these squares contribute with a term (again, we count the

words of  $\mathcal{T}_i(x'_i)$  on Layer 1, and free letters and free bits on Layer 4):

$$\begin{aligned} C_i &\leq (2^e)^{i^2} \times \sum_{t_i \in \mathcal{T}_i(x'_i)} \exp_2(i|t_i|_{\text{ON}}) \\ &\leq \exp_2(ei^2) \times \exp_2(i) \exp_2(i^2x'_i + O(i)) \\ &\leq \exp_2(i^2(e + x'_i) + O(i)) \end{aligned}$$

By Claim 12, such a square must be of index  $\geq \log n$ . This implies that all these squares contribute to  $\log N_X(pn, qn)$  with a term:

$$\begin{aligned} M_2 &\leq \log \left( \sum_{i=\log n}^k C_i \right) \\ &\leq \log \left( \sum_{i=\log n}^k \exp_2(i^2(e + x'_i) + O(i)) \right) \\ &\leq \log \left( k \exp_2 \left( k^2 \left( e + \max_{\log n \leq i \leq k} x'_i \right) + O(k) \right) \right) \\ &\leq \log \exp_2 \left( k^2 \left( e + \max_{\log n \leq i \leq k} x'_i \right) + O(k) \right) \\ &\leq k^2 \left( e + \max_{\log n \leq i \leq k} x'_i \right) + O(k) \\ &\leq \min(p, q)n \left( e + \max_{\log n \leq i \leq k} x'_i \right) + o(n) \quad \triangleleft \end{aligned}$$

**Contribution of the sparse squares of index  $i \geq k$**  Finally, we consider the sparse squares of indices  $i \geq k$ : these square only appear partially in a pattern of size  $pn \times qn$ . They contribute significantly to  $\log N_X(pn, qn)$ , as explained in the following claim:

▷ **Claim 15.** The sparse squares of indices  $i \geq k$  contribute to  $\log N_X(pn, qn)$  with a term:

$$M_3 \leq \min(p, q)n \left( e + \max_{k \leq i \leq \max(p, q)(k+1)^2} x'_i \right) + o(n)$$

*Proof.* This proof is organized as follows:

1. We provide an upper bound on the contribution  $C_i$  of the  $i^{\text{th}}$  sparse square to  $N_X(pn, qn)$ , for  $i \geq k$ , according to the number of free bits/free letters of these squares that can appear simultaneously in a pattern of size  $pn \times qn$ .
2. Then, we provide an upper bound on the contribution  $M_3$  of all these sparse squares of indices  $i \geq k$  to  $\log N_X(pn, qn)$ .
3. By studying the variations of two functions  $h : \mathbb{N} \mapsto \mathbb{N}$  and  $v : \mathbb{N} \mapsto \mathbb{N}$  we prove that for any  $i \geq k$  the number of simultaneously appearing free bits/free letters of the  $i^{\text{th}}$  sparse square is  $h(i)v(i) \leq \min(p, q)n + o(n)$ . This will conclude the proof.

**Contribution  $C_i$  of the sparse square of index  $i$**  First, we need to answer the following question: how many bits can appear simultaneously in a pattern of size  $pn \times qn$ ? Recall that there are exactly  $i$  bits in the square of index  $i$  per row (and per column). If  $h(i)$  denotes the number of horizontal bits that can appear simultaneously in a slice of width  $pn$  (and height 1), and  $v(i)$  the number of vertical bits in a slice of height  $qn$  (and width 1), then:

$$h(i) = \min\left(i, \left\lfloor \frac{pn-1}{i} \right\rfloor + 1\right) \quad v(i) = \min\left(i, \left\lfloor \frac{qn-1}{i} \right\rfloor + 1\right)$$

$h$  and  $v$  are eventually decreasing, and  $\lim_{i \rightarrow +\infty} h(i) = \lim_{i \rightarrow +\infty} v(i) = 1$ . There exists an integer  $J_k \leq \max(p, q)(k+1)^2$  such that for any  $i \geq J_k$ , one has  $v(i) = h(i) = 1$ .

We now count free letters and free bits to compute an upper bound on the contribution  $C_i$  of a square of index  $i \geq k$ . As opposed to the previous cases, in which the squares appeared entirely in the pattern of size  $pn \times qn$ , here we can only see at once  $h(i)$  different column: but thanks to the use of Toeplitz sequences, which have a very uniform distribution, the density of a Toeplitz subword of size  $h(i)$  is still  $h(i)x'_i + O(1)$ :

$$\begin{aligned} C_i &\leq \sum_{t_i \in \mathcal{T}(x'_i)^{h(i)}} \exp_2 [h(i)v(i)e + v(i)|t_i|_{\text{ON}}] \\ &\leq \exp_2 [h(i)v(i)(e + x'_i) + O(v(i))] \end{aligned}$$

Additionally, as  $v(i) \leq \frac{qn-1}{i} + 1$ ,  $i \geq k$  and  $k = \Theta(\sqrt{n})$ , one has  $O(v(i)) = O(\sqrt{n})$ .

**Contribution  $M_3$  of the sparse square of index  $i \geq k$**  This implies that the contribution  $M_3$  of all these squares of indices  $i \geq k$  is:

$$\begin{aligned} M_3 &\leq \log \left( \sum_{i=k}^{J_k} \exp_2 [h(i)v(i)(e + x'_i) + o(h(i)v(i))] \right) \\ &\leq \log \left( J_k \exp_2 \left( \left[ \max_{k \leq i \leq J_k} h(i)v(i) + o(h(i)v(i)) \right] \times \left[ e + \max_{k \leq i \leq J_k} x'_i \right] \right) \right) \\ &\leq \left[ \max_{k \leq i \leq J_k} h(i)v(i) + o(h(i)v(i)) \right] \times \left[ e + \max_{k \leq i \leq J_k} x'_i \right] + o(n) \end{aligned}$$

**Study of the product  $h(i)v(i)$  for  $k \geq i \geq J_k$**  We now have to study the product  $h(i)v(i)$  for  $k \leq i \leq J_k$ . Below, we will prove that  $\forall k \leq i \leq J_k, h(i)v(i) \leq \min(p, q)n + O(\sqrt{n})$ .

Without any loss of generality, assume  $q \geq p$ . We prove that:

$$h(i)v(i) \leq pn + O(\sqrt{n}), \quad \text{ie} \quad M_3 \leq pn \left( e + \max_{k \leq i \leq \max(p, q)(k+1)^2} x'_i \right) + o(n)$$

As  $p = \min(p, q)$ , one has  $h(i) = \lfloor \frac{pn-1}{i} \rfloor + 1$  and  $v(i) = \min\left(i, \left\lfloor \frac{qn-1}{i} \right\rfloor + 1\right)$ . For the first values of  $i$ ,  $h$  is an increasing function, and it then decreases for  $i$  large enough: below, we study these variations and conclude about the product  $h(i)v(i)$ .

- For any  $i \leq \lfloor \sqrt{qn-1} \rfloor$ , one has  $\lfloor \frac{qn-1}{i} \rfloor + 1 \geq i$  (which implies  $v(i) = i$ ). Indeed,

$$\begin{aligned} \left\lfloor \frac{qn-1}{i} \right\rfloor + 1 &\geq \left\lfloor \sqrt{qn-1} \right\rfloor + 1 \\ &\geq i \end{aligned}$$

- For any  $k \leq i \leq \lfloor \sqrt{qn-1} \rfloor$ , one has  $h(i)v(i) \leq pn + O(\sqrt{n})$ . Indeed,

$$\begin{aligned} h(i)v(i) = h(i)i &\leq \left( \left\lfloor \frac{pn-1}{i} \right\rfloor + 1 \right) i \\ &\leq i \left\lfloor \frac{pn-1}{i} \right\rfloor + i \\ &\leq pn - 1 + i \\ &\leq pn + O(\sqrt{n}) \end{aligned}$$

- For any  $i \geq \lfloor \sqrt{qn-1} \rfloor + 1$ , one has  $i \geq \lfloor \frac{qn-1}{i} \rfloor + 1$  (which implies  $v(i) = \lfloor \frac{qn-1}{i} \rfloor + 1$ ). Indeed,

$$\begin{aligned} \left\lfloor \frac{qn-1}{i} \right\rfloor + 1 &\leq \left\lfloor \frac{qn-1}{\sqrt{qn-1}} \right\rfloor + 1 \\ &\leq \left\lfloor \sqrt{qn-1} \right\rfloor + 1 \\ &\leq i \end{aligned}$$

- For any  $\lfloor \sqrt{qn-1} \rfloor + 1 \leq i \leq J_k$ , one has  $h(i)v(i) \leq pn + O(\sqrt{n})$ . Indeed,

$$\begin{aligned} h(i)v(i) &= h(i) \left( \left\lfloor \frac{qn-1}{i} \right\rfloor + 1 \right) = \left\lfloor \frac{pn-1}{i} + 1 \right\rfloor \left\lfloor \frac{qn-1}{i} + 1 \right\rfloor \\ &\leq \left\lfloor \frac{pn-1}{\sqrt{qn-1}} + 1 \right\rfloor \left\lfloor \frac{qn-1}{\sqrt{qn-1}} + 1 \right\rfloor \\ &\leq \left\lfloor \frac{pn-1}{\sqrt{qn-1}} + 1 \right\rfloor (\sqrt{qn-1} + 1) \\ &\leq pn + O(\sqrt{n}) \end{aligned}$$

With all of these computations, we conclude that

$$\max_{k \leq i \leq J_k} h(i)v(i) \leq pn + o(n)$$

In the case  $p \geq q$ , the computations are completely symmetric and one obtains:

$$\max_{k \leq i \leq J_k} h(i)v(i) \leq qn + O(\sqrt{n}) \quad \triangleleft$$

### A.2.6 Conclusion

We can now conclude the proof about the bounds of  $\log N_X(pn, qn)$ :

$$\min(p, q)n(e + x'_k) + o(n) \leq \log N_X(pn, qn) \leq \min(p, q)n \left( e + \sup_{\log n \leq i} x'_i \right) + o(n)$$

Indeed, we already proved the lower bound on the complexity function. In order to compute the upper bound, we can count the contributions of the different layers independently.

Layer 1 and 2 (the 1D subshift composed of the base line repeated vertically, and the tilings obtained with self-simulating tile sets) do not contribute to the surface entropy by Lemma 7. Indeed, the 1D entropy of the base line is  $h(Y_1) = 0$ . In other words, these two layers add  $o(n)$  to  $\log N_X(pn, qn)$ .

Degenerate configurations, Layer 3, along with considerations on the different shifts of the configurations, also contribute as  $o(n)$  to  $\log N_X(pn, qn)$  (they indeed contribute polynomially in  $n$  to  $N_X(pn, qn)$ , see the remark at the end of ‘‘Contribution of degenerate configurations’’). Finally, in the previous pages, we provided 3 quantities  $M_1, M_2, M_3$  whose sum is greater than the contribution of Layer 4 (and which take into account the different written words on non-degenerate configurations of Layer 1).

With these considerations, we conclude that:

$$\begin{aligned} \log N_X(pn, qn) &\leq o(n) + M_1 + M_2 + M_3 \\ &\leq o(n) + o(n) + \min(p, q)n \left( e + \max_{\log n \leq i \leq \max(p, q)(k+1)^2} x'_i \right) + o(n) \\ &\leq \min(p, q)n \left( e + \max_{\log n \leq i \leq \max(p, q)(k+1)^2} x'_i \right) + o(n) \quad \blacktriangleleft \end{aligned}$$