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A derivation on Jacobi forms: Oberdieck derivation

YOUNGJU CHOIE, FRANÇOIS DUMAS, FRANÇOIS MARTIN, AND EMMANUEL ROYER

ABSTRACT. The aim of this very short note is to give details on Oberdieck derivation. This is an unpublished companion to the work *Formal deformations of the algebra of Jacobi forms and Rankin-Cohen brackets* by the same authors.

We build a natural derivation on Jacobi forms that extends Serre derivation. Our construction has been influenced by a construction of some differential operator by Oberdieck in [Obe14] and hence we shall call this derivation the Oberdieck derivation (see also [DLM00, GK09, MTZ08]). References for the Weierstraß \wp and ζ functions are [Lan87, Ch. 18], [Sil94, Ch. 1] and [CS17, Ch. 2].

1. CONTEXT AND NOTATION

Let \mathcal{H} be the Poincaré upper half-plane, that is the set of complex numbers τ with $\text{Im}\tau > 0$. Let k be an integer and m a nonnegative integer. The multiplicative group $\text{SL}(2, \mathbb{Z})$ acts on \mathbb{Z}^2 by right multiplication. The semidirect product of $\text{SL}(2, \mathbb{Z})$ and \mathbb{Z}^2 with respect to this action is the Jacobi group: $\text{SL}(2, \mathbb{Z})^J = \text{SL}(2, \mathbb{Z}) \ltimes \mathbb{Z}^2$. Let \mathbf{F} be the set of functions from $\mathcal{H} \times \mathbb{C}$ to \mathbb{C} . Let k and m be two integers. We have the following actions of $\text{SL}(2, \mathbb{Z})$ and \mathbb{Z}^2 on \mathbf{F} . Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z})$, let $(\lambda, \mu) \in \mathbb{Z}^2$, let $\Phi \in \mathbf{F}$, then

$$\begin{aligned} \Phi|_{k,m} \begin{pmatrix} a & b \\ c & d \end{pmatrix} (\tau, z) &= (c\tau + d)^{-k} e^{-2i\pi \frac{mcz^2}{c\tau+d}} \Phi \left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d} \right) \\ \Phi|_m (\lambda, \mu) (\tau, z) &= e^{2i\pi m(\lambda^2 \tau + 2\lambda z)} \Phi(\tau, z + \lambda\tau + \mu) \end{aligned}$$

for all $(\tau, z) \in \mathcal{H} \times \mathbb{C}$. These two actions induce an action of $\text{SL}(2, \mathbb{Z})^J$ on \mathbf{F} the following way: if $(\gamma, (\lambda, \mu)) \in \text{SL}(2, \mathbb{Z})^J$, if $\Phi \in \mathbf{F}$, then we define

$$\Phi|_{k,m} (\gamma, (\lambda, \mu)) = \left(\Phi|_{k,m} \gamma \right)|_m (\lambda, \mu).$$

Explicitly, if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z})$ and $(\lambda, \mu) \in \mathbb{Z}^2$, then

$$\begin{aligned} f|_{k,m} (\gamma, (\lambda, \mu)) (\tau, z) &= \\ (c\tau + d)^{-k} \exp \left(2\pi i m \left(-\frac{c(z + \lambda\tau + \mu)^2}{c\tau + d} + \lambda^2 \tau + 2\lambda z \right) \right) & f \left(\frac{a\tau + b}{c\tau + d}, \frac{z + \lambda\tau + \mu}{c\tau + d} \right) \end{aligned}$$

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for all $(\tau, z) \in \mathcal{H} \times \mathbb{C}$. A function is invariant by the action of $\mathrm{SL}(2, \mathbb{Z})^J$ if and only if it is invariant by both the action of $\mathrm{SL}(2, \mathbb{Z})$ and the action of \mathbb{Z}^2 .

A Jacobi form of weight k and index m is a holomorphic function $\Phi: \mathcal{H} \times \mathbb{C} \rightarrow \mathbb{C}$ that is invariant by the action of the Jacobi group and that has a Fourier expansion of the form

$$\Phi(\tau, z) = \sum_{n=0}^{+\infty} \sum_{\substack{r \in \mathbb{Z} \\ r^2 \leq 4nm}} c(n, r) e^{2\pi i(n\tau + rz)}. \quad (1.1)$$

The vector space $\mathcal{J}_{k,m}$ of such functions is finite dimensional. We identify functions on $\mathcal{H} \times \mathbb{C}$ that are not depending on the second variable with functions on \mathcal{H} and define

$$\mathcal{J}_{k,0} = \mathcal{M}_k.$$

The space \mathcal{M}_k is the space of holomorphic modular forms of weight k on $\mathrm{SL}(2, \mathbb{Z})$ and we have

$$\mathcal{M}_* = \bigoplus_{\substack{k \in 2\mathbb{Z}_{\geq 0} \\ k \neq 2}} \mathcal{M}_k.$$

The action $\lfloor_{k,0}$ of $\mathrm{SL}(2, \mathbb{Z})^J$ on $\mathcal{J}_{k,0}$ induces an action of $\mathrm{SL}(2, \mathbb{Z})$ on \mathcal{M}_k . This action is $\lfloor_{k,0}$ and we shall simply write \lfloor_k .

The bigraded algebra

$$\mathcal{J}_{*,*} = \bigoplus_{k,m} \mathcal{J}_{k,m}$$

is *not* finitely generated and hence we introduce the notion of weak Jacobi form.

A weak Jacobi form of weight k and index m is a function invariant by the action of the Jacobi group but with a Fourier expansion of the form

$$\Phi(\tau, z) = \sum_{n=0}^{+\infty} \sum_{\substack{r \in \mathbb{Z} \\ r^2 \leq 4nm + m^2}} c(n, r) e^{2i\pi(n\tau + rz)}$$

instead of the one given in (1.1). For any given integer $n \geq 0$, the fact that the sum over r is limited to $r^2 \leq 4nm + m^2$ is a consequence of some periodicity of the coefficients [EZ85, p. 105]. The vector space $\tilde{\mathcal{J}}_{k,m}$ of such functions is still finite dimensional [EZ85, Theorem 9.2]. As a consequence, we obtain that

$$\tilde{\mathcal{J}}_{k,0} = \mathcal{M}_k.$$

Let $\mathbf{1}$ be the constant function taking value 1 everywhere (of one or two variables, depending on the context). The subgroup of the modular group $\mathrm{SL}(2, \mathbb{Z})$ of elements γ with $\mathbf{1}|_k \gamma = \mathbf{1}$ is

$$\mathrm{SL}(2, \mathbb{Z})_\infty = \left\{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\}.$$

The Eisenstein series of weight $k \in \mathbb{Z}_{\geq 4}$ is

$$E_k(\tau) = \sum_{\gamma \in \mathrm{SL}(2, \mathbb{Z})_\infty \setminus \mathrm{SL}(2, \mathbb{Z})} \mathbf{1}|_k \gamma(\tau) = \frac{1}{2} \sum_{\substack{(c,d) \in \mathbb{Z}^2 \\ (c,d)=1}} (c\tau + d)^{-k}.$$

Its Fourier expansion is given in terms of the divisor functions

$$\forall u \in \mathbb{C} \quad \forall n \in \mathbb{Z}_{\geq 0}^* \quad \sigma_u(n) = \sum_{d|n} d^u$$

by

$$\forall \tau \in \mathcal{H} \quad E_k(\tau) = 1 - \frac{2k}{B_k} \sum_{n=1}^{+\infty} \sigma_{k-1}(n) q^n$$

where $q = \exp(2\pi i \tau)$ and B_k is the Bernoulli number of order k . We use this Fourier expansion to define an Eisenstein series of weight two:

$$E_2(\tau) = 1 - 24 \sum_{n=1}^{+\infty} \sigma_1(n) q^n.$$

For all even $k \geq 2$, we shall sometimes use another normalisation:

$$G_k = -\frac{(2\pi i)^k}{k!} B_k E_k.$$

2. TWO INTERMEDIATE FUNCTIONS

For all $\tau \in \mathcal{H}$, let $\Lambda_\tau = \mathbb{Z} \oplus \tau \mathbb{Z}$. The ζ function associated to Λ_τ is defined by

$$\forall z \in \mathbb{C} - \Lambda_\tau \quad \zeta(\tau, z) = \frac{1}{z} + \sum_{\substack{\omega \in \Lambda_\tau \\ \omega \neq 0}} \left(\frac{1}{z - \omega} + \frac{1}{\omega} + \frac{z}{\omega^2} \right).$$

Sometimes, we shall use the notation $\zeta(\Lambda_\tau, z)$ instead of $\zeta(\tau, z)$. The function $z \mapsto \zeta(z, \tau)$ is meromorphic over \mathbb{C} . Its poles are the points of Λ_τ and they are simple.

We define J_1 by

$$\forall \tau \in \mathcal{H}, \forall z \in \mathbb{C} - \Lambda_\tau \quad J_1(\tau, z) = \frac{1}{2\pi i} \zeta(\tau, z) + \frac{\pi i}{6} z E_2(\tau).$$

To describe the transformation relations satisfied by J_1 , we define a function $X(M)$, for any $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z})$ by

$$X(M) : \mathcal{H} \times \mathbb{C} \rightarrow \mathbb{C} \\ (\tau, z) \mapsto \frac{cz}{c\tau + d}.$$

It satisfies

$$\forall (M, N) \in \text{SL}(2, \mathbb{Z})^2 \quad X(M)|_{1,0} N = X(MN) - X(N).$$

Lemma 1– *The function J_1 satisfies the following transformation properties:*

$$\forall (\lambda, \mu) \in \mathbb{Z}^2 \quad J_1|_0(\lambda, \mu) = J_1 - \lambda \\ \forall M \in \text{SL}(2, \mathbb{Z}) \quad J_1|_{1,0} M = J_1 + X(M).$$

The Fourier expansion of J_1 is

$$J_1(\tau, z) = -\frac{1}{2} + \frac{\xi}{\xi - 1} - \sum_{n=1}^{+\infty} \left(\sum_{d|n} (\xi^d - \xi^{-d}) \right) q^n$$

where $\xi = \exp(2\pi iz)$, valid if $\xi \neq 1$ and $|q| < |\xi| < |q|^{-1}$.

Its Laurent expansion around 0 is

$$J_1(\tau, z) = \frac{1}{2\pi iz} - \frac{1}{2\pi i} \sum_{n=0}^{+\infty} G_{2n+2}(\tau) z^{2n+1}$$

valid for all $\tau \in \mathcal{H}$ and z in any punctured neighborhood of 0 containing no point of Λ_τ .

Proof. We prove the transformation property by the action of \mathbb{Z}^2 . We have

$$J_1(\tau, z + \lambda\tau + \mu) - J_1(\tau, z) = \frac{1}{2\pi i} (\zeta(\tau, z + \lambda\tau + \mu) - \zeta(\tau, z)) + \frac{\pi i}{6} (\lambda\tau + \mu) E_2(\tau).$$

Let η be the quasi-period map associated to Λ_τ . Then,

$$\zeta(\tau, z + \lambda\tau + \mu) - \zeta(\tau, z) = \eta(\lambda\tau + \mu).$$

The map η is a homomorphism of the group Λ_τ and hence

$$\eta(\lambda\tau + \mu) = \lambda\eta(\tau) + \mu\eta(1).$$

The Legendre relation implies that $\tau\eta(1) - \eta(\tau) = 2\pi i$ so that

$$\eta(\lambda\tau + \mu) = (\lambda\tau + \mu)\eta(1) - 2\pi i\lambda.$$

We have also

$$\eta(1) = -\frac{(2\pi i)^2}{12} E_2(\tau).$$

We deduce

$$\frac{1}{2\pi i} (\zeta(\tau, z + \lambda\tau + \mu) - \zeta(\tau, z)) = -\frac{\pi i}{6} (\lambda\tau + \mu) E_2(\tau) - \lambda$$

and

$$J_1(\tau, z + \lambda\tau + \mu) - J_1(\tau, z) = -\lambda.$$

We prove the transformation property by the action of $\mathrm{SL}(2, \mathbb{Z})$. First, note that if $z \notin \Lambda_\tau$, then $\frac{z}{c\tau+d} \notin \Lambda_{M\tau}$. Let us show that it is sufficient to prove the result for $M \in \{S, T\}$. Let M and N be such that

$$J_1|_{1,0} M = J_1 + X(M) \quad \text{and} \quad J_1|_{1,0} N = J_1 + X(N).$$

Then,

$$\begin{aligned} J_1|_{1,0} MN &= (J_1|_{1,0} M)|_{1,0} N = (J_1 + X(M))|_{1,0} N = J_1 + X(N) + X(MN) - X(N) \\ &= J_1 + X(MN). \end{aligned}$$

The multiplicative group $\mathrm{SL}(2, \mathbb{Z})$ is generated by

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

We deduce that if $J_1|_{1,0} S = J_1 + X(S)$ and $J_1|_{1,0} T = J_1$ then $J_1|_{1,0} M = J_1 + X(M)$ for all $M \in \mathrm{SL}(2, \mathbb{Z})$.

Let us prove that $J_1|_{1,0} T = J_1$. We have

$$\begin{aligned} J_1(\tau+1, z) &= \frac{1}{2\pi i} \zeta(\Lambda_{\tau+1}, z) + \frac{\pi i}{6} z E_2(\tau+1) \\ &= \frac{1}{2\pi i} \zeta(\Lambda_\tau, z) + \frac{\pi i}{6} z E_2(\tau) = J_1(\tau, z) \end{aligned}$$

since $\Lambda_{\tau+1} = \Lambda_\tau$ and E_2 is periodic of period 1.

Finally, let us prove $J_1|_{1,0} S = J_1 + X(S)$. We have

$$J_1\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) = \frac{1}{2\pi i} \zeta\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) + \frac{\pi i}{6} \frac{z}{\tau} E_2\left(-\frac{1}{\tau}\right).$$

We compute

$$\begin{aligned} \zeta\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) &= \zeta(\tau^{-1} \Lambda_\tau, \tau^{-1} z) \quad \text{since } \Lambda_{-1/\tau} = \tau^{-1} \Lambda_\tau \\ &= \tau \zeta(\Lambda_\tau, z) \quad \text{by homogeneity} \\ &= \tau \zeta(\tau, z) \end{aligned}$$

and recall that

$$\tau^{-2} E_2\left(-\frac{1}{\tau}\right) = E_2(\tau) + \frac{6}{\pi i} \frac{1}{\tau}.$$

Finally,

$$\tau^{-1} J_1\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) = \frac{1}{2\pi i} \zeta(z, \tau) + \frac{\pi i}{6} z E_2(\tau) + \frac{z}{\tau}$$

or, equivalently,

$$J_1|_{1,0} S = J_1 + X(S).$$

The Fourier expansion of J_1 is a consequence of the following expansion for ζ :

$$\frac{1}{2\pi i} \zeta(\tau, z) = \sum_{n \geq 1} \left(\frac{\xi^{-1}}{1 - \xi^{-1} q^n} - \frac{\xi}{1 - \xi q^n} \right) q^n - \frac{i\pi}{6} z E_2(\tau) - \frac{1}{2} - \frac{\xi}{1 - \xi}.$$

The Laurent expansion of J_1 is a consequence of the following expansion for ζ :

$$\zeta(\tau, z) = \frac{1}{z} - \sum_{k=1}^{+\infty} G_{2k+2}(\tau) z^{2k+1}.$$

□

We define the J_2 function by

$$J_2 = D_z J_1 - \frac{1}{12} E_2 + J_1^2$$

where $D_z = \frac{\partial}{2\pi i \partial z}$. Similarly, we set $D_\tau = \frac{\partial}{2\pi i \partial \tau} \cdot s$

Lemma 2– *The function J_2 satisfies the following transformations properties:*

$$\begin{aligned} \forall (\lambda, \mu) \in \mathbb{Z}^2 \quad J_2|_{\lambda, \mu}(\lambda, \mu) &= J_2 - 2\lambda J_1 + \lambda^2 \\ \forall M \in \text{SL}(2, \mathbb{Z}) \quad J_2|_{2,0} M &= J_2 + 2J_1 X(M) + X(M)^2. \end{aligned}$$

The Fourier expansion of J_2 is

$$J_2(\tau, z) = \frac{1}{6} - 2 \sum_{n=1}^{+\infty} \left(\sum_{d|n} \frac{n}{d} (\xi^d - \xi^{-d}) \right) q^n$$

valid if $|q| < |\xi| < |q|^{-1}$.

Its Laurent expansion around 0 is

$$J_2(\tau, z) = -\frac{2}{(2\pi i)^2} G_2(\tau) - \sum_{n=0}^{+\infty} \frac{1}{n+1} D_\tau(G_{2n+2})(\tau) z^{2n+2}$$

valid for all $\tau \in \mathcal{H}$ and z in any punctured neighborhood of 0 containing no point of Λ_τ .

Proof. To prove the transformation properties, we apply D_z to the transformation relations satisfied by J_1 and get

$$D_z(J_1)|_{z,0} M = D_z(J_1) + \frac{1}{2\pi i z} X(M)$$

and

$$D_z(J_1)|_1(\lambda, \mu) = D_z(J_1).$$

The relations for J_2 follow from these equalities and the definition.

From the definition of J_2 and the Laurent expansion of J_1 , we have

$$(2\pi i)^2 J_2(\tau, z) = -2 G_2(\tau) + \sum_{k \geq 0} \left[-(2k+5) G_{2k+4}(\tau) + \sum_{a+b=k} G_{2a+2}(\tau) G_{2b+2}(\tau) \right] z^{2k+2}.$$

The Laurent expansion of J_2 follows then from an equality due to Ramanujan (see [Sko93, Eq. (1)]).

As a corollary of the Laurent expansions of J_1 and J_2 , we have that $D_z(J_2) = 2 D_\tau(J_1)$. We get from the Fourier expansion of J_1 the following

$$D_z(J_2)(\tau, z) = -2 \sum_{n \geq 1} n \sum_{d|n} (\xi^d - \xi^{-d}) q^n = -2 D_z \left(\sum_{n \geq 1} \sum_{d|n} \frac{n}{d} (\xi^d + \xi^{-d}) q^n \right).$$

We deduce that a function H exists such that

$$J_2(\tau, z) = -2 \sum_{n \geq 1} \sum_{d|n} \frac{n}{d} (\xi^d + \xi^{-d}) q^n + H(\tau).$$

We have

$$J_2(\tau, 0) = H(\tau) - 4 \sum_{n \geq 1} \sum_{d|n} \frac{n}{d} q^n = H(\tau) + \frac{1}{6} (E_2(\tau) - 1)$$

and hence

$$J_2(\tau, 0) = H(\tau) - \frac{1}{6} - \frac{2}{(2\pi i)^2} G_2(\tau).$$

The Laurent expansion of J_2 implies

$$J_2(\tau, 0) = -\frac{2}{(2\pi i)^2} G_2(\tau).$$

We deduce $H(\tau) = 1/6$. □

3. OBERDIECK'S DERIVATION

Let $(k, p) \in 2\mathbb{Z} \times \mathbb{Z}_{\geq 0}$. For $f \in \widetilde{\mathcal{J}}_{k,p}$, let

$$\text{Ob}(f) = D_\tau(f) - \frac{k}{12}f E_2 - J_1 D_z(f) + pJ_2 f.$$

Proposition 3– For $(k, p) \in 2\mathbb{Z} \times \mathbb{Z}_{\geq 0}$, the map Ob is linear from $\widetilde{\mathcal{J}}_{k,p}$ to $\widetilde{\mathcal{J}}_{k+2,p}$. Moreover, if $(\ell, q) \in 2\mathbb{Z} \times \mathbb{Z}_{\geq 0}$ and $(f, g) \in \widetilde{\mathcal{J}}_{k,p} \times \widetilde{\mathcal{J}}_{\ell,q}$ then

$$\text{Ob}(fg) = \text{Ob}(f)g + f \text{Ob}(g).$$

Proof. The computation of $\text{Ob}(fg)$ is left to the reader. Let $f \in \widetilde{\mathcal{J}}_{k,p}$ and $M \in \text{SL}(2, \mathbb{Z})$. We have

$$D_\tau(f|_{k,p} M) = \left(pX(M)^2 - \frac{k}{2\pi i z} X(M) \right) f|_{k,p} M - X(M) (D_z(f)|_{k+1,p} M) + D_\tau(f)|_{k+2,p} M$$

and

$$D_z(f|_{k,p} M) = -2pX(M)(f|_{k,p} M) + D_z(f)|_{k+1,p} M.$$

Since $f|_{k,p} M = f$ we deduce

$$D_\tau(f)|_{k+2,p} M = D_\tau(f) + \left(\frac{k}{2\pi i z} X(M) - pX(M)^2 \right) f + X(M) (D_z(f)|_{k+1,p} M)$$

and

$$D_z(f)|_{k+1,p} M = D_z(f) + 2pX(M)f.$$

In particular,

$$D_\tau(f)|_{k+2,p} M = D_\tau(f) + \left(D_z(f) + \frac{k}{2\pi i z} f \right) X(M) + pfX(M)^2. \quad (3.1)$$

From,

$$(J_1 D_z(f))|_{k+2,p} M = (J_1|_{1,0} M) (D_z(f)|_{k+1,p} M)$$

we get

$$(J_1 D_z(f))|_{k+2,p} M = J_1 D_z(f) + (D_z(f) + 2pJ_1 f) X(M) + 2pfX(M)^2. \quad (3.2)$$

Similarly,

$$\left(-\frac{k}{12} E_2 f \right)|_{k+2,p} M = -\frac{k}{12} E_2 f - \frac{k}{2\pi i z} f X(M) \quad (3.3)$$

and

$$(pJ_2 f)|_{k+2,p} M = pJ_2 f + 2pJ_1 f X(M) + pfX(M)^2. \quad (3.4)$$

Equations (3.1), (3.2), (3.3) and (3.4) lead to

$$\text{Ob}(f)|_{k+2,p} M = f.$$

Let $(\lambda, \mu) \in \mathbb{Z}^2$. Then

$$D_z(f)|_p(\lambda, \mu) = D_z(f) - 2pf\lambda$$

and

$$D_\tau(f)|_p(\lambda, \mu) = D_\tau(f) - D_z(f)\lambda + pf\lambda^2 \quad (3.5)$$

and so

$$(-J_1 D_z(f))|_p(\lambda, \mu) = -J_1 D_z(f) + (D_z(f) + 2pfJ_1)\lambda - 2pf\lambda^2.$$

We also have

$$(pJ_2 f)|_p(\lambda, \mu) = pJ_2 f - 2pJ_1 f\lambda + pf\lambda^2. \quad (3.6)$$

Equations (3.5)–(3.6) lead to

$$\text{Ob}(f)|_p(\lambda, \mu) = f.$$

Finally, let $\tau \in \mathcal{H}$. We prove that $\text{Ob}_\tau: z \mapsto \text{Ob}(f)(\tau, z)$ is holomorphic. By invariance by the action of \mathbb{Z}^2 , it is sufficient to prove that Ob_τ has no pole in $\mathcal{F}_\tau = \{a + b\tau: (a, b) \in [0, 1]^2\}$. The invariance of f by the action of $\text{SL}(2, \mathbb{Z})$ implies that the Laurent expansion of f around 0 is

$$f(\tau, z) = \sum_{v=0}^{+\infty} Q_{2v}(\tau)z^{2v}$$

where Q_{2v} is a quasimodular form of weight $k + 2v$ and depth less than or equal to v (see [Roy12], [MR05] or [Zag08]). The lack of odd powers in z is a consequence of the non existence of odd weight quasimodular form. The only pole of ζ in \mathcal{F}_τ is 0 and so J_1 has no other pole than 0 in \mathcal{F}_τ . The Laurent expansion of J_1 implies that the Laurent expansion of $J_1 D_z f$ around $z = 0$ is bounded and hence $J_1 D_z f$ has no pole in \mathcal{F}_τ . The function J_2 has no other pole in \mathcal{F}_τ than 0 as it can be seen from its definition. The Laurent expansion of J_2 implies than 0 is not a pole. Finally, Ob_τ is holomorphic. \square

REFERENCES

- [CS17] Henri Cohen and Fredrik Strömberg. *Modular forms*, volume 179 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2017. A classical approach.
- [DLM00] Chongying Dong, Haisheng Li, and Geoffrey Mason. Modular-invariance of trace functions in orbifold theory and generalized Moonshine. *Comm. Math. Phys.*, 214(1):1–56, 2000.
- [EZ85] Martin Eichler and Don Zagier. *The theory of Jacobi forms*, volume 55 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1985.
- [GK09] Matthias R. Gaberdiel and Christoph A. Keller. Differential operators for elliptic genera. *Commun. Number Theory Phys.*, 3(4):593–618, 2009.
- [Lan87] Serge Lang. *Elliptic functions*, volume 112 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1987. With an appendix by J. Tate.
- [MR05] François Martin and Emmanuel Royer. Formes modulaires et périodes. In *Formes modulaires et transcendance*, volume 12 of *Sémin. Congr.*, pages 1–117. Soc. Math. France, Paris, 2005.
- [MTZ08] Geoffrey Mason, Michael P. Tuite, and Alexander Zuevsky. Torus n -point functions for \mathbb{R} -graded vertex operator superalgebras and continuous fermion orbifolds. *Comm. Math. Phys.*, 283(2):305–342, 2008.
- [Obe14] Georg Oberdieck. A Serre derivative for even weight Jacobi Forms. arXiv:1209.5628, 2014.
- [Roy12] Emmanuel Royer. Quasimodular forms: an introduction. *Ann. Math. Blaise Pascal*, 19(2):297–306, 2012.
- [Sil94] Joseph H. Silverman. *Advanced topics in the arithmetic of elliptic curves*, volume 151 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1994.
- [Sko93] Nils-Peter Skoruppa. A quick combinatorial proof of Eisenstein series identities. *J. Number Theory*, 43(1):68–73, 1993.
- [Zag08] Don Zagier. Elliptic modular forms and their applications. In *The 1-2-3 of modular forms*, Universitext, pages 1–103. Springer, Berlin, 2008.

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