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A New Adaptive Control Chart for the Simultaneous Monitoring of the Mean and Variability of Multivariate Normal Processes

Hamed Sabahno^a, Amirhossein Amiri^{a*} and Philippe Castagliola^b

^aDepartment of Industrial Engineering, Faculty of Engineering, Shahed University, Tehran, Iran

^bUniversité de Nantes & LS2N UMR CNRS 6004, Nantes, France

*Corresponding author: amiri@shahed.ac.ir

Abstract

This paper considers adaptive schemes for the simultaneous monitoring of the mean and variability of a multivariate normal quality characteristic. At first, we extend an already existing bivariate non-adaptive simultaneous control chart to a multivariate one. Then, we develop several adaptive schemes, which will cover both previously bivariate and newly multivariate charts. After having designed adaptive schemes for the multivariate chart, eight performance measures are computed based on the run length, time to signal, number of observations to signal and number of switches to signal and evaluated using a new Markov chain model. With the developed performance measures, non-adaptive and adaptive schemes under different mean, variability, simultaneous shift sizes, and different number of quality characteristics are compared. Our scheme is also compared to one of the best methods available in the literature. A numerical example is also provided in order to demonstrate how the adaptive scheme can be implemented in practice.

Keywords: Adaptive control charts, Simultaneous monitoring, Multivariate normal processes, Single-chart monitoring, Performance measures, Markov chains.

1. Introduction

Multivariate control charts are widely used in order to monitor various kinds of processes and to detect assignable causes, in the case of more than one quality characteristic. The multivariate Hotelling (1947)'s T^2 control chart is a very simple and effective tool to monitor the process mean vector when large shifts occur. However, in the case of small to moderate mean shifts, this Shewhart type control chart cannot perform properly. Allowing the chart to adapt based on the previous process states makes it capable to detect small and moderate process shifts more effectively. In adaptive control charts, at least one of the chart's parameters (sample size, sampling interval and probability of Type-I error) varies throughout the monitoring process and, based on that, adaptive charts can be divided into four main categories; VSS (Variable Sample Sizes), VSI (Variable Sampling Intervals), VSSI (Variable Sample Sizes and Sampling Intervals) and VP (Variable Parameters, if all of the chart parameters are allowed to vary). Researchers such as Aparisi (1996), Faraz and Moghadam (2009), Khoo et al. (2013) and Sabahno, Amiri and Castagliola (2019) and Chong et al. (2019) considered VSS type adaptive Hotelling's T^2 charts. For VSI type Hotelling's T^2 control charts we can refer to Aparisi and Haro (2001), Faraz et al. (2011) and Sabahno, Amiri and Castagliola (2018) for example. VSSI type Hotelling's T^2 control charts have been studied by authors such as Aparisi and Haro (2003) and Faraz and Parsian (2006). Finally, VP type ones have been investigated by authors such as Chen (2007), Faraz et al. (2014) and Seif et al. (2016). Researchers such as Du et al. (2012) proposed some approaches for classifying the source of the mean shift in the Hotelling's T^2 control chart.

Multivariate control charts for monitoring the process variability have been considered by researchers as well. One of the common and simple method consists in using the sample's generalized variance ($|S|$) defined as the determinant of the sample variance-covariance matrix. Alt (1985) proposed a bivariate control chart based on $|S|$. Aparisi et al. (1999) extended the work of Alt (1985) and they proposed a multivariate control chart either with upper and lower control limits or with an upper control limit only. Aparisi et al. (2001) developed a VSS scheme for the bivariate case of the generalized variance. They studied the performance of their control chart with the average run length criterion using a Markov chain approach. Grigoryan and He (2005) developed a multivariate double sampling (MDS) shewhart type $|S|$ control chart scheme in order to monitor the variability. Using a genetic algorithm combined with simulations, they showed that their new method performs better than the FP (Fixed Parameters) and VSS charts. Lee and Khoo. (2015) added a VSI feature to the multivariate synthetic shewhart type $|S|$ control chart. Costa and Neto (2017) proposed the S chart with variable charting statistic (called VCS S chart) which is a Shewhart-type chart specially designed to control the covariance matrix of bi- and tri-variate processes. Though they found their method superior to the trivariate case, the only advantage of their method compared to the bi-variate generalized variance method was that it was easier to compute.

The simultaneous monitoring of the process parameters (mean and variability) instead of monitoring only one of them (which is more desired due to its simplicity), has caught some authors attention in the past years. The simultaneous monitoring has been proven to reduce the false alarm rates and to improve the performance of the monitoring procedure. In general, the simultaneous monitoring of control charts can be divided into single-chart and double-charts (one for each parameter) schemes. Concerning multivariate control charts for which two separate charts are used, one may refer to Yeh et al. (2003), Reynolds and Cho (2006), Hawkins and Maboudou-Tchao (2008) and Zhang and Chang (2008).

For multivariate single-chart schemes, the first contribution has been proposed by Yeh and Lin (2002). By using a probability integral transformation method, they introduced the box chart and they compared their single simultaneous chart with the traditional T^2 and $|S|$ (two) charts. However, they showed that their single chart, although simpler, does not usually perform better than the traditional T^2 and $|S|$ charts, especially in detecting small shifts. Khoo (2005) introduced a simple bivariate control chart for the simultaneous monitoring of the mean and variability. He used the Hotelling's T^2 statistic for the mean and Alt (1985)'s statistic, which is based on the generalized variance, for the variability. Then, based on those two statistics, he defined a new max-type statistic. Through simulations and by using the ARL performance measure, he evaluated the chart's performance under shifts in the mean, variability and both. Chen et al. (2005) also developed a single chart based on a max-type statistic containing two exponentially weighted moving average (EWMA) type statistics for each process parameters and they called it the Max-MEWMA chart. They also compared the performance of their chart to the traditional T^2 and $|S|$ charts and they concluded that the Max-MEWMA chart has better performances in detecting small to moderate shifts. Zhang et al. (2010) proposed a new single control chart which integrates an EWMA procedure with the generalized likelihood ratio (GLR) test and they called it ELR for jointly monitoring both multivariate process mean and variability. Their ELR chart, unlike the box and Max-MEWMA charts, can be applied for

individual observations as well. Wang et al. (2014) developed two simultaneous monitoring charts i) based on the generalized likelihood ratio test (GLRT) called PGLRT (penalized GLRT) and ii) based on the multivariate exponentially weighted moving average and covariance (MEWMAC) called PMEWMAC (penalized MEWMAC). The PGLRT chart only uses a statistic while the other one uses two statistics. Both methods are only used for individual observations.

For adaptive simultaneous double-chart schemes, Reynolds and Kim (2007) investigated the option of using sequential sampling, with the sample size at each sampling point depending on the process data, for the Shewhart and MEWMA-types chart combinations. They showed that the sequential sampling provides significant improvements to the charts' performance under most process shifts. Reynolds and Cho (2011) extended the work of Reynolds and Kim (2007) by adding a VSI feature. To the best of the authors' knowledge, adaptive schemes for multivariate simultaneous single-chart schemes have not been considered yet. In order to summarize the literature review and to show the limitations of the papers, we have classified them in Table 1, in terms of the control chart type, the number of quality characteristics, whether they have adaptive features or not, whether they can simultaneously monitor the process mean and variability or not, and finally, the computation method of performance measure(s).

"Insert Table 1 here"

According to our literature review (Table 1), adaptive features in multivariate simultaneous charts have only been considered in two papers and they both used double-charts and also they both used simulation runs in order to measure the chart performance. Using simulation methods to measure the charts performance is often very time consuming. If the distributions of the statistics used for monitoring are available, then the Markov chain is always the preferable method for computing performance measures.

As mentioned above, the work of Khoo (2005), although very simple and innovative, presented a simultaneous scheme capable of monitoring processes with only two quality characteristics, which is a very limited situation and, generally speaking, products might have more than two quality characteristics to be monitored. In addition, it is assumed that all the chart parameters are fixed throughout the monitoring process and no adaptive strategies were used. He also only used the ARL as the chart performance measure, obtained through very time consuming simulation runs. In addition, his chart was very slow specially in detecting small shifts.

In this paper, we improve the innovative approach of Khoo (2005) in four directions: 1) We extend his bivariate-only method to a multivariate one in order to be able to monitor products with more than two quality characteristics. 2) We add adaptive monitoring strategies to both bivariate and multivariate charts to increase the charts performance in detecting shifts. 3) We use 7 extra performance measures (other than just the classical ARL previously used by him) to measure the charts performance in many aspects. 4) We use a new Markov chain model to compute the performance measures instantaneously and more accurately rather than the very time consuming simulation method used by him.

The structure of this paper is as follows. In Section 2, we develop a single-type chart for simultaneously monitoring the mean and variability of a multivariate normal characteristic. We add adaptive features to the chart in Section 3. The performance measures are developed using a Markov chains methodology in Section 4. In Section 5, we perform simulation studies in order to compare the charts' performances and also we present an industrial application. Concluding remarks and suggestions for future researches are discussed in Section 6.

2. Proposed Multivariate Control Charts for Simultaneous Monitoring of Mean and Variability

When the in-control process parameters ($\boldsymbol{\mu}_0$ and $\boldsymbol{\Sigma}_0$) are known, in order to monitor the process mean the Hotelling's T^2 statistic is evaluated for each subgroup $i = 1, 2, 3, \dots$ as:

$$T_i^2 = n(\bar{\mathbf{X}}_i - \boldsymbol{\mu}_0)^T (\boldsymbol{\Sigma}_0)^{-1} (\bar{\mathbf{X}}_i - \boldsymbol{\mu}_0), \quad (1)$$

where n is the sample (subgroup) size, $\bar{\mathbf{X}}_i$ is the sample's mean and $T_i^2 \sim \chi^2(p)$, i.e. a chi-square distribution with p degrees of freedom and p is the number of quality characteristics.

For the process variability, as suggested by Gnanadesikan and Gupta (1970), we use the following statistic:

$$W_i = \frac{(n-1)|\mathbf{S}_i|^{1/p}}{|\boldsymbol{\Sigma}_0|^{1/p}}, \quad (2)$$

which follows approximately a gamma distribution $\Gamma(a, b)$ for $p > 2$, where $a = p(n-p)/2$ and

$$b = \left(\frac{2}{P} \left(1 - \frac{(p-1)(p-2)}{2n} \right) \right)^{-1/p}. \text{ Note that } W_i \text{ exactly follows a gamma distribution for } p=1 \text{ and } p=2.$$

In order to monitor the mean and variability simultaneously, we use the following transformations of T_i^2 and W_i :

$$M_i = \Phi^{-1} [H_p(T_i^2)] \sim N(0, 1), \quad i = 1, 2, \dots, \quad (3)$$

and

$$V_i = \Phi^{-1} [G_{(a,b)}(W_i)] \sim N(0, 1), \quad i = 1, 2, \dots, \quad (4)$$

where $\Phi(\cdot)$ is the standard normal cumulative distribution function, $H_p(\cdot)$ represents the chi-square cumulative distribution function with p degrees of freedom and $G_{a,b}(\cdot)$ denotes the gamma cumulative distribution function with shape parameter a and scale parameter b .

Note that, Khoo (2005) used a similar approach to monitor bivariate quality characteristics. However, he used the following statistic $V_i = \Phi^{-1} [H_{2n-4}(2W_i)] \sim N(0, 1)$, $i = 1, 2, \dots$, instead of the proposed statistic for monitoring the variability, where $2W_i$ follows a chi-square distribution with $2n-4$ degrees of freedom. Note that $p=2$ in the $2W_i$ statistic by Khoo (2005).

Then, we use the following max-type statistic:

$$C_i = \max \{ |M_i|, |V_i| \}. \quad (5)$$

The upper control limit (UCL) of this chart can be obtained by solving the following equation:

$$\begin{aligned} P(C_i \leq UCL) &= P(\max \{ |M_i|, |V_i| \} \leq UCL) \\ &= P(-UCL \leq M_i \leq UCL) P(-UCL \leq V_i \leq UCL) \\ &= (2\Phi(UCL) - 1)(2\Phi(UCL) - 1) = (2\Phi(UCL) - 1)^2 = 1 - \alpha, \end{aligned} \quad (6)$$

where α is the probability of Type-I error.

3. Proposed Adaptive Multivariate Control Charts for Monitoring the Mean and Variability Simultaneously

For adaptive control charts, at least one of the process parameters (sample size, sampling interval and probability of Type-I error) is allowed to vary throughout the sampling process. If all parameters vary, the adaptive chart is called a VP control chart. In this paper, we assume that there are:

- two types of sample sizes ($n_s : s \in \{1, 2\}$ with $n_1 < n_2$),
- two types of sampling intervals ($t_s : s \in \{1, 2\}$ with $t_2 < t_1$),
- two types of type I error probabilities ($\alpha_s : s \in \{1, 2\}$ where $\alpha_1 < \alpha_2$),

and, as our max-type control chart is one-sided, we also assume that there are:

- two upper control limits UCL_1 (related to α_1) and UCL_2 (related to α_2) As $\alpha_1 < \alpha_2$ we have $UCL_2 < UCL_1$.
- two upper warning limits UWL_1 and UWL_2 verifying $UWL_1 < UCL_1$ and $UWL_2 < UCL_2$, respectively.

The VP strategy for choosing the next sampling scheme is as follows:

- If $C_i \in [0, UWL]$, the process is declared as being in-control and the parameters for the next sample must be $\Omega_1 = (n_1, t_1, UCL_1, UWL_1)$, i.e. the small sample size, the long sampling interval and the large control and warning limits UCL and UWL .
- If $C_i \in (UWL, UCL]$, the process is also declared as being in-control but the parameters for the next sample must be $\Omega_2 = (n_2, t_2, UCL_2, UWL_2)$, i.e. the large sample size, the short sampling interval and the small control and warning limits UCL and UWL .
- If $C_i \in (UCL, \infty)$, the process is declared as being out-of-control and corrective actions have to be taken.

In the VP scheme, the following equations should be satisfied.

$$ASS = n_1 P_0 + n_2 (1 - P_0), \quad (7)$$

$$ASI = t_1 P_0 + t_2 (1 - P_0), \quad (8)$$

$$ATE = \alpha_1 P_0 + \alpha_2 (1 - P_0), \quad (9)$$

where ASS is the average sample size, ASI is the average sampling interval and ATE is the average Type-I error where P_0 is the probability of being in the safe state (Ω_1) while the process is in-control, i.e.:

$$\begin{aligned} P_0 &= \frac{P(C_i \leq UWL)}{P(C_i \leq UCL)} = \frac{P(\max\{|M_i|, |V_i|\} \leq UWL)}{P(\max\{|M_i|, |V_i|\} \leq UCL)} \\ &= \frac{P(|M_i| \leq UWL)P(|V_i| \leq UWL)}{P(|M_i| \leq UCL)P(|V_i| \leq UCL)} = \frac{[2\Phi(UWL) - 1]^2}{[2\Phi(UCL) - 1]^2} \end{aligned} \quad (10)$$

Note that for a VSS scheme, only Equation (7) must be kept and Equations (8) and (9) must be disregarded. Similarly, for a VSI scheme, only Equation (8) must be kept and Equations (7) and (9) have to be disregarded. Finally, for a VSSI scheme, only Equations (7) and (8) must be kept and Equation (9) must be ignored.

In order to solve Equations (7) to (9), some of the parameters' values should be fixed and some others have to be computed. The following steps can be taken to obtain the parameters' values.

Step 1. Fix α_1 and, using Equation (6), UCL_1 is obtained as:

$$UCL_1 = \Phi^{-1} \left(\frac{\sqrt{1-\alpha_1} + 1}{2} \right). \quad (11)$$

Step 2. Fix ASS , n_1, n_2 and, using Equation (7), P_0 is obtained as:

$$P_0 = \frac{ASS - n_2}{n_1 - n_2}. \quad (12)$$

Step 3. Fix ATE and, using Equations (9) and (12), α_2 is obtained as:

$$\alpha_2 = \frac{ATE(n_1 - n_2) - \alpha_1(ASS - n_2)}{n_1 - ASS}, \quad (13)$$

and UCL_2 is deduced from Equation (6) as:

$$UCL_2 = \Phi^{-1} \left(\frac{\sqrt{1-\alpha_2} + 1}{2} \right). \quad (14)$$

Step 4. Fix ASI , t_2 and, using Equations (8) and (12), t_1 is obtained as:

$$t_1 = \frac{ASI(n_1 - n_2) - t_2(n_1 - ASS)}{ASS - n_2}. \quad (15)$$

Step 5. By using Equation (10), UWL_1 and UWL_2 are obtained as follows:

$$UWL_s = \Phi^{-1} \left(\frac{(2\Phi(UCL_s) - 1)\sqrt{P_0} + 1}{2} \right) \quad (16)$$

4. Performance measures

For a FP control chart (for which the chart parameters are all fixed), performance measures like the Average Run Length (ARL) and the Standard Deviation of Run Length (SDRL) can be used and investigated. However, for adaptive schemes, six additional performance measures can be used: the Average Time to Signal (ATS), the Standard Deviation of Time to Signal (SDTS), the Average Number of Observations to Signal (ANOS), the Standard Deviation of Number of Observations to Signal (SDNOS), the Average Number of Switches to Signal (ANSW) and the Standard Deviation of Number of Switches to Signal (SDNSW).

The ARL and SDRL are the average and standard deviation of the *number* of samples taken before an out-of-control signal occurs, the ATS and SDTS are the average and standard deviation of the *time* needed until a control chart signals, the ANOS and SDNOS are the average and standard deviation of the *number* of products observed before an out-of-control signal and, finally, the ANSW and SDNSW are the average and standard deviation of the number of switches from Ω_1/Ω_2 to Ω_2/Ω_1 while the process is in-control. If the *number* of switches increases, the difficulty of the chart's administration increases as well. This is why these performance measures are worth to be investigated. Since there is

no switching in FP control charts, they always have an advantage in this regard and can be easily administrated.

It is reasonable to say that the most important criteria in order to measure the charts' performance is the time to signal (ATS/SDTS) and all charts' performances should be compared primarily via this time to signal criteria.

When the process is out-of-control, smaller ARL, ATS and ANOS values are preferable. On the contrary, when the process is in-control, larger ARL, ATS and ANOS values are expected. However, no matter if the process is in- or out-of-control, smaller SDRL, SDTS, SDNOS, ANSW and SDNSW values are always expected, especially in out-of-control situations.

In order to compute these quantities, we use a Markov Chain model. This approach requires to define the following three states:

$$\text{State 1: } C_i \in [0, UWL],$$

$$\text{State 2: } C_i \in (UWL, UCL],$$

$$\text{State 3: } C_i \in (UCL, \infty).$$

The first two states are transient while, the third one is absorbing. The Markov transition probability matrix \mathbf{P} corresponding to the VP chart is equal to

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix},$$

where p_{ij} is the probability of transition from state i to state j .

Since by definition T_i^2 and W_i are independent for a multivariate normal characteristic, consequently, M_i and V_i are also independent. Therefore, the transient state probabilities p_{11} , p_{12} , p_{21} and p_{22} for the proposed multivariate max-type chart are equal to:

$$\begin{aligned} p_{s1} &= P(C_i \leq UWL) = P(\max\{|M_i|, |V_i|\} \leq UWL) \\ &= P(-UWL \leq M_i \leq UWL)P(-UWL \leq V_i \leq UWL), \end{aligned} \quad (17)$$

$$\begin{aligned} p_{s2} &= P(UWL < C_i \leq UCL) = P(UWL < \max\{|M_i|, |V_i|\} \leq UCL) \\ &= P(-UCL \leq M_i < -UWL)P(-UCL \leq V_i < -UWL) + \\ &\quad P(-UCL \leq M_i < -UWL)P(-UWL < V_i < UWL) + \\ &\quad P(-UCL \leq M_i < -UWL)P(UWL < V_i \leq UCL) + \\ &\quad P(-UWL < M_i < UWL)P(-UCL \leq V_i < -UWL) + \\ &\quad P(-UWL < M_i < UWL)P(UWL < V_i \leq UCL) + \\ &\quad P(UWL < M_i \leq UCL)P(UWL < V_i \leq UCL) + \\ &\quad P(UWL < M_i \leq UCL)P(-UWL < V_i < UWL) + \\ &\quad P(UWL < M_i \leq UCL)P(-UCL \leq V_i < -UWL). \end{aligned} \quad (18)$$

Concerning the other probabilities of transition, we simply have $p_{13} = 1 - p_{11} - p_{12}$, $p_{23} = 1 - p_{21} - p_{22}$, $p_{31} = p_{32} = 0$ and $p_{33} = 1$.

As it can be seen, Equations (17) and (18) are both sums of products $P(m_1 \leq M_i \leq m_2)$ by $P(v_1 \leq V_i \leq v_2)$ where m_1, m_2, v_1 and v_2 stand for +/-UWL and +/-UCL. In general, these terms can be evaluated using Equations (21) and (23) when the process is out-of-control and using Equations (22) and (24) when the process is in-control. Let us first derive Equations (21) and (22) for $P(m_1 \leq M_i \leq m_2)$. We have:

$$\begin{aligned} P(m_1 \leq M_i \leq m_2) &= P(m_1 \leq \Phi^{-1} \left[H_p(T_i^2) \right] \leq m_2) \\ &= P(\Phi(m_1) \leq \left[H_p(T_i^2) \right] \leq \Phi(m_2)) \\ &= P(H_p^{-1}(\Phi(m_1)) \leq T_i^2 \leq H_p^{-1}(\Phi(m_2))). \end{aligned} \quad (19)$$

In order to compute Equation (19), let us assume that $p > 1$, i.e. $X_j \sim \text{MN}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, $j = 1, 2, \dots, n_s$. When the process is in-control, we have $\boldsymbol{\mu} = \boldsymbol{\mu}_0$ and $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_0$. Let

$$\mathbf{Z}_i = \sqrt{n_s} \boldsymbol{\Sigma}_0^{-1/2} (\bar{\mathbf{X}}_i - \boldsymbol{\mu}_0),$$

where $\boldsymbol{\Sigma}_0^{-1/2}$ is the matrix square root of the inverse of $\boldsymbol{\Sigma}_0$, i.e. $\boldsymbol{\Sigma}_0^{-1/2} \boldsymbol{\Sigma}_0^{-1/2} = \boldsymbol{\Sigma}_0^{-1}$. When the process is in-control, we have $\mathbf{Z}_i \sim \text{MN}(\mathbf{0}, \mathbf{I})$ and, therefore $T_i^2 = \mathbf{Z}_i^T \mathbf{Z}_i \sim \chi^2(p)$.

When the process is out-of-control, we have $\boldsymbol{\mu} = \boldsymbol{\mu}_1$, $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_1$ and the statistic

$$\mathbf{Z}_i \sim \text{MN}(\sqrt{n_s} \boldsymbol{\Sigma}_0^{-1/2} (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_0), \boldsymbol{\Sigma}_0^{-1/2} \boldsymbol{\Sigma}_1 \boldsymbol{\Sigma}_0^{-1/2}).$$

Let us define $\boldsymbol{\Lambda} = \boldsymbol{\Sigma}_0^{-1/2} \boldsymbol{\Sigma}_1 \boldsymbol{\Sigma}_0^{-1/2}$ and we deduce

$$\boldsymbol{\Lambda}^{-1/2} = ((\boldsymbol{\Sigma}_0^{-1/2} \boldsymbol{\Sigma}_1 \boldsymbol{\Sigma}_0^{-1/2})^{-1})^{1/2} = (\boldsymbol{\Sigma}_0^{1/2} \boldsymbol{\Sigma}_1^{-1} \boldsymbol{\Sigma}_0^{1/2})^{1/2}.$$

Consequently, we have

$$\boldsymbol{\Lambda}^{-1/2} \mathbf{Z}_i \sim \text{MN}(\sqrt{n_s} \boldsymbol{\Lambda}^{-1/2} \boldsymbol{\Sigma}_0^{-1/2} (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_0), \mathbf{I}),$$

and

$$\mathbf{Z}_i^T \boldsymbol{\Lambda}^{-1/2} \boldsymbol{\Lambda}^{-1/2} \mathbf{Z}_i = \mathbf{Z}_i^T \boldsymbol{\Lambda}^{-1} \mathbf{Z}_i \sim \chi^2(p, \lambda),$$

With the noncentrality parameter λ equals to

$$\begin{aligned} \lambda &= n_s (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_0)^T \boldsymbol{\Sigma}_0^{-1/2} \boldsymbol{\Lambda}^{-1/2} \boldsymbol{\Lambda}^{-1/2} \boldsymbol{\Sigma}_0^{-1/2} (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_0) \\ &= n_s (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_0)^T \boldsymbol{\Sigma}_0^{-1/2} \boldsymbol{\Lambda}^{-1} \boldsymbol{\Sigma}_0^{-1/2} (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_0). \end{aligned}$$

If we replace $\boldsymbol{\Lambda}$ with $\boldsymbol{\Sigma}_0^{-1/2} \boldsymbol{\Sigma}_1 \boldsymbol{\Sigma}_0^{-1/2}$, we have $\boldsymbol{\Lambda}^{-1} = \boldsymbol{\Sigma}_0^{1/2} \boldsymbol{\Sigma}_1^{-1} \boldsymbol{\Sigma}_0^{1/2}$ and the noncentrality parameter simplifies to

$$\lambda = n_s (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_0)^T (\boldsymbol{\Sigma}_1)^{-1} (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_0). \quad (20)$$

As it is mentioned above, for general variance-covariance matrices $\boldsymbol{\Sigma}_0$ and $\boldsymbol{\Sigma}_1$, we have $\mathbf{Z}_i^T \boldsymbol{\Lambda}^{-1} \mathbf{Z}_i \sim \chi^2(p, \lambda)$. However, in order to evaluate the terms $P(m_1 \leq M_i \leq m_2)$ in Equation (19), it is impossible to obtain the distribution of $T_i^2 = \mathbf{Z}_i^T \mathbf{Z}_i$ for an out-of-control condition. To overcome this problem, we will assume that $\boldsymbol{\Sigma}_1 = \tau \boldsymbol{\Sigma}_0$, with $\tau > 0$. Therefore, we have $\boldsymbol{\Lambda}^{-1} = \frac{1}{\tau} \mathbf{I}$ where \mathbf{I} is the

identity matrix and, consequently, $\frac{1}{\tau} T_i^2 = \frac{1}{\tau} \mathbf{Z}_i^T \mathbf{Z}_i \sim \chi^2(p, \lambda)$ with $\lambda = \frac{n_s}{\tau} (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_0)^T (\boldsymbol{\Sigma}_0)^{-1} (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_0)$.

As a result, we have

$$P(m_1 \leq M_i \leq m_2) = H_{p, \lambda}(H_p^{-1}(\Phi(m_2)) \times (1/\tau)) - H_{p, \lambda}(H_p^{-1}(\Phi(m_1)) \times (1/\tau)). \quad (21)$$

When the process is in-control, i.e. $\lambda = 0$ and $\boldsymbol{\Sigma}_1 = \boldsymbol{\Sigma}_0$, the previous equation reduces to:

$$P(m_1 \leq M_i \leq m_2) = H_p(H_p^{-1}(\Phi(m_2))) - H_p(H_p^{-1}(\Phi(m_1))) = \Phi(m_2) - \Phi(m_1) \quad (22)$$

Similarly, for $P(v_1 \leq V_i \leq v_2)$, we have:

$$\begin{aligned} P(v_1 \leq V_i \leq v_2) &= P(v_1 \leq \Phi^{-1}[G_{(a,b)}(W_i)] \leq v_2) \\ &= P(\Phi(v_1) \leq [G_{(a,b)}(W_i)] \leq \Phi(v_2)) \\ &= P(G_{(a,b)}^{-1}(\Phi(v_1)) \leq W_i \leq G_{(a,b)}^{-1}(\Phi(v_2))) \end{aligned}$$

In order to obtain the distribution of W_i when the process is out-of-control we multiply in the previous equation $|\Sigma_0|$ by $|\Sigma_1|/|\Sigma_0| = 1$, i.e.

$$P(G_{(a,b)}^{-1}(\Phi(v_1)) \leq \frac{(n_s - 1)|S_i|^{1/p}}{[|\Sigma_0||\Sigma_1|/|\Sigma_0|]^{1/p}} \leq G_{(a,b)}^{-1}(\Phi(v_2))).$$

Finally, all the terms are multiplied by $[|\Sigma_0|/|\Sigma_1|]^{1/p}$ and as a result, we have

$$\begin{aligned} P(v_1 \leq V_i \leq v_2) &= \\ &G_{(a,b)}(G_{(a,b)}^{-1}(\Phi(v_2)) \times [|\Sigma_0|/|\Sigma_1|]^{1/p}) - G_{(a,b)}(G_{(a,b)}^{-1}(\Phi(v_1)) \times [|\Sigma_0|/|\Sigma_1|]^{1/p}). \end{aligned} \quad (23)$$

When the process is in-control, i.e. $\Sigma_1 = \Sigma_0$, the previous equation reduces to:

$$\begin{aligned} P(v_1 \leq V_i \leq v_2) &= G_{(a,b)}(G_{(a,b)}^{-1}(\Phi(v_2))) - G_{(a,b)}(G_{(a,b)}^{-1}(\Phi(v_1))) \\ &= \Phi(v_2) - \Phi(v_1). \end{aligned} \quad (24)$$

One can easily compute Equations (17) and (18) using terms $P(m_1 \leq M_i \leq m_2)$ and $P(v_1 \leq V_i \leq v_2)$, replacing m_1, m_2, v_1 and v_2 in Equations (21) – (24) with $\pm UWL$ and $\pm UCL$.

In this paper, we consider two scenarios for the shift in the variance-covariance matrix as follows:

- Scenario 1- In this scenario we assume that all the elements (variance and covariance values) in Σ_0 shift with an equal multiplier τ .
- Scenario 2- In this scenario, each element of the variance-covariance matrix shifts independently and not necessarily in an equal way. In this case, we again consider the general form $\Sigma_1 = \tau \Sigma_0$ and estimate τ as $|\Lambda|^{1/p}$ through simulation runs. Note that τ is estimated such that the relationship $\Sigma_1 = \tau \Sigma_0$ is nearly satisfied. In this scenario, Σ_0 and Σ_1 can be any matrices.

Note that, the scenario 1 is a special case of scenario 2.

Then, from Jensen et al. (2008), we have the general forms of the performance measures as:

$$ARL = \mathbf{b}^T (\mathbf{I} - \mathbf{Q})^{-1} \mathbf{1}, \quad (25)$$

$$SDRL = \sqrt{\mathbf{b}^T (\mathbf{I} - \mathbf{Q})^{-1} (2\mathbf{I}(\mathbf{I} - \mathbf{Q})^{-1} \mathbf{1} - \mathbf{1}) - (ARL)^2}, \quad (26)$$

$$ATS = \mathbf{b}^T (\mathbf{I} - \mathbf{Q})^{-1} \mathbf{t}, \quad (27)$$

$$\text{SDTS} = \sqrt{\mathbf{b}^T (\mathbf{I} - \mathbf{Q})^{-1} (2\mathbf{D}_t (\mathbf{I} - \mathbf{Q})^{-1} \mathbf{t} - \mathbf{t}^{(2)}) - (\text{ATS})^2}, \quad (28)$$

$$\text{ANOS} = \mathbf{b}^T (\mathbf{I} - \mathbf{Q})^{-1} \mathbf{n}, \quad (29)$$

$$\text{SDNOS} = \sqrt{\mathbf{b}^T (\mathbf{I} - \mathbf{Q})^{-1} (2\mathbf{D}_n (\mathbf{I} - \mathbf{Q})^{-1} \mathbf{n} - \mathbf{n}^{(2)}) - (\text{ANOS})^2}, \quad (30)$$

$$\text{ANSW} = \mathbf{b}^T (\mathbf{I} - \mathbf{Q})^{-1} \mathbf{p}, \quad (31)$$

$$\text{SDNSW} = \sqrt{\mathbf{b}^T (\mathbf{I} - \mathbf{Q})^{-1} (2\mathbf{D}_p (\mathbf{I} - \mathbf{Q})^{-1} \mathbf{p} - \mathbf{p}^{(2)}) - (\text{ANSW})^2}, \quad (32)$$

where $\mathbf{b}^T = (b_1, b_2)$ is the vector of starting probabilities such that $b_1 + b_2 = 1$, \mathbf{I} is the identity matrix of order 2, \mathbf{Q} is a 2×2 transition probability matrix for the transient states, $\mathbf{1}$ is a 2×1 unit column vector, $\mathbf{t} = (t_1, t_2)^T$ is the vector of sample sizes, \mathbf{D}_t is the 2×2 diagonal matrix with the diagonal elements of \mathbf{t} , $\mathbf{t}^{(2)}$ contains the squares of the elements of vector \mathbf{t} , $\mathbf{n} = (n_1, n_2)^T$ is the vector of sample sizes, \mathbf{D}_n is the 2×2 diagonal matrix with the diagonal elements of \mathbf{n} , $\mathbf{n}^{(2)}$ contains the squares of the elements of vector \mathbf{n} , $\mathbf{p} = (p_{12}, p_{21})^T$ is the vector of the probabilities of switching from one sample size to another, \mathbf{D}_p is the 2×2 diagonal matrix with the diagonal elements of \mathbf{p} and, finally, $\mathbf{p}^{(2)}$ contains the squares of the elements of vector \mathbf{p} .

At the beginning, when the process is assumed to be in-control ($\boldsymbol{\mu}_1 = \boldsymbol{\mu}_0$ & $\boldsymbol{\Sigma}_1 = \boldsymbol{\Sigma}_0$), b_1 and b_2 are obtained as follows:

$$b_1 = \frac{p_{11}}{p_{11} + p_{12}},$$

$$b_2 = \frac{p_{22}}{p_{21} + p_{22}}.$$

After computing the transition probabilities via the proposed method, the general formulas in Equations (25) – (32) can be used to compute the performance measures.

5. Numerical analyses

In this section, for scenarios 1 and 2 defined in the previous section, by using the Matlab software, we perform numerical analyses in order to compare the adaptive schemes with each other as well as with the FP one. We perform our analyses for several number of quality characteristics p . We also compare our proposed scheme to one of the best existing methods. Finally, in this section, through an illustrative example we show how our scheme can be implemented in practice.

5.1 Performance comparison of the proposed methods: Scenario-1

First, for $p=2$, we assume that $\boldsymbol{\Sigma}_0 = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix} \rightarrow \boldsymbol{\Sigma}_1 = \begin{bmatrix} \tau & 0.5\tau \\ 0.5\tau & \tau \end{bmatrix}$; $\tau > 0$. Since the in-control correlation between the quality characteristics is 0.5 and their variances are equal, then, as it was

numerically tested as well, the following shifts $\mu_0 = (0,0) \rightarrow \mu_1 = (\delta,0)$, $\mu_0 = (0,0) \rightarrow \mu_1 = (0,\delta)$ and $\mu_0 = (0,0) \rightarrow \mu_1 = (\delta,\delta)$, will correspond to the same performance measures values. We also assume that $\delta \in \{0, 0.1, 0.3, 0.7, 2\}$ and $\tau \in \{1, 1.05, 1.2, 1.5, 3\}$.

For the FP scheme, assuming $n=10$, $\alpha=0.005$ (Equation (6) results in $UCL = 3.023$) and $t = 1$, the results for all performance measures can be seen in Table 2. In this Table, the values $ANSW=0$ and $SDNSW=0$ correspond to the FP scheme for which no switching occurs.

For the FP scheme, assuming $n=10$, $\alpha=0.005$ (Equation (6) results in $UCL = 3.023$) and $t = 1$, the results for all performance measures can be seen in Table 1. In this Table, the values $ANSW=0$ and $SDNSW=0$ correspond to the FP scheme for which no switching occurs.

For the VP scheme, assuming $ASS=10$, $ASI=1$, $ATE=0.005$, $\alpha_1 = 0.004$, $t_2 = 0.1$, $n_1 = 5$ and $n_2 = 15$, from Equations (11) – (16) we have: $UCL_1 = 3.0899$, $p_0 = 0.5$, $\alpha_2 = 0.006$, $UCL_2 = 2.9673$, $t_1 = 1.9$, $UWL_1 = 1.0487$ and $UWL_2 = 1.0472$. The results for all performance measures in the VP chart can be seen in Table 2, as well.

For the VSSI scheme, assuming $ASS=10$, $ASI=1$, $\alpha = 0.005$, $t_2 = 0.1$, $n_1 = 5$ and $n_2 = 15$, from Equations (6), (7), (8) and (10) we have: $UCL = 3.023$, $p_0 = 0.5$, $t_1 = 1.9$ and $UWL = 1.0479$.

For the VSS scheme, assuming $ASS=10$, $t = 1$, $\alpha = 0.005$, $n_1 = 5$ and $n_2 = 15$, from Equations (6), (7) and (10) we have: $UCL = 3.023$, $p_0 = 0.5$ and $UWL = 1.0479$.

For the VSI scheme, assuming $ASI=1$, $\alpha = 0.005$, $t_1 = 1.9$, $t_2 = 0.1$, $n = 10$, from Equations (6), (8) and (10) we have: $UCL = 3.023$, $p_0 = 0.5$ and $UWL = 1.0479$.

The results for all performance measures in the VSSI, VSS and VSI charts can also be seen in Table 2. Note that, although the computations have been done for both average and standard deviation types performance measures, in order to decrease the size/number of tables, we only present in this paper the results for the average type performance measures. However, the results for standard deviation type performance measures can be easily available from the corresponding author.

"Insert Table 2 here"

From the results shown in Table 2, as the shift sizes increase, the performance measures' values decrease. Moreover, under small to moderate simultaneous and separate shifts, in terms of time to signal, VP, VSSI, VSI, VSS, and FP charts have the best to the worst performances, meaning the VP chart is the best scheme and the FP chart is the worst one. With the exception of 0.3 mean shift, in which VSS chart has a better time to signal performance than the VSI chart.

Under small to moderate simultaneous and separate shifts, in terms of the run length and number of observations to signal, VP, VSSI=VSS, VSI=FP have the best to the worst performances. In terms of the number of switches to signal, the order is FP, VP, VSSI=VSS and VSI.

Under large shifts, all the adaptive and non-adaptive charts have almost the same performances.

For $p=3$, we assume $\Sigma_0 = \begin{bmatrix} 1 & 0.5 & 0.5 \\ 0.5 & 1 & 0.5 \\ 0.5 & 0.5 & 1 \end{bmatrix} \rightarrow \Sigma_1 = \begin{bmatrix} \tau & 0.5\tau & 0.5\tau \\ 0.5\tau & \tau & 0.5\tau \\ 0.5\tau & 0.5\tau & \tau \end{bmatrix}; \tau > 0$. Again, as for the case

$p=2$, the following shifts, $\mu_0 = (0,0,0) \rightarrow \mu_1 = (\delta,0,0)$, $\mu_0 = (0,0,0) \rightarrow \mu_1 = (0,\delta,0)$, $\mu_0 = (0,0,0) \rightarrow \mu_1 = (0,0,\delta)$ and $\mu_0 = (0,0,0) \rightarrow \mu_1 = (\delta,\delta,\delta)$ correspond to the same performance measures values.

With the same methodology used for $p=2$, one can see the results for the FP, VP, VSSI, VSS and VSI charts, for $p=3$ in Table 3. All the conclusions for $p=2$ (Table 2) can be drawn for the $p=3$ case from Table 3 as well. Moreover, as it is clear from the results presented in Tables 2 and 3, as the number of quality characteristics increases, under small to relatively large mean shifts, the performance measures worsen. However, under small to large variance shifts, the charts performance improves and also, in most of the cases, under simultaneous shifts, the charts performances improve, except under 1.05 variance and 0.3 mean shifts, in which the performances deteriorate and also under 1.05 variance and 0.7 mean shifts, for which the performances only deteriorate for the FP and VSI charts.

"Insert Table 3 here"

5.2 Performance comparison of the proposed methods: Scenario-2

In this scenario, we assume that all the assumptions are the same as for the first scenario, except for Σ_1 . In this scenario, we assume two cases for the shift in variance-covariance matrix.

5.2.1 Scenario-2. Case 1: Mean and Variance Shifts

In the first case, only the variance components increase in the variance-covariance matrix such as:

$$\text{For } p=2: \Sigma_0 = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix} \rightarrow \Sigma_1 = \begin{bmatrix} \tau_1 & 0.5 \\ 0.5 & \tau_2 \end{bmatrix}; \theta_1, \theta_2 \geq 1 \text{ and}$$

$$\text{For } p=3: \Sigma_0 = \begin{bmatrix} 1 & 0.5 & 0.5 \\ 0.5 & 1 & 0.5 \\ 0.5 & 0.5 & 1 \end{bmatrix} \rightarrow \Sigma_1 = \begin{bmatrix} \tau_1 & 0.5 & 0.5 \\ 0.5 & \tau_2 & 0.5 \\ 0.5 & 0.5 & \tau_3 \end{bmatrix}; \tau_1, \tau_2, \tau_3 \geq 1.$$

For $p=2$, we assume that:

$$\tau = (\tau_1, \tau_2) \in \{(1,1), (1.05,1), (1.05,1.05), (1.05,1.3), (1.4,1.4), (3,3)\}.$$

We also assume that:

$$\mu_0 = (0,0) \rightarrow \mu_1 = (\delta_1, \delta_2) \text{ so that } (\delta_1, \delta_2) \in \{(0,0), (0.1,0), (0.1,0.3), (0.5,0.8), (2,2)\}.$$

For $p=3$ we have:

$$\tau = (\tau_1, \tau_2, \tau_3) \in \{(1,1,1), (1.05,1,1), (1.05,1.05,1.05), (1.05,1.3,1.05), (1.4,1.4,1.05), (3,3,1.4)\},$$

$$\mu_0 = (0,0,0) \rightarrow \mu_1 = (\delta_1, \delta_2, \delta_3) \in \{(0,0,0), (0.1,0,0.1), (0.1,0.3,0.3), (0.5,0.8,0.5), (2,2,2)\}.$$

The results for the scenario 2 case 1 for $p=2$ and 3 can be seen in Tables 4 and 5, respectively. The conclusions that have been drawn for scenario 1 (Tables 2 and 3) can be applied for this scenario as well (see Tables 4 and 5). Since in this scenario the shift sizes for $p=2$ and 3 are assumed different, there is no systematic trend in the charts' performances when the number of quality characteristics changes.

"Insert Tables 4 and 5 here"

5.2.2 Scenario-2. Case 2: Mean and Covariance shifts

In the second case, we assume that only the covariance component shifts in the variance-covariance matrix, such as:

$$\Sigma_0 = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix} / \begin{bmatrix} 1 & 0.5 & 0.5 \\ 0.5 & 1 & 0.5 \\ 0.5 & 0.5 & 1 \end{bmatrix} / \begin{bmatrix} 1 & 0.5 & 0.5 & 0.5 \\ 0.5 & 1 & 0.5 & 0.5 \\ 0.5 & 0.5 & 1 & 0.5 \\ 0.5 & 0.5 & 0.5 & 1 \end{bmatrix} \rightarrow \Sigma_1 = \begin{bmatrix} 1 & \theta \\ \theta & 1 \end{bmatrix} / \begin{bmatrix} 1 & \theta & \theta \\ \theta & 1 & \theta \\ \theta & \theta & 1 \end{bmatrix} / \begin{bmatrix} 1 & \theta & \theta & \theta \\ \theta & 1 & \theta & \theta \\ \theta & \theta & 1 & \theta \\ \theta & \theta & \theta & 1 \end{bmatrix};$$

$\theta \in \{0.01, 0.02, \dots, 0.99\}$, for $p=2, 3$ and 4 , respectively.

For this analysis, since the trends are the same for all performance measures, in order to decrease the size of the paper we only used the most important, i.e. the ATS, performance measure in order to compare the charts. We also assume that the mean shifts for all quality characteristics are equal $\mu_0 = (0, 0) / (0, 0, 0) / (0, 0, 0, 0) \rightarrow \mu_1 = (\delta, \delta) / (\delta, \delta, \delta) / (\delta, \delta, \delta, \delta)$, under none ($\delta = 0$), small ($\delta = 0.2$), medium ($\delta = 0.8$) and large ($\delta = 1.5$) mean shifts. All other assumptions are as assumed similar as before. In Figures 1 to 3, the performances of all non-adaptive and adaptive charts are compared for $p=2, 3$ and 4 , respectively. Since the in-control covariance is 0.5, and the variances are assumed fixed, logically, any deviation from the in-control covariance, should result in less (better) performance measure value than the in-control one. This happens for all the charts for covariance shifts less than 0.5. However, for covariance shifts more than the in-control value (0.5), the charts' performance values increase up to some point and decreases afterward. That point, goes further away from 0.5 as the mean shifts increase and gets near to 0.5 as p increases (see Figures 1 to 3). Moreover, in case of large mean shifts, the performances of all the charts remain almost fixed with shifts in the covariance.

"Insert Figures 1 to 3 here"

5.3 Performance comparison of the proposed methods with the ELR Chart

In this section we compare our proposed scheme with the one proposed by Zhang et al. (2010) which should be considered as one of the most powerful scheme available for the simultaneous monitoring of the process mean and variability as it based on the combination of the EWMA and GLR frameworks. In their paper, they compared their single-chart scheme with some available charts and they concluded that their chart surpasses them in many situations. However, as for any EWMA schemes, the choice of the optimal value for the smoothing parameter λ is crucial for the chart to have an optimal efficiency and, unfortunately this optimal value actually depends on the size of the shift which is generally unknown. These kinds of issues do not exist in our proposed scheme. For all the charts considered for the comparison we assumed that the in-control ATS=370 and $p=2$. Also, for the FP chart we used: $\alpha = 0.0027$ (to have the in-control ATS=370), $n=5, t=1$, for the VP chart we used: $ASS=5, ASI=1, ATE=0.0027$ (to have the in-control ATS=370), $\alpha_1 = 0.0017, t_2 = 0.1, n_1 = 3, n_2 = 7$. The ATS for the ELR scheme is also obtained with $n=5, t=1$ and for different λ values (0.1, 0.2, 0.3 and 0.4). In addition, the parameters used in Table 5 to represent the shifts are:

$$\mu_0 = (0, 0) \rightarrow \mu_1 = (\delta_1, \delta_2) \text{ and } \Sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow \Sigma_1 = \begin{bmatrix} \tau_1 & \rho\sqrt{\tau_1\tau_2} \\ \rho\sqrt{\tau_1\tau_2} & \tau_2 \end{bmatrix}.$$

Moreover, the performance of the ELR chart has been obtained based on 10000 simulation runs. We used 30 different separate as well as simultaneous shifts and, as it can be easily seen in Table 6, in many situations the proposed scheme surpasses the ELR chart. More precisely, among the 30 situations investigated, the scheme proposed in this paper is better than the ELR chart in 16 cases, i.e.

about 50% of the time. However, unlike the ELR scheme (as explained above), our scheme can be easily implemented in practice.

"Insert Tables 6 here"

5.4 An illustrative example

For illustration of the proposed adaptive method, and to show how it works, the illustrative example presented by Aparisi (1996) is adopted. We have a mechanical part with $p=3$ main quality characteristics as it is shown in Figure 4. The quality characteristics are the width X_1 , the inner diameter X_2 and the length X_3 .

"Insert Figure 4 here"

Based on Aparisi (1996)'s assumptions, the average Type-I error is $ATE = 0.005$ and the mean vector and the variance-covariance matrix of the quality characteristics are:

$$\boldsymbol{\mu}_0 = (7, 3, 15)', \boldsymbol{\Sigma}_0 = \begin{bmatrix} 0.2 & 0.054 & 0.162 \\ 0.054 & 0.09 & 0.042 \\ 0.162 & 0.042 & 0.31 \end{bmatrix}.$$

Since it was proven in the previous sections that the VP scheme has the best performance, we adopt this scheme for our process monitoring. We use the VP strategy explained in section 3 and all the other chart parameters values related to the adaptive scheme are assumed as (to be determined by quality experts): $ASS=10$, $ASI=1hr$, $\alpha_1=0.004$, $t_2=0.1$, $n_1=5$ and $n_2=15$. Having the assumed parameters values, by using Equation (11) we obtain $UCL_1 = 3.0899$, by using Equation (14) we obtain $UCL_2 = 2.9673$, by using Equation (15) we obtain $t_1 = 1.9 hrs$, and at last, by using Equation (16) we obtain $UWL_1 = 1.0487$ and $UWL_2 = 1.0472$.

For the second to the last columns of Table 7, respectively we have: the number of samples taken for the current sample, the cumulative number of samples taken up to the current sample, the sampling interval adopted for reaching the current sample, the cumulative sampling intervals up to the current sample, the total number of switches between the in-control states up to the current sample, the M_i , V_i and C_i statistics values for the current sample (which are computed by using Equations (3), (4) and (5), respectively), UWL and UCL adopted for the current sample, and finally, the state of the process after taking the current sample.

Before beginning the process monitoring, we randomly shift the mean and variability as:

$$\boldsymbol{\mu}_1 = (7.05, 3.01, 15)', \boldsymbol{\Sigma}_1 = \begin{bmatrix} 0.2 & 0.0545 & 0.162 \\ 0.0545 & 0.095 & 0.041 \\ 0.162 & 0.041 & 0.31 \end{bmatrix}.$$

We have generated 20 consecutive samples for the process monitoring. As it can be seen in Table 7, the proposed VP chart was able to find an out-of-control situation in sample #17; after 10.6 hours, 205 samples measured and with three shifts between in-control states. After finding an out-of-control situation, quality practitioners should investigate the cause and find the problem source.

"Insert Table 7 here"

In addition, the proposed VP control chart can also be seen graphically (plotted using MS Excel) in Figure 5.

"Insert Figure 5 here"

6. Conclusions and future research

A new multivariate control chart based on an already existing bivariate chart has been introduced in order to monitor both process parameters (mean and variation) simultaneously and by using only a single max-type chart. After establishing the multivariate control chart, adaptive schemes have been proposed to make the chart's performance better in detecting small and moderate shifts. In order to measure the charts' performances and to compare the non-adaptive with the adaptive charts, eight performance measures have been used, each measuring one aspect of the performance. The average run length and its standard deviation, the average time to signal its standard deviation, the average number of observations to signal and its standard deviation, the average number of switches to signal and its standard deviation are the performance measures used. In order to compute these performance measures, a new Markov chain model has been proposed with the derivation of the corresponding transition probabilities. We used two scenarios for the variance-covariance matrix shifts. In the first scenario, the shifted variance-covariance matrix is a constant multiplier of the in-control one. In the second scenario, the more general one, the elements in the variance-covariance matrix are allowed to change independently. We also considered two cases: in the first, only the variances shift but the covariance components stay unchanged. However, in the second case, only the covariance components shift. Through extensive numerical analysis, adaptive schemes were found to perform much better under small to moderate separate and simultaneous shifts in all scenarios; VP and VSSI charts, respectively, having the best performances. However, the adaptive and FP charts almost have the same performance under large shifts. In the case two of the second scenario, in which only the covariance shifts and the variances remain fixed, we concluded that an increasing shift in the covariance components, would result in negative performance of all the charts up to some point (covariance value). However, that point gets closer to the in-control value as the shift in the mean decreases (except for large mean shifts in which the chart performance is almost insensible to the covariance change) or, also, as the number of quality characteristics increases. Moreover, our results showed that in the first scenario, in which the variance and covariance components shift with an equal multiplier, with the increase in the number of quality characteristics, as the mean shift increases to a relatively large one, the performance measures' values increase. However, as the variability shift increases and also in most simultaneous mean and variability shifts, the performance measures improve. In addition, we compared the proposed scheme with the ELR scheme and the results showed that our scheme performs at least as good as the ELR method and can be easily implemented in practice, unlike the ELR scheme. Finally, through a numerical example, the implementation of the proposed adaptive scheme was shown in practice.

Concerning possible extensions of this paper, researchers can apply the proposed adaptive schemes and also the proposed Markov chain model for other types simultaneous control charts, especially max-type ones. Moreover, the effect of estimated parameters, measurement errors as well as autocorrelation can be considered on the performance of the proposed control charts as future research.

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Table 1 Literature review in comparison with our paper

Paper	Control chart type	Variables	Adaptive	Simultaneous	Computation method of performance measures
Aparisi (1996)	VSS Hotelling's T^2	Multivariate	✓	-	Markov chains
Aparisi et al. (1999)	$ s $	Multivariate	-	-	Probabilistic relations
Aparisi and Haro (2001)	VSI Hotelling's T^2	Multivariate	✓	-	Markov chains
Aparisi et al. (2001)	VSS $ s $	Bivariate	✓	-	Markov chains
Yeh and Lin (2002)	Box	Multivariate	-	✓	Monte Carlo simulation
Yeh et al. (2003)	EWMA-V , EWMA-M	Multivariate	-	✓	Monte Carlo simulation
Aparisi and Haro (2003)	VSSI Hotelling's T^2	Multivariate	✓	-	Markov chains
Khoo (2005)	a Max-Type Chart	Bivariate	-	✓	Monte Carlo simulation
Grigoryan and He (2005)	MDS $ s $	Multivariate	✓	-	Monte Carlo simulation
Chen et al. (2005)	Max-MEWMA	Multivariate	-	✓	Monte Carlo simulation
Reynolds and Cho (2006)	MEWMA-SSDM	Multivariate	-	✓	Monte Carlo simulation
Faraz and Parsian (2006)	VSSI-DWL Hotelling T^2	Multivariate	✓	-	Markov chains
Chen (2007)	VP Hotelling T^2	Multivariate	✓	-	Probabilistic relations
Reynolds and Kim (2007)	SS Shewharts&MEWMA	Multivariate	✓	✓	Monte Carlo simulation
Hawkins and Maboudou-Tchao (2008)	MEWMA-MEWMC	Multivariate	-	✓	Monte Carlo simulation
Zhang and Chang (2008)	two MEWMA (individual observations)	Multivariate	-	✓	Monte Carlo simulation
Faraz and Moghadam (2009)	VSS Hotelling's T^2	Multivariate	✓	-	Markov chains
Zhang et al. (2010)	ELR	Multivariate	-	✓	Monte Carlo simulation
Faraz et al. (2011)	VSI Hotelling's T^2	Multivariate	✓	-	Markov chains
Reynolds and Cho (2011)	VSI-SS Shewharts&MEWMA	Multivariate	✓	✓	Monte Carlo simulation
Khoo et al. (2013)	Synthetic Double Sampling Hotelling's T^2	Multivariate	✓	-	Markov chains
Faraz et al. (2014)	VP Hotelling's T^2	Multivariate	✓	-	Probabilistic relations
Wang et al. (2014)	pGLR-pMEWMA (individual observations)	Multivariate	-	✓	Monte Carlo simulation
Lee and Khoo (2015)	VSI Synthetic $ s $	Multivariate	✓	-	Probabilistic relations
Seif et al. (2016)	VP-VSSCL-VSICL Hotelling T^2	Multivariate	✓	-	Markov chains
Costa and Neto (2017)	VCS S	Trivariate	✓	-	Probabilistic relations
Sabahno et al. (2018)	VSI Hotelling's T^2 (with measurement errors)	Multivariate	✓	-	Markov chains
Sabahno et al. (2019)	VSS Hotelling's T^2 (with measurement errors)	Multivariate	✓	-	Markov chains
Chong et al. (2019)	VSS Hotelling T^2 (with finite horizon)	Multivariate	✓	-	Markov chains
Current Paper	a new VP / VSS / VSS / VSI Max Type Chart	Multivariate	✓	✓	a new Markov chain model

Table 2- Performance measures, for adaptive and non-adaptive schemes and for $p=2$ using Scenario I.

τ	I						1.05						δ
	VP	VSSI	VSS	VSI	FP	VP	VSSI	VSS	VSI	FP			
	200.0000	200.0000	200.0000	200.0000	200.0000	162.3187	163.7648	163.7648	163.7649	163.7649	ARL	0	
	200.0000	200.0000	200.0000	200.0000	200.0000	158.4435	159.8506	163.7648	159.8549	163.7649	ATS	0	
	2000	2000	2000	2000	2000	1644.7	1659.4	1659.4	1637.6	1637.649	ANOS	0	
	99.5000	99.5000	99.5	99.5000	0	80.3043	81.0088	81.0088	81.3248	0	ANSW	0	
	178.7717	180.6405	180.6405	181.1620	181.1620	144.2625	146.9438	146.9438	147.6008	147.6008	ARL	0.1	
	176.5693	178.4073	180.6405	178.9487	181.1620	138.9601	141.5306	146.9438	142.2147	147.6008	ATS	0.1	
	1800	1818.8	1818.8	1811.6	1811.620	1472.1	1499.5	1499.5	1476	1476.008	ANOS	0.1	
	88.2583	89.1778	89.1778	90.0643	0	70.7472	72.0492	72.0492	73.1791	0	ANSW	0.1	
	59.7137	63.9442	63.9442	74.3715	74.3715	49.9569	53.4155	53.4155	61.9614	61.9614	ARL	0.3	
	51.9333	55.5488	63.9442	65.3187	74.3715	42.2747	45.1392	53.4155	53.0002	61.9614	ATS	0.3	
	640.3619	686.0836	686.0836	743.7149	743.715	542.2483	580.1342	580.1342	619.6145	619.614	ANOS	0.3	
	26.7014	28.5766	28.5766	36.0055	0	22.0569	23.5638	23.5638	29.6807	0	ANSW	0.3	
	3.4748	3.6072	3.6072	5.6445	5.6445	3.4100	3.5337	3.5337	5.3831	5.3831	ARL	0.7	
	1.9980	2.0291	3.6072	2.8686	5.6445	1.9729	2.0020	3.5337	2.7654	5.3831	ATS	0.7	
	42.9527	44.8388	44.8388	56.4446	56.444	42.0845	43.8458	43.8458	53.8309	53.831	ANOS	0.7	
	0.7548	0.7630	0.7630	1.4795	0	0.7494	0.7565	0.7565	1.4058	0	ANSW	0.7	
	1.0295	1.0261	1.0261	1.0001	1.0001	1.0314	1.0279	1.0279	1.0001	1.0001	ARL	2	
	1.0032	1.0029	1.0261	1.0000	1.0001	1.0035	1.0031	1.0279	1.0000	1.0001	ATS	2	
	10.4414	10.3896	10.3896	10.0009	10.0009	10.4698	10.4164	10.4164	10.0013	10.001	ANOS	2	
	0.0294	0.0259	0.0259	0.0004	0	0.0313	0.0277	0.0277	0.0000	0	ANSW	2	

Table 2(Continued).

T	1.2						1.5						3					
	VP	VSSI	VSS	VSI	FP		VP	VSSI	VSS	VSI	FP		VP	VSSI	VSS	VSI	FP	
61.2081	64.3712	64.3712	67.7209	67.7209	67.7209		10.4080	11.0533	11.0533	14.2970	14.2970		1.4418	1.4372	1.4372	1.3940	1.3940	
52.6587	55.3380	64.3712	58.5297	67.7209	67.7209		6.4337	6.7692	11.0533	9.0462	14.2970		1.1315	1.1309	1.4372	1.0735	1.3940	
659.5779	693.8959	693.8959	677.2086	677.209	677.209		126.1587	134.3329	134.3329	142.9695	142.970		16.1410	16.0739	16.0739	13.9404	13.940	
28.1096	29.5358	29.5358	32.5904	0	0		3.3386	3.5190	3.5190	5.4581	0		0.3114	0.3009	0.3009	0.1515	0	
56.1523	59.2546	59.2546	62.7476	62.7476	62.7476		10.0821	10.7093	10.7093	13.8628	13.8628		1.4395	1.4349	1.4349	1.3908	1.3908	
47.5871	50.1704	59.2546	53.4869	62.7476	62.7476		6.1621	6.4815	10.7093	8.6776	13.8628		1.1305	1.1299	1.4349	1.0726	1.3908	
609.1077	643.0133	643.0133	627.4757	627.476	627.476		122.5992	130.5807	130.5807	138.6276	138.628		16.1113	16.0434	16.0434	13.9076	13.908	
25.4679	26.8482	26.8482	30.0301	0	0		3.1839	3.3551	3.3551	5.2342	0		0.3104	0.2999	0.2999	0.1505	0	
26.5336	28.4255	28.4255	33.5879	33.5879	33.5879		7.6318	8.0942	8.0942	10.6539	10.6539		1.4215	1.4165	1.4165	1.3649	1.3649	
19.7460	21.0867	28.4255	25.5019	33.5879	33.5879		4.3215	4.5192	8.0942	6.1604	10.6539		1.1227	1.1221	1.4165	1.0656	1.3649	
303.0453	325.0268	325.0268	335.8786	335.879	335.879		94.7093	100.8027	100.8027	106.5389	106.539		15.8751	15.8007	15.8007	13.6487	13.649	
10.6145	11.3447	11.3447	15.0923	0	0		2.1251	2.2263	2.2263	3.6570	0		0.3028	0.2923	0.2923	0.1419	0	
3.1811	3.2830	3.2830	4.6837	4.6837	4.6837		2.6130	2.6788	2.6788	3.4703	3.4703		1.3399	1.3328	1.3328	1.2500	1.2500	
1.8468	1.8696	3.2830	2.4207	4.6837	4.6837		1.5421	1.5534	2.6788	1.7859	3.4703		1.0908	1.0900	1.3328	1.0393	1.2500	
39.2232	40.6814	40.6814	46.8367	46.837	46.837		32.0789	33.0406	33.0406	34.7031	34.703		14.7824	14.6774	14.6774	12.5002	12.500	
0.6993	0.7025	0.7025	1.1669	0	0		0.5501	0.5461	0.5461	0.7305	0		0.2655	0.2548	0.2548	0.1031	0	
1.0368	1.0330	1.0330	1.0003	1.0003	1.0003		1.0456	1.0414	1.0414	1.0009	1.0009		1.0542	1.0501	1.0501	1.0037	1.0037	
1.0043	1.0039	1.0330	1.0000	1.0003	1.0003		1.0058	1.0054	1.0414	1.0001	1.0009		1.0083	1.0079	1.0501	1.0004	1.0037	
10.5488	10.4917	10.4917	10.0029	10.003	10.003		10.6767	10.6149	10.6149	10.0093	10.009		10.7966	10.7352	10.7352	10.0366	10.037	
0.0365	0.0327	0.0327	0.0001	0	0		0.0449	0.0408	0.0408	0.0004	0		0.0525	0.0484	0.0484	0.0018	0	

Table 3- Performance measures, for adaptive and non-adaptive schemes and for $p=3$ using Scenario1.

τ	1						1.05						δ
	VP	VSSI	VSS	VSI	FP	FP	VP	VSSI	VSS	VSI	FP	FP	
200.0000	200.0000	200.0000	200.0000	200.0000	200.0000	200.0000	158.6859	160.4929	160.4929	160.5663	160.5663	160.5663	0
200.0000	200.0000	200.0000	200.0000	200.0000	200.0000	200.0000	154.5204	156.2734	160.4929	156.3547	160.5663	160.5663	0
2000	2000	2000	2000	2000	2000	2000	1610	1628.4	1628.4	1605.7	1605.663	1605.663	0
99.5	99.5000	99.5	99.5	99.5	0	0	78.3376	79.2186	79.2186	79.7150	0	0	0
182.7049	184.2645	184.2645	184.7168	184.7168	184.7168	184.7168	143.8069	146.5934	146.5934	147.2521	147.2521	147.2521	0.1
180.9167	182.4547	184.2645	182.9291	184.7168	184.7168	184.7168	138.4799	141.1501	146.5934	141.8394	147.2521	147.2521	0.1
1837	1852.7	1852.7	1847.2	1847.2	1847.2	1847.2	1467.7	1496.2	1496.2	1472.5	1472.521	1472.521	0.1
90.3386	91.1073	91.1073	91.1073	91.8477	0	0	70.4667	71.8193	71.8193	73.0032	0	0	0
66.4058	70.9646	70.9646	82.1712	82.1712	82.1712	82.1712	53.7655	57.4441	57.4441	66.4516	66.4516	66.4516	0.3
58.7842	62.7584	70.9646	73.4416	82.1712	82.1712	82.1712	46.0415	49.1304	57.4441	57.5226	66.4516	66.4516	0.3
706.4002	755.2359	755.2359	821.7118	821.712	821.712	821.712	580.5663	620.6279	620.6279	664.5162	664.516	664.516	0.3
30.0454	32.0931	32.0931	40.0131	40.0131	0	0	23.9231	25.5407	25.5407	31.9852	0	0	0
3.5011	3.6336	3.6336	5.8219	5.8219	5.8219	5.8219	3.4098	3.5320	3.5320	5.4875	5.4875	5.4875	0.7
2.0358	2.0675	3.6336	2.9877	5.8219	5.8219	5.8219	1.9912	2.0201	3.5320	2.8351	5.4875	5.4875	0.7
43.1516	45.0364	45.0364	58.2188	58.219	58.219	58.219	41.9794	43.7194	43.7194	54.8749	54.875	54.875	0.7
0.7631	0.7718	0.7718	1.5593	0	0	0	0.7500	0.7572	0.7572	1.4523	0	0	0
1.0243	1.0214	1.0214	1.0000	1.0000	1.0000	1.0000	1.0258	1.0228	1.0228	1.0001	1.0001	1.0001	2
1.0026	1.0023	1.0214	1.0000	1.0000	1.0000	1.0000	1.0028	1.0025	1.0228	1.0000	1.0001	1.0001	2
10.3632	10.3195	10.3195	10.0004	10.0004	10.0004	10.0004	10.3856	10.3407	10.3407	10.0006	10.001	10.001	2
0.0242	0.0213	0.0213	0.0000	0.0000	0	0	0.0257	0.0227	0.0227	0.0000	0	0	2

Table 3(Continued).

T	1.2						1.5						3					
	VP	VSSI	VSS	VSI	FP		VP	VSSI	VSS	VSI	FP		VP	VSSI	VSS	VSI	FP	
50.5153	53.6914	53.6914	58.4969	58.4969	58.4969		7.2150	7.6364	7.6364	10.7879	10.7879		1.3558	1.3470	1.3470	1.1980	1.1980	
42.1338	44.7286	53.6914	49.1617	58.4969	58.4969		4.1563	4.3353	7.6364	6.2654	10.7879		1.1044	1.1034	1.3470	1.0294	1.1980	
551.7162	586.7073	586.7073	584.9686	584.969	584.969		89.1423	94.7039	94.7039	107.8789	107.879		14.9552	14.8238	14.8238	11.9803	11.980	
22.4155	23.8000	23.8000	27.8288	0	0		1.9787	2.0690	2.0690	3.7236	0		0.2948	0.2840	0.2840	0.0844	0	
47.1492	50.2188	50.2188	54.9973	54.9973	54.9973		7.0566	7.4688	7.4688	10.5383	10.5383		1.3543	1.3454	1.3454	1.1964	1.1964	
38.7828	41.2506	50.2188	45.6435	54.9973	54.9973		4.0302	4.2017	7.4688	6.0638	10.5383		1.1037	1.1027	1.3454	1.0291	1.1964	
517.9728	552.0109	552.0109	549.9726	549.973	549.973		87.3790	92.8390	92.8390	105.3828	105.383		14.9350	14.8032	14.8032	11.9641	11.964	
20.6747	21.9954	21.9954	26.0166	0	0		1.9086	1.9946	1.9946	3.5964	0		0.2939	0.2831	0.2831	0.0838	0	
24.6425	26.4978	26.4978	31.9631	31.9631	31.9631		5.7735	6.0980	6.0980	8.5651	8.5651		1.3418	1.3328	1.3328	1.1836	1.1836	
17.9113	19.1871	26.4978	23.7985	31.9631	31.9631		3.1257	3.2391	6.0980	4.6073	8.5651		1.0977	1.0967	1.3328	1.0266	1.1836	
283.8208	305.5935	305.5935	319.6313	319.631	319.631		72.4453	76.8630	76.8630	85.6511	85.651		14.7738	14.6394	14.6394	11.8360	11.836	
9.5539	10.2466	10.2466	14.1942	0	0		1.4023	1.4547	1.4547	2.6536	0		0.2861	0.2753	0.2753	0.0789	0	
3.1011	3.1982	3.1982	4.6093	4.6093	4.6093		2.4054	2.4604	2.4604	3.1679	3.1679		1.2835	1.2739	1.2739	1.1257	1.1257	
1.8071	1.8277	3.1982	2.3609	4.6093	4.6093		1.4544	1.4625	2.4604	1.6196	3.1679		1.0729	1.0718	1.2739	1.0165	1.1257	
38.2006	39.5961	39.5961	46.0932	46.093	46.093		29.3377	30.1484	30.1484	31.6791	31.679		14.0051	13.8614	13.8614	11.2568	11.257	
0.6724	0.6746	0.6746	1.1276	0	0		0.5019	0.4963	0.4963	0.6168	0		0.2476	0.2368	0.2368	0.0563	0	
1.0300	1.0268	1.0268	1.0001	1.0001	1.0001		1.0366	1.0331	1.0331	1.0004	1.0004		1.0409	1.0376	1.0376	1.0012	1.0012	
1.0034	1.0031	1.0268	1.0000	1.0001	1.0001		1.0045	1.0042	1.0331	1.0000	1.0004		1.0059	1.0055	1.0376	1.0001	1.0012	
10.4471	10.3992	10.3992	10.0014	10.001	10.001		10.5439	10.4922	10.4922	10.0045	10.004		10.6033	10.5537	10.5537	10.0125	10.012	
0.0297	0.0265	0.0265	0.0000	0	0		0.0361	0.0327	0.0327	0.0002	0		0.0399	0.0366	0.0366	0.0006	0	

Table 4- Performance measures, for adaptive and non-adaptive schemes and for $p=2$ using Scenario2.

τ	1.05,1										1.05,1.05									
	VP	VSSI	VSS	VSI	FP	VP	VSSI	VSS	VSI	FP	VP	VSSI	VSS	VSI	FP	VP	VSSI	VSS	VSI	FP
200.0000	200.0000	200.0000	200.0000	200.0000	200.0000	176.3709	177.2259	177.2259	177.1545	177.1545	148.8315	150.8209	150.8209	150.9839	150.9839	AKL				
200.0000	200.0000	200.0000	200.0000	200.0000	200.0000	173.8063	174.6469	177.2259	174.5736	177.1545	143.8324	145.7475	150.8209	145.9231	150.9839	ATS				
2000	2000	2000	2000	2000	2000	1778	1786.6	1786.6	1771.5	1771.5	1516.1	1536.4	1536.4	1509.8	1509.8	ANOS				
99.5	99.5	99.5000	99.5	0	0	87.5190	87.9359	87.9359	88.0540	0	73.3484	74.3149	74.3149	74.8872	0	ANSW				
178.7717	180.6405	180.6405	181.1620	181.1620	181.1620	156.8174	159.1694	159.1694	159.7190	159.7190	132.3925	135.3658	135.3658	136.1955	136.1955	AKL				
176.5693	178.4073	180.6405	178.9487	181.1620	181.1620	152.5422	154.8202	159.1694	155.3920	159.7190	126.2244	129.0441	135.3658	129.9060	136.1955	ATS				
1800	1818.8	1818.8	1811.6	1811.620	1811.620	1591.9	1615.9	1615.9	1597.2	1597.2	1358.2	1388.8	1388.8	1362	1362	ANOS				
88.2583	89.1778	89.1778	90.0643	0	0	77.1660	78.3136	78.3136	79.2871	0	64.6517	66.0880	66.0880	67.4185	0	ANSW				
78.6411	83.5169	83.5169	92.6841	92.6841	92.6841	69.5098	73.8061	73.8061	81.8971	81.8971	60.4841	64.2722	64.2722	71.4168	71.4168	AKL				
70.8768	75.2174	83.5169	84.0437	92.6841	92.6841	61.5943	65.3480	73.8061	73.0501	81.8971	52.4538	55.6847	64.2722	62.4015	71.4168	ATS				
829.5462	881.2768	881.2768	926.8413	926.8413	926.8413	739.0720	785.0498	785.0498	818.9714	818.9714	649.4534	690.4304	690.4304	714.1683	714.1683	ANOS				
36.2590	38.4926	38.4926	45.3448	0	0	31.8427	33.7932	33.7932	39.8586	0	27.3992	29.0949	29.0949	34.5059	0	ANSW				
3.4748	3.6072	3.6072	5.6445	5.6445	5.6445	3.4330	3.5596	3.5596	5.4709	5.4709	3.3878	3.5088	3.5088	5.3025	5.3025	AKL				
1.9980	2.0291	3.6072	2.8686	5.6445	5.6445	1.9828	2.0126	3.5596	2.8020	5.4709	1.9625	1.9910	3.5088	2.7303	5.3025	ATS				
42.9527	44.8388	44.8388	56.4446	56.4446	56.4446	42.3869	44.1895	44.1895	54.7088	54.7088	41.7968	43.5209	43.5209	53.0252	53.0252	ANOS				
0.7548	0.7630	0.7630	1.4795	0	0	0.7520	0.7595	0.7595	1.4318	0	0.7462	0.7529	0.7529	1.3810	0	ANSW				
1.0295	1.0261	1.0261	1.0001	1.0001	1.0001	1.0308	1.0273	1.0273	1.0001	1.0001	1.0320	1.0285	1.0285	1.0001	1.0001	AKL				
1.0032	1.0029	1.0261	1.0000	1.0001	1.0001	1.0034	1.0030	1.0273	1.0000	1.0001	1.0035	1.0032	1.0285	1.0000	1.0001	ATS				
10.4414	10.3896	10.3896	10.0009	10.0009	10.0009	10.4602	10.4073	10.4073	10.0011	10.0011	10.4788	10.4249	10.4249	10.0014	10.0014	ANOS				
0.0294	0.0259	0.0259	0.0000	0	0	0.0306	0.0271	0.0271	0.0000	0	0.0319	0.0283	0.0283	0.0000	0	ANSW				

Table 5- Performance measures, for adaptive and non-adaptive schemes and for $p=3$ using Scenario 2.

τ	1.0,5,1,1										1.0,5,1.05										μ_1
	VP	VSSI	VSS	VSI	FP	VP	VSSI	VSS	VSI	FP	VP	VSSI	VSS	VSI	FP	VP	VSSI	VSS	VSI	FP	
200	200.0000	200.0000	200.0000	200.0000	200.0000	181.7259	182.4323	182.4323	182.34946	182.3494	135.7811	138.5664	138.5664	139.1338	139.1338	AKL	0,				
200	200.0000	200.0000	200.0000	200.0000	200.0000	179.7153	180.4123	182.4323	180.3264	182.3494	129.8405	132.4904	138.5664	133.0807	139.1338	AKL	0,				
2000	2000	20000	2000	2000	2000	1828.4	1835.5	1835.5	1823.5	1823.5	1390.8	1419.4	1419.4	1391.3	1391.3	AKL	0,				
99.5000	99.5000	99.5000	99.5000	0	0	90.2241	90.5691	90.5691	90.6609	0	66.4422	67.7898	67.7898	68.9043	0	AKL	0				
176.6876	178.7349	178.7349	179.5145	179.5145	179.5145	159.3611	161.8201	161.8201	162.5939	162.5939	119.0124	122.5791	122.5791	123.9999	123.9999	AKL	0.1,				
174.3430	176.3544	178.7349	177.1689	179.5145	179.5145	155.3729	157.7597	161.8201	158.5653	162.5939	112.0182	115.3536	122.5791	116.8132	123.9999	AKL	0,				
1779.9	1800.6	1800.6	1795.1	1795.1	1795.1	1615.8	1640.8	1640.8	1625.9	1625.9	1229	1265.9	1265.9	1240.0	1240.0	AKL	0.1				
87.1614	88.1681	88.1681	89.2383	0	0	78.3909	79.5921	79.5921	80.7353	0	57.5963	59.3054	59.3054	61.2428	0	AKL	0.1				
66.4058	70.9646	70.9646	82.1712	82.1712	82.1712	60.1903	64.2920	64.2920	74.3673	74.3673	47.9112	51.2287	51.2287	59.3564	59.3564	AKL	0.1,				
58.7842	62.7584	70.9646	73.4416	82.1712	82.1712	52.5167	56.0347	64.2920	65.5379	74.3673	40.1793	42.8991	51.2287	50.3767	59.3564	AKL	0.3,				
706.4002	755.2359	755.2359	821.7118	821.7118	821.7118	644.5	688.8	688.8	743.7	743.7	522.1	558.6	558.6	593.6	593.6	AKL	0.3				
30.0454	32.0931	32.0931	40.0131	0	0	27.0646	28.8918	28.8918	36.0365	0	21.0277	22.4627	22.4627	28.3396	0	AKL	0.5,				
4.1623	4.3468	4.3468	7.1352	7.1352	7.1352	4.0937	4.2701	4.2701	6.8958	6.8958	3.9476	4.1090	4.1090	6.4367	6.4367	AKL	0.8,				
2.3903	2.4432	4.3468	3.7938	7.1352	7.1352	2.3560	2.4063	4.2701	3.6722	6.8958	2.2728	2.3178	4.1090	3.4198	6.4367	AKL	0.5				
51.4678	54.0432	54.0432	71.3519	71.3519	71.3519	50.6	53.1	53.1	69	69	48.8	51	51	64.4	64.4	AKL	2,				
0.9418	0.9620	0.9620	2.1017	0	0	0.9307	0.9495	0.9495	2.0177	0	0.8987	0.9145	0.9145	1.8455	0	AKL	2,				
1.0243	1.0214	1.0214	1.0000	1.0000	1.0000	1.0250	1.0221	1.0221	1.0000	1.0000	1.0265	1.0235	1.0235	1.0001	1.0001	AKL	2,				
1.0026	1.0023	1.0214	1.0000	1.0000	1.0000	1.0027	1.0024	1.0221	1.0000	1.0000	1.0029	1.0026	1.0235	1.0000	1.0001	AKL	2,				
10.3632	10.3195	10.3195	10.0004	10.0004	10.0004	10.4	10.3	10.3	10	10	10.4	10.4	10.4	10.0	10.0	AKL	2,				
0.0242	0.0213	0.0213	0.0000	0	0	0.0249	0.0220	0.0220	0.0000	0.0000	0.0264	0.0233	0.0233	0.0000	0	AKL	2				

Table 5(Continued).

τ	1.05,1.3,1.05				1.4,1.4,1.05				3.3,1.4				μ_1			
	VP	VSSI	VSS	VSI	FP	VP	VSSI	VSS	VSI	FP	VP	VSSI		VSS	VSI	FP
56.8366	60.2258	60.2258	60.2258	64.6986	64.6986	12.7767	13.6631	13.6631	18.1039	18.1039	1.4094	1.4022	1.4022	1.2855	1.2855	ARKL
48.3020	51.1317	60.2258	55.3201	64.6986	64.6986	8.1756	8.6710	13.6631	11.8940	18.1039	1.1262	1.1253	1.4022	1.0456	1.2855	ATIS
615.7810	652.7814	652.7814	646.9855	646.9855	646.9855	153.3282	164.3651	164.3651	181.0392	181.0392	15.6669	15.5595	15.5595	12.8554	12.8554	ANOS
25.6183	27.1215	27.1215	31.0101	0	0	4.2324	4.4996	4.4996	7.2371	0	0.3204	0.3097	0.3097	0.1151	0	ANSW
51.5313	54.7821	54.7821	59.3116	59.3116	59.3116	12.2188	13.0701	13.0701	17.3344	17.3344	1.4068	1.3995	1.3995	1.2823	1.2823	ARKL
42.9923	45.6500	54.7821	49.8716	59.3116	59.3116	7.6903	8.1536	13.0701	11.2168	17.3344	1.1249	1.1240	1.3995	1.0448	1.2823	ATIS
562.7520	598.5558	598.5558	593.1161	593.1161	593.1161	147.3463	158.0144	158.0144	173.3439	173.3439	15.6338	15.5257	15.5257	12.8234	12.8234	ANOS
22.8703	24.2880	24.2880	28.2283	0	0	3.9656	4.2150	4.2150	6.8345	0	0.3190	0.3084	0.3084	0.1140	0	ANSW
26.6838	28.6805	28.6805	34.3427	34.3427	34.3427	9.0925	9.7098	9.7098	13.1345	13.1345	1.3919	1.3844	1.3844	1.2640	1.2640	ARKL
19.7578	21.1646	28.6805	26.0029	34.3427	34.3427	5.2509	5.5345	9.7098	7.8112	13.1345	1.1175	1.1166	1.3844	1.0407	1.2640	ATIS
305.3159	328.5604	328.5604	343.4275	343.4275	343.4275	112.2670	120.2935	120.2935	131.3449	131.3449	15.4439	15.3310	15.3310	12.6404	12.6404	ANOS
10.5376	11.3000	11.3000	15.4212	0	0	2.6024	2.7509	2.7509	4.7354	0	0.3112	0.3004	0.3004	0.1077	0	ANSW
3.6017	3.7350	3.7350	5.5290	5.5290	5.5290	2.9283	3.0186	3.0186	4.1210	4.1210	1.3282	1.3194	1.3194	1.1871	1.1871	ARKL
2.0550	2.0885	3.7350	2.8748	5.5290	5.5290	1.6741	1.6912	3.0186	2.0606	4.1210	1.0891	1.0881	1.3194	1.0260	1.1871	ATIS
44.6100	46.4979	46.4979	55.2904	55.2904	55.2904	36.2509	37.5596	37.5596	41.2096	41.2096	14.6109	14.4793	14.4793	11.8706	11.8706	ANOS
0.7996	0.8088	0.8088	1.4780	0	0	0.6096	0.6089	0.6089	0.9246	0	0.2749	0.2639	0.2639	0.0800	0	ANSW
1.0296	1.0264	1.0264	1.0001	1.0001	1.0001	1.0344	1.0311	1.0311	1.0003	1.0003	1.0421	1.0387	1.0387	1.0014	1.0014	ARKL
1.0034	1.0030	1.0264	1.0000	1.0001	1.0001	1.0041	1.0038	1.0311	1.0000	1.0003	1.0060	1.0057	1.0387	1.0001	1.0014	ATIS
10.4416	10.3939	10.3939	10.0013	10.0013	10.0013	10.5128	10.4622	10.4622	10.0031	10.0031	10.6211	10.5700	10.5700	10.0136	10.0136	ANOS
0.0294	0.0262	0.0262	0.0001	0	0	0.0341	0.0307	0.0307	0.0002	0	0.0411	0.0377	0.0377	0.0007	0	ANSW

Table 6 ATS comparison of the proposed schemes versus the ELR scheme when $p=2$, $ASS=5$, $ASI=1$ and in-control $ATS=370$.

$(\delta_1, \delta_2, \sqrt{\tau_1}, \sqrt{\tau_1}, \rho)$	ELR				FP	VP
	$\lambda = 0.1$	$\lambda = 0.2$	$\lambda = 0.3$	$\lambda = 0.4$		
(0.2,0.2,1,1,0)	<u>32.1</u>	42.5	59.3	77.0	260.4	232.9
(0.5,0.5,1,1,0)	7.6	<u>7.1</u>	7.3	8.2	44.8	20.6
(0.8,0,1,1,0)	6.5	<u>5.9</u>	<u>5.9</u>	6.2	29.2	11.3
(0,0,1.2,1.2,0)	<u>17.5</u>	21.6	27.2	33.5	51.5	32.3
(0.7,0.7,1,1,0)	5.0	4.3	4.2	4.2	13.0	<u>3.9</u>
(0,0,1.5,1,0)	<u>7.6</u>	<u>7.6</u>	8.4	9.1	41.1	24.0
(0.2,0.2,1.5,1,0)	6.8	<u>6.6</u>	7.0	7.9	33.7	18.5
(0.5,0,1.5,1,0)	5.6	<u>5.3</u>	5.4	5.7	22.7	10.9
(0.5,0.2,1.5,1.2,0)	5.0	4.6	4.7	4.9	10.8	<u>4.3</u>
(1.1,1.1,1,1,0)	2.8	2.4	2.2	2.1	2.4	<u>1.2</u>
(0,0,1.5,1.5,0)	4.7	4.2	4.3	4.4	6.9	<u>2.7</u>
(0.2,0.2,1.5,1.5,0)	4.4	3.9	3.9	4.2	6.5	<u>2.5</u>
(1.5,1,1,1,0)	2.3	2.0	1.9	1.8	1.6	<u>1.1</u>
(0,0,1.75,1.75,0)	2.8	2.5	2.4	2.3	3.0	<u>1.5</u>
(1.5,1.5,1,1,0)	1.9	1.7	1.5	1.5	1.2	<u>1.0</u>
(2.2,1,1,0)	1.3	1.1	1.1	<u>1</u>	<u>1</u>	<u>1</u>
(0,0,1.5,1,0.2)	<u>7.3</u>	7.0	7.6	8.2	7.1	<u>2.8</u>
(0.2,0,1.5,1,0.2)	6.9	<u>6.7</u>	7	7.5	41.4	24.3
(0.2,0.2,1.5,1,0.2)	6.5	<u>6.2</u>	6.6	7.1	37.3	21.2
(0.5,0,1.5,1,0.2)	5.5	<u>5.1</u>	<u>5.1</u>	5.5	24.6	12.1
(0.5,0.5,1.5,1,0.2)	4.6	<u>4.1</u>	<u>4.1</u>	<u>4.1</u>	14.5	6.0
(0.5,0.2,1.5,1.2,0.2)	5.0	<u>4.5</u>	<u>4.5</u>	4.7	11.6	4.7
(0,0,1.5,1.5,0.2)	4.5	4.1	4.2	4.3	7.4	<u>2.9</u>
(0.2,0,1.5,1.5,0.2)	4.4	3.9	4.0	4.1	7.2	<u>2.8</u>
(0.2,0.2,1.5,1.5,0.2)	4.3	3.9	3.9	4.0	6.9	<u>2.7</u>
(0.5,0,1.5,1.5,0.2)	3.9	3.5	3.4	3.4	5.9	<u>2.3</u>
(0.5,0.5,1.5,1.5,0.2)	3.5	3.1	3.0	3.0	4.9	<u>2.0</u>
(0.5,0.5,1.5,1,0.5)	4.1	3.6	3.6	<u>3.5</u>	20.9	9.1
(0.2,0.2,1.5,1.5,0.5)	3.8	<u>3.4</u>	<u>3.4</u>	<u>3.4</u>	10.7	4.3
(0.5,0.5,1.5,1.5,0.5)	3.2	2.8	2.7	2.7	6.6	<u>2.6</u>

Table 7- Details of the first 20 samples taken for the illustrative example

i	n_i	$\sum n_i$	t_i	$\sum t_i$	Sum of Switches	M_i	V_i	C_i	UWL_i	UCL_i	Status
1	5	5	1.9	1.9	0	0.1318	0.0732	0.1318	1.0487	3.0899	In-control
2	5	10	1.9	3.8	0	-0.0463	-0.0322	0.0463	1.0487	3.0899	In-control
3	5	15	1.9	5.7	0	1.1533	0.5718	1.1533	1.0487	3.0899	In-control
4	15	30	0.1	5.8	1	1.2119	-0.0515	1.2119	1.0472	2.9673	In-control
5	15	45	0.1	5.9	1	1.4744	-1.2172	1.4744	1.0472	2.9673	In-control
6	15	60	0.1	6	1	0.1525	0.0259	0.1525	1.0472	2.9673	In-control
7	5	65	1.9	7.9	2	-0.4480	0.4541	0.4480	1.0487	3.0899	In-control
8	5	70	1.9	9.7	2	0.8321	1.4539	1.4539	1.0487	3.0899	In-control
9	15	85	0.1	9.8	3	1.1209	-0.3267	1.1209	1.0472	2.9673	In-control
10	15	100	0.1	9.9	3	2.5332	0.7967	2.5332	1.0472	2.9673	In-control
11	15	115	0.1	10	3	-1.6425	0.0807	1.6425	1.0472	2.9673	In-control
12	15	130	0.1	10.1	3	2.0549	-0.0206	2.0549	1.0472	2.9673	In-control
13	15	145	0.1	10.2	3	1.5673	2.3796	2.3796	1.0472	2.9673	In-control
14	15	160	0.1	10.3	3	2.2500	-1.6017	2.2500	1.0472	2.9673	In-control
15	15	175	0.1	10.4	3	1.2529	-2.5033	2.5033	1.0472	2.9673	In-control
16	15	190	0.1	10.5	3	-2.1290	-1.3716	2.1290	1.0472	2.9673	In-control
17	15	205	0.1	10.6	3	0.0905	2.9908	2.9908	1.0472	2.9673	Out-of-control
18	15	220	0.1	10.7	3	-1.2114	-1.1186	1.2114	1.0472	2.9673	In-control
19	15	235	0.1	10.8	3	1.5181	-0.6905	1.5181	1.0472	2.9673	In-control
20	15	250	0.1	10.9	3	3.5460	1.8565	3.5460	1.0472	2.9673	Out-of-control

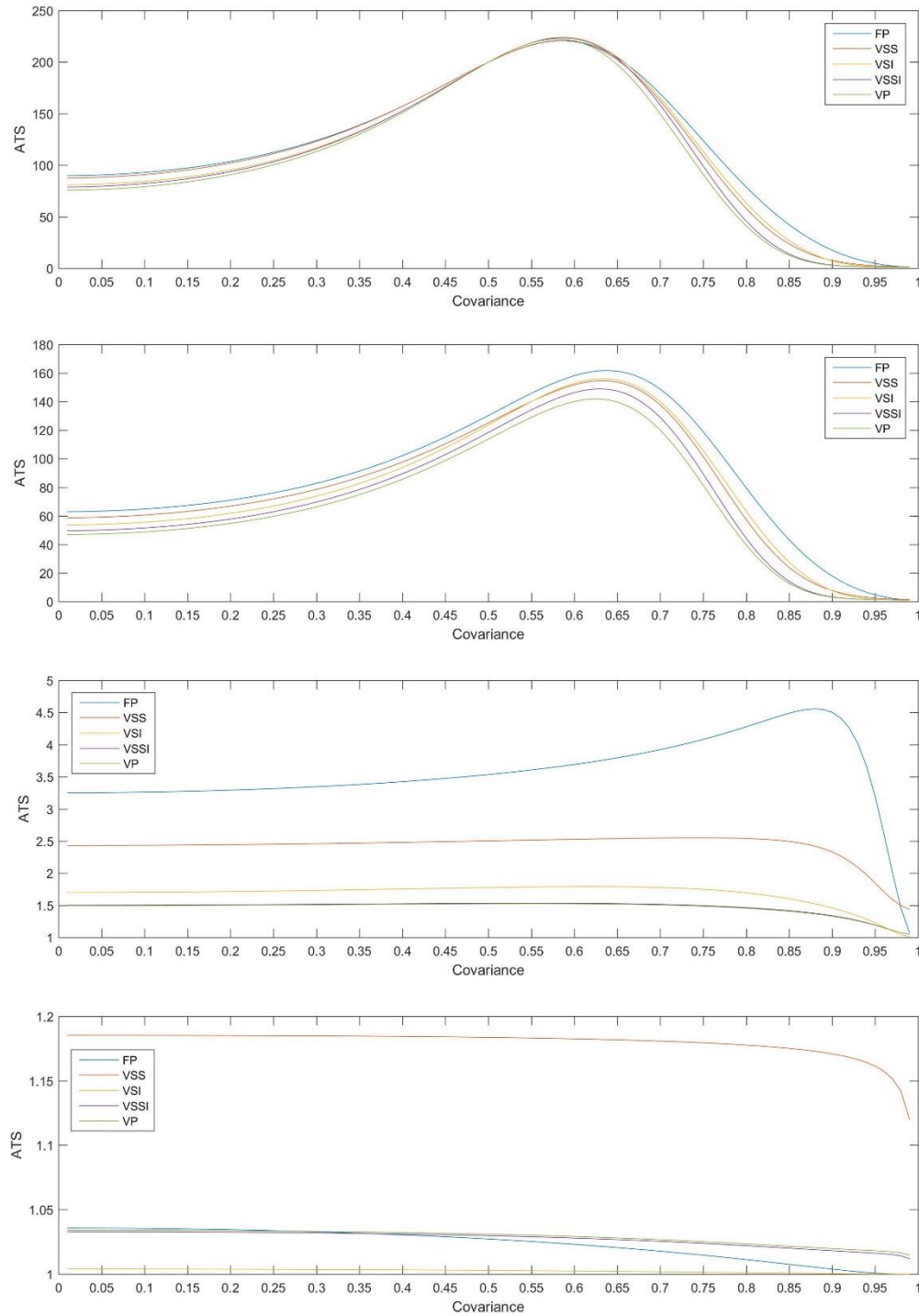


Figure1-ATS vs Covariance (θ) for zero, small (0.2), medium (0.8) and large (1.5) mean shifts, respectively, for $p = 2$ using first scenario case 2.

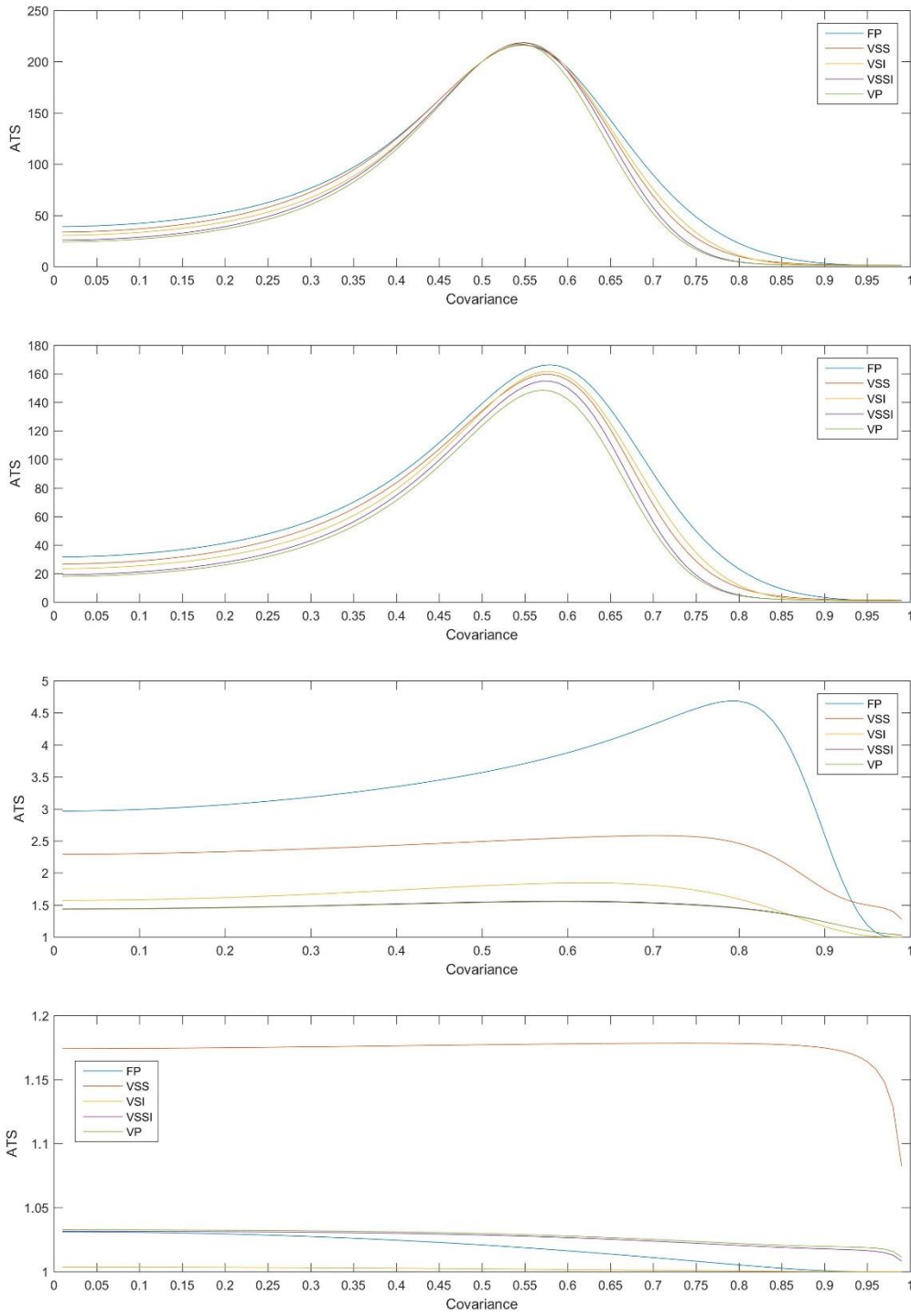


Figure2-ATS vs Covariance (θ) for zero, small (0.2), medium (0.8) and large (1.5) mean shifts, respectively, for $p = 3$ using first scenario case 2.

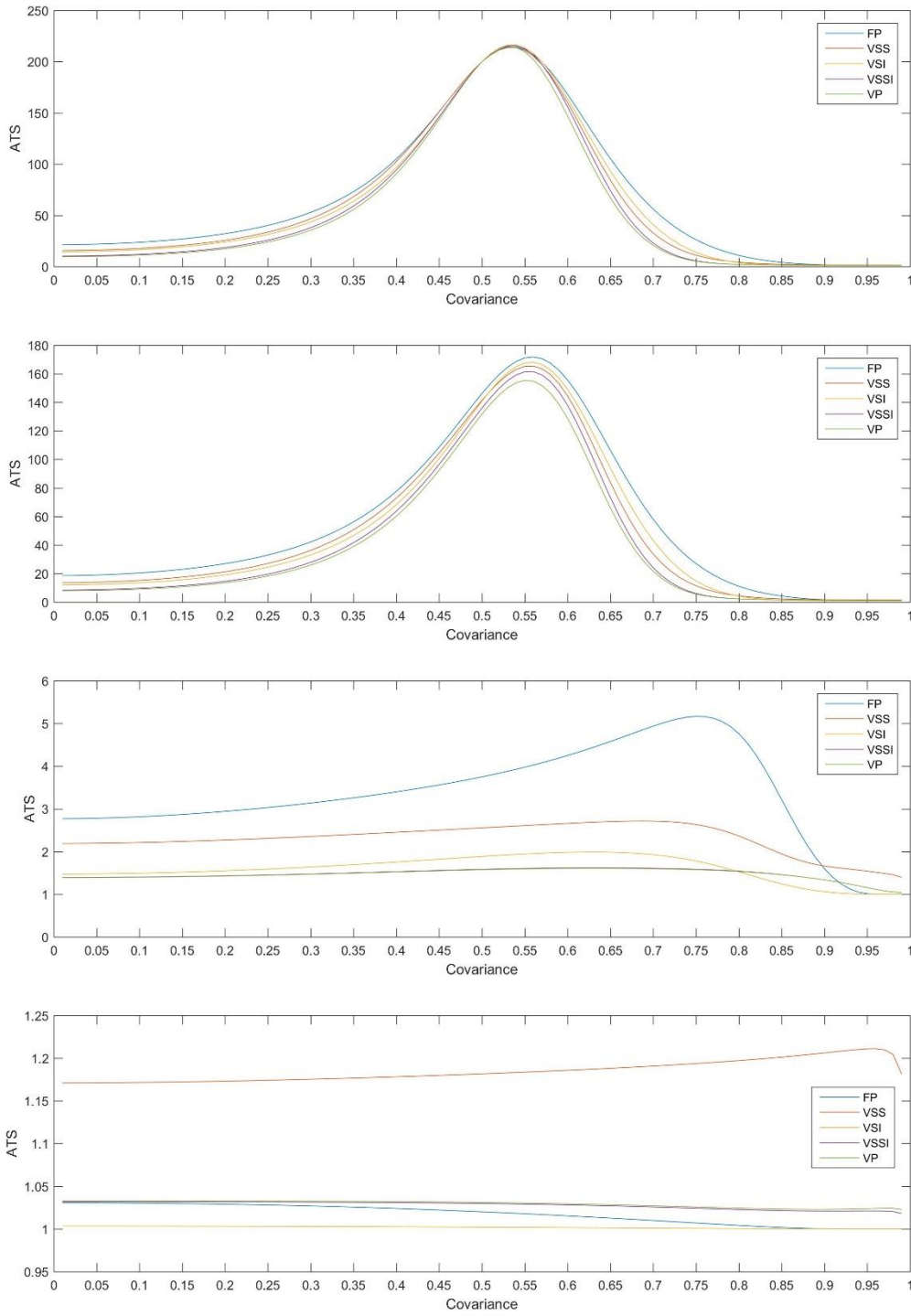


Figure3-ATS vs Covariance (θ) for zero, small (0.2), medium (0.8) and large (1.5) mean shifts, respectively, for $p = 4$ using first scenario case 2.

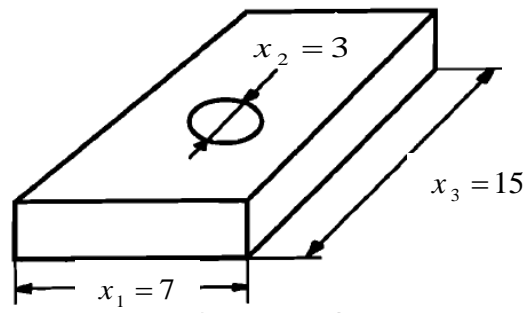


Figure 4- A mechanical part for an illustrative example

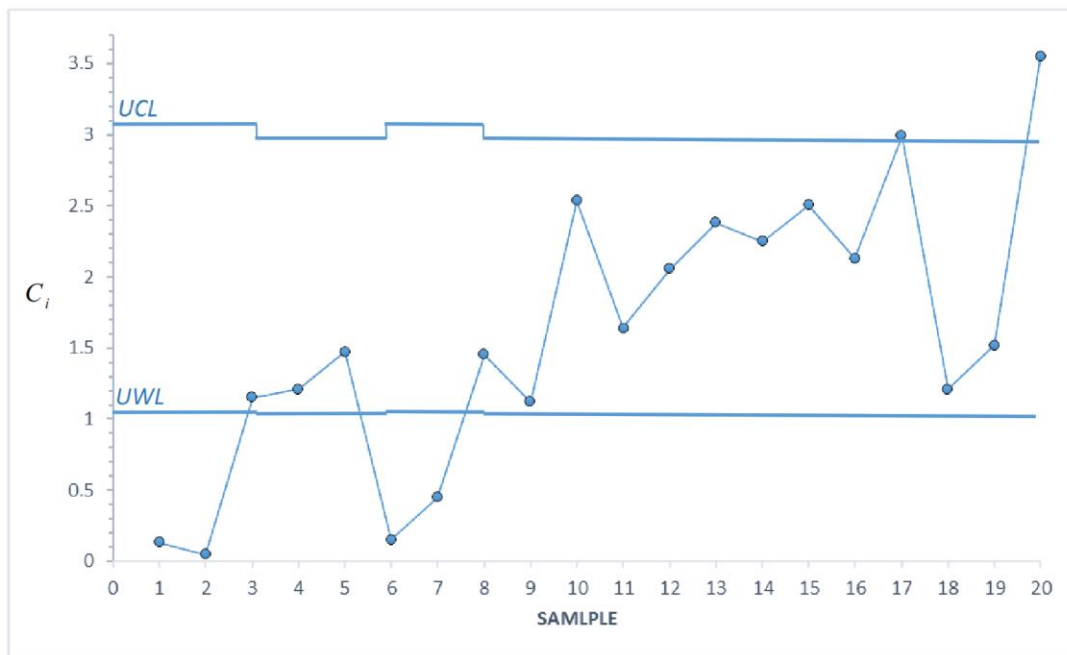


Figure 5- The monitoring VP chart for the illustrative example