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# Sharp stability for finite difference approximations of hyperbolic equations with boundary conditions

Jean-François COULOMBEL & Grégory FAYE\*

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## Abstract

In this article, we consider a class of finite rank perturbations of Toeplitz operators that have simple eigenvalues on the unit circle. Under a suitable assumption on the behavior of the essential spectrum, we show that such operators are power bounded. The problem originates in the approximation of hyperbolic partial differential equations with boundary conditions by means of finite difference schemes. Our result gives a positive answer to a conjecture by Trefethen, Kreiss and Wu that only a weak form of the so-called Uniform Kreiss-Lopatinskii Condition is sufficient to imply power boundedness.

**AMS classification:** 65M06, 65M12, 47B35, 35L04, 35L20.

**Keywords:** hyperbolic equations, difference approximations, stability, boundary conditions, semigroup estimates, Toeplitz operators.

Throughout this article, we use the notation

$$\begin{aligned}\mathcal{U} &:= \{\zeta \in \mathbb{C}, |\zeta| > 1\}, & \mathbb{D} &:= \{\zeta \in \mathbb{C}, |\zeta| < 1\}, & \mathbb{S}^1 &:= \{\zeta \in \mathbb{C}, |\zeta| = 1\}, \\ \overline{\mathcal{U}} &:= \mathcal{U} \cup \mathbb{S}^1, & \overline{\mathbb{D}} &:= \mathbb{D} \cup \mathbb{S}^1.\end{aligned}$$

If  $w$  is a complex number, the notation  $B_r(w)$  stands for the open ball in  $\mathbb{C}$  centered at  $w$  and with radius  $r > 0$ , that is  $B_r(w) := \{z \in \mathbb{C} / |z - w| < r\}$ . We let  $\mathcal{M}_{n,k}(\mathbb{C})$  denote the set of  $n \times k$  matrices with complex entries. If  $n = k$ , we simply write  $\mathcal{M}_n(\mathbb{C})$ .

Eventually, we let  $C$ , resp.  $c$ , denote some (large, resp. small) positive constants that may vary throughout the text (sometimes within the same line). The dependance of the constants on the various involved parameters is made precise throughout the article.

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# 1 Introduction

This article is devoted to the proof of power boundedness for a class of finite rank perturbations of some Toeplitz operators. The problem originates in the discretization of initial boundary value problems for hyperbolic partial differential equations. From the standard approach in numerical analysis, convergence of numerical schemes follows from stability and consistency. We focus here on stability. For discretized hyperbolic problems with numerical boundary conditions, several possible definitions of stability have been explored. From a historic perspective, the first stability definition introduced for instance in [Kre68, Osh69b, Osh69a] is a power boundedness property and reads (here  $T$  denotes the discrete evolution operator which gives the solution at each time step, and the norm in (1) below corresponds to an operator norm on  $\ell^2(\mathbb{N})$  - the numerical boundary conditions are incorporated in the definition of the functional space):

$$\sup_{n \in \mathbb{N}} \|T^n\| < +\infty. \quad (1)$$

The notion of *strong stability* later introduced in the fundamental contribution [GKS72] amounts to proving a strengthened version of the resolvent condition :

$$\sup_{z \in \mathcal{U}} (|z| - 1) \|(zI - T)^{-1}\| < +\infty. \quad (2)$$

We refer to [SW97] for a detailed exposition of the links between the conditions (1) and (2). Both conditions (1) and (2) preclude the existence of unstable eigenvalues  $z \in \mathcal{U}$  for the operator  $T$ , the so-called Godunov-Ryabenkii condition [GKO95].

The notion of strong stability analyzed in [GKS72] has the major advantage of being stable with respect to perturbations. It is an *open* condition, hence suitable for nonlinear analysis. However, it is restricted to zero initial data and is therefore not so convenient in practical applications. A long line of research has dealt with proving that strong stability implies power boundedness<sup>1</sup>. As far as we know, the most complete answers in the discrete case are [Wu95] (for scalar 1D problems and one time step schemes), [CG11] (for multidimensional systems and one time step schemes) and [Cou15] (for scalar multidimensional problems and multistep schemes). In the continuous setting, that is for hyperbolic partial differential equations, the reader is referred to [Rau72, Aud11, Mét17] and to references therein. All the above mentioned works are based on the fact that strong stability (or equivalently, the fulfillment of the so-called Uniform Kreiss-Lopatinskii Condition) provides with a sharp *trace estimate* of the solution in terms of the data. Summarizing the methodology in the strongly stable case, the goal is to control the time derivative (the time difference in the discrete case) of the solution in terms of its trace. All these techniques thus break down if the considered problem is not strongly stable and a trace estimate is not available.

However, it has been noted that several numerical boundary conditions do not yield strongly stable problems, see for instance [Tre84]. As observed in [Tre84] and later made more formal in [KW93], even though the Uniform Kreiss-Lopatinskii Condition may not be fulfilled, it does seem that some numerical schemes remain stable in the sense that their associated (discrete) semigroup is bounded (property (1)). This is precisely such a result that we aim at proving here, in the case where the Uniform Kreiss-Lopatinskii Condition breaks down because of simple, isolated eigenvalues on the unit circle<sup>2</sup>. Up to our knowledge,

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<sup>1</sup>Recall that strong stability is actually stronger than just verifying (2). It is known that in general, (2) does not imply (1) in infinite dimension, see [SW97].

<sup>2</sup>This is not the only possible breakdown for the Uniform Kreiss-Lopatinskii Condition, see [Tre84] or [BGS07] for the analogous continuous problem. However, the case we deal here with is the simplest and therefore the first to tackle in view of future generalizations.

this is the first general result of this type. Our analysis is based on pointwise semigroup bounds in the spirit of a long series of works initiated in [ZH98] and devoted to the stability analysis of viscous shock profiles. We thus restrict, in this work, to finite difference approximations of the transport operator that are stable in  $\ell^1(\mathbb{Z})$  (or equivalently  $\ell^\infty(\mathbb{Z})$ ) without any boundary condition. By the result in [Tho65], see more recent developments in [Des08, DSC14], we thus base our analysis on the dissipation Assumption 1 below. This does seem restrictive at first glance, but it is very likely that our methodology is flexible enough to handle more general situations, up to refining some steps in the analysis. We shall explore such extensions in the future.

## 1.1 The framework

We consider the scalar transport equation

$$\partial_t u + a \partial_x u = 0, \quad (3)$$

in the half-line  $\{x > 0\}$ , and restrict from now on to the case of an incoming velocity, that is,  $a > 0$ . The transport equation (3) is supplemented with Dirichlet boundary conditions:

$$u(t, 0) = 0, \quad (4)$$

and a Cauchy datum at  $t = 0$ . Our goal in this article is to explore the stability of finite difference approximations of the continuous problem (3), (4). We thus introduce a time step  $\Delta t > 0$  and a space step  $\Delta x > 0$ , assuming from now on that the ratio  $\lambda := \Delta t / \Delta x$  is always kept fixed. The solution to (3), (4) is meant to be approximated by a sequence<sup>3</sup>  $(u_j^n)$ . We consider some fixed integers  $r, p$  with  $\min(r, p) \geq 1$ . The *interior* cells are then the intervals  $[(j-1)\Delta x, j\Delta x)$  with  $j \in \mathbb{N}^*$ , and the *boundary* cells are the intervals  $[(\nu-1)\Delta x, \nu\Delta x)$  with  $\nu = 1-r, \dots, 0$ . The numerical scheme in the interior domain  $\mathbb{N}^*$  reads:

$$u_j^{n+1} = \sum_{\ell=-r}^p a_\ell u_{j+\ell}^n, \quad j \geq 1, \quad (5)$$

where the coefficients  $a_{-r}, \dots, a_p$  are real and may depend only on  $\lambda$  and  $a$ , but not on  $\Delta t$  (or  $\Delta x$ ). The numerical boundary conditions that we consider in this article take the form:

$$\forall \nu = 1-r, \dots, 0, \quad u_\nu^{n+1} = \sum_{\ell=1}^{p_b} b_{\ell, \nu} u_\ell^{n+1}, \quad (6)$$

where the coefficients  $b_{\ell, \nu}$  in (6) are real and may also depend on  $\lambda$  and  $a$ , but not on  $\Delta t$  (or  $\Delta x$ ). We assume for simplicity that the (fixed) integer  $p_b$  in (6) satisfies  $p_b \leq p$ . This is used below to simplify some minor technical details (when we rewrite high order *scalar* recurrences as first order *vectorial* recurrences).

An appropriate vector space for the stability analysis of (5)-(6) is the Hilbert space  $\mathcal{H}$  defined by:

$$\mathcal{H} := \left\{ (w_j)_{j \geq 1-r} \in \ell^2 \quad / \quad \forall \nu = 1-r, \dots, 0, \quad w_\nu = \sum_{\ell=1}^{p_b} b_{\ell, \nu} w_\ell \right\}. \quad (7)$$

Sequences in  $\mathcal{H}$  are assumed to be complex valued (even though, in practice, the numerical scheme (5)-(6) applies to real sequences). Since any element  $w$  of  $\mathcal{H}$  is uniquely determined by its interior values (those

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<sup>3</sup>As usual, we identify the sequence  $(u_j^n)$  with its associated step function on the time-space grid.

$w_j$ 's with  $j \geq 1$ ), we use the following norm on  $\mathcal{H}$ :

$$\forall w \in \mathcal{H}, \quad \|w\|_{\mathcal{H}}^2 := \sum_{j \geq 1} |w_j|^2.$$

The numerical scheme (5)-(6) can be then rewritten as:

$$\forall n \in \mathbb{N}, \quad u^{n+1} = \mathcal{T} u^n, \quad u^0 \in \mathcal{H},$$

where  $\mathcal{T}$  is the bounded operator on  $\mathcal{H}$  defined by:

$$\forall w \in \mathcal{H}, \quad \forall j \geq 1, \quad (\mathcal{T} w)_j := \sum_{\ell=-r}^p a_\ell w_{j+\ell}. \quad (8)$$

Recall that a sequence in  $\mathcal{H}$  is uniquely determined by its interior values so (8) determines  $\mathcal{T} w \in \mathcal{H}$  unambiguously. We introduce the following terminology.

**Definition 1** (Stability [Kre68, Osh69a]). *The numerical scheme (5)-(6) is said to be stable if there exists a constant  $C > 0$  such that, for any  $f \in \mathcal{H}$ , the solution  $(u^n)_{n \in \mathbb{N}}$  to (5)-(6) with initial condition  $u^0 = f$  satisfies:*

$$\sup_{n \in \mathbb{N}} \|u^n\|_{\mathcal{H}} \leq C \|f\|_{\mathcal{H}}.$$

*This means equivalently that the operator  $\mathcal{T}$  in (8) is power bounded by the same constant  $C$ :*

$$\sup_{n \in \mathbb{N}} \|\mathcal{T}^n\|_{\mathcal{H} \rightarrow \mathcal{H}} \leq C.$$

Our goal in this article is to show that the scheme (5)-(6) is stable under some spectral assumptions on the operator  $\mathcal{T}$ .

## 1.2 Assumptions and main result

We make two major assumptions: one on the finite difference scheme (5), and one on the compatibility between the scheme (5) and the numerical boundary conditions (6).

**Assumption 1.** *The finite difference approximation (5) is consistent with the transport equation (3):*

$$\sum_{\ell=-r}^p a_\ell = 1, \quad \sum_{\ell=-r}^p \ell a_\ell = -\lambda a < 0, \quad (\text{consistency}). \quad (9)$$

*Moreover, the coefficients  $a_\ell$  in (5) satisfy  $a_{-r} a_p \neq 0$  and the dissipativity condition:*

$$\forall \theta \in [-\pi, \pi] \setminus \{0\}, \quad \left| \sum_{\ell=-r}^p a_\ell e^{i\ell\theta} \right| < 1, \quad (10)$$

*and for some nonzero integer  $\mu$  and some positive real number  $\beta > 0$ , there holds:*

$$\sum_{\ell=-r}^p a_\ell e^{i\ell\theta} = \exp\left(-i\lambda a \theta - \beta \theta^{2\mu} + O(\theta^{2\mu+1})\right), \quad (11)$$

*as  $\theta$  tends to 0.*

An important consequence of Assumption 1 is the following Bernstein type inequality, which we prove in Appendix A.

**Lemma 1.** *Under Assumption 1, there holds  $\lambda a < r$ .*

The relevance of (11) for the  $\ell^1$  stability of (5) on  $\mathbb{Z}$  is the major result in [Tho65] (see [CF20, Des08, DSC14] for recent developments in this direction). This stability property will greatly simplify the final steps of the proof of our main result, which is Theorem 1 below. Relaxing (10) and (11) in order to encompass a wider class of finite difference schemes is postponed to some future works. We now state two Lemma whose proofs, which are relatively standard, can also be found in Appendix A. These two Lemma will allow us to introduce our second spectral assumption on the operator  $\mathcal{T}$ .

**Lemma 2.** *There exists a constant  $c_0 > 0$  such that, if we define the set:*

$$\mathcal{C} := \left\{ \rho e^{i\varphi} \in \mathbb{C} / \varphi \in [-\pi, \pi] \quad \text{and} \quad 0 \leq \rho \leq 1 - c_0 \varphi^{2\mu} \right\},$$

then  $\mathcal{C}$  is a compact star-shaped subset of  $\overline{\mathbb{D}}$ , and the curve:

$$\left\{ \sum_{\ell=-r}^p a_\ell e^{i\ell\theta} / \theta \in [-\pi, \pi] \right\} \quad (12)$$

is contained in  $\mathcal{C}$ .

The above Lemma 2 provides an estimate on the location of the essential spectrum of the operator  $\mathcal{T}$  and shows that it is contained in  $\mathcal{C}$  (see the reminder below on the spectrum of Toeplitz operators). Next, we introduce the following matrix:

$$\forall z \in \mathbb{C}, \quad \mathbb{M}(z) := \begin{pmatrix} \frac{\delta_{p-1,0} z - a_{p-1}}{a_p} & \dots & \dots & \frac{\delta_{-r,0} z - a_{-r}}{a_p} \\ 1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ 0 & 0 & 1 & 0 \end{pmatrix} \in \mathcal{M}_{p+r}(\mathbb{C}). \quad (13)$$

Since  $\min(r, p) \geq 1$ , the upper right coefficient of  $\mathbb{M}(z)$  is always nonzero (it equals  $-a_{-r}/a_p$ ), and  $\mathbb{M}(z)$  is invertible. We shall repeatedly use the inverse matrix  $\mathbb{M}(z)^{-1}$  in what follows.

**Lemma 3** (Spectral splitting). *Let  $z \in \mathbb{C}$  and let the matrix  $\mathbb{M}(z)$  be defined as in (13). Let the set  $\mathcal{C}$  be defined by Lemma 2. Then for  $z \notin \mathcal{C}$ ,  $\mathbb{M}(z)$  has:*

- no eigenvalue on  $\mathbb{S}^1$ ,
- $r$  eigenvalues in  $\mathbb{D} \setminus \{0\}$ ,
- $p$  eigenvalues in  $\mathcal{U}$  (eigenvalues are counted with multiplicity).

Furthermore,  $\mathbb{M}(1)$  has 1 as a simple eigenvalue, it has  $r - 1$  eigenvalues in  $\mathbb{D}$  and  $p$  eigenvalues in  $\mathcal{U}$ .

We introduce some notation. For  $z \notin \mathcal{C}$ , Lemma 3 shows that the so-called stable subspace, which is spanned by the generalized eigenvectors of  $\mathbb{M}(z)$  associated with eigenvalues in  $\mathbb{D}$ , has constant dimension  $r$ . We let  $\mathbb{E}^s(z)$  denote the stable subspace of  $\mathbb{M}(z)$  for  $z \notin \mathcal{C}$ . Because of the spectral splitting shown

in Lemma 3,  $\mathbb{E}^s(z)$  depends holomorphically on  $z$  in the complementary set of  $\mathcal{C}$ . We can therefore find, near every point  $z \notin \mathcal{C}$ , a basis  $e_1(z), \dots, e_r(z)$  of  $\mathbb{E}^s(z)$  that depends holomorphically on  $z$ . Similarly, the unstable subspace, which is spanned by the generalized eigenvectors of  $\mathbb{M}(z)$  associated with eigenvalues in  $\mathcal{U}$ , has constant dimension  $p$ . We denote it by  $\mathbb{E}^u(z)$ , and it also depends holomorphically on  $z$  in the complementary set of  $\mathcal{C}$ . With obvious notation, the projectors associated with the decomposition:

$$\forall z \notin \mathcal{C}, \quad \mathbb{C}^{p+r} = \mathbb{E}^s(z) \oplus \mathbb{E}^u(z),$$

are denoted  $\pi^s(z)$  and  $\pi^u(z)$ .

Let us now examine the situation close to  $z = 1$ . Since 1 is a simple eigenvalue of  $\mathbb{M}(1)$ , we can extend it holomorphically to a simple eigenvalue  $\kappa(z)$  of  $\mathbb{M}(z)$  in a neighborhood of 1. This eigenvalue is associated with the eigenvector:

$$E(z) := \begin{bmatrix} \kappa(z)^{p+r-1} \\ \vdots \\ \kappa(z) \\ 1 \end{bmatrix} \in \mathbb{C}^{p+r},$$

which also depends holomorphically on  $z$  in a neighborhood of 1. Furthermore, the unstable subspace  $\mathbb{E}^u(1)$  associated with eigenvalues in  $\mathcal{U}$  has dimension  $p$ . It can be extended holomorphically to a neighborhood of 1 thanks to the Dunford formula for spectral projectors. This holomorphic extension coincides with the above definition for  $\mathbb{E}^u(z)$  if  $z$  is close to 1 and  $z \notin \mathcal{C}$ . Eventually, the stable subspace of  $\mathbb{M}(1)$  associated with eigenvalues in  $\mathbb{D}$  has dimension  $r - 1$ . For the sake of clarity, we denote it by  $\mathbb{E}^{ss}(1)$  (the double  $s$  standing for *strongly stable*). Using again the Dunford formula for spectral projectors, we can extend this “strongly stable” subspace holomorphically with respect to  $z$ ; for  $z$  close to 1,  $\mathbb{E}^{ss}(z)$  has dimension  $r - 1$  and is either all or a hyperplane within the stable subspace of  $\mathbb{M}(z)$ . Namely, the situation has no ambiguity: for  $z \notin \mathcal{C}$  close to 1, the eigenvalue  $\kappa(z)$  necessarily belongs to  $\mathbb{D}$  and the stable subspace  $\mathbb{E}^s(z)$  of  $\mathbb{M}(z)$  (which has been defined above and has dimension  $r$ ) splits as:

$$\mathbb{E}^s(z) = \mathbb{E}^{ss}(z) \oplus \text{Span } E(z). \quad (14)$$

Since the right hand side in (14) depends holomorphically on  $z$  in a whole neighborhood of 1 and not only in  $\mathcal{C}^c$ , the stable subspace  $\mathbb{E}^s(z)$  extends holomorphically to a whole neighborhood of 1 as an invariant subspace of dimension  $r$  for  $\mathbb{M}(z)$ . In particular, we shall feel free to use below the notation  $\mathbb{E}^s(1)$  for the  $r$ -dimensional vector space:

$$\mathbb{E}^s(1) := \mathbb{E}^{ss}(1) \oplus \text{Span} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \quad (15)$$

which is, in our case, the direct sum of the stable and central subspaces of  $\mathbb{M}(1)$ .

For future use, it is convenient to introduce the following matrix:

$$\mathcal{B} := \begin{pmatrix} 0 & \cdots & 0 & -b_{p_b,0} & \cdots & -b_{1,0} & 1 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots & 0 & \vdots & \ddots & \vdots \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -b_{p_b,1-r} & \cdots & -b_{1,1-r} & 0 & \cdots & 0 & 1 \end{pmatrix} \in \mathcal{M}_{r,p+r}(\mathbb{R}). \quad (16)$$

We can now state our final assumption.

**Assumption 2.** For any  $z \in \mathcal{U} \cup \{1\}$ , there holds:

$$\mathbb{C}^{p+r} = \text{Ker } \mathcal{B} \oplus \mathbb{E}^s(z),$$

or, in other words,  $\mathcal{B}|_{\mathbb{E}^s(z)}$  is an isomorphism from  $\mathbb{E}^s(z)$  to  $\mathbb{C}^r$ . Moreover, choosing a holomorphic basis  $e_1(z), \dots, e_r(z)$  of  $\mathbb{E}^s(z)$  near every point  $z \in \mathbb{S}^1 \setminus \{1\}$ , the function:

$$\Delta : z \mapsto \det [\mathcal{B}e_1(z) \quad \cdots \quad \mathcal{B}e_r(z)]$$

has finitely many simple zeroes in  $\mathbb{S}^1 \setminus \{1\}$ .

Let us recall that for  $z \in \overline{\mathcal{U}} \setminus \{1\}$ ,  $\mathbb{E}^s(z)$  denotes the stable subspace of the matrix  $\mathbb{M}(z)$  in (13) since then  $z \in \mathcal{C}^c$ . At the point  $z = 1$ ,  $\mathbb{E}^s(1)$  denotes the holomorphic extension of  $\mathbb{E}^s(z)$  at 1 and it is furthermore given by (15).

Of course, the function  $\Delta$  in Assumption 2 depends on the choice of the (holomorphic) basis  $e_1(z), \dots, e_r(z)$  of  $\mathbb{E}^s(z)$ . However, the location of its zeroes and their multiplicity does not depend on that choice, which means that Assumption 2 is an intrinsic property of the operator  $\mathcal{T}$ . We shall refer later on to the function  $\Delta$  as the *Lopatinskii determinant* associated with (5)-(6). It plays the role of a characteristic polynomial for  $\mathcal{T}$  which detects the eigenvalues in  $\mathcal{C}^c$ . This object already appears in [Kre68, Osh69b, Osh69a]. Its analogue in the study of discrete shock profiles is the so-called Evans function, see [God03]. Our main result is the following.

**Theorem 1.** Under Assumptions 1 and 2, the operator  $\mathcal{T}$  in (8) is power bounded, that is, the numerical scheme (5)-(6) is stable.

If the function  $\Delta$  in Assumption 2 does not vanish on  $\overline{\mathcal{U}}$ , the Uniform Kreiss-Lopatinskii Condition is said to hold and the main result in [Wu95] implies that  $\mathcal{T}$  is power bounded, see also [Kre68, Osh69b, Osh69a]. The novelty here is to allow  $\Delta$  to vanish on  $\mathbb{S}^1$ . The Uniform Kreiss-Lopatinskii Condition thus breaks down. Power boundedness of  $\mathcal{T}$  in this case was conjectured in [Tre84, KW93].

The remainder of this article is organized as follows. The proof of Theorem 1 follows the same strategy as in [CF20]. In Section 2, we clarify the location of the spectrum of  $\mathcal{T}$  and give accurate bounds on the so-called *spatial* Green's function (that is, the Green's function for the operator  $zI - \mathcal{T}$  with  $z \notin \sigma(\mathcal{T})$ ). This preliminary analysis is used in Section 3 to give an accurate description of the so-called *temporal* Green's function (that is, the Green's function for the original problem (5)-(6)). Power boundedness of  $\mathcal{T}$  easily follows by classical inequalities. An example of operator for which Theorem 1 applies is given in Section 4.

## 2 Spectral analysis

For later use, we let  $z_1, \dots, z_K \in \mathbb{S}^1 \setminus \{1\}$  denote the pairwise distinct roots of the Lopatinskii determinant  $\Delta$  introduced in Assumption 2. We recall that these roots are simple. We first locate the spectrum of the operator  $\mathcal{T}$  and then give an accurate description of the so-called spatial Green's function. Precise definitions are provided below.



## 2.1 A reminder on the spectrum of Toeplitz operators

The operator  $\mathcal{T}$  is a finite rank (hence compact) perturbation of the Toeplitz operator on  $\ell^2(\mathbb{N})$  represented by the semi-infinite matrix:

$$\begin{pmatrix} a_0 & \cdots & a_p & 0 & \cdots & \cdots & \cdots \\ \vdots & \ddots & & \ddots & \ddots & & \\ a_{-r} & \cdots & a_0 & \cdots & a_p & \ddots & \\ 0 & \ddots & & \ddots & & \ddots & \ddots \\ \vdots & \ddots & \ddots & & \ddots & & \ddots \end{pmatrix}.$$

Therefore  $\mathcal{T}$  shares the same essential spectrum as the Toeplitz operator [Con90]. (The latter Toeplitz operator corresponds to enforcing the Dirichlet boundary conditions  $u_{1-r}^{n+1} = \cdots = u_0^{n+1} = 0$  instead of the more general form (6)). The spectrum of Toeplitz operators is well-known, see for instance [Dur64] and further developments in [TE05]. The resolvent set of the above Toeplitz operator consists of all points  $z \in \mathbb{C}$  that do not belong to the curve (12) and that have index 0 with respect to it. Moreover, any point on the curve (12) is in the essential spectrum. In the particular case we are interested in, Assumption 1 implies that the essential spectrum of  $\mathcal{T}$  is located in the set  $\mathcal{C}$  defined by Lemma 2 and that 1 belongs to the essential spectrum of  $\mathcal{T}$ . There remains to clarify the point spectrum of  $\mathcal{T}$ . The situation which we consider here and that is encoded in Assumption 2 is that where the finite rank perturbation of the Toeplitz operator generates finitely many simple eigenvalues on the unit circle (there may also be eigenvalues within  $\mathcal{C}$  but we are mainly concerned here with the eigenvalues of largest modulus). A precise statement is the following.

**Lemma 4** (The resolvent set). *Let the set  $\mathcal{C}$  be defined by Lemma 2. Then there exists  $\varepsilon > 0$  such that  $(\mathcal{C}^c \setminus \{\underline{z}_1, \dots, \underline{z}_K\}) \cap \{\zeta \in \mathbb{C} / |\zeta| > 1 - \varepsilon\}$  is contained in the resolvent set of the operator  $\mathcal{T}$ . Moreover, each zero  $\underline{z}_k$  of the Lopatinskii determinant is an eigenvalue of  $\mathcal{T}$ .*

*Proof.* The proof of Lemma 4 is first useful to clarify the location of the spectrum of the operator  $\mathcal{T}$  and it is also useful to introduce some of the tools used in the construction of the spatial Green's function which we shall perform below.

Let therefore  $z \in \mathcal{C}^c \setminus \{\underline{z}_1, \dots, \underline{z}_K\}$  and let  $f \in \mathcal{H}$ . We are going to explain why we can uniquely solve the equation:

$$(zI - \mathcal{T})w = f, \tag{17}$$

with  $w \in \mathcal{H}$  (up to assuming  $|z| > 1 - \varepsilon$  for some sufficiently small  $\varepsilon > 0$ ). Using the definitions (7) and (8), we wish to solve the system:

$$\begin{cases} z w_j - \sum_{\ell=-r}^p a_\ell w_{j+\ell} = f_j, & j \geq 1, \\ w_\nu = \sum_{\ell=1}^{p_b} b_{\ell,\nu} w_\ell, & \nu = 1 - r, \dots, 0. \end{cases}$$

We introduce, for any  $j \geq 1$ , the augmented vector:

$$W_j := \begin{bmatrix} w_{j+p-1} \\ \vdots \\ w_{j-r} \end{bmatrix} \in \mathbb{C}^{p+r},$$

which must satisfy the problem<sup>4</sup>:

$$\begin{cases} W_{j+1} = \mathbb{M}(z) W_j - a_p^{-1} f_j \mathbf{e}, & j \geq 1, \\ \mathcal{B} W_1 = 0, \end{cases} \quad (18)$$

where we have used the notation  $\mathbf{e}$  to denote the first vector of the canonical basis of  $\mathbb{C}^{p+r}$ , namely:

$$\mathbf{e} := \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{C}^{p+r}.$$

Our goal now is to solve the spatial dynamical problem (18). Since  $z \in \mathcal{C}^c$ , we know that  $\mathbb{M}(z)$  enjoys a hyperbolic dichotomy between its unstable and stable eigenvalues. We first solve for the unstable components of the sequence  $(W_j)_{j \geq 1}$  by integrating from  $+\infty$  to any integer  $j \geq 1$ , which gives:

$$\forall j \geq 1, \quad \pi^u(z) W_j := a_p^{-1} \sum_{\ell \geq 0} f_{j+\ell} \mathbb{M}(z)^{-1-\ell} \pi^u(z) \mathbf{e}. \quad (19)$$

In particular, we get the ‘‘initial value’’:

$$\pi^u(z) W_1 := a_p^{-1} \sum_{\ell \geq 0} f_{1+\ell} \mathbb{M}(z)^{-1-\ell} \pi^u(z) \mathbf{e}. \quad (20)$$

The initial value for the stable components is obtained by using Assumption 2. Namely, if  $z \in \mathcal{U}$ , we know that the linear operator  $\mathcal{B}|_{\mathbb{E}^s(z)}$  is an isomorphism, and this property remains true near every point of  $\mathbb{S}^1 \setminus \{1\}$  except at the  $z_k$ 's. We can thus find some  $\varepsilon > 0$  such that, for any  $z \in \mathcal{C}^c \setminus \{z_1, \dots, z_K\}$  verifying  $|z| > 1 - \varepsilon$ , the linear operator  $\mathcal{B}|_{\mathbb{E}^s(z)}$  is an isomorphism. For such  $z$ 's, we can therefore define the vector  $\pi^s(z) W_1 \in \mathbb{E}^s(z)$  through the formula:

$$\pi^s(z) W_1 := -a_p^{-1} \left( \mathcal{B}|_{\mathbb{E}^s(z)} \right)^{-1} \mathcal{B} \sum_{\ell \geq 0} f_{1+\ell} \mathbb{M}(z)^{-1-\ell} \pi^u(z) \mathbf{e}, \quad (21)$$

which is the only way to obtain both the linear constraint  $\mathcal{B} W_1 = 0$  and the decomposition  $W_1 = \pi^s(z) W_1 + \pi^u(z) W_1$  in agreement with (20). Once we have determined the stable components  $\pi^s(z) W_1$  of the initial value  $W_1$ , the only possible way to solve (18) for the stable components is to set:

$$\forall j \geq 1, \quad \pi^s(z) W_j := \mathbb{M}(z)^{j-1} \pi^s(z) W_1 - a_p^{-1} \sum_{\ell=1}^{j-1} f_\ell \mathbb{M}(z)^{j-1-\ell} \pi^s(z) \mathbf{e}. \quad (22)$$

Since the sequences  $(\mathbb{M}(z)^{-\ell} \pi^u(z))_{\ell \geq 1}$  and  $(\mathbb{M}(z)^\ell \pi^s(z))_{\ell \geq 1}$  are exponentially decreasing, we can define a solution  $(W_j)_{j \geq 1} \in \ell^2$  to (18) by decomposing along the stable and unstable components and using the defining equations (19) and (22). This provides us with a solution  $w \in \mathcal{H}$  to the equation (17)

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<sup>4</sup>This is the place where we use the assumption  $p_b \leq p$  in order to rewrite the numerical boundary conditions as a linear constraint on the first element  $W_1$  of the sequence  $(W_j)_{j \geq 1}$ . The case  $p_b > p$  can be dealt with quite similarly but is just heavier in terms of notation.

by going back to the scalar components of each vector  $W_j$ . Such a solution is necessarily unique since if  $w \in \mathcal{H}$  is a solution to (17) with  $f = 0$ , then the augmented vectorial sequence  $(W_j)_{j \geq 1} \in \ell^2$  satisfies:

$$\begin{cases} W_{j+1} = \mathbb{M}(z) W_j, & j \geq 1, \\ \mathcal{B} W_1 = 0. \end{cases}$$

This means that the vector  $W_1$  belongs to  $\mathbb{E}^s(z)$  and to the kernel of the matrix  $\mathcal{B}$ , and therefore vanishes. Hence the whole sequence  $(W_j)_{j \geq 1} \in \ell^2$  vanishes. We have thus shown that  $z$  belongs to the resolvent set of  $\mathcal{T}$ .

The fact that each  $z_k$  is an eigenvalue of  $\mathcal{T}$  follows from similar arguments. At a point  $z_k$ , the intersection  $\text{Ker } \mathcal{B} \cap \mathbb{E}^s(z_k)$  is not trivial, so we can find a nonzero vector  $\underline{W}_1 \in \text{Ker } \mathcal{B}$  for which the sequence  $(\underline{W}_j)_{j \geq 1}$  defined by:

$$\forall j \geq 1, \quad \underline{W}_{j+1} := \mathbb{M}(z_k) \underline{W}_j,$$

is square integrable (it is even exponentially decreasing). Going back to scalar components, this provides with a nonzero solution to the eigenvalue problem:

$$\begin{cases} z_k w_j - \sum_{\ell=-r}^p a_\ell w_{j+\ell} = 0, & j \geq 1, \\ w_\nu = \sum_{\ell=1}^{p_b} b_{\ell,\nu} w_\ell, & \nu = 1 - r, \dots, 0. \end{cases}$$

The proof of Lemma 4 is complete.  $\square$

We are now going to define and analyze the so-called spatial Green's function. The main point, as in [ZH98, God03, CF20] and related works, is to be able to “pass through” the essential spectrum close to 1 and extend the spatial Green's function holomorphically to a whole neighborhood of 1. This was already achieved with accurate bounds in [CF20] on the whole line  $\mathbb{Z}$  (with no numerical boundary condition) and we apply similar arguments here, while adding the difficulty of the eigenvalues on  $\mathbb{S}^1$ . Near all such eigenvalues, we isolate the precise form of the singularity in the Green's function and show that the remainder admits a holomorphic extension at the eigenvalue. All these arguments are made precise in the following paragraph.

## 2.2 The spatial Green's function

For any  $j_0 \geq 1$ , we let  $\delta_{j_0}$  denote the only element of the space  $\mathcal{H}$  in (7) that satisfies:

$$\forall j \geq 1, \quad (\delta_{j_0})_j := \begin{cases} 1, & \text{if } j = j_0, \\ 0, & \text{otherwise.} \end{cases}$$

The boundary values of  $\delta_{j_0}$  are defined accordingly. Then as long as  $z$  belongs to the resolvent set of the operator  $\mathcal{T}$ , the spatial Green's function, which we denote  $G_z(\cdot, \cdot)$  is defined by the relation:

$$\forall j_0 \geq 1, \quad (zI - \mathcal{T}) G_z(\cdot, j_0) = \delta_{j_0}, \quad (23)$$

together with the numerical boundary conditions  $G_z(\cdot, j_0) \in \mathcal{H}$ . We give below an accurate description of  $G_z$  in order to later obtain an accurate description of the temporal Green's function, that is obtained by applying the iteration (5)-(6) to the initial condition  $\delta_{j_0}$ . The analysis of the spatial Green's function splits between three cases:

- The behavior near regular points (away from the spectrum of  $\mathcal{T}$ ),

- The behavior near the point 1 (the only point where the essential spectrum of  $\mathcal{T}$  meets  $\mathbb{S}^1$ ),
- The behavior near the eigenvalues  $\underline{z}_1, \dots, \underline{z}_K$ .

Let us start with the easiest case.

**Lemma 5** (Bounds away from the spectrum). *Let  $\underline{z} \in \overline{\mathcal{U}} \setminus \{1, \underline{z}_1, \dots, \underline{z}_K\}$ . Then there exists an open ball  $B_r(\underline{z})$  centered at  $\underline{z}$  and there exist two constants  $C > 0$ ,  $c > 0$  such that, for any couple of integers  $j, j_0 \geq 1$ , there holds:*

$$\forall z \in B_r(\underline{z}), \quad |G_z(j, j_0)| \leq C \exp(-c|j - j_0|).$$

*Proof.* Almost all ingredients have already been set in the proof of Lemma 4. Let therefore  $\underline{z} \in \overline{\mathcal{U}} \setminus \{1, \underline{z}_1, \dots, \underline{z}_K\}$ , and let us first fix  $r > 0$  small enough such that the closed ball  $\overline{B_r(\underline{z})}$  is contained both in  $\mathcal{C}^c$  and in the resolvent set of  $\mathcal{T}$ . All complex numbers  $z$  below are assumed to lie within  $\overline{B_r(\underline{z})}$ . Then the problem (23) can be recast under the vectorial form (18) with:

$$\forall j \geq 1, \quad f_j := \begin{cases} 1, & \text{if } j = j_0, \\ 0, & \text{otherwise.} \end{cases}$$

Let us therefore consider the spatial dynamics problem (18) with the above Dirac mass type source term. The unstable components of the sequence  $(W_j)_{j \geq 1}$  solution to (18) are given by (19), which gives here:

$$\forall j \geq 1, \quad \pi^u(z) W_j = \begin{cases} 0, & \text{if } j > j_0, \\ a_p^{-1} \mathbb{M}(z)^{-(j_0+1-j)} \pi^u(z) \mathbf{e}, & \text{if } 1 \leq j \leq j_0. \end{cases}$$

In particular, we get the following uniform bounds with respect to  $z, j, j_0$ :

$$\forall z \in \overline{B_r(\underline{z})}, \quad \forall j \geq 1, \quad |\pi^u(z) W_j| \leq \begin{cases} 0, & \text{if } j > j_0, \\ C \exp(-c(j_0 - j)), & \text{if } 1 \leq j \leq j_0. \end{cases} \quad (24)$$

The initial value  $\pi^s(z) W_1$  of the stable components is then obtained by the relation (21), which immediately gives the bound<sup>5</sup>:

$$|\pi^s(z) W_1| \leq C \exp(-c j_0).$$

The stable components are then determined for any integer  $j \geq 1$  by the general formula (22), which gives here:

$$\forall j \geq 1, \quad \pi^s(z) W_j = \begin{cases} \mathbb{M}(z)^{j-1} \pi^s(z) W_1, & \text{if } 1 \leq j \leq j_0, \\ \mathbb{M}(z)^{j-1} \pi^s(z) W_1 - a_p^{-1} \mathbb{M}(z)^{j-j_0-1} \pi^s(z) \mathbf{e}, & \text{if } j > j_0. \end{cases}$$

By using the exponential decay of the sequence  $(\mathbb{M}(z)^j \pi^s(z))_{j \geq 1}$ , we get the following bounds for the stable components:

$$\forall j \geq 1, \quad |\pi^s(z) W_j| \leq \begin{cases} C \exp(-c(j_0 + j)), & \text{if } 1 \leq j \leq j_0, \\ C \exp(-c(j_0 + j)) + C \exp(-c(j - j_0)), & \text{if } j > j_0. \end{cases} \quad (25)$$

Adding (24) with (25), and examining for each situation which among the terms is the largest, we get the conclusion of Lemma 5 (recall that the scalar component  $G_z(j, j_0)$  is just one among the coordinates of the vector  $W_j$  considered above).  $\square$

<sup>5</sup>Here we use the fact that the linear map  $\mathcal{B}|_{\mathbb{E}^s(z)}$  is an isomorphism for all  $z \in \overline{B_r(\underline{z})}$ .

We are now going to examine the behavior of the spatial Green's function  $G_z$  close to 1. Let us first recall that the exterior  $\mathcal{U}$  of the unit disk belongs to the resolvent set of  $\mathcal{T}$ . Hence, for any  $j_0 \geq 1$ , the sequence  $G_z(\cdot, j_0)$  is well-defined in  $\mathcal{H}$  for  $z \in \mathcal{U}$ . Lemma 6 below shows that each individual sequence  $G_z(j, j_0)$  can be holomorphically extended to a whole neighborhood of 1.

**Lemma 6** (Bounds close to 1). *There exists an open ball  $B_\varepsilon(1)$  centered at 1 and there exist two constants  $C_1 > 0$  and  $c_1 > 0$  such that, for any couple of integers  $(j, j_0)$ , the component  $G_z(j, j_0)$  defined on  $B_\varepsilon(1) \cap \mathcal{U}$  extends holomorphically to the whole ball  $B_\varepsilon(1)$  with respect to  $z$ , and the holomorphic extension satisfies the bound:*

$$\forall z \in B_\varepsilon(1), \quad |G_z(j, j_0)| \leq \begin{cases} C_1 \exp(-c_1 |j - j_0|), & \text{if } 1 \leq j \leq j_0, \\ C_1 |\kappa(z)|^{|j - j_0|}, & \text{if } j > j_0, \end{cases}$$

where  $\kappa(z)$  denotes the (unique) holomorphic eigenvalue of  $\mathbb{M}(z)$  that satisfies  $\kappa(1) = 1$ .

*Proof.* Most ingredients of the proof are similar to what we have already done in the proof of Lemma 5. The novelty is that there is one stable component which behaves more and more singularly as  $z \in \mathcal{U}$  gets close to 1 since one stable eigenvalue, namely  $\kappa(z)$ , gets close to  $\mathbb{S}^1$  (its exponential decay is thus weaker and weaker). We thus recall that on some suitably small neighborhood  $B_\varepsilon(1)$  of 1, we have the (holomorphic in  $z$ ) decomposition:

$$\mathbb{C}^{p+r} = \mathbb{E}^u(z) \oplus \mathbb{E}^{ss}(z) \oplus \text{Span } E(z),$$

where all the above spaces are invariant by  $\mathbb{M}(z)$ , the spectrum of  $\mathbb{M}(z)$  restricted to  $\mathbb{E}^u(z)$  lies in  $\mathcal{U}$ , the spectrum of  $\mathbb{M}(z)$  restricted to  $\mathbb{E}^{ss}(z)$  lies in  $\mathbb{D}$ , and  $E(z)$  is an eigenvector for  $\mathbb{M}(z)$  associated with the eigenvalue  $\kappa(z)$ . With obvious notation, we use the corresponding decomposition:

$$X = \pi^u(z)X + \pi^{ss}(z)X + \mu E(z).$$

Let us from now on consider some complex number  $z \in B_\varepsilon(1) \cap \mathcal{U}$  so that the Green's function  $G_z(\cdot, j_0)$  is well-defined in  $\mathcal{H}$  for any  $j_0 \geq 1$ . As in the proof of Lemma 5, the Green's function is defined by solving the spatial dynamics problem (18) with the Dirac mass datum:

$$\forall j \geq 1, \quad f_j := \begin{cases} 1, & \text{if } j = j_0, \\ 0, & \text{otherwise.} \end{cases}$$

The unstable components are uniquely determined by:

$$\forall j \geq 1, \quad \pi^u(z)W_j = \begin{cases} 0, & \text{if } j > j_0, \\ a_p^{-1} \mathbb{M}(z)^{-(j_0+1-j)} \pi^u(z) \mathbf{e}, & \text{if } 1 \leq j \leq j_0, \end{cases}$$

and we readily observe that the latter right hand side depends holomorphically on  $z$  in the whole ball  $B_\varepsilon(1)$  and not only in  $B_\varepsilon(1) \cap \mathcal{U}$ . This already allows to extend the unstable components  $\pi^u(z)W_j$  to  $B_\varepsilon(1)$ , with the corresponding uniform bound similar to (24), that is:

$$\forall z \in B_\varepsilon(1), \quad \forall j \geq 1, \quad |\pi^u(z)W_j| \leq \begin{cases} 0, & \text{if } j > j_0, \\ C \exp(-c(j_0 - j)), & \text{if } 1 \leq j \leq j_0. \end{cases} \quad (26)$$

We can then use the fact that  $\mathcal{B}|_{\mathbb{E}^s(1)}$  is an isomorphism from  $\mathbb{E}^s(1)$  to  $\mathbb{C}^r$ , which implies that, up to restricting the radius  $\varepsilon$ , the matrix  $\mathcal{B}$  restricted to the holomorphically extended stable subspace:

$$\mathbb{E}^{ss}(z) \oplus \text{Span } E(z),$$

is an isomorphism. We can thus uniquely determine some vector  $\pi^{ss}(z)W_1 \in \mathbb{E}^{ss}(z)$  and a scalar  $\mu_1$  such that:

$$\mathcal{B} \left( \pi^{ss}(z)W_1 + \mu_1 E(z) \right) = -a_p^{-1} \mathcal{B} \mathbb{M}(z)^{-j_0} \pi^u(z) \mathbf{e}.$$

In particular, we have the bound:

$$\forall z \in B_\varepsilon(1), \quad \left| \pi^{ss}(z)W_1 \right| + |\mu_1| \leq C \exp(-c j_0).$$

For  $z \in B_\varepsilon(1) \cap \mathcal{U}$ , the strongly stable components of  $(W_j)_{j \geq 1}$  are then defined by the formula:

$$\forall j \geq 1, \quad \pi^{ss}(z)W_j = \begin{cases} \mathbb{M}(z)^{j-1} \pi^{ss}(z)W_1, & \text{if } 1 \leq j \leq j_0, \\ \mathbb{M}(z)^{j-1} \pi^{ss}(z)W_1 - a_p^{-1} \mathbb{M}(z)^{j-j_0-1} \pi^{ss}(z) \mathbf{e}, & \text{if } j > j_0, \end{cases}$$

and the coordinate of  $(W_j)_{j \geq 1}$  along the eigenvector  $E(z)$  is defined by the formula:

$$\forall j \geq 1, \quad \mu_j = \begin{cases} \kappa(z)^{j-1} \mu_1, & \text{if } 1 \leq j \leq j_0, \\ \kappa(z)^{j-1} \mu_1 - a_p^{-1} \kappa(z)^{j-j_0-1} \mu(\mathbf{e}), & \text{if } j > j_0. \end{cases}$$

As for the unstable components, we observe that for each couple of integers  $j, j_0$ , the above components of  $W_j$  extend holomorphically to the whole ball  $B_\varepsilon(1)$  since the spectral projectors of  $\mathbb{M}(z)$  along  $\mathbb{E}^{ss}(z)$  and  $\text{Span } E(z)$  do so. We thus consider from now on the holomorphic extension of  $W_j$  for  $z \in B_\varepsilon(1)$  and collect the three pieces of the vector  $W_j$ . For  $1 \leq j \leq j_0$ , we have:

$$W_j = a_p^{-1} \mathbb{M}(z)^{-(j_0+1-j)} \pi^u(z) \mathbf{e} + \mathbb{M}(z)^{j-1} \pi^{ss}(z)W_1 + \kappa(z)^{j-1} \mu_1 E(z),$$

which satisfies the bound:

$$\left| W_j \right| \leq C \exp(-c(j_0 - j)) + C \exp(-c(j_0 + j)) + C |\kappa(z)|^j \exp(-c j_0),$$

for some constants  $C > 0$  and  $c > 0$  that are uniform with respect to  $z$ . Since  $\kappa(1) = 1$ , we can always assume that there holds  $|\kappa(z)| \leq \exp c$  on the ball  $B_\varepsilon(1)$ , and we are then left with the estimate:

$$\left| W_j \right| \leq C \exp(-c(j_0 - j)),$$

as claimed in the statement of Lemma 6.

It remains to examine the case  $j > j_0$  for which we have the decomposition:

$$W_j = \mathbb{M}(z)^{j-1} \pi^{ss}(z)W_1 - a_p^{-1} \mathbb{M}(z)^{j-j_0-1} \pi^{ss}(z) \mathbf{e} + \kappa(z)^{j-1} \mu_1 E(z) - a_p^{-1} \kappa(z)^{j-j_0-1} \mu(\mathbf{e}) E(z),$$

and we can thus derive the bound:

$$\left| W_j \right| \leq C \exp(-c(j_0 + j)) + C \exp(-c(j - j_0)) + C |\kappa(z)|^j \exp(-c j_0) + C |\kappa(z)|^{|j-j_0|}.$$

Since we can always assume that the ball  $B_\varepsilon(1)$  is so small that  $|\kappa(z)|$  takes its values within the interval  $[\exp(-c), \exp c]$ , it appears that the largest term on the above right hand side is the last term, which completes the proof of Lemma 6.  $\square$

Let us observe that we can extend holomorphically each scalar component  $G_z(j, j_0)$  but that does not mean that we can extend holomorphically  $G_z(\cdot, j_0)$  in  $\mathcal{H}$ . As a matter of fact, the eigenvalue  $\kappa(z)$  starts contributing to the unstable subspace of  $\mathbb{M}(z)$  as  $z$  (close to 1) crosses the curve (12). The holomorphic extension  $G_z(\cdot, j_0)$  then ceases to be in  $\ell^2$  for it has an exponentially growing mode in  $j$ . The last case to examine is that of the neighborhood of each eigenvalue  $\underline{z}_k$ .

**Lemma 7** (Bounds close to the eigenvalues). *For any eigenvalue  $\underline{z}_k \in \mathbb{S}^1$  of  $\mathcal{T}$ , there exists an open ball  $B_\varepsilon(\underline{z}_k)$  centered at  $\underline{z}_k$ , there exists a sequence  $(\underline{w}_k(j, j_0))_{j, j_0 \geq 1}$  with  $\underline{w}_k(\cdot, j_0) \in \mathcal{H}$  for all  $j_0 \geq 1$ , and there exist two constants  $\underline{C}_k > 0$  and  $\underline{c}_k > 0$  such that for any couple of integers  $(j, j_0)$ , the component  $G_z(j, j_0)$  defined on  $B_\varepsilon(\underline{z}_k) \setminus \{\underline{z}_k\}$  is such that:*

$$R_z(j, j_0) := G_z(j, j_0) - \frac{\underline{w}_k(j, j_0)}{z - \underline{z}_k},$$

extends holomorphically to the whole ball  $B_\varepsilon(\underline{z}_k)$  with respect to  $z$ , and the holomorphic extension satisfies the bound:

$$\forall z \in B_\varepsilon(\underline{z}_k), \quad |R_z(j, j_0)| \leq \underline{C}_k \exp(-\underline{c}_k |j - j_0|).$$

Moreover, the sequence  $(\underline{w}_k(j, j_0))_{j, j_0 \geq 1}$  satisfies the pointwise bound:

$$\forall j, j_0 \geq 1, \quad |\underline{w}_k(j, j_0)| \leq \underline{C}_k \exp(-\underline{c}_k (j + j_0)).$$

*Proof.* Many ingredients for the proof of Lemma 7 are already available in the proof of Lemma 5. Namely, let us consider an eigenvalue  $\underline{z}_k \in \mathbb{S}^1$  of  $\mathcal{T}$ . Since  $\underline{z}_k \in \mathcal{C}^c$ , the matrix  $\mathbb{M}(z)$  enjoys the hyperbolic dichotomy between its stable and unstable eigenvalues in the neighborhood of  $\underline{z}_k$ . Moreover, for a sufficiently small radius  $\varepsilon > 0$ , the pointed ball  $B_\varepsilon(\underline{z}_k) \setminus \{\underline{z}_k\}$  lies in the resolvent set of  $\mathcal{T}$ . In particular, for any  $z \in B_\varepsilon(\underline{z}_k) \setminus \{\underline{z}_k\}$ , the spatial Green's function is obtained by selecting the appropriate scalar component of the vector sequence  $(W_j)_{j \geq 1}$  defined by:

$$\forall j \geq 1, \quad \pi^u(z) W_j := \begin{cases} 0, & \text{if } j > j_0, \\ a_p^{-1} \mathbb{M}(z)^{-(j_0+1-j)} \pi^u(z) \mathbf{e}, & \text{if } 1 \leq j \leq j_0, \end{cases} \quad (27)$$

and:

$$\forall j \geq 1, \quad \pi^s(z) W_j := \begin{cases} \mathbb{M}(z)^{j-1} \pi^s(z) W_1, & \text{if } 1 \leq j \leq j_0, \\ \mathbb{M}(z)^{j-1} \pi^s(z) W_1 - a_p^{-1} \mathbb{M}(z)^{j-j_0-1} \pi^s(z) \mathbf{e}, & \text{if } j > j_0, \end{cases} \quad (28)$$

where the vector  $\pi^s(z) W_1 \in \mathbb{E}^s(z)$  is defined by (see (21)):

$$\pi^s(z) W_1 := -a_p^{-1} \left( \mathcal{B}|_{\mathbb{E}^s(z)} \right)^{-1} \mathcal{B} \mathbb{M}(z)^{-j_0} \pi^u(z) \mathbf{e}. \quad (29)$$

(Here we use the fact that for every  $z$  in the pointed ball  $B_\varepsilon(\underline{z}_k) \setminus \{\underline{z}_k\}$ , the linear map  $\mathcal{B}|_{\mathbb{E}^s(z)}$  is an isomorphism.)

The unstable component  $\pi^u(z) W_j$  in (27) obviously extends holomorphically to the whole ball  $B_\varepsilon(\underline{z}_k)$  and the estimate (24) shows that this contribution to the remainder term  $R_z(j, j_0)$  satisfies the desired uniform exponential bound with respect to  $z$ . We thus focus from now on on the stable components defined by (28), (29). We first observe that, as in the unstable component (27), the contribution:

$$-a_p^{-1} \mathbb{M}(z)^{j-j_0-1} \pi^s(z) \mathbf{e}$$

appearing in the definition (28) for  $j > j_0$  also extends holomorphically to the ball  $B_\varepsilon(\underline{z}_k)$  and contributes to the remainder term  $R_z(j, j_0)$  with an  $O(\exp(-c|j - j_0|))$  term. We thus focus on the sequence:

$$\left( \mathbb{M}(z)^{j-1} \pi^s(z) W_1 \right)_{j \geq 1},$$

where the vector  $\pi^s(z) W_1 \in \mathbb{E}^s(z)$  is defined by (29) for  $z \in B_\varepsilon(\underline{z}_k) \setminus \{\underline{z}_k\}$ . The singularity in the Green's function comes from the fact that  $\mathcal{B}|_{\mathbb{E}^s(\underline{z}_k)}$  is no longer an isomorphism. We now make this singularity explicit.

We pick a basis  $e_1(z), \dots, e_r(z)$  of the stable subspace  $\mathbb{E}^s(z)$  that depends holomorphically on  $z$  near  $\underline{z}_k$ . Since the Lopatinskii determinant factorizes as:

$$\Delta(z) = (z - \underline{z}_k) \vartheta(z),$$

where  $\vartheta$  is a holomorphic function that does not vanish at  $\underline{z}_k$ , we can therefore write:

$$\left( \mathcal{B} e_1(z) \quad \cdots \quad \mathcal{B} e_r(z) \right)^{-1} = \frac{1}{z - \underline{z}_k} D(z),$$

where  $D(z)$  is a matrix in  $\mathcal{M}_r(\mathbb{C})$  that depends holomorphically on  $z$  near  $\underline{z}_k$ . We then define the vector:

$$\underline{\mathcal{W}}(j_0) := \left( e_1(\underline{z}_k) \quad \cdots \quad e_r(\underline{z}_k) \right) D(\underline{z}_k) \left( -a_p^{-1} \mathcal{B} \mathbb{M}(\underline{z}_k)^{-j_0} \pi^u(\underline{z}_k) \mathbf{e} \right), \quad (30)$$

which satisfies the bound:

$$|\underline{\mathcal{W}}(j_0)| \leq C \exp(-c j_0), \quad (31)$$

for some positive constants  $C$  and  $c$ , uniformly with respect to  $j_0 \geq 1$ . Moreover, since we have the relation:

$$\mathcal{B} \left( e_1(\underline{z}_k) \quad \cdots \quad e_r(\underline{z}_k) \right) D(\underline{z}_k) = 0,$$

the vector  $\underline{\mathcal{W}}(j_0)$  belongs to  $\text{Ker } \mathcal{B} \cap \mathbb{E}^s(\underline{z}_k)$ . Hence, by selecting the appropriate coordinate, the geometric sequence (which is valued in  $\mathbb{C}^{p+r}$ ):

$$\left( \mathbb{M}(\underline{z}_k)^{j-1} \underline{\mathcal{W}}(j_0) \right)_{j \geq 1},$$

provides with a scalar sequence  $(\underline{w}_k(j, j_0))_{j, j_0 \geq 1}$  with  $\underline{w}_k(\cdot, j_0) \in \mathcal{H}$  for all  $j_0 \geq 1$ , and that satisfies the bound:

$$\forall j, j_0 \geq 1, \quad |\underline{w}_k(j, j_0)| \leq \underline{C}_k \exp(-\underline{c}_k(j + j_0)),$$

as stated in Lemma 7. It thus only remains to show that the remainder term:

$$\mathcal{R}_z(j, j_0) := \mathbb{M}(z)^{j-1} \pi^s(z) W_1 - \frac{\mathbb{M}(\underline{z}_k)^{j-1} \underline{\mathcal{W}}(j_0)}{z - \underline{z}_k} \quad (32)$$

extends holomorphically to  $B_\varepsilon(\underline{z}_k)$  and satisfies a suitable exponential bound.

We decompose the vector  $\pi^s(z) W_1$  in (29) along the basis  $e_1(z), \dots, e_r(z)$  of the stable subspace  $\mathbb{E}^s(z)$  and write:

$$\pi^s(z) W_1 = \frac{1}{z - \underline{z}_k} \left( e_1(z) \quad \cdots \quad e_r(z) \right) D(z) \left( -a_p^{-1} \mathcal{B} \mathbb{M}(z)^{-j_0} \pi^u(z) \mathbf{e} \right).$$



Using the definitions (30) and (32), we can decompose the remainder  $\mathcal{R}_z(j, j_0)$  as follows:

$$\begin{aligned} \mathcal{R}_z(j, j_0) &= \frac{1}{z - \underline{z}_k} \left( (\mathbb{M}(z) \pi^s(z))^{j-1} - (\mathbb{M}(\underline{z}_k) \pi^s(\underline{z}_k))^{j-1} \right) \underline{\mathcal{W}}(j_0) \\ &\quad - \frac{a_p^{-1}}{z - \underline{z}_k} (\mathbb{M}(z) \pi^s(z))^{j-1} \left( (e_1(z) \ \cdots \ e_r(z)) D(z) \mathcal{B} (\mathbb{M}(z) \pi^u(z))^{-j_0} \mathbf{e} \right. \\ &\quad \left. - (e_1(\underline{z}_k) \ \cdots \ e_r(\underline{z}_k)) D(\underline{z}_k) \mathcal{B} (\mathbb{M}(\underline{z}_k) \pi^u(\underline{z}_k))^{-j_0} \mathbf{e} \right). \end{aligned}$$

Both terms (the first line, and the difference between the second and third lines) in the above decomposition are dealt with by applying the following result combined with the hyperbolic dichotomy of  $\mathbb{M}(z)$  near  $\underline{z}_k$ .

**Lemma 8.** *Let  $M$  be a holomorphic function on the open ball  $B_\delta(0)$  with values in  $\mathcal{M}_N(\mathbb{C})$  for some  $\delta > 0$  and integer  $N$ , that satisfies:*

$$\exists C > 0, \quad \exists r \in (0, 1), \quad \forall j \in \mathbb{N}, \quad \forall z \in B_\delta(0), \quad \|M(z)^j\| \leq C r^j.$$

*Then up to diminishing  $\delta$  and for some possibly new constants  $C > 0$  and  $r \in (0, 1)$ , there holds:*

$$\forall j \in \mathbb{N}, \quad \forall z_1, z_2 \in B_\delta(0), \quad \|M(z_1)^j - M(z_2)^j\| \leq C |z_1 - z_2| r^j.$$

Applying Lemma 8 to the above decomposition of  $\mathcal{R}_z(j, j_0)$ , and using the exponential decay of the sequences  $(\mathbb{M}(z)^j \pi^s(z))_{j \in \mathbb{N}}$  and  $(\mathbb{M}(z)^{-j} \pi^u(z))_{j \in \mathbb{N}}$ , we get the bound:

$$|\mathcal{R}_z(j, j_0)| \leq C \exp(-c_k(j + j_0)),$$

which means that  $\mathcal{R}_z(j, j_0)$  remains bounded on  $B_\varepsilon(\underline{z}_k) \setminus \{\underline{z}_k\}$  and can therefore be extended holomorphically to the whole ball  $B_\varepsilon(\underline{z}_k)$ . The proof of Lemma 7 is complete.  $\square$

*Proof of Lemma 8.* The argument is a mere application of the Taylor formula. Let us recall that the differential of the mapping:

$$\Psi_j \quad : \quad A \in \mathcal{M}_N(\mathbb{C}) \quad \longmapsto \quad A^j,$$

is given by:

$$d\Psi_j(A) \cdot B = \sum_{\ell=0}^{j-1} A^\ell B A^{j-1-\ell},$$

so we have:

$$M(z_1)^j - M(z_2)^j = (z_1 - z_2) \int_0^1 \sum_{\ell=0}^{j-1} M(z_2 + t(z_1 - z_2))^\ell M'(z_2 + t(z_1 - z_2)) M(z_2 + t(z_1 - z_2))^{j-1-\ell} dt.$$

The result follows by using a uniform bound for the first derivative  $M'$ , up to diminishing  $\delta$ , and using the exponential decay of the sequence  $(j r^j)_{j \in \mathbb{N}}$ .  $\square$

### 2.3 Summary

Collecting the results of Lemma 5, Lemma 6 and Lemma 7, we can obtain the following bound for the spatial Green's function away from the spectrum of  $\mathcal{T}$ .

**Corollary 1.** *There exist a radius  $\varepsilon > 0$ , some width  $\eta_\varepsilon > 0$  and two constants  $C_0 > 0$ ,  $c_0 > 0$  such that, for all  $z$  in the set:*

$$\left\{ \zeta \in \mathbb{C} / e^{-\eta_\varepsilon} < |\zeta| \leq e^\pi \right\} \setminus \left( \bigcup_{k=1}^K B_\varepsilon(\underline{z}_k) \cup B_\varepsilon(1) \right),$$

and for all  $j_0 \geq 1$ , the Green's function  $G_z(\cdot, j_0) \in \mathcal{H}$  solution to (23) satisfies the pointwise bound:

$$\forall j \geq 1, \quad |G_z(j, j_0)| \leq C_0 \exp(-c_0 |j - j_0|).$$

Moreover, for  $z$  inside the ball  $B_\varepsilon(1)$ , the Green's function component  $G_z(j, j_0)$  depends holomorphically on  $z$  and satisfies the bound given in Lemma 6, and for  $k = 1, \dots, K$  and  $z$  in the pointed ball  $B_\varepsilon(\underline{z}_k) \setminus \{\underline{z}_k\}$ ,  $G_z(j, j_0)$  has a simple pole at  $\underline{z}_k$  with the behavior stated in Lemma 7.

## 3 Temporal Green's function and proof of Theorem 1

The starting point of the analysis is to use inverse Laplace transform formula to express the temporal Green's function  $\mathcal{G}^n(\cdot, j_0) := \mathcal{T}^n \delta_{j_0}$  as the following contour integral:

$$\forall n \in \mathbb{N}^*, \quad \forall j \geq 1, \quad \forall j_0 \geq 1, \quad \mathcal{G}^n(j, j_0) = (\mathcal{T}^n \delta_{j_0})_j = \frac{1}{2\pi \mathbf{i}} \int_{\tilde{\Gamma}} z^n G_z(j, j_0) dz, \quad (33)$$

where  $\tilde{\Gamma}$  is a closed curve in the complex plane surrounding the unit disk  $\mathbb{D}$  lying in the resolvent set of  $\mathcal{T}$  and  $G_z(\cdot, j_0) \in \mathcal{H}$  is the spatial Green's function defined in (23).

Following our recent work [CF20], the idea will be to deform  $\tilde{\Gamma}$  in order to obtain sharp pointwise estimates on the temporal Green's function using our pointwise estimates on the spatial Green's function summarized in Corollary 1 above. To do so, we first change variable in (33), by setting  $z = \exp(\tau)$ , such that we get

$$\mathcal{G}^n(j, j_0) = \frac{1}{2\pi \mathbf{i}} \int_{\Gamma} e^{n\tau} \mathbf{G}_\tau(j, j_0) d\tau, \quad (34)$$

where without loss of generality  $\Gamma = \{s + \mathbf{i}\ell \mid \ell \in [-\pi, \pi]\}$  for some (and actually any)  $s > 0$ , and  $\mathbf{G}_\tau(\cdot, j_0) \in \mathcal{H}$  is given by

$$\forall \tau \in \Gamma, \quad \forall j \geq 1, \quad \forall j_0 \geq 1, \quad \mathbf{G}_\tau(j, j_0) := G_{e^\tau}(j, j_0) e^\tau.$$

It is already important to remark that as  $\mathcal{T}$  is a recurrence operator with finite stencil, for each  $n \geq 1$ , there holds

$$\mathcal{G}^n(j, j_0) = 0, \quad \text{for } j - j_0 > rn \text{ or } j - j_0 < -pn.$$

As a consequence, throughout this section, we assume that  $j, j_0$  and  $n$  satisfy

$$-pn \leq j - j_0 \leq rn.$$

The very first step in the analysis of the temporal Green's function defined in (34) is to translate the pointwise estimates from Corollary 1 for the spatial Green's function  $G_z(j, j_0)$  to pointwise estimates

for  $\mathbf{G}_\tau(j, j_0)$ . We let be  $\underline{\tau}_k = \mathbf{i}\underline{\theta}_k := \log(z_k)$  for  $\underline{\theta}_k \in [-\pi, \pi] \setminus \{0\}$  for each  $k = 1, \dots, K$ . Finally, we also set  $\alpha := \lambda a > 0$ . As in [CF20], the temporal Green's function is expected to have a leading order contribution concentrated near  $j - j_0 \sim \alpha n$ . An important feature of the situation we deal here with is that the temporal Green's function should also incorporate the contribution of the eigenvalues  $\underline{z}_k$ . These contributions will not decay with respect to  $n$  since the  $\underline{z}_k$ 's have modulus 1.

**Lemma 9.** *There exist a radius  $\varepsilon > 0$ , some width  $\eta_\varepsilon > 0$  and constants  $0 < \beta_* < \beta < \beta^*$  and  $C > 0$ ,  $c > 0$  such that, for all  $z$  in the set:*

$$\Omega_\varepsilon := \left\{ \tau \in \mathbb{C} \mid -\eta_\varepsilon < \operatorname{Re}(\tau) \leq \pi \right\} \setminus \left( \bigcup_{k=1}^K B_\varepsilon(\mathbf{i}\underline{\theta}_k) \cup B_\varepsilon(0) \right),$$

and for all  $j_0 \geq 1$ , the Green's function  $\mathbf{G}_\tau(\cdot, j_0) \in \mathcal{H}$  satisfies the pointwise bound:

$$\forall j, j_0 \geq 1, \quad |\mathbf{G}_\tau(j, j_0)| \leq C \exp(-c|j - j_0|).$$

Moreover, for  $\tau$  inside the ball  $B_\varepsilon(0)$ , the Green's function component  $\mathbf{G}_\tau(j, j_0)$  depends holomorphically on  $\tau$  and satisfies the bound

$$\forall \tau \in B_\varepsilon(0), \quad \forall j, j_0 \geq 1, \quad |\mathbf{G}_\tau(j, j_0)| \leq \begin{cases} C \exp(-c|j - j_0|), & \text{if } 1 \leq j \leq j_0, \\ C \exp(|j - j_0| \operatorname{Re}(\varpi(\tau))), & \text{if } j > j_0, \end{cases}$$

with

$$\varpi(\tau) = -\frac{1}{\alpha} \tau + (-1)^{\mu+1} \frac{\beta}{\alpha^{2\mu+1}} \tau^{2\mu} + O(|\tau|^{2\mu+1}), \quad \forall \tau \in B_\varepsilon(0),$$

together with

$$\operatorname{Re}(\varpi(\tau)) \leq -\frac{1}{\alpha} \operatorname{Re}(\tau) + \frac{\beta^*}{\alpha^{2\mu+1}} \operatorname{Re}(\tau)^{2\mu} - \frac{\beta_*}{\alpha^{2\mu+1}} \operatorname{Im}(\tau)^{2\mu}, \quad \forall \tau \in B_\varepsilon(0).$$

At last, for any  $k = 1, \dots, K$  and  $\tau$  in the pointed ball  $B_\varepsilon(\mathbf{i}\underline{\theta}_k) \setminus \{\mathbf{i}\underline{\theta}_k\}$ ,  $\mathbf{G}_\tau(j, j_0)$  has a simple pole at  $\mathbf{i}\underline{\theta}_k$  with the following behavior. There exists a sequence  $(\underline{\mathbf{w}}_k(j, j_0))_{j, j_0 \geq 1}$  with  $\underline{\mathbf{w}}_k(\cdot, j_0) \in \mathcal{H}$  for all  $j_0 \geq 1$ , such that:

$$\forall j, j_0 \geq 1, \quad \mathbf{R}_\tau(j, j_0) := \mathbf{G}_\tau(j, j_0) - \frac{\underline{\mathbf{w}}_k(j, j_0)}{\tau - \mathbf{i}\underline{\theta}_k},$$

extends holomorphically to the whole ball  $B_\varepsilon(\mathbf{i}\underline{\theta}_k)$  with respect to  $\tau$ , and the holomorphic extension satisfies the bound:

$$\forall \tau \in B_\varepsilon(\mathbf{i}\underline{\theta}_k), \quad |\mathbf{R}_\tau(j, j_0)| \leq C \exp(-c|j - j_0|).$$

Moreover, the sequence  $(\underline{\mathbf{w}}_k(j, j_0))_{j, j_0 \geq 1}$  satisfies the pointwise bound:

$$\forall j, j_0 \geq 1, \quad |\underline{\mathbf{w}}_k(j, j_0)| \leq C \exp(-c(j + j_0)). \quad (35)$$

*Proof.* The proof simply relies on writing  $\kappa(z) = \exp(\omega(z))$  and using  $z = \exp(\tau)$ , such that after identification we have  $\varpi(\tau) := \omega(\exp(\tau))$ . Next, using our assumption (11), we obtain the desired expansion for  $\varpi(\tau)$  near  $\tau = 0$ . From this expansion, we get

$$\begin{aligned} \operatorname{Re}(\varpi(\tau)) &= -\frac{1}{\alpha} \operatorname{Re}(\tau) - \frac{\beta}{\alpha^{2\mu+1}} \operatorname{Im}(\tau)^{2\mu} - \frac{\beta}{\alpha^{2\mu+1}} (-1)^\mu \operatorname{Re}(\tau)^{2\mu} \\ &\quad - \frac{\beta}{\alpha^{2\mu+1}} \sum_{m=1}^{\mu-1} (-1)^m \binom{2\mu}{2m} \operatorname{Re}(\tau)^{2m} \operatorname{Im}(\tau)^{2(\mu-m)} + O(|\tau|^{2\mu+1}). \end{aligned}$$

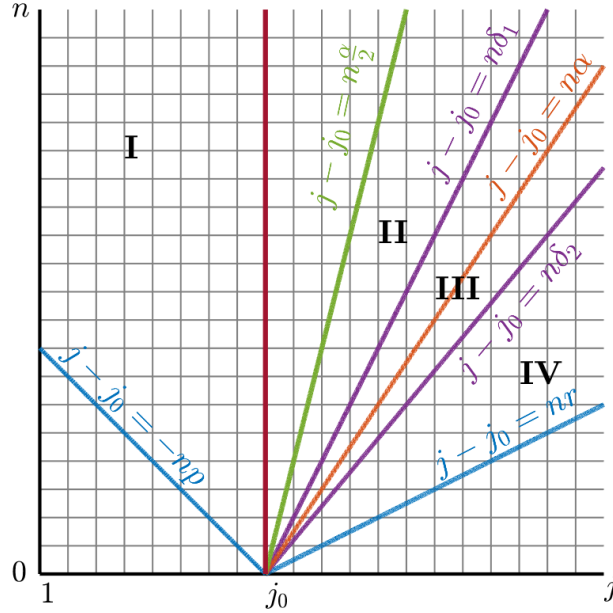


Figure 1: Schematic illustration of the different domains in the  $(j, n)$  plane used in the analysis. In each domain, we use a different contour integral in (34). In domain **I**, that is for  $-np \leq j - j_0 \leq n\frac{\alpha}{2}$ , we can push the contour of integration  $\Gamma$  to  $\text{Re}(\tau) = -\eta$  for some well chosen  $\eta > 0$ . For values in domains **II**, **III** and **IV**, we can also push the contour of integration  $\Gamma$  to  $\text{Re}(\tau) = -\eta$  but this time we have to use a “parabolic” contour near the origin. We refer to Figure 2 for an illustration of such contours in domain **III**. Note that below the lines  $j - j_0 = -np$  and  $j - j_0 = nr$  (blue) the Green’s function  $\mathcal{G}^n(j, j_0)$  vanishes.

for all  $\tau \in B_\varepsilon(0)$ . We crucially note that the term  $\text{Im}(\tau)^{2\mu}$  comes with a negative sign such that both  $O(|\tau|^{2\mu+1})$  and each term of the sum, using Young’s inequality, can be absorbed and we arrive at the desired estimate for two uniform constants  $0 < \beta_* < \beta < \beta^*$ . The remainder of the proof is a simple transposition of Lemma 5, 6 and 7 in the new variable  $\tau$ .  $\square$

With the notations introduced in the above Lemma, we can summarize in the following proposition the results that we will prove in this section. Why Proposition 1 is sufficient to get the result of Theorem 1 is explained at the end of this Section.

**Proposition 1.** *There exist two constants  $C > 0$  and  $\omega > 0$  such that for any  $n \geq 1$  the temporal Green’s function  $\mathcal{G}^n$  satisfies the pointwise estimate*

$$\forall j, j_0 \geq 1, \quad \left| \mathcal{G}^n(j, j_0) - \sum_{k=1}^K \mathbf{w}_k(j, j_0) e^{n\mathbf{i}\ell_k} \right| \leq \frac{C}{n^{\frac{1}{2\mu}}} \exp\left(-\omega \left(\frac{|j - j_0 - \alpha n|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right).$$

From now on, we fix  $0 < \eta < \eta_\varepsilon$  such that the segment  $\{-\eta + \mathbf{i}\ell \mid \ell \in [-\pi, \pi]\}$  intersects  $\partial B_\varepsilon(0)$  outside the curve (12) of essential spectrum of  $\mathcal{T}$  near the origin. We are going to distinguish several cases depending on the relative position between  $j - j_0$  and  $n$ , as sketched in Figure 1. Formally, we will use different contours of integration in (34) depending if  $j - j_0$  is near  $n\alpha$  or away from  $n\alpha$ . Indeed, when  $j - j_0 \sim n\alpha$ , we expect to have Gaussian-like bounds coming from the contribution in  $B_\varepsilon(0)$  near the origin

where the essential spectrum of  $\mathcal{T}$  touches the imaginary axis. In that case, we will use contours similar to [God03, CF20] and that were already introduced in the continuous setting in [ZH98]. Let us note that unlike in [CF20], we have isolated poles on the imaginary axis given by the  $\underline{\tau}_k = \mathbf{i}\underline{\theta}_k$ ,  $k = 1, \dots, K$ , whose contributions in (34) will be handled via Cauchy's formula and the residue theorem. We thus divide the analysis into a medium range, that is for those values of  $j - j_0$  away from  $n\alpha$ , and short range when  $j - j_0$  is near  $n\alpha$ . More specifically, we decompose our domain as

- Medium range:  $-np \leq j - j_0 < n\frac{\alpha}{2}$ ;
- Short range:  $n\frac{\alpha}{2} \leq j - j_0 \leq nr$ ;

where we recall that  $\alpha = \lambda a > 0$  from our consistency condition.

### 3.1 Medium range

In this section, we consider the medium range where  $-np \leq j - j_0 < n\frac{\alpha}{2}$ . In order to simplify the presentation, we first treat the case where  $-np \leq j - j_0 \leq 0$  and then consider the range  $1 \leq j - j_0 < n\frac{\alpha}{2}$ .

**Lemma 10.** *There exist constants  $C > 0$  and  $c > 0$ , such that for all integers  $j, j_0, n$  satisfying  $-np \leq j - j_0 \leq 0$ , the temporal Green's function satisfies*

$$\left| \mathcal{G}^n(j, j_0) - \sum_{k=1}^K \mathbf{w}_k(j, j_0) e^{n\mathbf{i}\underline{\theta}_k} \right| \leq C e^{-n\eta - c|j - j_0|}.$$

*Proof.* We first recall that

$$\mathcal{G}^n(j, j_0) = \frac{1}{2\pi\mathbf{i}} \int_{\Gamma} e^{n\tau} \mathbf{G}_{\tau}(j, j_0) d\tau,$$

with  $\Gamma = \{s + \mathbf{i}\ell \mid \ell \in [-\pi, \pi]\}$  for any  $0 < s \leq \pi$ . Next, we denote  $\Gamma_{-\eta} = \{-\eta + \mathbf{i}\ell \mid \ell \in [-\pi, \pi]\}$ . Using the residue theorem, we obtain that

$$\frac{1}{2\pi\mathbf{i}} \int_{\Gamma} e^{n\tau} \mathbf{G}_{\tau}(j, j_0) d\tau = \underbrace{\frac{1}{2\pi\mathbf{i}} \int_{\Gamma_{-\eta}} e^{n\tau} \mathbf{G}_{\tau}(j, j_0) d\tau}_{:= \tilde{\mathcal{G}}^n(j, j_0)} + \sum_{k=1}^K \text{Res}\left(\tau \mapsto e^{n\tau} \mathbf{G}_{\tau}(j, j_0), \underline{\tau}_k\right),$$

where we readily have that

$$\sum_{k=1}^K \text{Res}\left(\tau \mapsto e^{n\tau} \mathbf{G}_{\tau}(j, j_0), \underline{\tau}_k\right) = \sum_{k=1}^K \mathbf{w}_k(j, j_0) e^{n\mathbf{i}\underline{\theta}_k},$$

from Lemma 9. Here, and throughout, we use the fact that the integrals along  $\{-v \pm \mathbf{i}\pi \mid v \in [-\eta, s]\}$  compensate each other. Now,  $\Gamma_{-\eta}$  intersects each ball  $B_{\varepsilon}(\mathbf{i}\underline{\theta}_k)$  and we denote  $\Gamma_{-\eta}^k := \Gamma_{-\eta} \cap B_{\varepsilon}(\mathbf{i}\underline{\theta}_k)$ . Using once again Lemma 9, we have for each  $\tau \in \Gamma_{-\eta}^k$

$$|\mathbf{G}_{\tau}(j, j_0)| \leq |\mathbf{R}_{\tau}(j, j_0)| + \left| \frac{\mathbf{w}_k(j, j_0)}{\tau - \mathbf{i}\underline{\theta}_k} \right| \leq C e^{-c|j - j_0|} + C_0 e^{-c|j + j_0|} \leq C_1 e^{-c|j - j_0|},$$

for some positive constants  $C_{0,1} > 0$ . Finally, we remark that for  $-np \leq j - j_0 \leq 0$ , we have

$$\forall \tau \in \left\{ \omega \in \mathbb{C} \mid -\eta_{\varepsilon} < \text{Re}(\omega) \leq \pi \right\} \setminus \left( \bigcup_{k=1}^K B_{\varepsilon}(\mathbf{i}\underline{\theta}_k) \right), \quad |\mathbf{G}_{\tau}(j, j_0)| \leq C e^{-c|j - j_0|},$$

such that in fact, for all  $\tau \in \Gamma_{-\eta}$  and  $-np \leq j - j_0 \leq 0$  we have the following bound

$$|\mathbf{G}_\tau(j, j_0)| \leq C e^{-c|j-j_0|}.$$

The estimate on  $\tilde{\mathcal{G}}^n(j, j_0)$  easily follows and concludes the proof.  $\square$

Next, we consider the range  $1 \leq j - j_0 < n \frac{\alpha}{2}$ . This time, the spatial Green's function  $\mathbf{G}_\tau(j, j_0)$  satisfies a different bound in  $B_\varepsilon(0)$ . Nevertheless, we can still obtain some strong decaying estimates which are summarized in the following lemma.

**Lemma 11.** *There exists a constant  $C > 0$  such that for all integers  $j, j_0, n$  satisfying  $1 \leq j - j_0 < n \frac{\alpha}{2}$ , the temporal Green's function satisfies*

$$\left| \mathcal{G}^n(j, j_0) - \sum_{k=1}^K \mathbf{w}_k(j, j_0) e^{n\mathbf{i}\varrho_k} \right| \leq C e^{-n \frac{\eta}{4}}.$$

*Proof.* The beginning of the proof follows similar lines as the ones in the proof of Lemma 10. We deform the initial contour  $\Gamma$  to  $\Gamma_{-\eta}$ , and using the residue theorem we get

$$\mathcal{G}^n(j, j_0) = \sum_{k=1}^K \mathbf{w}_k(j, j_0) e^{n\mathbf{i}\varrho_k} + \frac{1}{2\pi\mathbf{i}} \int_{\Gamma_{-\eta}} e^{n\tau} \mathbf{G}_\tau(j, j_0) d\tau.$$

We denote by  $\Gamma_{-\eta}^{in}$  and  $\Gamma_{-\eta}^{out}$  the portions of  $\Gamma_{-\eta}$  which lie either inside or outside  $B_\varepsilon(0)$ . Note that the analysis along  $\Gamma_{-\eta}^{out}$  is similar as in Lemma 10, and we already get the estimate

$$\left| \frac{1}{2\pi\mathbf{i}} \int_{\Gamma_{-\eta}^{out}} e^{n\tau} \mathbf{G}_\tau(j, j_0) d\tau \right| \leq C e^{-n\eta - c|j-j_0|} \leq C e^{-n \frac{\eta}{4}}.$$

Along  $\Gamma_{-\eta}^{in}$ , we compute

$$\left| \frac{1}{2\pi\mathbf{i}} \int_{\Gamma_{-\eta}^{in}} e^{n\tau} \mathbf{G}_\tau(j, j_0) d\tau \right| \leq C e^{-n\eta} \int_{\Gamma_{-\eta}^{in}} e^{|j-j_0| \operatorname{Re}(\varpi(\tau))} |d\tau|.$$

Next, for all  $\tau \in \Gamma_{-\eta}^{in}$  we have

$$\begin{aligned} \operatorname{Re}(\varpi(\tau)) &\leq -\frac{1}{\alpha} \operatorname{Re}(\tau) - \underbrace{\frac{\beta^*}{\alpha^{2\mu+1}} \operatorname{Im}(\tau)^{2\mu}}_{\leq 0} + \frac{\beta^*}{\alpha^{2\mu+1}} \operatorname{Re}(\tau)^{2\mu} \\ &\leq \frac{\eta}{\alpha} + \frac{\beta^*}{\alpha^{2\mu+1}} \eta^{2\mu}. \end{aligned}$$

As a consequence,

$$\begin{aligned} -n\eta + |j - j_0| \operatorname{Re}(\varpi(\tau)) &\leq n\eta \left( -1 + \frac{|j - j_0|}{n\alpha} + \frac{|j - j_0|}{n} \frac{\beta^*}{\alpha^{2\mu+1}} \eta^{2\mu-1} \right) \\ &\leq n\eta \left( -\frac{1}{2} + \frac{\beta^*}{2\alpha^{2\mu}} \eta^{2\mu-1} \right) \leq -n \frac{\eta}{4}, \end{aligned}$$

provided that  $\eta$  is chosen small enough (the choice only depends on  $\beta^*$  and  $\alpha$ ).  $\square$

### 3.2 Short range

Throughout this section, we assume that  $n \geq 1$  and  $n \frac{\alpha}{2} \leq j - j_0 \leq nr$ . Following [ZH98, God03, CF20], we introduce a family of parametrized curves given by

$$\Gamma_p := \left\{ \operatorname{Re}(\tau) - \frac{\beta^*}{\alpha^{2\mu}} \operatorname{Re}(\tau)^{2\mu} + \frac{\beta_*}{\alpha^{2\mu}} \operatorname{Im}(\tau)^{2\mu} = \Psi(\tau_p) \mid -\eta \leq \operatorname{Re}(\tau) \leq \tau_p \right\} \quad (36)$$

with  $\Psi(\tau_p) := \tau_p - \frac{\beta^*}{\alpha^{2\mu}} \tau_p^{2\mu}$ . Note that these curves intersect the real axis at  $\tau_p$ . We also let

$$\zeta := \frac{j - j_0 - n\alpha}{2\mu n}, \quad \text{and} \quad \gamma := \frac{j - j_0}{n} \frac{\beta^*}{\alpha^{2\mu}} > 0,$$

and define  $\rho\left(\frac{\zeta}{\gamma}\right)$  as the unique real root to the equation

$$-\zeta + \gamma x^{2\mu-1} = 0,$$

that is

$$\rho\left(\frac{\zeta}{\gamma}\right) := \operatorname{sgn}(\zeta) \left(\frac{|\zeta|}{\gamma}\right)^{\frac{1}{2\mu-1}}.$$

The specific value of  $\tau_p$  is now fixed depending on the ratio  $\frac{\zeta}{\gamma}$  as follows

$$\tau_p := \begin{cases} -\frac{\eta}{2} & \text{if } \rho\left(\frac{\zeta}{\gamma}\right) < -\frac{\eta}{2}, \\ \rho\left(\frac{\zeta}{\gamma}\right) & \text{if } -\frac{\eta}{2} \leq \rho\left(\frac{\zeta}{\gamma}\right) \leq \varepsilon_0, \\ \varepsilon_0 & \text{if } \rho\left(\frac{\zeta}{\gamma}\right) > \varepsilon_0, \end{cases}$$

where  $0 < \varepsilon_0 < \varepsilon$  is chosen such that  $\Gamma_p$  with  $\tau_p = \varepsilon_0$  intersects the segment  $\{-\eta + \mathbf{i}\ell \mid \ell \in [-\pi, \pi]\}$  precisely on the boundary<sup>6</sup> of  $B_\varepsilon(0)$ . Finally, let us note that as  $n \frac{\alpha}{2} \leq j - j_0 \leq nr$ , we have  $-\frac{\alpha}{4\mu} \leq \zeta \leq \frac{r-\alpha}{2\mu}$ . As  $r > \alpha = \lambda a$  (see Lemma 1), the region where  $-\frac{\eta}{2} \leq \rho\left(\frac{\zeta}{\gamma}\right) \leq \varepsilon_0$  holds is not empty. From now on, we will treat each subcase separately.

**Lemma 12.** *There exist constants  $C > 0$  and  $M > 0$  such that for  $n \geq 1$  and  $-\frac{\eta}{2} \leq \rho\left(\frac{\zeta}{\gamma}\right) \leq \varepsilon_0$ , the following estimate holds:*

$$\left| \mathcal{G}^n(j, j_0) - \sum_{k=1}^K \mathbf{w}_k(j, j_0) e^{n\mathbf{i}\theta_k} \right| \leq \frac{C}{n^{\frac{1}{2\mu}}} \exp\left(-\frac{1}{M} \left(\frac{|j - j_0 - \alpha n|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right).$$

*Proof.* We will consider a contour depicted in Figure 2 which consists of the parametrized curve  $\Gamma_p$  near the origin and otherwise is the segment  $\Gamma_{-\eta}$ . We will denote  $\Gamma_{-\eta}^{in}$  and  $\Gamma_{-\eta}^{out}$ , the portions of the segment  $\Gamma_{-\eta}$  which lie either inside or outside  $B_\varepsilon(0)$  with  $|\operatorname{Im}(\tau)| \leq \pi$ . Using the residue theorem, we have that

$$\mathcal{G}^n(j, j_0) = \sum_{k=1}^K \mathbf{w}_k(j, j_0) e^{n\mathbf{i}\theta_k} + \frac{1}{2\pi \mathbf{i}} \int_{\Gamma_{-\eta}^{in} \cup \Gamma_{-\eta}^{out}} e^{n\tau} \mathbf{G}_\tau(j, j_0) d\tau + \frac{1}{2\pi \mathbf{i}} \int_{\Gamma_p} e^{n\tau} \mathbf{G}_\tau(j, j_0) d\tau.$$

<sup>6</sup>This is possible because the curves  $\Gamma_p$  are symmetric with respect to the real axis.

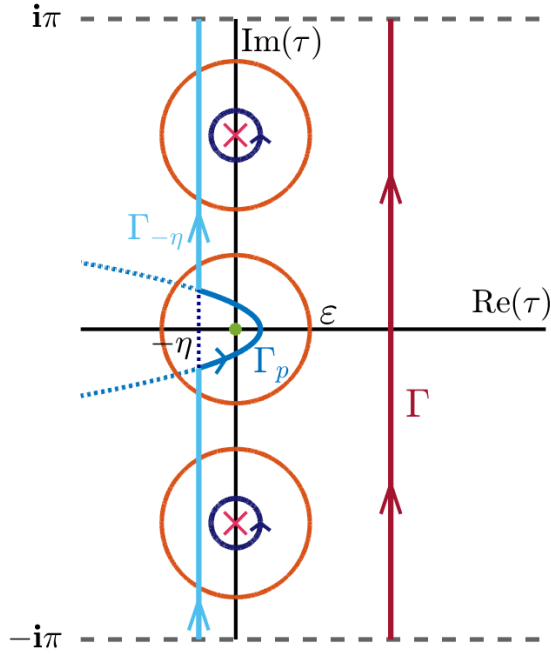


Figure 2: Illustration of the contour used in the case  $-\frac{\eta}{2} \leq \rho\left(\frac{\zeta}{\gamma}\right) \leq \varepsilon_0$  when  $n\frac{\alpha}{2} \leq j - j_0 \leq nr$ . We deform the initial contour  $\Gamma$  (dark red) into the contour which consists of the parametrized curve  $\Gamma_p$  near the origin (blue) and a portion of the segment  $\Gamma_{-\eta} = \{-\eta + \mathbf{i}\ell \mid \ell \in [-\pi, \pi]\}$  (light blue). Note that  $\Gamma_{-\eta} \cap B_\varepsilon(0) \neq \emptyset$ . Near each pole on the imaginary axis (pink cross), we use the residue theorem which is symbolized by the small oriented circles (dark blue) surrounding each of them.

Computations along  $\Gamma_{-\eta}^{out}$  are similar to the previous cases, and we directly get

$$\left| \frac{1}{2\pi\mathbf{i}} \int_{\Gamma_{-\eta}^{out}} e^{n\tau} \mathbf{G}_\tau(j, j_0) d\tau \right| \leq C e^{-n\eta - c|j-j_0|}.$$

For all  $\tau \in \Gamma_{-\eta}^{in}$ , we use that  $\text{Im}(\tau)^2 \geq \text{Im}(\tau_*)^2$  where  $\tau_* = -\eta + \mathbf{i}\ell_*$  and  $\ell_* > 0$  is the positive root of

$$-\eta - \frac{\beta^*}{\alpha^{2\mu}} \eta^{2\mu} + \frac{\beta_*}{\alpha^{2\mu}} \ell_*^{2\mu} = \Psi(\tau_p).$$

That is, the point  $\tau_* = -\eta + \mathbf{i}\ell_*$  lies at the intersection of  $\Gamma_p$  and the segment  $\{-\eta + \mathbf{i}\ell \mid \ell \in [-\pi, \pi]\}$  with  $\tau_* \in B_\varepsilon(0)$ . As a consequence, for all  $\tau \in \Gamma_{-\eta}^{in}$  we have

$$\begin{aligned} \text{Re}(\varpi(\tau)) &\leq \frac{\eta}{\alpha} + \frac{\beta^*}{\alpha^{2\mu+1}} \eta^{2\mu} - \frac{\beta_*}{\alpha^{2\mu+1}} \text{Im}(\tau)^{2\mu} \\ &= -\frac{\tau_p}{\alpha} + \frac{\beta^*}{\alpha^{2\mu+1}} \tau_p^{2\mu} - \frac{\beta_*}{\alpha^{2\mu+1}} \underbrace{(\text{Im}(\tau)^{2\mu} - \ell_*^{2\mu})}_{\geq 0} \\ &\leq -\frac{\tau_p}{\alpha} + \frac{\beta^*}{\alpha^{2\mu+1}} \tau_p^{2\mu}. \end{aligned}$$



Thus, we have

$$\begin{aligned}
n \operatorname{Re}(\tau) + (j - j_0) \operatorname{Re}(\varpi(\tau)) &\leq -n\eta + (j - j_0) \left( -\frac{\tau_p}{\alpha} + \frac{\beta^*}{\alpha^{2\mu+1}} \tau_p^{2\mu} \right) \\
&= \frac{n}{\alpha} \left[ -\eta\alpha + \frac{(j - j_0)}{n} \left( -\tau_p + \frac{\beta^*}{\alpha^{2\mu}} \tau_p^{2\mu} \right) \right] \\
&= \frac{n}{\alpha} \left[ -(\eta + \tau_p)\alpha - 2\mu\zeta\tau_p + \gamma\tau_p^{2\mu} \right] \\
&= \frac{n}{\alpha} \left[ -(\eta + \tau_p)\alpha + (1 - 2\mu)\gamma \left( \frac{|\zeta|}{\gamma} \right)^{\frac{2\mu}{2\mu-1}} \right],
\end{aligned}$$

for all  $\tau \in \Gamma_{-\eta}^{in}$ . Finally, as  $-\frac{\eta}{2} \leq \rho\left(\frac{\zeta}{\gamma}\right) = \tau_p$  we have  $\eta + \tau_p \geq \frac{\eta}{2}$ , and we obtain an estimate of the form

$$\left| \frac{1}{2\pi\mathbf{i}} \int_{\Gamma_{-\eta}^{in}} e^{n\tau} \mathbf{G}_\tau(j, j_0) d\tau \right| \leq C e^{-n\frac{\eta}{2} - \frac{n}{\alpha}(2\mu-1)\gamma\left(\frac{|\zeta|}{\gamma}\right)^{\frac{2\mu}{2\mu-1}}} \leq C e^{-n\frac{\eta}{2} - cn|\zeta|^{\frac{2\mu}{2\mu-1}}},$$

since  $\gamma$  is bounded from below and from above by positive constants.

We now turn our attention to the integral along  $\Gamma_p$ . We first notice that for all  $\tau \in \Gamma_p \subset B_\varepsilon(0)$ , we have

$$\operatorname{Re}(\tau) \leq \tau_p - c_* \operatorname{Im}(\tau)^{2\mu},$$

for some constant  $c_* > 0$ . As a consequence, we obtain the upper bound

$$\begin{aligned}
n \operatorname{Re}(\tau) + (j - j_0) \operatorname{Re}(\varpi(\tau)) &\leq n \operatorname{Re}(\tau) - \frac{j - j_0}{\alpha} \operatorname{Re}(\tau) + \frac{\beta^*(j - j_0)}{\alpha^{2\mu+1}} \operatorname{Re}(\tau)^{2\mu} - \frac{\beta_*(j - j_0)}{\alpha^{2\mu+1}} \operatorname{Im}(\tau)^{2\mu} \\
&\leq n(\operatorname{Re}(\tau) - \tau_p) + \frac{n}{\alpha} \left[ -2\mu\zeta\tau_p + \gamma\tau_p^{2\mu} \right] \\
&\leq -nc_* \operatorname{Im}(\tau)^{2\mu} - \frac{n}{\alpha} (2\mu - 1)\gamma \left( \frac{|\zeta|}{\gamma} \right)^{\frac{2\mu}{2\mu-1}},
\end{aligned}$$

for all  $\tau \in \Gamma_p \subset B_\varepsilon(0)$ . As a consequence, we can derive the following bound

$$\begin{aligned}
\left| \frac{1}{2\pi\mathbf{i}} \int_{\Gamma_p} e^{n\tau} \mathbf{G}_\tau(j, j_0) d\tau \right| &\leq C \int_{\Gamma_p} e^{n \operatorname{Re}(\tau) + (j - j_0) \operatorname{Re}(\varpi(\tau))} |d\tau| \\
&\leq C e^{-\frac{n}{\alpha}(2\mu-1)\gamma\left(\frac{|\zeta|}{\gamma}\right)^{\frac{2\mu}{2\mu-1}}} \int_{\Gamma_p} e^{-nc_* \operatorname{Im}(\tau)^{2\mu}} |d\tau| \\
&\leq C \frac{e^{-\frac{n}{\alpha}(2\mu-1)\gamma\left(\frac{|\zeta|}{\gamma}\right)^{\frac{2\mu}{2\mu-1}}}}{n^{\frac{1}{2\mu}}} \leq C \frac{e^{-cn|\zeta|^{\frac{2\mu}{2\mu-1}}}}{n^{\frac{1}{2\mu}}},
\end{aligned}$$

where we use again that  $\gamma$  is bounded from below and from above by positive constants. At the end of the day, we see that the leading contribution is the one coming from the integral along  $\Gamma_p$ .  $\square$

Finally, we treat the last two cases altogether.

**Lemma 13.** *There exist constants  $C > 0$  and  $c_* > 0$  such that for  $n \geq 1$  and  $\rho\left(\frac{\zeta}{\gamma}\right) < -\frac{\eta}{2}$  or  $\rho\left(\frac{\zeta}{\gamma}\right) > \varepsilon_0$  there holds:*

$$\left| \mathcal{G}^n(j, j_0) - \sum_{k=1}^K \mathbf{w}_k(j, j_0) e^{n \mathbf{i} \theta_k} \right| \leq C e^{-n c_*}.$$

*Proof.* We only present the proof in case  $\rho(\zeta/\gamma) > \varepsilon_0$  as the proof for  $\rho\left(\frac{\zeta}{\gamma}\right) < -\frac{\eta}{2}$  follows similar lines. We deform the contour  $\Gamma$  into  $\Gamma_p \cup \Gamma_{-\eta}^{out}$  where  $\Gamma_{-\eta}^{out}$  are the portions of  $\Gamma_{-\eta}$  which lie outside  $B_\varepsilon(0)$  with  $|\operatorname{Im}(\tau)| \leq \pi$ . We recall that we choose  $\tau_p = \varepsilon_0$  here, so the curve  $\Gamma_p$  intersects  $\partial B_\varepsilon(0)$  precisely at  $\operatorname{Re}(\tau) = -\eta$ . In that case, we have that for all  $\tau \in \Gamma_p$

$$n \operatorname{Re}(\tau) + (j - j_0) \operatorname{Re}(\varpi(\tau)) \leq -n c_* \operatorname{Im}(\tau)^{2\mu} + \frac{n}{\alpha} \left( -2\mu \zeta \varepsilon_0 + \gamma \varepsilon_0^{2\mu} \right).$$

But as  $\rho(\zeta/\gamma) > \varepsilon_0$  we get that  $\zeta > 0$  and  $\zeta > \varepsilon_0^{2\mu-1} \gamma$ , the last term in the previous inequality is estimated via

$$-2\mu \zeta \varepsilon_0 + \gamma \varepsilon_0^{2\mu} < (1 - 2\mu) \gamma \varepsilon_0^{2\mu}.$$

As a consequence, we can derive the following bound

$$\left| \frac{1}{2\pi \mathbf{i}} \int_{\Gamma_p} e^{n\tau} \mathbf{G}_\tau(j, j_0) d\tau \right| \leq C \frac{e^{-\frac{n}{\alpha}(2\mu-1)\gamma \varepsilon_0^{2\mu}}}{n^{\frac{1}{2\mu}}}.$$

With our careful choice of  $\varepsilon_0 > 0$ , the remaining contribution along segments  $\Gamma_{-\eta}^{out}$  can be estimated as usual as

$$\left| \frac{1}{2\pi \mathbf{i}} \int_{\Gamma_{-\eta}^{out}} e^{n\tau} \mathbf{G}_\tau(j, j_0) d\tau \right| \leq C e^{-n\eta - c(j-j_0)},$$

as  $|\tau| \geq \varepsilon$  for  $\tau \in \Gamma_{-\eta}^{out}$ . The conclusion of Lemma 13 follows.  $\square$

We can now combine Lemma 10, Lemma 11, Lemma 12 and Lemma 13 to obtain the result of Proposition 1. Indeed, we observe that in Lemma 10, Lemma 11 and Lemma 13, the obtained exponential bounds can always be subsumed into Gaussian-like estimates. (Lemma 12 yields the worst estimate of all.) For instance, in Lemma 11, the considered integers  $j, j_0, n$  satisfy  $1 \leq j - j_0 \leq n\alpha/2$ , which implies

$$-n \leq \frac{j - j_0 - n\alpha}{\alpha} \leq -\frac{n}{2},$$

and therefore:

$$-n \leq -\omega \left( \frac{|j - j_0 - n\alpha|}{n^{\frac{1}{2\mu}}} \right)^{\frac{2\mu}{2\mu-1}},$$

for some sufficiently small constant  $\omega > 0$ . It remains to explain why Proposition 1 implies Theorem 1.

### 3.3 Proof of the main result

We let  $w \in \mathcal{H}$ , and first remark that for any integer  $n$ , the sequence  $\mathcal{F}^n w \in \mathcal{H}$  is given by:

$$\forall j \geq 1, \quad (\mathcal{F}^n w)_j = \sum_{j_0 \geq 1} \mathcal{G}^n(j, j_0) w_{j_0}.$$

From Proposition 1, we can decompose  $\mathcal{G}^n(j, j_0)$  into two pieces

$$\mathcal{G}^n(j, j_0) = \sum_{k=1}^K \mathbf{w}_k(j, j_0) e^{n \mathbf{i} \theta_k} + \tilde{\mathcal{G}}^n(j, j_0),$$

where the remainder term  $\tilde{\mathcal{G}}^n(j, j_0)$  satisfies the generalized Gaussian estimate of Proposition 1. From the exponential bound (35) and Proposition 1, we have:

$$\sum_{j_0 \geq 1} \left| \mathcal{G}^n(j, j_0) w_{j_0} \right| \leq C e^{-cj} \sum_{j_0 \geq 1} e^{-cj_0} |w_{j_0}| + \frac{C}{n^{\frac{1}{2\mu}}} \sum_{j_0 \geq 1} \exp \left( -\omega \left( \frac{|j - j_0 - \alpha n|}{n^{\frac{1}{2\mu}}} \right)^{\frac{2\mu}{2\mu-1}} \right) |w_{j_0}|.$$

Noting that the sequence  $(e^{-cj})_{j \geq 1}$  is in  $\ell^2$ , we get that

$$\sum_{j \geq 1} \left( e^{-cj} \sum_{j_0 \geq 1} e^{-cj_0} |w_{j_0}| \right)^2 \leq C \|w\|_{\mathcal{H}}^2.$$

Now for the second term, we observe that the sequence defined as

$$\forall j \in \mathbb{Z}, \quad \mathbf{g}_j := \frac{1}{n^{\frac{1}{2\mu}}} \exp \left( -\omega \left( \frac{|j - \alpha n|}{n^{\frac{1}{2\mu}}} \right)^{\frac{2\mu}{2\mu-1}} \right),$$

is bounded (with respect to  $n \in \mathbb{N}^*$ ) in  $\ell^1(\mathbb{Z})$ . Using the Young's convolution inequality  $\ell^1(\mathbb{Z}) \star \ell^2(\mathbb{Z}) \rightarrow \ell^2(\mathbb{Z})$ , we thus obtain the uniform in time bound:

$$\sum_{j \geq 1} \left( \frac{1}{n^{\frac{1}{2\mu}}} \sum_{j_0 \geq 1} \exp \left( -\omega \left( \frac{|j - j_0 - \alpha n|}{n^{\frac{1}{2\mu}}} \right)^{\frac{2\mu}{2\mu-1}} \right) |w_{j_0}| \right)^2 \leq C \|w\|_{\mathcal{H}}^2.$$

This completes the proof that our operator  $\mathcal{T}$  is power bounded on  $\mathcal{H}$ .

## 4 An illustrative example

We illustrate our main result by considering the modified Lax-Friedrichs numerical scheme which reads

$$u_j^{n+1} = u_j^n + \frac{D}{2} (u_{j-1}^n - 2u_j^n + u_{j+1}^n) - \frac{\lambda a}{2} (u_{j+1}^n - u_{j-1}^n), \quad j \geq 1, \quad (37)$$

where  $D > 0$  and  $\lambda a > 0$ , along with some specific boundary condition at  $j = 0$  which we shall specify later. Using our formalism from (5), we have  $p = r = 1$  and

$$a_{-1} = \frac{D + \lambda a}{2}, \quad a_0 = 1 - D, \quad \text{and} \quad a_1 = \frac{D - \lambda a}{2}.$$

We readily note that our consistency conditions (9) are satisfied. Next, if we denote

$$F(\theta) := \sum_{\ell=-1}^1 a_\ell e^{\mathbf{i} \theta \ell}, \quad \theta \in [-\pi, \pi],$$

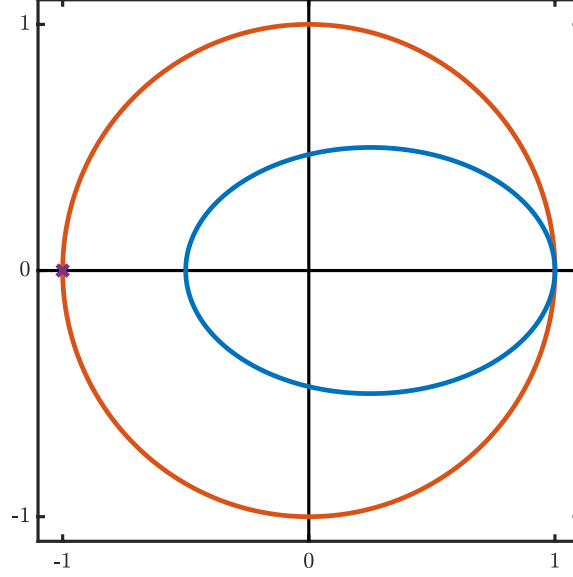


Figure 3: Spectrum of the modified Lax-Friedrichs numerical scheme (37)-(38) with  $\lambda a = 1/2$ ,  $D = 3/4$  and  $b = -1 - \frac{2\sqrt{5}}{5}$ . The blue curve is the essential spectrum of  $\mathcal{T}$  and we have indicated by a cross the presence of eigenvalue at  $\underline{z} = -1$ . Our Assumptions 1-2 are satisfied in that case.

then we have

$$F(\theta) = 1 - D + D \cos(\theta) - \mathbf{i} \lambda a \sin(\theta).$$

As a consequence, provided that  $0 < \lambda a < 1$  and  $(\lambda a)^2 < D < 1$ , we get

$$\forall \theta \in [-\pi, \pi] \setminus \{0\}, \quad |F(\theta)| < 1,$$

such that the dissipativity condition (10) is also verified. Next, we compute that

$$F(\theta) = 1 - \mathbf{i} \lambda a \theta - \frac{D}{2} \theta^2 + O(\theta^3),$$

as  $\theta$  tends to 0. We thus deduce that (11) is satisfied with

$$\mu := 1, \quad \text{and} \quad \beta := \frac{D - (\lambda a)^2}{2} > 0.$$

Assumption 1 is thus satisfied provided that we have  $0 < \lambda a < 1$  and  $(\lambda a)^2 < D < 1$ . We also assume from now on  $D \neq \lambda a$  so that the coefficient  $a_1$  is nonzero.

We now prescribe a boundary condition for (37) which will ensure that our Assumption 2 on the Lopatinskii determinant is satisfied. That is, we want to find  $\underline{z} \in \mathbb{S}^1 \setminus \{1\}$  which is an eigenvalue for  $\mathcal{T}$ . This means that at this point  $\underline{z}$  the boundary condition must be adjusted so as to have  $\text{Ker } \mathcal{B} \cap \mathbb{E}^s(\underline{z}) \neq \{0\}$ . We use a boundary condition of the form given in (6) with  $p_b = p = 1$ :

$$u_0^n = b u_1^n, \quad n \geq 1,$$

where  $b \in \mathbb{R}$  is a constant. In order to ensure that  $\text{Ker } \mathcal{B} \cap \mathbb{E}^s(\underline{z}) \neq \emptyset$  is satisfied, we impose that

$$1 = b \kappa_s(\underline{z}),$$

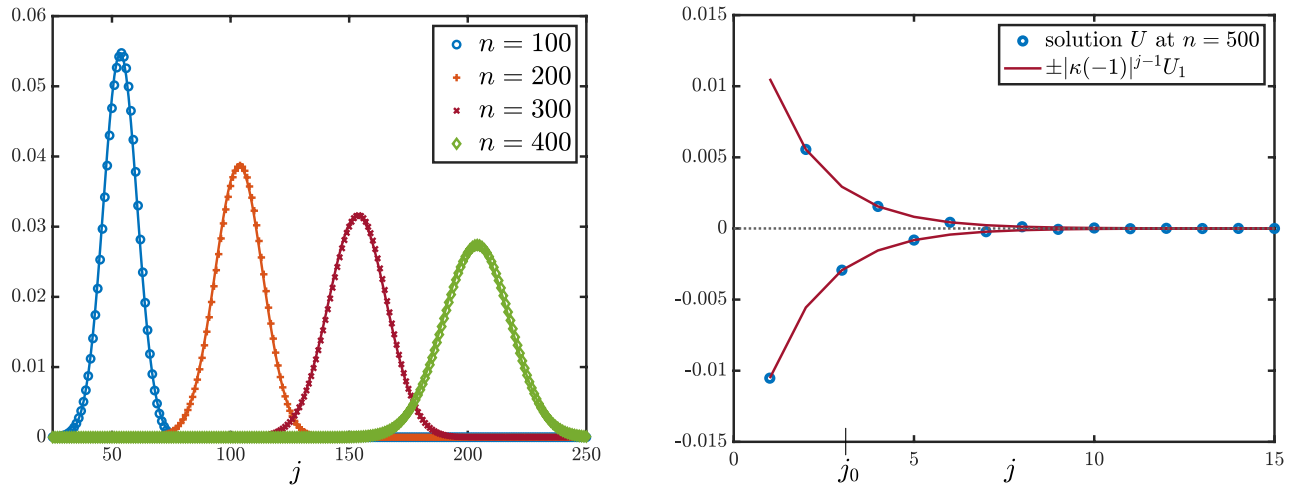


Figure 4: We illustrate our main Theorem 1 and Proposition 1 for the modified Lax-Friedrichs numerical scheme (37)-(38) with  $\lambda a = 1/2$ ,  $D = 3/4$  and  $b = -1 - \frac{2\sqrt{5}}{5}$ . We started with an initial condition given by the Dirac mass at  $j_0 = 3$ . In the left figure, we represent the Green's function at different time iterations and compare with a fixed Gaussian profile centered at  $j - j_0 = \lambda a n$  away from the boundary  $j = 1$ . In the right figure, we highlight the behavior of the Green's function near the boundary. We represent the solution (blue circles) after 500 time iterations and show that it corresponds to a so-called surface wave given by the eigenvalue at  $\underline{z} = -1$  of  $\mathcal{T}$ .

where  $\kappa_s(\underline{z})$  refers to the (unique) stable eigenvalue of  $\mathbb{M}(\underline{z})$ . Finally, we select  $\underline{z} = -1$ . This is the only value on the unit circle, apart from  $z = 1$ , which ensures that  $\kappa_s(\underline{z})$  is real. Note that  $\kappa_s(-1)$  has the exact expression

$$\kappa_s(-1) = \frac{-1 - a_0}{2 a_1} + \sqrt{\left(\frac{-1 - a_0}{2 a_1}\right)^2 - \frac{a_{-1}}{a_1}} \neq 0.$$

Our actual boundary condition is thus

$$u_0^n = \frac{1}{\kappa_s(-1)} u_1^n, \quad n \geq 1. \quad (38)$$

With that specific choice, we easily see that  $\text{Ker } \mathcal{B} \cap \mathbb{E}^s(z)$  is nontrivial for  $z \in \overline{\mathcal{U}} \setminus \{1\}$  if and only if  $z = -1$ , for the Lopatinskii determinant equals  $1 - \kappa_s(z)/\kappa_s(-1)$ , and the equation  $\kappa_s(z) = \kappa_s(-1)$  has a unique solution given precisely by  $z = \underline{z} = -1$ . Moreover,  $-1$  is a simple root of the Lopatinskii determinant. Hence Assumption 2 is satisfied with the choice (38).

Note that the modified Lax-Friedrichs numerical scheme (37)-(38) is (formally) consistent with discretization of the transport equation

$$\begin{cases} \partial_t u + a \partial_x u = 0, & t > 0, \quad x > 0, \\ u(t, 0) = 0, & t > 0, \\ u(0, x) = u_0(x), & x > 0, \end{cases}$$

for some given (smooth) initial condition  $u_0$ .

We present in Figure 3 the spectrum of  $\mathcal{T}$  associated to the modified Lax-Friedrichs numerical scheme (37)-(38) with  $\lambda a = 1/2$ ,  $D = 3/4$  and  $b = -1 - \frac{2\sqrt{5}}{5}$ . In Figure 4, we illustrate the decomposition given

in Proposition 1 where the temporal Green's function decomposes into two parts: a boundary layer part given by  $\underline{\mathbf{w}}_1(j, j_0) (-1)^n$  which is exponentially localized in both  $j$  and  $j_0$  and a generalized Gaussian part which is advected away along  $j - j_0 = \lambda a n$ . We start with an initial condition given by the Dirac mass at  $j_0 = 3$ . We remark that the Green's function at different time iterations compares well with a fixed Gaussian profile centered at  $j - j_0 = \lambda a n$  away from the boundary  $j = 1$ . We also visualize the behavior of the solution near the boundary for  $1 \leq j \leq 15$  and shows that up to a constant, depending on  $j_0$ , the envelope of the Green's function is given by  $\pm |\kappa_s(-1)|^{j-1}$ .

## A Proofs of intermediate results

This Appendix is devoted to the proof of several intermediate results, which are recalled here for the reader's convenience.

### A.1 The Bernstein type inequality

**Lemma 14.** *Under Assumption 1, there holds  $\lambda a < r$ .*

*Proof.* We introduce the polynomial function:

$$\forall z \in \mathbb{C}, \quad P(z) := \sum_{\ell=-r}^p a_\ell z^{\ell+r}.$$

Assumption 1 implies that  $P$  is a nonconstant holomorphic function on  $\mathbb{C}$  and that the modulus of  $P$  is not larger than 1 on  $\mathbb{S}^1$ . By the maximum principle for holomorphic functions,  $P$  maps  $\mathbb{D}$  onto  $\mathbb{D}$ . In particular, since  $P$  has real coefficients,  $P$  achieves its maximum on  $[0, 1]$  at 1, and we thus have  $P'(1) \geq 0$ . From (9), we thus have  $P'(1) = r - \lambda a \geq 0$ . It remains to explain why  $\lambda a$  can not equal  $r$ .

We assume from now on  $\lambda a = r$  and explain why this leads to a contradiction. Multiplying (11) by  $\exp(\mathbf{i} r \theta)$ , we obtain:

$$P(e^{\mathbf{i}\theta}) = \exp(-\beta \theta^{2\mu} + O(\theta^{2\mu+1})),$$

for  $\theta$  close to 0. By the unique continuation theorem for holomorphic functions, the latter expansion holds for either real or complex values of  $\theta$ . We thus choose  $\theta = \varepsilon \exp(\mathbf{i}\pi/(2\mu))$  for any sufficiently small  $\varepsilon > 0$ . We have:

$$P(e^{\mathbf{i}\varepsilon \exp(\mathbf{i}\pi/(2\mu))}) = \exp(\beta \varepsilon^{2\mu} + O(\varepsilon^{2\mu+1})),$$

which is a contradiction since  $P$  maps  $\mathbb{D}$  onto  $\mathbb{D}$  and  $\beta > 0$ . We have thus proved  $\lambda a < r$ .  $\square$

### A.2 Proof of Lemma 2

**Lemma 15.** *Under Assumption 1, there exists  $c_0 > 0$  such that, if we define the set:*

$$\mathcal{C} := \left\{ \rho e^{\mathbf{i}\varphi} \in \mathbb{C} / \varphi \in [-\pi, \pi] \quad \text{and} \quad 0 \leq \rho \leq 1 - c_0 \varphi^{2\mu} \right\},$$

*then  $\mathcal{C}$  is a compact star-shaped subset of  $\overline{\mathbb{D}}$ , and the curve:*

$$\left\{ \sum_{\ell=-r}^p a_\ell e^{\mathbf{i}\ell\theta} / \theta \in [-\pi, \pi] \right\}$$

*is contained in  $\mathcal{C}$ .*

*Proof.* We first choose the constant  $c_0$  such that for any sufficiently small  $\theta$ , the point:

$$\sum_{\ell=-r}^p a_\ell e^{i\ell\theta}$$

lies in  $\mathcal{C}$ . To do so, we use (11) from Assumption 1 and thus write for any sufficiently small  $\theta$ :

$$\sum_{\ell=-r}^p a_\ell e^{i\ell\theta} = \rho(\theta) e^{i\varphi(\theta)},$$

with:

$$0 \leq \rho(\theta) \leq 1 - \frac{\beta}{2} \theta^{2\mu}, \quad \text{and} \quad \frac{\lambda a}{2} |\theta| \leq |\varphi(\theta)| \leq \frac{3\lambda a}{2} |\theta|.$$

Hence there exists  $c_0 > 0$  and  $\theta_0 > 0$  small enough such that, for  $|\theta| \leq \theta_0$ , there holds:

$$0 \leq \rho(\theta) \leq 1 - c_0 \varphi(\theta)^{2\mu}.$$

Let us now examine the case  $\theta_0 \leq |\theta| \leq \pi$ . By continuity and compactness, (10) yields:

$$\sup_{\theta_0 \leq |\theta| \leq \pi} \left| \sum_{\ell=-r}^p a_\ell e^{i\ell\theta} \right| = 1 - \delta_0,$$

for some  $\delta_0 > 0$ . Up to choosing  $c_0$  smaller, we can always assume  $c_0 \pi^{2\mu} \leq \delta_0$ , so for any angle  $\theta$  with  $\theta_0 \leq |\theta| \leq \pi$ , the point:

$$\sum_{\ell=-r}^p a_\ell e^{i\ell\theta}$$

lies in  $\mathcal{C}$ . The proof is thus complete. □

### A.3 Proof of Lemma 3 on the spectral splitting

**Lemma 16.** *Under Assumption 1, let  $z \in \mathbb{C}$  and let the matrix  $\mathbb{M}(z)$  be defined as in (13). Let the set  $\mathcal{C}$  be defined by Lemma 2. Then for  $z \notin \mathcal{C}$ ,  $\mathbb{M}(z)$  has:*

- no eigenvalue on  $\mathbb{S}^1$ ,
- $r$  eigenvalues in  $\mathbb{D} \setminus \{0\}$ ,
- $p$  eigenvalues in  $\mathcal{U}$  (eigenvalues are counted with multiplicity).

Furthermore,  $\mathbb{M}(1)$  has 1 as a simple eigenvalue, it has  $r - 1$  eigenvalues in  $\mathbb{D}$  and  $p$  eigenvalues in  $\mathcal{U}$ .

*Proof.* We are first going to show that for  $z \notin \mathcal{C}$ ,  $\mathbb{M}(z)$  has no eigenvalue on the unit circle  $\mathbb{S}^1$  (this is a classical observation that dates back to [Kre68]). From the definition (13), we first observe that for any  $z \in \mathbb{C}$ ,  $\mathbb{M}(z)$  is invertible (its kernel is trivial since  $r \geq 1$  and  $a_{-r} \neq 0$  so the upper right coefficient of  $\mathbb{M}(z)$  is nonzero). Therefore, for any  $z \in \mathbb{C}$ , the eigenvalues of  $\mathbb{M}(z)$  are those  $\kappa \neq 0$  such that:

$$z = \sum_{\ell=-r}^p a_\ell \kappa^\ell. \tag{39}$$

In particular, Lemma 2 shows that for  $z \notin \mathcal{C}$ ,  $\mathbb{M}(z)$  cannot have an eigenvalue  $\kappa$  on the unit circle for otherwise the right hand side of (39) would belong to  $\mathcal{C}$ .

Since  $\mathcal{C}$  is closed and star-shaped, its complementary is pathwise-connected hence connected. Therefore, the number of eigenvalues of  $\mathbb{M}(z)$  in  $\mathbb{D}$  is independent of  $z \notin \mathcal{C}$  (same for the number of eigenvalues in  $\mathcal{U}$ ). Following [Kre68] (see also [Cou13] for the complete details), this number is computed by letting  $z$  tend to infinity for in that case, the eigenvalues of  $\mathbb{M}(z)$  in  $\mathbb{D}$  tend to zero (the eigenvalues in  $\mathbb{D}$  cannot remain uniformly away from the origin for otherwise the right hand side of (39) would remain bounded while the left hand side tends to infinity).

The final argument is the following. For any  $z \notin \mathcal{C}$ , the eigenvalues of  $\mathbb{M}(z)$  are those  $\kappa \neq 0$  such that:

$$\kappa^r = \frac{1}{z} \sum_{\ell=-r}^p a_\ell \kappa^{r+\ell},$$

which is just an equivalent way of writing (39). Hence for  $z$  large, the small eigenvalues of  $\mathbb{M}(z)$  behave at the leading order like the roots of the reduced equation:

$$\kappa^r = \frac{a_{-r}}{z},$$

and there are exactly  $r$  distinct roots close to 0 of that equation. Hence  $\mathbb{M}(z)$  has  $r$  eigenvalues in  $\mathbb{D}$  for any  $z \notin \mathcal{C}$ .

There remains to examine the spectral situation for  $z = 1$ . Using (39) again, the eigenvalues of  $\mathbb{M}(1)$  are exactly the roots  $\kappa \neq 0$  to the equation:

$$1 = \sum_{\ell=-r}^p a_\ell \kappa^\ell. \quad (40)$$

Thanks to Assumption 1 (see (9) and (10)), the only root of (40) on the unit circle is  $\kappa = 1$  and it is a simple root. This simple eigenvalue can therefore be extended holomorphically with respect to  $z$  as a simple eigenvalue of  $\mathbb{M}(z)$  for  $z$  in a neighborhood of 1. Differentiating (39) with respect to  $z$ , we obtain the Taylor expansion:

$$\kappa(z) = 1 - \frac{1}{\lambda a} (z - 1) + O((z - 1)^2),$$

so we necessarily have  $\kappa(z) \in \mathbb{D}$  for  $z \notin \mathcal{C}$  close to 1. This means that the eigenvalues of  $\mathbb{M}(1)$  that are different from 1 split as follows:  $r - 1$  of them belong to  $\mathbb{D}$  and  $p$  belong to  $\mathcal{U}$  (for otherwise the spectral splitting between  $\mathbb{D}$  and  $\mathcal{U}$  for  $z \notin \mathcal{C}$  would not persist for  $z$  close to 1. The proof of Lemma 3 is now complete.  $\square$ )

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