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El Hassan Youssfi

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# POLYANALYTIC REPRODUCING KERNELS IN $\mathbb{C}^n$ .

E. H. YOUSSEFI

ABSTRACT. In this note we establish integral formulas for polyanalytic functions in several variables. More precisely, given a positive integer  $q$ , we provide explicit expressions for the reproducing kernels of the weighted Bergman spaces of  $q$ -analytic functions on the unit ball in  $\mathbb{C}^n$  and that of  $q$ -analytic Fock space in  $\mathbb{C}^n$ .

## 1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

Polyanalytic function theory in higher dimensions is still not understood. This is essentially due to the lack of explicit formula for related integral kernels. On the other hand, despite this extensive study, there were only two examples of explicit polyanalytic reproducing kernels; the first one was established by Koshelev [10] in the disc, and the second one was given in [4] and [1], and proved in [7], [8]. It was only in the very recent work by Hachadi and the author [6] that a wide source of examples was constructed. There it is proved that to a given a positive integer  $q$  and sequence of orthogonal polynomials associated to a non-negative measure in the half real line  $[0, +\infty$  whose support contains at least  $q$  strictly positive points, is naturally associated a Hilbert space of  $q$ -analytic functions and established a general formula for its reproducing kernel. It is also shown that the formula generalizes to the higher dimensional setting of  $(q_1, \dots, q_n)$ -analytic functions, where  $q_1, \dots, q_n$  are positive integers. These functions were considered by Avanissian and Traor [2] and [3]. We recall that a function  $f$  is said to be polyanalytic of order  $(q_1, \dots, q_n)$  or just  $((q_1, \dots, q_n)$ -analytic) in the domain  $\Omega \subset \mathbb{C}^n$  if in this domain it satisfies the generalized Cauchy-Riemann equation

$$\frac{\partial^{q_1+\dots+q_n} f}{\partial \bar{z}_1^{q_1} \dots \partial \bar{z}_n^{q_n}} = 0. \quad (1.1)$$

They can be uniquely expressed as

$$f(z) = \sum_{j=(0, \dots, 0)}^{(q_1-1, \dots, q_n-1)} \bar{z}^j \phi_j(z) \quad (1.2)$$

where the  $\phi_j(z)$  are holomorphic in  $\Omega$  where for  $j = (j_1, \dots, j_p), k = (k_1, \dots, k_n) \in \mathbb{N}^p$  and  $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ , the inequality  $j \leq k$  means that  $j_l \leq k_l$  for all  $l = 1, \dots, p$  and  $z^j := z_1^{j_1} \dots z_n^{j_n}$ .

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Another notion of polyanalyticity was considered recently by Daghighi [5]. Namley, for a positive integer  $q$ , a function  $f$  defined on an open set  $\Omega$  in  $\mathbb{C}^n$  is called analytic of absolute order  $q$  (or just  $q$ -analytic) if it is of class  $C^q$  that satisfies the generalized Cauchy-Riemann equation

$$\frac{\partial^\alpha f}{\partial \bar{z}^m}(z) = 0,$$

on  $\Omega$  for all multi-indices  $m = (m_1, \dots, m_n) \in \mathbb{N}^n$  with length  $|m| := m_1 + \dots + m_n \leq q$ . He proved that these functions  $f$  are precisely the slice  $q$ -analytic functions in the sense that for all  $(a, \xi) \in \Omega \times \mathbb{C}^n$ , the one variable functions  $\lambda \mapsto f(a + \lambda\xi)$  are  $q$ -analytic in some open neighborhood of 0 in  $\mathbb{C}$ . It is clear that an absolute  $q$ -analytic function  $f$  on  $\mathbb{B}$  can be expressed as a sum

$$f(z) = \sum_{|\alpha| < q} \bar{z}^\alpha f_\alpha(z), \quad (1.3)$$

where for all  $m \in \mathbb{N}^n$  with  $|m| \leq q - 1$  the functions  $f_m$  are holomorphic on  $\Omega$  called the holomorphic components of the polyanalytic function  $f$ .

To the best of our knowledge, no integral representations are available for  $q$ -analytic functions in  $\mathbb{C}^n$ . The main goal of this work is to consider this question when  $\Omega$  is  $\mathbb{C}^n$  or the unit ball in  $\mathbb{C}^n$ .

In this note, we let  $\mathbb{B} := \{z \in \mathbb{C}^n : |z| < 1\}$  be the unit ball in  $\mathbb{C}^n$  and denote by  $dV(z)$  the Lebesgue measure on  $\mathbb{C}^n$  normalized so that  $V(\mathbb{B}) = 1$ .

A first result of this paper is to establish an explicit expression of the weighted polyanalytic (in the absolute sense) Bergman kernel of the unit ball  $\mathbb{B}$ . More precisely, for  $s > -1$ , we consider the space  $\mathcal{A}_{s,q}^2(\mathbb{B})$  of all square integrable  $q$ -analytic functions with respect to the measure  $d\nu_s(z) : (1 - |z|^2)^s dV(z)$ . More precisely, we will prove the following

**Theorem A.** *The space  $\mathcal{A}_{s,q}^2(\mathbb{B})$  is Hilbert space which coincides with the closure of the  $q$ -analytic polynomials in  $L^2(\nu_s)$  and its reproducing kernel is given by*

$$K_{s,q,n}(z, w) = \frac{\Gamma(q+s+n)}{n!\Gamma(s+n)} \frac{(1-\langle w, z \rangle)^{q-1}}{(1-\langle z, w \rangle)^{s+q+n}} P_{q-1}^{(n,s)}(1 - 2|\varphi_z(w)|^2),$$

for all  $z, w \in \mathbb{B}$ , where  $P_{q-1}^{(n,s)}$  is the classical Jacobi polynomial with parameters  $n, s$  and degree  $q - 1$ , and  $\varphi_w$  is the biholomorphic automorphism of  $\mathbb{B}$  exchanging 0 and  $w$  and

We point out that when  $n = 1$ , this result was established recently by Hachadi and the author [6]. In this case, the one dimensional weighted polyanalytic Bergman kernel reduces

$$K_{s,q,1}(z, w) = (q + s) \frac{(1-\langle w, z \rangle)^{q-1}}{(1-\langle z, w \rangle)^{s+q+n}} P_{q-1}^{(1,s)}(1 - 2|\varphi_z(w)|^2), \quad (1.4)$$

where

$$\varphi_z(w) := \frac{z - w}{1 - \langle z, w \rangle}$$

for all elements  $z, w$  of the unit disc  $\mathbb{D}$  in  $\mathbb{C}$ .

To state our second result, we consider the Gaussian measure  $\mu$  defined on  $\mathbb{C}^n$  by

$$d\mu(z) := e^{-|z|^2} dv(z) \quad (1.5)$$

and denote by  $\mathcal{F}_q^2(\mathbb{C}^n)$  the weighted  $q$ -polyanalytic Fock space on  $\mathbb{C}^n$  where  $q$  is a positive integer. This is the space of all  $q$ -analytic functions  $f$  on  $\mathbb{C}^n$  which are square integrable with respect to  $d\mu(z)$ .

**Theorem B.** *The space  $\mathcal{F}_q^2(\mathbb{C}^n)$  is a Hilbert space which coincides with the closure of the  $q$ -analytic polynomials in  $L^2(\mu)$  and its reproducing kernel is given by*

$$K_{\mathcal{F},q}(z, w) = \frac{e^{\langle z, w \rangle}}{(n-1)!} L_{q-1}^n(|z|^2) \quad (1.6)$$

for all  $z, w \in \mathbb{C}^n$ , where  $L_{q-1}^n$  is the classical weighted Laguerre polynomial with weight  $n$  and degree  $q-1$ ,

## 2. PRELIMINARIES

Let  $\mathbb{D}^n$  be the unit polydisc in  $\mathbb{C}^n$ . This, the open set unit ball in  $\mathbb{C}^n$  with respect to the norm

$$\|z\|_\infty := \max_{1 \leq j \leq n} |z_j|, \quad z = (z_1, \dots, z_n) \in \mathbb{C}^n.$$

For a positive integer  $q$ , we let  $\mathcal{A}_q^2(\mathbb{D}^n)$  denote the space of those  $(q, \dots, q)$ -analytic functions on  $\mathbb{D}^n$  which are square integrable with respect to the Lebesgue measure  $dv(z)$  on  $\mathbb{D}^n$ . As a consequence of Theorem D' of [6], we have the following

**Lemma 2.1.** *The space  $\mathcal{A}_q^2(\mathbb{D}^n)$  is the closure of the  $(q, \dots, q)$ -analytic polynomials in the space  $L^2(\mathbb{D}^n)$  of all square integrable functions on  $\mathbb{D}^n$  with respect to the measure  $dv(z)$ . Furthermore, its reproducing kernel is given by*

$$K_{\mathbb{D}^n, q}(z, w) = \prod_{j=1}^n K_{0, q, 1}(z_j, w_j)$$

for all  $z = (z_1, \dots, z_n), w = (w_1, \dots, w_n) \in \mathbb{D}^n$ , where  $K_{0, q, 1}$  is the one dimensional polyanalytic weighted Bergman kernel of the unit disc given by (1.4) with  $s = 0$ .

**Lemma 2.2.** *Let  $\Omega$  be an open set in  $\mathbb{C}^n$  and let  $\varrho : \Omega \rightarrow ]0, +\infty[$  be a continuous weight function. For each compact subset  $C$  of  $\Omega$ , and each  $m \in \mathbb{N}_0^n$ , there is a positive constant  $c_m > 0$  such that*

$$\max_C \left| \frac{\partial^m f}{\partial z^m} \right| \leq c_m \|f\|_{L^2(\Omega, \varrho dv)},$$

for all  $q$ -analytic functions in  $L^2(\Omega, \varrho dv)$ . In particular, the space  $\mathcal{A}_{\varrho, q}^2(\Omega)$  of all such functions, equipped the  $L^2(\Omega, \varrho dv)$  inner product is Hilbert space and for all  $z \in \Omega$ , the evaluation functional  $f \mapsto f(z)$  is continuous on  $\mathcal{A}_{\varrho, q}^2(\Omega)$ .

*Proof.* It is sufficient to consider compact polydiscs  $C = a + r\overline{\mathbb{D}^n}$ , for  $a \in \Omega$  and  $r > 0$  such that  $a + 2r\overline{\mathbb{D}^n} \subset \Omega$ . Considering this, we let  $f \in \mathcal{A}_{\varrho, q}^2(\Omega)$ . Then the function  $g(z) := f(a + 2rz)$  is well-defined  $(q, \dots, q)$ -analytic function on a neighborhood of  $\mathbb{D}^n$ . By Lemma 2.1

$$g(z) = \int_{\mathbb{D}^n} g(w) K_{\mathbb{D}^n, q}(z, w) dv(w),$$

from which it follows that

$$\frac{\partial^m g(z)}{\partial z^m} = \int_{\mathbb{D}^n} g(w) \frac{\partial^m}{\partial z^m} (K_{\mathbb{D}^n, q}(z, w)) dv(w),$$

so that by Cauchy-Schwarz inequality we see that

$$\left| \frac{\partial^m f}{\partial z^m}(a + rz) \right| \leq \frac{\|f\|_{L^2(\Omega, \varrho dv)}}{r^{|m|}} \left[ \int_{\mathbb{D}^n} \frac{\left| \frac{\partial^m}{\partial z^m} (K_{\mathbb{D}^n, q}(z/2, w)) \right|^2}{\varrho(a + rz)} dv(w) \right]^{1/2}.$$

Hence

$$\max_C \left| \frac{\partial^m f}{\partial z^m} \right| \leq c_m \|f\|_{L^2(\Omega, \varrho dv)}$$

where

$$c_m := \sup_{z \in \mathbb{D}^n} \left[ \int_{\mathbb{D}^n} \frac{\left| \frac{\partial^m}{\partial z^m} (K_{\mathbb{D}^n, q}(z/2, w)) \right|^2}{r^{|m|} \varrho(a + rz)} dv(w) \right]^{1/2}.$$

This completes the proof of the lemma.  $\square$

**Lemma 2.3.** *Let  $\Omega$  be a domain in  $\mathbb{C}^n$  and  $\mathcal{D}$  a domain in  $\mathbb{C}^p$ . Suppose that  $\psi : \Omega \rightarrow \mathcal{D}$  is a holomorphic map of the form*

$$\psi(z) = \frac{g(z)}{h(z)}$$

where  $g : \Omega \rightarrow \mathbb{C}^p$  and  $h : \Omega \rightarrow \mathbb{C}$  are polynomials of respective degrees  $d_g$  and  $d_h$  such that  $d_g \leq d_h$  and  $h(z) \neq 0$  for all  $z \in \Omega$ . Suppose that  $q$  and  $l$  are positive integers such that  $l \geq q$ . If  $f$  is a  $q$ -analytic function on  $\mathcal{D}$ , then the function  $\bar{h}^l(f \circ \psi)$  is also  $(ld_h + 1)$ -analytic on  $\Omega$ ,

*Proof.* Writing  $f$  in the form  $f(w) = \sum_{|\alpha| < q} \bar{w}^\alpha f_\alpha(w)$ , it follows for all  $z \in \mathcal{D}$ , we have

$$\bar{h}^l(z)(f \circ \varphi)(z) = \sum_{|\alpha| < q} (\bar{h}(z))^{l-|\alpha|} \overline{(g(z))^\alpha} f_\alpha(\psi(z))$$

which is  $q$ -analytic since for each  $\alpha$ , the function  $(\bar{h}(z))^{l-|\alpha|} \overline{(g(z))^\alpha}$  is a polynomial of degree at most  $ld_h + |\alpha|(d_g - d_h) \leq ld_h$ . This completes the proof.  $\square$

Let  $\text{Aut}(\mathbb{B})$  be the group of all biholomorphic automorphisms of the unit ball  $\mathbb{B}$ . Let  $\varphi \in \text{Aut}(\mathbb{B})$  and let  $a := \varphi^{-1}(0)$ . Its is well-known (see [12], p28) that  $\varphi$  can be written in the form

$$\varphi = U\varphi_a \tag{2.1}$$

where  $U : \mathbb{C}^n \rightarrow \mathbb{C}^n$  is a unitary linear transformation and  $\varphi_0 := -I$  and for  $a \neq 0$ ,

$$\varphi_a(z) = \frac{a - (1 - |a|^2)P_a z - \sqrt{1 - |a|^2}(Q_a(z))}{1 - \langle z, a \rangle}, \text{ for all } z \in \mathbb{B}. \tag{2.2}$$

where  $P_a z := \langle z, a \rangle a / |a|^2$  and  $Q_a := I - P_a$ .

**Lemma 2.4.** *Suppose that  $\varphi$  is an automorphism of the unit ball and let  $J_\varphi$  be its complex jacobian. If  $f$  is a  $q$ -analytic function on  $\mathbb{B}$ , then the function  $(\overline{J_\varphi})^{(1-q)/(n+1)}(f \circ \varphi)$  is also  $q$ -analytic on  $\mathbb{B}$ ,*

*Proof.* Suppose that  $\varphi \in \text{Aut}(\mathbb{B})$  and let  $a := \varphi^{-1}(0)$ . Then it follows from Theorem 2.2 in [12] that

$$J_\varphi(z) = \left( \frac{1 - |a|^2}{1 - \langle z, a \rangle} \right)^{n+1}, \quad \text{for all } z \in \mathbb{B}. \quad (2.3)$$

In addition, in view of (2.2) we see that there are a  $\mathbb{C}$ -linear mapping  $L : \mathbb{C}^n \rightarrow \mathbb{C}^n$  and a vector  $b \in \mathbb{C}^n$  such that for all  $z \in \mathbb{B}$ ,

$$\varphi(z) = \frac{L(z) + b}{1 - \langle z, a \rangle}, \quad \text{for all } z \in \mathbb{B}.$$

Applying Lemma 2.3, with  $g(z) = L(z) + b$  and  $h(z) = 1 - \langle z, a \rangle$  shows that if  $f$  is  $q$ -analytic function on  $\mathbb{B}$ , then

$$\left( \overline{J_\varphi(z)} \right)^{(1-q)/(n+1)} (f \circ \varphi)(z) = \frac{1}{(1 - |a|^2)^{q-1}} (\bar{h}(z))^{q-1} (f \circ \varphi)(z)$$

which also  $q$ -analytic. This completes the proof.  $\square$

### 3. PROOF OF THEOREM A

In this section we will prove the following transformation rule stated for the unit ball.

**Lemma 3.1.** *Suppose that  $\varphi$  is an automorphism of the unit ball and let  $J_\varphi$  be its complex jacobian. Then the Bergman kernel  $K_{q,s}$  of  $\mathcal{A}_q^s(\mathbb{B})$  follows the transformation rule*

$$K_{q,s}(z, \xi) = \frac{\left( J_\varphi(z) \overline{J_\varphi(\xi)} \right)^{\frac{s+q+1}{n+1}}}{\left( \overline{J_\varphi(z)} J_\varphi(\xi) \right)^{\frac{q-1}{n+1}}} K_{q,s}(\varphi(z), \varphi(\xi)) \quad (3.1)$$

for all  $z, \xi \in \mathbb{B}$ ,

*Proof.* It is sufficient to assume that  $\varphi \circ \varphi(z) = z$ , for all  $z \in \mathbb{B}$ . We recall that the measure  $\frac{dV(z)}{(1-|z|^2)^{n+1}}$  is invariant under the action of the automorphism group of the unit ball. We also observe that for any fixed  $\xi \in \mathbb{B}$ , the function

$$z \mapsto \frac{\left( J_\varphi(z) \right)^{\frac{s+q+1}{n+1}}}{\left( \overline{J_\varphi(z)} \right)^{\frac{q-1}{n+1}}} K_{q,s}(\varphi(z), \xi)$$

is an element of  $\mathcal{A}_q^s(\mathbb{B})$ . By the reproducing property and change of variables formula we see that

$$\begin{aligned} \frac{(J_\varphi(z))^{\frac{s+q+1}{n+1}}}{(\overline{J_\varphi(z)})^{\frac{q-1}{n+1}}} K_{q,s}(\varphi(z), \xi) &= \int_{\mathbb{B}} \frac{(J_\varphi(w))^{\frac{s+q+1}{n+1}}}{(\overline{J_\varphi(w)})^{\frac{q-1}{n+1}}} K_{q,s}(\varphi(w), \xi) K_{q,s}(z, w) dV_s(w) \\ &= \int_{\mathbb{B}} \frac{(\overline{J_\varphi(w)})^{(s+q+1)/(n+1)}}{(J_\varphi(w))^{(q-1)/2}} K_{q,s}(w, \xi) K_{q,s}(z, \varphi(w)) dV_s(w) \\ &= \frac{(\overline{J_\varphi(\xi)})^{(s+q+1)/(n+1)}}{(J_\varphi(\xi))^{(q-1)/(n+1)}} K_{q,s}(z, \varphi(\xi)) \end{aligned}$$

Replacing  $\xi$  by  $\varphi(\xi)$  the latter equalities yield

$$\frac{(\overline{J_\varphi(z)J_\varphi(\xi)})^{(s+q+1)/(n+1)}}{(\overline{J_\varphi(z)J_\varphi(\xi)})^{(q-1)/(n+1)}} K_{q,s}(\varphi(z), \varphi(\xi)) = K_{q,s}(z, \xi)$$

This completes the proof.  $\square$

**Lemma 3.2.** *The Bergman kernel  $K_{q,s}$  of  $\mathcal{A}_q^s(\mathbb{B})$  satisfies the equality*

$$K_{q,s}(z, w) = \frac{(1 - \langle w, z \rangle)^{q-1}}{(1 - \langle z, w \rangle)^{s+n+q}} K_{q,s}(\varphi_z(w), 0). \quad (3.2)$$

for all  $z, \xi \in \mathbb{B}$ ,

*Proof.* Let  $z, w \in \mathbb{B}$ . Applying Lemma 3.1 with  $\varphi = \varphi_z$  and  $\xi = \varphi_z(w)$  and using (2.3) yields

$$\begin{aligned} K_{q,s}(z, w) &= \frac{(\overline{J_\varphi(z)J_\varphi(\xi)})^{\frac{s+q+1}{n+1}}}{(\overline{J_\varphi(z)J_\varphi(\xi)})^{\frac{q-1}{n+1}}} K_{q,s}(0, \varphi_z(w)) \\ &= \frac{(1 - \langle w, z \rangle)^{q-1}}{(1 - \langle z, w \rangle)^{s+n+q}} K_{q,s}(0, \varphi_z(w)). \end{aligned}$$

Since  $K_{q,s}(z, w) = \overline{K_{q,s}(w, z)}$ , for all  $z, w \in \mathbb{B}$ , the lemma follows.  $\square$

We shall make use of the classical Jacobi polynomials  $P_d^{(s,a)}$  with parameters  $(s, a)$  and degree  $d$ . An explicit formula for these polynomials is given by

$$P_m^{(s,d)}(x) = \frac{1}{2^m} \sum_{k=0}^d \binom{s+m}{k} \binom{d+m}{m-k} (x-1)^{m-k} (x+1)^k. \quad (3.3)$$

This can be found in [9]. It is well-known by formula (3.96) in ([11], p. 71) that these polynomials verify the equality

$$P_m^{(s,d)}(1-2x) = \frac{\Gamma(m+s+1)}{m!} \sum_{j=0}^m \frac{(-1)^j \binom{m}{j} \Gamma(m+j+s+d+1)}{\Gamma(m+s+d+1)\Gamma(j+s+1)} x^j. \quad (3.4)$$

The Jacobi polynomials satisfy the orthogonality condition

$$\int_0^1 P_m^{(s,d)}(2x-1) P_{m'}^{(s,d)}(2x-1) x^d (1-x)^s dx = \delta_{m,m'} h_m^{s,d} \quad (3.5)$$

where

$$h_m^{s,d} := \frac{\Gamma(s+m+1) \Gamma(d+m+1)}{m! \Gamma(s+d+m+1) (s+d+2m+1)}. \quad (3.6)$$

**Lemma 3.3.** *For each  $z \in \mathbb{B}$ , we have*

$$K_{q,s}(z, 0) = \frac{\Gamma(q+s+n)}{n! \Gamma(q+s)} P_{q-1}^{n,s}(1-2|z|^2) \quad (3.7)$$

*Proof.* Due to Lemma 3.1 we see that the  $q$ -analytic function  $z \mapsto K_{q,s}(z, 0)$  invariant under unitary linear transformations and hence it is of the form  $K_{q,s}(z, 0) = \sum_{j=0}^{q-1} c_j |z|^{2j}$  for some complex coefficients  $c_0, \dots, c_{q-1}$ . Set

$$Q_{q-1}^{s,n}(x) := \sum_{j=0}^{q-1} c_j x^j, \quad x \in ]0, 1[. \quad (3.8)$$

We equip the space  $\mathcal{P}_{q-1}$  of polynomials with degree at most  $q-1$  with respect to the  $L^2$ -inner product associated to the measure  $x^n(1-x)^s dx$  in the interval  $]0, 1[$ . Integrating in polar coordinates shows that for all  $j = 0, \dots, q-2$ , one has that

$$\begin{aligned} \int_0^1 x^j Q_{q-1}^{s,n}(x) x^n (1-x)^s dx &= 2 \int_0^1 x^{2j} Q_{q-1}^{s,n}(x^2) x^{2n} (1-x^2)^s dx \\ &= \frac{1}{n} \int_{\mathbb{B}} |z|^{2j} Q_{q-1}^{s,n}(|z|^2) |z|^2 (1-|z|^2)^s dV(z) \\ &= \frac{1}{n} \int_{\mathbb{B}} |z|^{2j+2} K_{q-1}^{s,n}(z, 0) (1-|z|^2)^s dV(z) \\ &= 0, \end{aligned}$$

where the latter equality holds due to the reproducing property at 0 of the  $q$  analytic polynomial  $|z|^{2j+2}$ . Hence  $Q_{q-1}^{s,n}$  is orthogonal in  $L^2(0, 1[, x^n(1-x)^s dx)$  to all polynomials with strictly smaller than  $q-1$  so that by (3.5) we see that  $Q_{q-1}^{s,n}(x)$  is proportional to the polynomial  $P_{q-1}^{(s,n)}(2x-1)$  and thus for some complex constant  $C(s, n, q-1)$  we have

$$Q_{q-1}^{s,n}(x) = C(s, n, q) (-1)^{q-1} P_{q-1}^{(s,n)}(2x-1) = C(s, n, q) P_{q-1}^{(n,s)}(1-2x).$$



To compute the constant, we apply the reproducing property of the constant function 1 and get

$$\begin{aligned}
\int_0^1 Q_{q-1}^{s,n}(x)x^{n-1}(1-x)^s dx &= 2 \int_0^1 Q_{q-1}^{s,n}(x^2)x^{2n-1}(1-x^2)^s dx \\
&= \frac{1}{n} \int_{\mathbb{B}} Q_{q-1}^{s,n}(|z|^2)|z|^{2n-1}(1-|z|^2)^s dV(z) \\
&= \frac{1}{n} \int_{\mathbb{B}} K_{q-1}^{s,n}(z,0)(1-|z|^2)^s dV(z) \\
&= \frac{1}{n},
\end{aligned}$$

and thus

$$\frac{1}{n} = C(s, n, q)(-1)^{q-1} \int_0^1 P_{q-1}^{(s,n)}(2x-1)x^{n-1}(1-x)^s dx.$$

To compute the constant  $C(s, n, q)$ , let

$$I(s, n-1, q-1) := \int_0^1 P_{q-1}^{(s,n)}(2x-1)x^{n-1}(1-x)^s dx.$$

The the orthogonality property of Jacobi polynomials, combined with integration by parts, yields

$$I(s, n-1, q-1) = -\frac{q+s+n}{n} I(s+1, n, q-2)$$

so that by induction we get

$$I(s, n-1, q-1) = (-1)^{q-1} \frac{(n-1)! \Gamma(2q+s+n-1)}{\Gamma(q+s+n) \Gamma(n+q-1)} I(s+q-1, n+q-2, 0).$$

Since

$$I(s+q-1, n+q-2, 0) = \frac{\Gamma(q+s) \Gamma(n+q-1)}{\Gamma(2q+s+n-1)}$$

it follows that

$$I(s, n-1, q-1) = (-1)^{q-1} \frac{(n-1)! \Gamma(q+s)}{\Gamma(q+s+n)}$$

showing that

$$C(s, n, q) = \frac{\Gamma(q+s+n)}{n! \Gamma(q+s)}$$

and hence

$$Q_{q-1}^{s,n}(x) = \frac{\Gamma(q+s+n)}{n! \Gamma(q+s)} P_{q-1}^{(n,s)}(1-2x).$$

This completes the proof of the lemma.  $\square$

**Proof of Theorem A.** Combining Lemmas 3.2 and 3.3 gives the expression of the reproducing kernel  $K_{q,s}$ . To show that the  $q$ -analytic polynomials are dense in  $\mathcal{A}_q^s(\mathbb{B})$  it suffices to observe that for all  $\alpha, \beta \in \mathbb{N}_0^n$ , and any  $q$ -analytic function  $f \in \mathcal{A}_q^s(\mathbb{B})$  we have

$$\frac{\partial^{\alpha+\beta} f}{\partial z^\alpha \partial \bar{z}^\beta}(0) = \langle f, P \rangle$$

where

$$P(w) := \left[ \frac{\partial^{\alpha+\beta}}{\partial z^\alpha \partial \bar{z}^\beta} K_{q,s}(\cdot, w) \right]_{z=0}.$$

On the other hand, a little computing shows that  $P$  is a  $q$ -analytic polynomial provided that  $|\beta| < q$ . Therefore, if  $f$  is orthogonal to  $q$ -analytic polynomials in  $\mathcal{A}_q^s(\mathbb{B})$ , then  $f$  must vanish everywhere, showing that  $q$ -analytic polynomials are dense in  $\mathcal{A}_q^s(\mathbb{B})$ .

#### 4. THE POLYANALYTIC FOCK SPACE

We consider the Gaussian measure  $\mu$  given by (1.5) and the corresponding Fock space  $\mathcal{F}_q^2(\mathbb{C}^n)$  of all  $q$ -analytic functions  $f$  on  $\mathbb{C}^n$  which are square integrable with respect to  $d\mu(z)$ . From Lemma 2.2, we see that  $\mathcal{F}_q^2(\mathbb{C}^n)$  furnished with the  $L^2(\mu)$  scalar product turns out to be a Hilbert space for which the evaluation functional  $f \mapsto f(z)$  is continuous for all  $z \in \mathbb{C}^n$ . Hence  $\mathcal{F}_q^2(\mathbb{C}^n)$  admits a reproducing kernel  $K_{\mathcal{F},q}(\cdot, \cdot)$ .

On the other hand, by the change of variable formula, it is clear that if  $a \in \mathbb{C}^n$  and  $f \in \mathcal{F}_q^2(\mathbb{C}^n)$ , then the function  $z \mapsto e^{-\langle z, a \rangle} f(z + a)$  remains in  $\mathcal{F}_q^2(\mathbb{C}^n)$ .

**Lemma 4.1.** *The reproducing kernel  $K_{\mathcal{F},q}$  of  $\mathcal{F}_q(\mathbb{C}^n)$  follows the transformation rule*

$$e^{-\langle z, a \rangle} K_{\mathcal{F},q}(Uz + a, \xi) = e^{\langle a, \xi \rangle} K_{\mathcal{F},q}(z, U\xi - Ua) \quad (4.1)$$

for all  $z, \xi \in \mathbb{C}^n$ , and all unitary transformations  $U : \mathbb{C}^n \rightarrow \mathbb{C}^n$ . In particular,

$$K_{\mathcal{F},q}(z, w) = e^{\langle z, w \rangle} K_{\mathcal{F},q}(z - w, 0) \quad (4.2)$$

and

$$K_{\mathcal{F},q}(Uz, Uw) = K_{\mathcal{F},q}(z, w) \quad (4.3)$$

for all  $z, \xi \in \mathbb{C}^n$  and all unitary transformations  $U : \mathbb{C}^n \rightarrow \mathbb{C}^n$ .

*Proof.* If  $a, z, \xi \in \mathbb{C}^n$ , then applying twice the reproducing formula and a change of variable yield

$$\begin{aligned} e^{-\langle z, a \rangle} K_{\mathcal{F},q}(Uz + a, \xi) &= \int_{\mathbb{C}^n} e^{-\langle Uw, a \rangle} K_{\mathcal{F},q}(Uw + a, \xi) K_{\mathcal{F},q}(z, w) d\mu(w) \\ &= \int_{\mathbb{C}^n} e^{-\langle a, w \rangle} K_{\mathcal{F},q}(z, Uw - Ua) K_{\mathcal{F},q}(\xi, w) d\mu(w) \\ &= e^{-\langle a, w \rangle} K_{\mathcal{F},q}(z, U\xi - Ua). \end{aligned}$$

This completes the proof.  $\square$

In view of the latter lemma, in order compute the Fock reproducing kernel it is sufficient to calculate  $K_{\mathcal{F},q}(z, 0)$  for all  $z \in \mathbb{C}^n$ . To do so, we shall make use of the classical weighted Laguerre polynomials  $L_d^m$  of degree  $d$  and weight  $s$ . These polynomials satisfy,

$$\int_0^{+\infty} L_d^m(x) L_{d'}^m(x) x^m e^{-x} dx = \frac{\Gamma(d+m+1)}{d!} \delta_{d,d'}. \quad (4.4)$$

They have the following explicit representation

$$L_d^m(x) = \sum_{l=0}^d \frac{(n+m)!}{l! \Gamma(d+m+1-l)} \frac{(-x)^{n-l}}{(n-l)!}. \quad (4.5)$$

**Lemma 4.2.** *For all  $z \in \mathbb{C}^n$ , we have*

$$K_{\mathcal{F},q}(z, 0) = \frac{1}{n!} L_{q-1}^n(|z|^2). \quad (4.6)$$

*Proof.* In virtue of (4.3) we see that  $z \mapsto K_{\mathcal{F},q}(z, 0)$  is invariant under unitary linear transformations and hence it is of the form

$$K_{\mathcal{F},q}(z, 0) = \sum_{j=0}^{q-1} c_j |z|^{2j}$$

for some complex coefficients  $c_0, \dots, c_{q-1}$ . To compute the coefficients  $c_j$ , we consider the space  $\mathcal{P}_{q-1}$  of polynomials with degree at most  $q-1$  and equip it with the  $L^2$ -inner product associated to the measure  $e^{-x} dx$  in the interval  $[0, +\infty[$ . Set

$$R_q(x) := \sum_{j=0}^{q-1} c_j x^j, \quad x \geq 0.$$

Integrating in polar coordinates shows that for all  $j = 0, \dots, q-2$ , one has that

$$\begin{aligned} \int_0^{+\infty} x^j R_q(x) x^n e^{-x} dx &= \frac{1}{n} \int_{\mathbb{C}^n} |z|^{2j+2} R_{q-1}(|z|^2) d\mu(z) \\ &= \frac{1}{n} \int_{\mathbb{C}^n} |z|^{2j+2} K_{\mathcal{F},q}(z, 0) d\mu(z) \\ &= 0, \end{aligned}$$

where the latter equality holds due to the reproducing property at 0 of the  $q$  analytic polynomial  $|z|^{2j+2}$ . Hence  $R_{q-1}(x)$  is orthogonal in  $L^2(\mu)$  to all polynomials with strictly smaller than  $q-1$  so that by (4.4) we see that  $R_{q-1}$  is proportional to the polynomial  $L_{q-1}^n$  and thus for some complex constant  $c(n, q)$  we have

$$R_{q-1}(x) = c(n, q) L_{q-1}^n(x).$$

To compute the constant, we apply the reproducing property of the constant function 1 and get

$$\begin{aligned} \int_0^{+\infty} R_{q-1}(x) x^{n-1} e^{-x} dx &= \frac{1}{n} \int_{\mathbb{C}^n} K_{\mathcal{F},q}(z, 0) d\mu(z) \\ &= \frac{1}{n}, \end{aligned}$$

and thus

$$\frac{1}{n} = c(n, q) \int_0^{+\infty} L_{q-1}^n(x) x^{n-1} e^{-x} dx.$$

To compute the constant  $c(n, q)$ , let

$$I(n, q) := \int_0^{+\infty} L_{q-1}^n(x) x^{n-1} e^{-x} dx.$$

Using the observation that

$$\frac{d}{dx} (e^{-x} L_q^{n-1}(x)) = -e^{-x} L_{q-1}^n(x)$$

integrations by parts shows that

$$I(n, q) = (n-1)! I(0, q+n-1) = (n-1)!.$$

The lemma follows.  $\square$

**Proof of Theorem B.** Combining Lemmas 4.1 and 4.2 gives the expression of the reproducing kernel  $K_{\mathcal{F}, q}$ . To show that the  $q$ -analytic polynomials are dense in  $\mathcal{F}_q(\mathbb{C}^n)$  it suffices to observe that for all  $\alpha, \beta \in \mathbb{N}_0^n$ , and any  $q$ -analytic function  $f \in \mathcal{F}_q^2(\mathbb{C}^n)$  we have

$$\frac{\partial^{\alpha+\beta} f}{\partial z^\alpha \partial \bar{z}^\beta}(0) = \langle f, P \rangle$$

where

$$P(w) := \left[ \frac{\partial^{\alpha+\beta}}{\partial z^\alpha \partial \bar{z}^\beta} K_{\mathcal{F}, q}(\cdot, w) \right]_{z=0}.$$

On the other hand, a little computing shows that  $P$  is a  $q$ -analytic polynomial provided that  $|\beta| < q$ . Therefore, if  $f$  is orthogonal to  $q$ -analytic polynomials in  $\mathcal{F}_q^2(\mathbb{C}^n)$ , then  $f$  must vanish everywhere, showing that  $q$ -analytic polynomials are dense in  $\mathcal{F}_q^2(\mathbb{C}^n)$ .

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YOUSSEFI: I2M, U.M.R. C.N.R.S. 7373, CMI, UNIVERSITÉ D'AIX-MARSEILLE, 39 RUE F-JOLIOT-CURIE,  
13453 MARSEILLE CEDEX 13, FRANCE

*E-mail address:* el-hassan.youssfi@univ-amu.fr