

On the curved exponential family in the Stochastic Approximation Expectation Maximization Algorithm - Supplementary material

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ANNEX: PROOF OF THEOREM 2 FOR $m \geq 2$

Proof We prove the theorem 2 in the case $m \geq 2$ and $l = 2$. For $l \geq 3$, the proof could be obtained using Taylor Lagrange formula at a higher order.

We recall the following equation verified by the limit (theorem 1):

$$\int_{\mathbb{R}^m} v_k g(\theta_\infty, \bar{\psi}_\sigma + v) \exp(-\frac{\|v\|^2}{2\sigma^2}) dv = 0, \quad (1)$$

First step: $d(\bar{\psi}_\sigma, \mathcal{L}) \xrightarrow{\sigma \rightarrow 0} 0$

We suppose that $d(\bar{\psi}_\sigma, \mathcal{L})$ does not converge towards 0. Even if it means extracting, we can suppose that, $\exists c > 0, \forall \sigma > 0, d(\bar{\psi}_\sigma, \mathcal{L}) > 3c$. As for the one-dimensional proof, we forget the θ in g and write $g(\psi) = g(\theta_\infty, \psi)$. We also set $K_0 = K \setminus \{y \mid d(y, \mathcal{L}) < c\}$.

We want to show that

$$\exists c_0 > 0, \exists c_1 > 0, \forall y \in K_0, \exists 1 \leq i \leq m, \forall x \text{ verifying } \|x - y\| \leq c_1, \left| \frac{\partial g}{\partial \psi_i}(x) \right| > c_0. \quad (2)$$

By contradiction, we can take $c_0 = 1/n$ and extract a converging subsequence in the compact K_0 to find:

$$\forall c_1 > 0, \exists y \in K_0, \forall 1 \leq i \leq m, \exists x \text{ verifying } \|x - y\| \leq c_1, \left| \frac{\partial g}{\partial \psi_i}(x) \right| = 0.$$

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However, because $y \notin \mathcal{L}$, there exists $1 \leq j \leq m$ such that $|\frac{\partial g}{\partial \psi_j}(y)| \neq 0$. If we take c_1 small enough then, for all x such that $\|y - x\| \leq c_1$, $|\frac{\partial g}{\partial \psi_j}(x)| \neq 0$ and we find a contradiction. Hence, the condition (2) is verified.

Hence,

$$\exists c_0 > 0, \exists c_1 > 0, \forall \sigma > 0, \exists k \in [1, m], \forall v \text{ verifying } \|v - \bar{\psi}_\sigma\| \leq c_1, |\frac{\partial g}{\partial \psi_k}(v)| > c_0.$$

As for the proof in dimension 1, we split up our integral in two parts: $I_1 = \{v \mid \forall i \in [1, n], v_i \leq c_2\}$ and $I_2 = \{v \mid \exists i \in [1, n], v_i \geq c_2\}$ where c_2 is chosen such that $\{v \mid \forall i \in [1, n], v_i \leq c_2\} \subset \{v \mid \|v\| \leq c_1\}$.

First, on I_2 ,

$$\begin{aligned} \int_{I_2} v_k g(\bar{\psi}_\sigma + v) \exp(-\frac{\|v\|^2}{2\sigma^2}) dv &\geq \int_{I_2, v_k \leq 0} v_k g(\bar{\psi}_\sigma + v) \exp(-\frac{\|v\|^2}{2\sigma^2}) dv \\ &\geq \|g\|_\infty \left(\int_{\mathbb{R}^m, v_k \leq 0} v_k \exp(-\frac{\|v\|^2}{2\sigma^2}) dv - \int_{I_1, v_k \leq 0} v_k \exp(-\frac{\|v\|^2}{2\sigma^2}) dv \right) \\ &\geq -\sigma^2 (\sqrt{2\pi}\sigma)^{m-1} \|g\|_\infty \left(1 - \left(1 - \exp(-\frac{c_2^2}{2\sigma^2}) \right) \operatorname{erf}\left(\frac{c_2}{\sqrt{2}\sigma}\right)^{m-1} \right) \end{aligned}$$

where erf is the error function defined by $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$.

We now integrate on I_1 . We write v^{-k} the vector such that, $\forall i \neq k, (v^{-k})_i = v_i$ and $(v^{-k})_k = -v_k$. Then, using the mean value theorem, we have

$$\begin{aligned} \int_{I_1} v_k g(\bar{\psi}_\sigma + v) \exp(-\frac{\|v\|^2}{2\sigma^2}) dv &= \int_{I_1, v_k \leq 0} v_k (g(\bar{\psi}_\sigma + v) - g(\bar{\psi}_\sigma + v^{-k})) \exp(-\frac{\|v\|^2}{2\sigma^2}) dv \\ &= \int_{I_1} 2v_k^2 \frac{\partial g}{\partial \psi_k}(\tilde{\psi}_v) \exp(-\frac{\|v\|^2}{2\sigma^2}) dv \end{aligned}$$

where, for $i \neq k$, $(\tilde{\psi}_v)_i = (\bar{\psi}_\sigma)_i + v_i$ and $(\tilde{\psi}_v)_k \in [(\bar{\psi}_\sigma)_k - v_k, (\bar{\psi}_\sigma)_k + v_k]$. But we know that $\frac{\partial g}{\partial \psi_k}$ does not cancel on I_1 . Hence, it is either positive or negative. If it is positive, we find:

$$\begin{aligned} \int_{I_1} v_k g(\bar{\psi}_\sigma + v) \exp(-\frac{\|v\|^2}{2\sigma^2}) dv &\geq 2c_0 \int_{I_1, v_k \leq 0} v_k^2 \exp(-\frac{\|v\|^2}{2\sigma^2}) dv \\ &\geq 2c_0 (\sqrt{2\pi}\sigma)^{m-1} \sigma^2 \left[-c_2 \exp(-\frac{c_2^2}{2\sigma^2}) + \sigma \frac{\sqrt{\pi}}{2} \operatorname{erf}\left(\frac{c_2}{\sqrt{2}\sigma}\right) \right] \operatorname{erf}\left(\frac{c_2}{\sqrt{2}\sigma}\right)^{m-1} \end{aligned}$$

Finally, using (1), we have:

$$2c_0 \left[-\frac{c_2}{\sigma} \exp\left(-\frac{c_2^2}{2\sigma^2}\right) + \frac{\sqrt{\pi}}{2} \operatorname{erf}\left(\frac{c_2}{\sqrt{2}\sigma}\right) \right] \operatorname{erf}\left(\frac{c_2}{\sqrt{2}\sigma}\right)^{m-1} \\ \leq \|g\|_\infty \left[\frac{1 - \operatorname{erf}\left(\frac{c_2}{\sqrt{2}\sigma}\right)^{m-1}}{\sigma} + \frac{1}{\sigma} \exp\left(-\frac{c_2^2}{2\sigma^2}\right) \operatorname{erf}\left(\frac{c_2}{\sqrt{2}\sigma}\right)^{m-1} \right] \quad (3)$$

But,

$$\operatorname{erf}(x) =_{x \rightarrow \infty} 1 - \frac{\exp(-x^2)}{\sqrt{\pi}x} + o\left(\frac{\exp(-x^2)}{x}\right).$$

Hence, when σ goes to 0, the left-hand side of the inequality goes to $\sqrt{\pi}c_0$ while the right-hand side goes to 0. We thus find a contradiction.

Finally, if $\frac{\partial g}{\partial \psi_k}$ is not positive on I_1 (as supposed here) but negative, we can use the same method to find an upper bound on the integral on I_2 and on the integral on I_1 . We would then find the same inequality as in (3).

Hence, in all cases, we have proved that $d(\bar{\psi}_\sigma, \mathcal{L}) \xrightarrow{\sigma \rightarrow 0} 0$.

Second step: Choice of the basis

The upper bound using second derivatives is more complex to obtain for $m > 1$ as crossed partial derivatives appear that can be either positive or negative. To control those parts, the choice of the compact is more complex. We will first show that we can express our vector v and our function g in any orthonormal basis and still have the equation (1).

Indeed, let P be a change-of-basis matrix. Then, because the equation (1) is linear on v_k and true for all $k \in \llbracket 1, m \rrbracket$, we still have:

$$\int_{\mathbb{R}^m} (Pv)_k g(\bar{\psi}_\sigma + v) \exp\left(-\frac{\|v\|^2}{2\sigma^2}\right) dv = 0.$$

Using the change of variable $u = Pv$, we then find, for any $k \in \llbracket 1, m \rrbracket$:

$$\int_{\mathbb{R}^m} u_k g(P^{-1}(P\bar{\psi}_\sigma + u)) \exp\left(-\frac{\|u\|^2}{2\sigma^2}\right) dv = 0.$$

We write $h : u \mapsto g(P^{-1}u)$. Hence, h verifies the equation (1).

We can thus choose to express our function g in any base. In particular, we write ψ_M the nearest maximum of $\bar{\psi}_\sigma$. Then, the Hessian of g at ψ_M is a negative symmetric matrix. Hence, it is diagonal in an orthonormal basis. We choose to express g in that basis. With a change of notation, we can

hence assume that the hessian of g at ψ_M is diagonal. In particular, for all $i \neq j \in [[1, m]]$,

$$\frac{\partial^2 g}{\partial \psi_i^2}(\psi_M) < 0 \text{ and } \frac{\partial^2 g}{\partial \psi_i \partial \psi_j}(\psi_M) = 0.$$

In particular, we are now able to impose a condition between the second derivatives of g on a compact centered around ψ_M . There exists K_0 compact such that,

$$\begin{aligned} -\sup_{K_0} \frac{\partial^2 g}{\partial \psi_k^2} &> (m-1) \sup \left\{ \frac{\partial^2 g}{\partial \psi_k \partial \psi_j}(v) \mid v \in K_0, j \neq k, \frac{\partial^2 g}{\partial \psi_k \partial \psi_j}(v) > 0 \right\} \\ &\quad - \frac{m-1}{2} \inf \left\{ \frac{\partial^2 g}{\partial \psi_k \partial \psi_j}(v) \mid v \in K_0, j \neq k, \frac{\partial^2 g}{\partial \psi_k \partial \psi_j}(v) < 0 \right\} \end{aligned} \quad (4)$$

Third step: Search of the upper bound

As for the proof in 1D, we will split our integral into two parts and say that neither can be too big for the complete integral to be equal to 0. More precisely, for $\sigma > 0$, let k be the coordinate such that $|(\bar{\psi}_\sigma)_k - (\psi_M)_k| = \max |(\bar{\psi}_\sigma)_i - (\psi_M)_i|$. We write for $i \in [[1, m]]$, $(\alpha_\sigma)_i = |(\bar{\psi}_\sigma)_i - (\psi_M)_i|$. The goal is to show that $(\alpha_\sigma)_k$ goes to 0 when σ goes to 0.

Let $c > 0$ and σ small enough such that

$$I_1 := \{v \in \mathbb{R}^m \mid v_k \in [-(\alpha_\sigma)_k, (\alpha_\sigma)_k] \text{ and, for } i \neq k, v_i \in [-c, c]\} \subset K_0.$$

On I_1^c , we use the same upper bounds as in the first step to find:

$$\begin{aligned} \int_{I_1^c} v_k g(\bar{\psi}_\sigma + v) \exp\left(-\frac{\|v\|^2}{2\sigma^2}\right) dv &\geq -\sigma^2 (\sqrt{2\pi}\sigma)^{m-1} \|g\|_\infty \\ &\quad \cdot \left(1 - \left(1 - \exp\left(-\frac{(\alpha_\sigma)_k^2}{2\sigma^2}\right)\right) \operatorname{erf}\left(\frac{c}{\sqrt{2}\sigma}\right)^{m-1}\right) \end{aligned} \quad (5)$$

On I_1 we will use once again the mean value theorem, first between $\bar{\psi}_\sigma + v^{-k}$ and $\bar{\psi}_\sigma + v$ to find $\tilde{\psi}_v \in K_0$ such that, for $i \neq k$, $(\tilde{\psi}_v)_i = (\bar{\psi}_\sigma + v)_i$, $(\tilde{\psi}_v)_k \in [(\bar{\psi}_\sigma - v)_k, (\bar{\psi}_\sigma + v)_k]$ and

$$g(\bar{\psi}_\sigma + v) - g(\bar{\psi}_\sigma + v^{-k}) = 2v_k \frac{\partial g}{\partial \psi_k}(\tilde{\psi}_v)$$

and then between $\tilde{\psi}_v$ and ψ_M to find $\psi_v^1 \in K_0$ such that:

$$\frac{\partial g}{\partial \psi_k}(\tilde{\psi}_v) = \sum_{i=1}^m (\tilde{\psi}_v - \psi_M)_i \frac{\partial^2 g}{\partial \psi_k \partial \psi_i}(\psi_v^1).$$

Even if it means changing basis, we can assume that, $\forall i \in [1, m], (\alpha_\sigma)_i = |(\bar{\psi}_\sigma)_i - (\psi_M)_i| = (\psi_M)_i - (\bar{\psi}_\sigma)_i$ without modifying the hypothesis (4).

The difficulty to find upper bounds is that $\frac{\partial^2 g}{\partial \psi_k \partial \psi_i}(\psi_v^1)$ and also $(\bar{\psi}_v - \psi_M)_i$ can be either positive or negative.

Using those previous equalities and the facts that $\frac{\partial^2 g}{\partial \psi_k^2} < 0$ on K_0 and $(\bar{\psi}_v - \psi_M)_k \leq (\bar{\psi}_\sigma + v - \psi_M)_k = (v - \alpha_\sigma)_k$, we have:

$$\int_{I_1} v_k g(\bar{\psi}_\sigma + v) \exp\left(-\frac{\|v\|^2}{2\sigma^2}\right) dv \geq 2 \int_{I_1, v_k \geq 0} v_k^2 \sum_{i=1}^m (v - \alpha_\sigma)_i \frac{\partial^2 g}{\partial \psi_k \partial \psi_i}(\psi_v^1) \exp\left(-\frac{\|v\|^2}{2\sigma^2}\right) dv$$

We will study the different terms of the sum differently according to $i = k$, or $i \neq k$.

First, for $i = k$, using the fact that $(v - \alpha_\sigma)_k \leq 0$ on I_1 , we can compute the integral using integration per part and the function erf defined above to find:

$$\begin{aligned} \int_{I_1, v_k \geq 0} 2v_k^2 (v - \alpha_\sigma)_k \frac{\partial^2 g}{\partial \psi_k^2}(\psi_v^1) \exp\left(-\frac{\|v\|^2}{2\sigma^2}\right) dv &\geq \sup_{K_0} \left(\frac{\partial^2 g}{\partial \psi_k^2} \right) \int_{I_1, v_k \geq 0} 2v_k^2 (v - \alpha_\sigma)_k \exp\left(-\frac{\|v\|^2}{2\sigma^2}\right) dv \\ &= -2 \sup_{K_0} \left(\frac{\partial^2 g}{\partial \psi_k^2} \right) (\sqrt{2\pi}\sigma)^{m-1} \operatorname{erf}\left(\frac{c}{\sqrt{2}\sigma}\right)^{m-1} (\alpha_\sigma)_k^4 \left[\sqrt{\frac{\pi}{2}} \left(\frac{\sigma}{(\alpha_\sigma)_k} \right)^3 \operatorname{erf}\left(\frac{(\alpha_\sigma)_k}{\sqrt{2}\sigma}\right) \right. \\ &\quad \left. - 2 \left(\frac{d}{(\alpha_\sigma)_k} \right)^4 \left(1 - \exp\left(-\frac{(\alpha_\sigma)_k}{2\sigma^2}\right) \right) \right] \end{aligned}$$

For $i \neq k$, we do similar computations remarking that:

- if $\frac{\partial^2 g}{\partial \psi_k \partial \psi_i}(\psi_v^1) > 0$ and $(v - \alpha_\sigma)_i > 0$,

$$2v_k^2 (v - \alpha_\sigma)_i \frac{\partial^2 g}{\partial \psi_k \partial \psi_i}(\psi_v^1) \exp\left(-\frac{\|v\|^2}{2\sigma^2}\right) > 0$$

- if $\frac{\partial^2 g}{\partial \psi_k \partial \psi_i}(\psi_v^1) > 0$ and $(v - \alpha_\sigma)_i < 0$, with $K_0^+ = \left\{ v \in K_0 \mid \frac{\partial^2 g}{\partial \psi_k \partial \psi_i}(v) > 0 \right\}$,

$$2v_k^2 (v - \alpha_\sigma)_i \frac{\partial^2 g}{\partial \psi_k \partial \psi_i}(\psi_v^1) \exp\left(-\frac{\|v\|^2}{2\sigma^2}\right) > 2v_k^2 (v - \alpha_\sigma)_i \exp\left(-\frac{\|v\|^2}{2\sigma^2}\right) \sup_{K_0^+, i} \frac{\partial^2 g}{\partial \psi_k \partial \psi_i}$$

- if $\frac{\partial^2 g}{\partial \psi_k \partial \psi_i}(\psi_v^1) < 0$ and $(v - \alpha_\sigma)_i < 0$

$$2v_k^2 (v - \alpha_\sigma)_i \frac{\partial^2 g}{\partial \psi_k \partial \psi_i}(\psi_v^1) \exp\left(-\frac{\|v\|^2}{2\sigma^2}\right) > 0$$

- if $\frac{\partial^2 g}{\partial \psi_k \partial \psi_i}(\psi_v^1) < 0$ and $(v - \alpha_\sigma)_i > 0$, with $K_0^- = \left\{ v \in K_0 \mid \frac{\partial^2 g}{\partial \psi_k \partial \psi_i}(v) < 0 \right\}$,

$$2v_k^2 (v - \alpha_\sigma)_i \frac{\partial^2 g}{\partial \psi_k \partial \psi_i}(\psi_v^1) \exp\left(-\frac{\|v\|^2}{2\sigma^2}\right) > 2v_k^2 (v - \alpha_\sigma)_i \exp\left(-\frac{\|v\|^2}{2\sigma^2}\right) \inf_{K_0^-, i} \frac{\partial^2 g}{\partial \psi_k \partial \psi_i}$$

Hence, for $i \neq k$, we write $I_1^- = \{v \in I_1 \mid v_k \geq 0, v_i \leq (\alpha_\sigma)_i\}$ and:

$$\begin{aligned} \int_{I_1^-} 2v_k^2(v - \alpha_\sigma)_i \frac{\partial^2 g}{\partial \psi_k \partial \psi_i}(\psi_v^1) \exp\left(-\frac{\|v\|^2}{2\sigma^2}\right) dv &\geq \sup_{K_{0,i}^+, i} \frac{\partial^2 g}{\partial \psi_k \partial \psi_i} \int_{I_1^-} 2v_k^2(v - \alpha_\sigma)_i \exp\left(-\frac{\|v\|^2}{2\sigma^2}\right) dv \\ &= 2 \sup_{K_{0,i}^+, i} \frac{\partial^2 g}{\partial \psi_k \partial \psi_i} (\sqrt{2\pi}\sigma)^{m-1} \operatorname{erf}\left(\frac{c}{\sqrt{2}\sigma}\right)^{m-1} (\alpha_\sigma)_k^4 \\ &\quad \cdot \left[\sqrt{\frac{\pi}{2}} \left(\frac{\sigma}{(\alpha_\sigma)_k}\right)^4 \operatorname{erf}\left(\frac{(\alpha_\sigma)_k}{\sqrt{2}\sigma}\right) - \left(\frac{\sigma}{(\alpha_\sigma)_k}\right)^3 \exp\left(-\frac{(\alpha_\sigma)_k^2}{2\sigma^2}\right) \right] \\ &\quad \cdot \left[\frac{\exp\left(\frac{-c^2}{2\sigma^2}\right) - \exp\left(\frac{-(\alpha_\sigma)_j^2}{2\sigma^2}\right)}{\sqrt{2\pi}\operatorname{erf}\left(\frac{c}{\sqrt{2}\sigma}\right)} - \frac{1}{2} \frac{(\alpha_\sigma)_j}{d} \left(1 + \frac{\operatorname{erf}\left(\frac{(\alpha_\sigma)_j}{\sqrt{2}\sigma}\right)}{\operatorname{erf}\left(\frac{c}{\sqrt{2}\sigma}\right)}\right) \right] \end{aligned}$$

Similarly, with $I_1^+ = \{v \in I_1 \mid v_k \geq 0, v_i \geq (\alpha_\sigma)_i\}$,

$$\begin{aligned} \int_{I_1^+} 2v_k^2(v - \alpha_\sigma)_i \frac{\partial^2 g}{\partial \psi_k \partial \psi_i}(\psi_v^1) \exp\left(-\frac{\|v\|^2}{2\sigma^2}\right) dv &\geq \inf_{K_{0,i}^-, i} \frac{\partial^2 g}{\partial \psi_k \partial \psi_i} \int_{I_1^+} 2v_k^2(v - \alpha_\sigma)_i \exp\left(-\frac{\|v\|^2}{2\sigma^2}\right) dv \\ &= 2 \inf_{K_{0,i}^-, i} \frac{\partial^2 g}{\partial \psi_k \partial \psi_i} (\sqrt{2\pi}\sigma)^{m-1} \operatorname{erf}\left(\frac{c}{\sqrt{2}\sigma}\right)^{m-1} (\alpha_\sigma)_k^4 \\ &\quad \cdot \left[\sqrt{\frac{\pi}{2}} \left(\frac{\sigma}{(\alpha_\sigma)_k}\right)^4 \operatorname{erf}\left(\frac{(\alpha_\sigma)_k}{\sqrt{2}\sigma}\right) - \left(\frac{\sigma}{(\alpha_\sigma)_k}\right)^3 \exp\left(-\frac{(\alpha_\sigma)_k^2}{2\sigma^2}\right) \right] \\ &\quad \cdot \left[\frac{\exp\left(\frac{-(\alpha_\sigma)_j^2}{2\sigma^2}\right) - \exp\left(\frac{-c^2}{2\sigma^2}\right)}{\sqrt{2\pi}\operatorname{erf}\left(\frac{c}{\sqrt{2}\sigma}\right)} - \frac{1}{2} \frac{(\alpha_\sigma)_j}{d} \left(1 - \frac{\operatorname{erf}\left(\frac{(\alpha_\sigma)_j}{\sqrt{2}\sigma}\right)}{\operatorname{erf}\left(\frac{c}{\sqrt{2}\sigma}\right)}\right) \right] \end{aligned}$$

Those three upper bounds are quite complex, but we can remark that they can be written as $d^{m-1} \operatorname{erf}\left(\frac{c}{\sqrt{2}\sigma}\right)^{m-1} (\alpha_\sigma)_k^4 h\left(\frac{(\alpha_\sigma)_k}{\sigma}\right)$. We will now see that this function h is strictly positive at infinity.

Indeed, using all the previous upper bounds presented previously, equation (1) and using the fact that $\operatorname{erf}\left(\frac{c}{\sqrt{2}\sigma}\right) \geq 1/2$ for σ small enough, we can write:

$$(\alpha_\sigma)_k^4 \operatorname{erf}\left(\frac{c}{\sqrt{2}\sigma}\right)^{m-1} \left(h\left(\frac{\alpha_k}{\sigma}\right) - \frac{1 - \operatorname{erf}(c/\sqrt{2}\sigma)}{\sigma^2 \operatorname{erf}(c/\sqrt{2}\sigma)} \right) \leq \frac{\|g\|_\infty}{2} \sigma^2$$

with:

$$h(x) = \frac{e^{x^2/2}}{x^4} \left[-\sup_{K_0} \frac{\partial^2 g}{\partial \psi_k^2} \left(\frac{\sqrt{\pi}}{2} \operatorname{erf}(x/\sqrt{2}) - 2(1 - e^{-x^2/2}) \right) \right. \\ \left. + \left(\frac{\sqrt{\pi}}{2} \operatorname{erf}(x/\sqrt{2}) - x^2 e^{-x^2/2} \right) \cdot \left[(m-1) \left(\inf_{K_0^-,i} \frac{\partial^2 g}{\partial \psi_k \partial \psi_i} - \sup_{K_0^+,i} \frac{\partial^2 g}{\partial \psi_k \partial \psi_i} \right) \left(\frac{2}{\sqrt{2\pi}x} + \frac{1}{2} \right) \right. \right. \\ \left. \left. - \frac{m-1}{2} \sup_{K_0^+,i} \frac{\partial^2 g}{\partial \psi_k \partial \psi_i} \right] \right]$$

is a function independent of α_σ and σ .

In particular, when x goes to infinity, $h(x)$ is equivalent to:

$$\frac{e^{x^2/2}}{x^4} \frac{\sqrt{\pi}}{2} x \left[-\sup_{K_0} \frac{\partial^2 g}{\partial \psi_k^2} + \frac{m-1}{2} \inf_{K_0^-,i} \frac{\partial^2 g}{\partial \psi_k \partial \psi_i} - (m-1) \sup_{K_0^+,i} \frac{\partial^2 g}{\partial \psi_k \partial \psi_i} \right]$$

But, according to the hypothesis done on the compact K_0 , this is strictly positive.

Hence, there exist $c_0 > 0, c_1 > 0$ such that, if $x \geq c_1$, $h(x) > c_0 > 0$. We will now suppose that $(\alpha_\sigma)_k \geq c_1 \sigma$. Then, $h\left(\frac{\alpha_k}{\sigma}\right) \geq c_0 > 0$. Moreover,

$$\frac{1 - \operatorname{erf}(c/\sqrt{2}\sigma)}{\sigma^2 \operatorname{erf}(c/\sqrt{2}\sigma)} \xrightarrow{\sigma \rightarrow 0} 0.$$

So, for σ small enough, it is smaller than $c_0/2$ and we finally find:

$$(\alpha_\sigma)_k^4 \leq c_0 \|g\|_\infty \frac{\sigma^2}{\operatorname{erf}\left(\frac{c}{\sqrt{2}\sigma}\right)^{m-1}}$$

Using the fact that $\operatorname{erf}\left(\frac{c}{\sqrt{2}\sigma}\right)^{m-1} \xrightarrow{\sigma \rightarrow 0} 1$ gives us finally the existence of a constant $c > 0$ such that

$$d(\bar{\psi}_\sigma, \mathcal{L}) \leq (c_1 \sigma) \vee (c\sqrt{\sigma})$$

which allows us to conclude for σ small enough.

Fourth step: Approximation when σ goes to infinity

The last step follows the exact same steps as for $m = 1$. It is copied here. We use again the equation (1). For all $\sigma \in \mathbb{R}, \forall 1 \leq k \leq m$,

$$\int_{\mathbb{R}^m} v_k g(\bar{\psi}_\sigma + v) \exp\left(-\frac{\|v\|^2}{2\sigma^2}\right) dv = 0$$

Using the change of variable $\bar{\psi}_\sigma + v$, we find:

$$(\bar{\psi}_\sigma)_k = \frac{\int_{\mathbb{R}^m} v_k g(v) \exp\left(-\frac{\|v - \bar{\psi}_\sigma\|^2}{2\sigma^2}\right) dv}{\int_{\mathbb{R}^m} g(v) \exp\left(-\frac{\|v - \bar{\psi}_\sigma\|^2}{2\sigma^2}\right) dv}$$

But $\bar{\psi}_\sigma$ is supposed to stay in a compact so, $\forall v \in \mathbb{R}^m$, $\exp\left(-\frac{\|v - \bar{\psi}_\sigma\|^2}{2\sigma^2}\right) \xrightarrow{\sigma \rightarrow \infty} 1$. Using the integrability of g and $v \mapsto v_k g(v)$, it is easy to conclude using the dominated convergence theorem.