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► **To cite this version:**

Alain Kuzniak, Assia Nechache. On forms of geometric work: a study with pre-service teachers based on the theory of Mathematical Working Spaces. *Educational Studies in Mathematics*, 2021, 106 (2), pp.271-289. 10.1007/s10649-020-10011-2 . hal-03128307

HAL Id: hal-03128307

<https://hal.science/hal-03128307>

Submitted on 3 Apr 2022

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On forms of geometric work: A study with pre-service teachers based on the Theory of Mathematical Working Spaces

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In this paper, we identify various forms of geometric work carried out by student teachers who were asked to perform a geometric task for the estimation of a land area. The theory of Mathematical Working Spaces is used to analyze and characterize the work produced. This study provides evidence that students developed forms of geometric work that are compliant with at least two distinct geometric paradigms, one characterized by the utilization of measuring and drawing tools, and the other by a property-based discourse on proof. Significantly, a sizable number of students also developed work forms that do not correspond to any geometrical paradigm. A broader purpose of this paper is to highlight three criteria born by the theory and shown to be useful for the description and evaluation of geometric work: compliance, completeness, and correctness.

Keywords: Geometrical paradigms, Teachers training, Mathematical Working Spaces, Geometry training, geometric work form.

1. A study of how pre-service teachers practice geometric work

This study is concerned with the mathematical and didactic knowledge that future teachers should have in order to teach geometry to their students. It centers on student teachers' geometric work and expands on research begun earlier (Houdement & Kuzniak, 1999; Kuzniak & Rauscher, 2011). In particular, Kuzniak and Rauscher (2011) proposed a classification of the solutions given by pre-service teachers to a geometry task. That study examined the difficulties that future primary school teachers encountered teaching geometry, often resulting from a deficient or dated knowledge of the field.

On-going research on mathematical and didactic knowledge for teacher training builds on Shulman's work (1986) that introduced the notion of pedagogical content knowledge (PCK) to complement subject content knowledge. Based on this idea, various refinements have been made that delineate knowledge directly relevant in the teaching of mathematics. One of the most notable is Ball's proposal (Hill, Ball, & Schilling, 2008), developed around the notions of knowledge of contents and students (KCS) and knowledge of content and teaching (KCT) to organize mathematical knowledge for teaching (MKT).

In the particular case of geometry, studies have exposed the many difficulties encountered by students. Some studies, using Van Hiele's levels, differentiate profiles in geometry thinking by students (Mayberry, 1983; Guttierrez, Jaime, & Fortuny, 1991). Without denying the interest of these studies, we have already criticized this approach (Rauscher & Kuzniak, 2005), which overlooks the fact that the students we are discussing had already followed the entire curriculum and had on occasion demonstrated skills far superior to those of their current Van Hiele's level. We agree with Swafford, Jones, and Thornton (1997, p. 481) who conclude their study by emphasizing that Van Hiele levels have low reliability for adults who have been away from learning for years

and whose performance is sensitive to knowledge recall, which is exactly the case for our students. To overcome these criticisms, Manizade and Martinovic (2018) combine Van Hiele's levels and PCK and SMK (Spezialized Mathematical Knowledge), which leads them to propose student profiles based on their PCK and their understanding of geometry. In the same vein, in a study with content closer ours, Chinnappan, White, and Trenholm (2018, p. 145) consider construction problems and conclude with these suggestions that we share:

Our analysis suggests future research needs to consider (a) the particular characteristics of the discipline of geometry and (b) the developmental knowledge trajectories of teachers of geometry in order to better understand how teachers' SMK influences and [is] influenced by PCK.

In a distinct area of research on geometry teaching, several studies have also focused on the use of drawing tools, considering dynamic geometry software, and have shown how a teacher's SMK is essential to managing effectively geometry teaching situations in all their complexity (Laborde, Kynigos, Hollebrands, & Straesser, 2006).

In this context of on-going debates and high level of complexity of the subject matter, we intend to contribute to a better understanding of geometry teaching training by focusing the study on geometric work in an educational context. Specifically, our objective is to identify the geometric work forms that student teachers actually produce. We consider it an essential first step before being eventually able to guide students' work and enable them to be more effective as geometry teachers. In this study, a geometric task on the estimation of the area of a land parcel is assigned to student teachers. The analysis of their geometric work is based on a previously used research framework (Kuzniak & Rauscher, 2011) which has significantly evolved towards what is now referred to as the theory of Mathematical Working Spaces (MWS) (Kuzniak, Tanguay, & Elia, 2016). More generally, this paper also purports to highlight our analysis method and make explicit some criteria for the description and evaluation of the geometric work produced by students.

2. Identifying and understanding the mathematical work through the lens of the MWS theory

The theory of Mathematical Working Spaces is primarily concerned with the interpretation of mathematical work developed during the execution of mathematical tasks by students or teachers in a specific educational setting (Kuzniak, Tanguay, & Elia, 2016).

2.1 On mathematical work

In the theory, "mathematical work" must be understood as a on-going production process that combines syntactic and semantic elements to generate statements and practices applying to tasks at hand. Mathematical work implies consideration of three aspects of its execution:

- The goal of work. This attribute helps us distinguish work from a simple activity. This is done by assigning a purpose to an action, clarifying the stakes associated with it, and inscribing it in the long term by showing its general relevance. In this study, the stakes associated with the pursuit of that work are specifically mathematical.
- Processes of the work. These are related to the procedures and constraints of the implementation on given tasks. The appropriation and the respect of the rules of functioning are important points for the observation of the work.
- Results of the work. They should be valid and correct within the mathematical domain under consideration.

We will pay particular attention to two aspects: process and result. Two of the main constructs of the theory, MWSs and geometrical paradigms, are invoked to set forth criteria on which the characterization of the forms of work carried out by the students will be based.

2.2 Mathematical Working Spaces

A Mathematical Working Space (Kuzniak, Tanguay, & Elia, 2016) refers to an abstract structure organized in such a way as to enable individuals to conduct mathematical work operations with problems in a specific domain (geometry, probability, etc.). In the case of school mathematics, these individuals are generally not experts but students, more or less experienced.

Both the epistemological and cognitive facets of problem solving in mathematics are taken into account to grasp the specificity of students' performance. These facets are articulated in the MWS diagram into two planes: an epistemological plane directly associated with the mathematical content of the field of study (here geometry), and a cognitive plane related to the thinking of individuals solving mathematical tasks.

Three components in interaction are characterized for the purpose of describing the work in its epistemological dimension, organized according to specifically mathematical criteria: a set of concrete and tangible objects, referred to as *representamen*; a set of *artifacts*, such as drawing instruments or software; and, a *theoretical referential* based on definitions, properties and theorems.

In close relation to the components of the epistemological level, three cognitive processes are taken into account: visualization, which addresses the deciphering and interpreting of signs; construction, which refers to the mode of reasoning and is contingent upon the artifacts utilized and the associated techniques; and proving, which is conveyed through validation processes and based on a theoretical referential.

Bridging the epistemological plane and the cognitive plane is part of mathematical work according our perspective and can be characterized as a generating process (in the sense of generating specific forms of reasoning, decision making, etc.) along three geneses, i.e., semiotic, instrumental, and discursive, corresponding to the components constitutive of the theory.

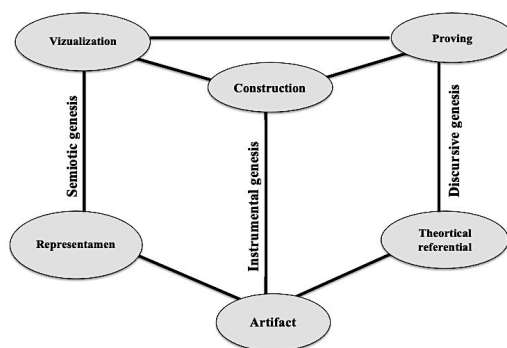


Fig. 1: The MWS diagram

In order to understand these intricate interrelationships, we use a diagram (Fig. 2) to identify the work forms in terms of the type of generating processes they rely on. The diagram identifies three vertical planes that correspond to the dimensions along which geometric work can unfold, thus allowing analysts to distinguish between semiotic-instrumental [Sem-Ins], instrumental-discursive

[Ins-Dis], and semiotic-discursive [Sem-Dis] articulations (Coutat & Richard, 2011). The use of these different vertical planes helps to specify the manner in which geometric work is carried out.

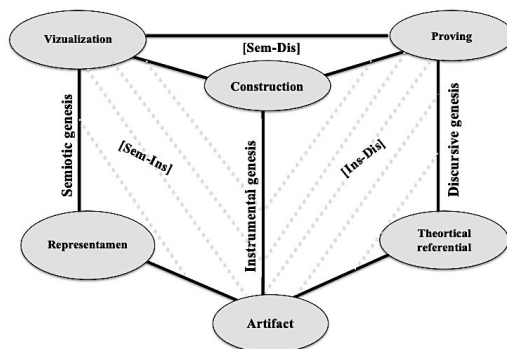


Fig. 2: The three vertical planes in the MWS

The contribution of each genesis to the work process can be assessed by studying how certain objects of the epistemological plane operate as tools that will be semiotic, technological, or theoretical, depending on which type of genesis of the MWS they relate to (Kuzniak, Nechache, & Drouhard, 2016). It is thus possible to describe precisely the genesis actually mobilized by the student.

When a single aspect of the work process, i.e. a single genesis or single vertical plane, is activated, the work is said to be confined to one dimension or plane. Work can be confined but effective and may remain relevant in certain cases and for certain tasks. When all the genesis MWS are in play, work is said to be complete. The degree of completion achieved can be used to characterize geometric work.

2.3 Geometrical paradigms

In geometry education, geometrical paradigms have been introduced by Houdement and Kuzniak (1999) to clarify and organize the various and conflicting points of view prevailing in education around geometry. Drawn from Kuhn (1966), a paradigm stands for the entire constellation of beliefs, values, techniques, practices shared by the members of a given scientific community. According to Houdement and Kuzniak (1999), in an educational context, it is possible to identify three geometrical paradigms named respectively Geometry-I, Geometry-II and Geometry-III and we refer the reader to Kuzniak (2018) for a recent and synthetic presentation.

In what follows, we present the characteristics of geometric paradigms and the elements related to area and area measurement that are relevant to the present research. The paradigm called Geometry-I is concerned with the practical use of technology. In this geometry, valid assertions are generated using arguments based upon perception, experiment, and deduction. Mechanical and experimental proofs are acceptable and the effective area measurement of a figure is a primary consideration and can be based on the actual constructed figure. The area estimation can also be based on a formula and the values used for the calculation may also be obtained by a direct measurement on the figure. This approach implies a reflection on the approximation and accuracy of measures and figure constructions. The numbers used are mostly decimal numbers and fractions.

The paradigm called Geometry II, whose archetype is classic Euclidean geometry, is built on a model that may closely approach reality. Once the axioms are set up, proofs have to be developed within the system of axioms to be valid. But the process of axiomatization is not finished and

certain notions remain intuitive, such as the concept of area, which is based on a notion of equality not defined in the Euclidean tradition: two figures are considered to have the same area when they have the same space content (Hartshorne, 2000). It is possible to verify these equalities by decomposing and reassembling the figure. It is also possible to use formulas, but these must be justified in their generality. In general, direct measurement on the drawing is prohibited and the numbers used are the real numbers constructible with ruler and compass.

Lastly, Geometry-III is usually not present in compulsory schooling, but it is the implicit reference of mathematics teachers who are trained in advanced mathematics. In Geometry-III, the system of axioms itself is disconnected from reality, but central. In the Geometry-III paradigm, the theoretical question of area and measure of function area becomes the first issue. It follows that theoretical bases are theorems on equi-decomposability and axioms of the measure on a set. Real numbers are the numbers involved in this measure, which is more theoretical than practical.

Identification of geometrical paradigms and their interplay contributes to the understanding of what guides the work actually performed in a school setting and helps characterize the form of this work. In the following, paradigms will be used to capture the diversity of ways of thinking and doing exhibited by students in geometry. This will help us clarify the extent to which the work performed complies with expectations.

2.4 Settings and issue

Our study is carried out in the context of the training of primary school (age 3 to 11) pre-service teachers in France. From an institutional point of view, primary school teachers are now trained at the university and must obtain a Master's degree in education. The students participating in the study are first-year Master's students who are also preparing a selective competitive examination which will allow them, after obtaining their diploma, to secure a position as school teachers for the rest of their professional life, if they so desire. The stakes of this first year are therefore very important and the pressure that students are under is very high.

Most of these students have graduated in literary studies and have not taken any mathematics courses after completing the first year of senior high school (age 16). Moreover, as numerous studies have shown (Section 1), they have difficulties in mathematics, especially geometry. They are generally familiar with geometric properties and theorems, but they do not know how to use them well because they lack sufficient mastery in this domain.

In this context, students are expected to develop a personal geometric work that is sufficiently advanced to enable them to teach primary school pupils a geometrical content that tends towards the Geometry-I paradigm, with figure construction and measure aided by drawings tools. Yet, they must be able to solve problems relevant to Geometry-II to succeed in the competitive examination. This part depends on secondary school curriculum and it rests essentially on the mostly-mechanical calculations and applications of the Pythagoras and Thales theorems, mainly to practice algebraic computation techniques. Also, more emphasis is put on modeling activities. Lastly, this teaching relies on triangle and quadrilateral properties and area estimation for the formulation of some formal proofs (Geometry-II) or the construction of experimental proofs as part of the primary school curriculum (Geometry-I).

However, the distinction between paradigms remains implicit to the extent that the French syllabus is set up according to class years, not geometric work forms. Typically, however, the tasks given to them play on these two paradigms, but that is not explicit and students need to articulate both paradigms by themselves.

In summary, our study purports to identify and characterize the geometric work forms that these teacher students actually perform. To achieve this goal, we will observe how students tackle a task, paying special attention to the use of tools and the form of genesis activated. The objective is to assess the completeness of the work; the conformity of the work processes to certain paradigms that we will specify; and finally the correctness of the results obtained.

3. On the method of identifying the students' geometric work forms

3.1 The task given to the students and its implementation: Alphonse's land

The task statement is given in the form of a text to be read.

Alphonse has just returned from a trip in Périgord where he saw a parcel of land in the shape of a quadrilateral that had interested his family. He would like to estimate its area. To do this, during his trip, he successively measured the four sides of the plot and found, approximately, 300 m, 900 m, 610 m, 440 m.

He's having a hard time finding the area. Can you help him by showing him the method to be followed?

The task was implemented in two phases. The objective of the first phase was to conclude that there was a lack of data to fix the shape of the quadrilateral. Next, information was given about the length of one of the diagonals.

Alphonse asked a Périgord friend to help him and she sent him back only the length of one of the diagonals: 630 m.

The objective of the second phase was to explore the different possible shapes of the land parcel (convex or concave) and to estimate the land area according to the shape of the quadrilateral selected.

During the first phase and after ten minutes of investigation, students were asked to reflect on their approaches by answering two questions. Students were given five minutes to respond.

1. If you did not have enough time to finish this part, can you briefly describe what you intended to continue doing?
2. What doubts or difficulties did you encounter in solving this exercise?

In posing these questions, we expected the students to explain their certainties, their doubts, and their difficulties and blockages in completing the task. They were then invited to discuss their solutions and reflections with the whole group during the pooling of ideas session. The same process was followed for the second phase.

In this article, we limit our report to the analysis of students' geometric work in solving Alphonse task during the first phase, which was highly significant because almost all the students, in their search for the land area, spontaneously added new data while failing to point out that necessary information was absent in the task statement.

3.2. Preview of expected task performance outcomes (*A priori* task analysis)

The task assigned to the students has the merit of potentially producing a wide variety of geometric work forms because it is not linked to a predefined instructional situation (Herbst, Boileau, & Gürsel, 2018) such as constructing a figure, exploring a figure or doing a proof. In the *a priori* analysis, we have considered different possible procedures taking into account students' knowledge and institutional expectations for this level of training. This led us to consider the possible use of the false and well-known theorem-in-action (Vergnaud, 2009) about the relationship between perimeter and area: figures with the same perimeter have the same area (named TiA in this paper). The list of possible procedure sequences follows:

P1. After reading the statement, students immediately identify the lack of data to fix the quadrilateral shape and ask for a complement of information. Once they have the information, they can move on to the second phase and/or give a P2a or P3 solution.

P2. Students begin with a construction of the figure to scale with drawing tools.

P2a. They succeed in constructing a quadrilateral then try, through measurement, to find the area.

P2b. They spontaneously add data in constructing the figure and continue as in P2a.

P3. Students working freehand or with drawing tools, avoid all measurements.

P3a. They try to solve the problem by breaking up the quadrilateral into sub-figures and notice the absence of necessary data. .

P3b. They construct several figures and find that there are several area values possible.

P4. Students resort to the theorem-in-action TiA.

P4a. They start a P2a or P3b procedure by adding data.

P4b. They select a particular figure for their purpose, a quadrilateral as a rectangle or square with the same perimeter from which they deduce the area.

P1 procedure can be validated on the grounds of the property that a quadrilateral shape is not determined when only side lengths are provided. The P2 procedure is based on the quadrilateral properties taken directly from the figure constructed with the ruler and compass. The work is compliant with Geometry-I paradigm. The work in P3 procedure conforms with Geometry-II through its rejection of any recourse to drawing tools and the priority given to the process of discursive genesis. As for the P4 procedure, it also seems to fit within the Geometry-II paradigm but relies on a false theorem.

The students were familiar with solving problems with missing data and warned against the customary didactic contract that assumes that all data must be included in the task statement.

3.3 Data collection method

The study took place over a two-year period. Eighty-five students were enlisted at the end of the second semester of the first year of the Master's program. The courses were given to four groups of students by the second author and an experienced teachers' trainer, each of whom having taught to two groups. The students carried out the task individually and were directed not to discuss their work among themselves and with the teacher. Their written productions were collected by the teachers.

The classroom proceedings were taped through a voice recorder and two sessions were videotaped. This provided us with a record of the discussions that took place during the pooling sessions between the students and the teacher. All discussions were transcribed in their entirety.

3.4 Data analysis method

The particularity of geometric work is that it is mainly a type of intellectual work that involves significant cognitive activity. Leplat (2004) points out that since cognitive activities are not observable, they must be inferred from visible actions. According to this viewpoint, mathematical actions are the real and observable traces of the subject's activity. We use cognitive analysis to reconstruct and characterize the mathematical work performed by the students. It is based on the collected data during the resolution of the task. We designed an original data analysis method inspired by Cognitive Task Analysis (CTA) (Darses 2001) and based on elements of the MWS theory. CTA-methods are used in work psychology and have been developed for the study of the cognitive activity of people performing complex tasks in production environment. Authors using CTA methods emphasize the difficulties encountered both in describing the particular field in which the work occurs and in providing and developing appropriate analytical tools. In the field of mathematics, MWS theory associated with the notion of geometric paradigms provides us with a set of appropriate tools to analyze this work.

Our analysis of geometric work forms starts with the observation of students' performance and will develop in three steps. The first step is a top-down analysis that seeks to identify the different episodes planned by the student to achieve the prescribed task. In this view, each episode corresponds to a sub-task self-prescribed by a student and includes the sequence of mathematical actions used to solve the task. Mathematical actions are then described in terms of the cognitive and epistemological components and processes of the MWS planes. The top-down analysis thus consists of a "zoom in" on the students' production in order to provide a detailed analysis in terms of MWS. The grain size of the analysis depends on the grade of students who are performing the task. For example, for primary school student-teachers, it is sufficient to consider mathematical actions such as "drawing the height of a triangle ABC from B". Whereas, for primary school students, it may be useful to specify the action "place the right angle of square on the line AC". Indeed, at this level of education, this process is not necessarily routinized and mastered.

Starting from the results given by the top-down analysis, the second step is a bottom-up analysis conducted to provide a synthetic overview of the episodes planned by a student, and infer the logical organization of his or her mathematical actions and thus uncover his or her cognitive path. The analysis seeks to identify how the work generating processes interact within the MWS diagram. This then allows us to understand what guided the student in the performance of the task.

In a third step, we use observations from our analyses to examine the extent to which work processes are compliant with all or part of one of the geometrical paradigms. We furthermore determine whether the work is complete and the degree to which it is confined to a particular genesis or a particular plane. Finally, we evaluate whether the results given by a student are mathematically correct or not. The analysis ends with a characterization of the geometric work forms that we have observed.

4. The different forms of geometric work

We identified five forms of geometric work. Twenty-one student works out of 85 were set aside because of manifest breakdowns as a result of blockages and inconsistencies (such as using the triangle area formula to evaluate the area of a quadrilateral or calculating the perimeter instead of the area).

Five forms of work were identified, each corresponding to a type of geometric work encountered in solving the task; they are: Dissector, Surveyor, Explorer, Constructor, and Calculator. They are derived from an assessment of the *completeness* of the work, its *compliance* with a geometrical paradigm or interacting paradigms, and the *correctness* of the results achieved from a mathematical viewpoint. These categories are not intended as statements on a student's profile.

In what follows, we describe the five forms of work by reporting the actions actually undertaken by students in connection with each of these forms. The examples given may have been produced by different students.

4.1 Dissectors' geometric work

The solving method introduced by dissectors students (16 out of 85) is based on the decomposition of the figure into sub-figures for which they know the formula for area calculation. It is close to the P3 procedure of our *a priori* analysis. Of the 16 students, eight propose a triangulation and the other eight decompositions combining triangles and quadrilaterals.

Two episodes were identified through the top-down analysis. In Episode-1, the student uses a freehand drawing (Action-1) to dissect the quadrilateral (representamen) into simple sub-figures for which the formula (theoretical referential) is known (Action-2).

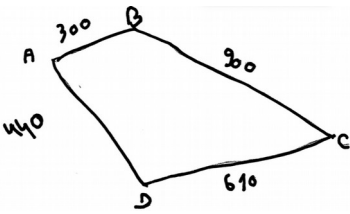
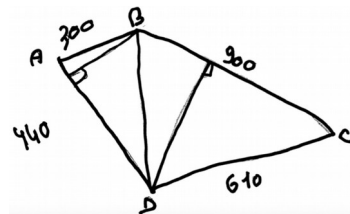
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| <p>Action-1. Freehand drawing of a quadrilateral</p>  | <p>Action-2. Dissection of the quadrilateral into relevant sub-figures.</p>  <p>Pauline: He could try to decompose the figure into figures of which we can calculate the area (square, rectangle, right triangle).</p> |
|--|--|

Table 1. Dissectors' work top-down analysis – Episode-1

In Episode-2, students ponder which theoretical tools could help them determine what data will be needed to calculate the area of the figure. To that end, some at once consider resorting to the possible application of the Pythagoras theorem or using formulas to calculate the area of the sub-figures obtained after dissection (Action-3). In connection with this action and based on the decomposition obtained, some students reach the conclusion that they need numerical data on heights in order to progress towards a solution (Action-4).

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| <p><i>Action-3. Exploration of the theoretical referential to find the area.</i></p> <p>Corine: Determine the area of the "square" and the area of the two triangles that form the quadrilateral using Pythagoras or a 2 unknown equation.</p> | <p>Action-4. Identifying the data necessary to do the calculations.</p> <p>Pauline: He can separate his land lot into two triangles ABD and BCD and estimate the area of the two triangles but he needs the heights.</p> |
|--|--|

Table 2. Dissectors' work top-down analysis – Episode-2

Bottom-up analysis. In Episode-1, students project on the representamen various visualizations of the decomposition of the quadrilateral into sub-figures with known area formulas. Thus the semiotic genesis is supported by the properties. In the second episode, visualization is associated with a discursive exploration on possible use of Pythagoras theorem to calculate some of the values needed to achieve the task. Discursive genesis is thus related to visualization of well-known sub-figures.

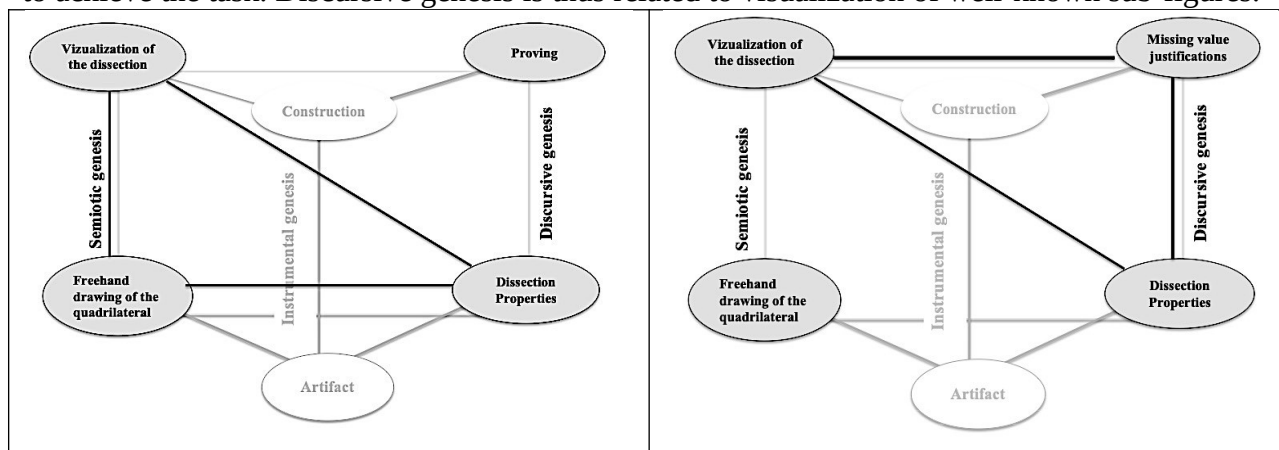


Table 3. Dissectors' work - Bottom-up analysis

In the end, the efforts by students did not result in a successful outcome and led them to a predictable dead-end because some information is actually missing. However, students did not explicitly point out the lack of sufficient data. Only one student, Manon, expressed the need for supplementary data.

Manon: In order to calculate the area of the land, Alphonse would have to measure one of its diagonals. Thus, he could calculate the area of the two triangles.

When the professor probed Manon about her response to the problem, she said she had a flash of inspiration:

Manon: After that, I had a flash of inspiration. I thought that Alphonse had to measure one of the two diagonals of the land, and so he has two triangles and adds them both.

The use of the word inspiration indicates that the student was not immediately aware that necessary information was missing. That realization eventually occurred within the 10-minute time period granted to solve the problem.

In conclusion, the work is compliant with the Geometry-II paradigm, in which construction and measuring tools are prohibited as means of proof. But, for these students, this compliance to Geometry-II paradigm inhibits the heuristic potential that constructions with drawing tools allow for exploring the existence properties of figures and their work is confined in the semiotic-discursive plane (see Table 3). It follows that the work can be considered correct even though it remains unfinished since the reason for the blockage, i.e. the absence of sufficient data, was not clearly understood or explicitly formulated.

4.2 Surveyors' geometric work

The method adopted by Surveyors (6 out to 85) follows closely procedure P2. It is based on area calculations with some measurements taken from a drawing constructed to scale.

After top-down analysis, two episodes are identified. In Episode-1 (Table 5), students first draw freehand a convex quadrilateral (representamen) (Action-1) and adding an assumption on the shape of the quadrilateral. This assumption (Action-2) is formulated and then a particular quadrilateral is constructed to scale using drawing tools (Action-3). For some students, the construction of a specific quadrilateral is justified by the theorem in action TiA that Ivana reformulates in this way during the pooling session: "The shape of the land does not matter since the area remains the same because the sides have the same measures".

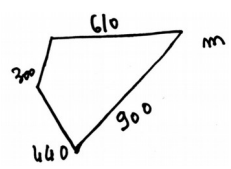
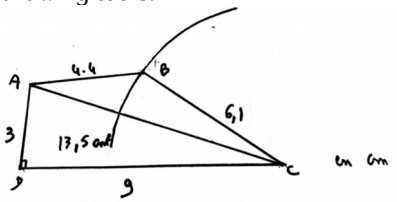
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| <p>Action-1. Freehand drawing of a quadrilateral</p>  | <p>Action-2. Expression of a assumption for the construction of a particular figure.</p> <p>Ivana: In fact I have constructed the figure to have a right angle, and so I could calculate the area of the first triangle by using Pythagoras theorem.</p> | <p>Action-3. Construction of the figure with drawing tools.</p>  |
|--|--|--|

Table 4. Surveyors' work - Top-down analysis - Episode-1

In Episode-2, the quadrilateral area is calculated with certain measurements taken on the figure (representamen) and other resulting from the use of a theorem. Different actions were carried out depending on the shapes chosen by the students. For example, in the case of Ivana, we can point to five actions described in Table 6.

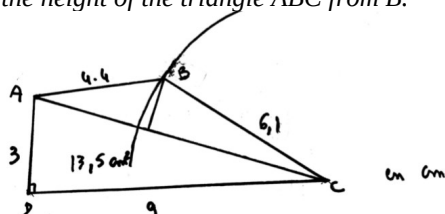
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| <p>Action-4. Application of the triangle area formula to calculate the area of the triangle ADC</p> <p>We calculate the area of the ADC right triangle at D</p> $\mathcal{A}_{ADC} = \frac{3 \times 9}{2} = 13,5 \text{ cm}^2$ | <p>Action-5: Calculation of the length AC by applying the Pythagoras theorem in the triangle ADC</p> <p>We know that ADC is right at D and AD = 3 cm et DC = 9 cm</p> <p>According to Pythagoras</p> $AC^2 = AD^2 + DC^2$ $AC^2 = 90$ $AC = \sqrt{90} = 3\sqrt{10}$ |
| <p>Action-6: Drawing and measuring with a graduated ruler the height of the triangle ABC from B.</p>  | <p>Action-7. Application of the triangle area formula for calculating the area of triangle ABC</p> $\mathcal{A}_{ABC} = \frac{B \times h}{2} = \frac{3\sqrt{10} \times 1,7}{2} \approx 8,06$ <p>Action-8 (potential). Sum of areas and conversion in m²</p> <p>I did not have time to finish my calculation and to convert the result from cm² to m²</p> |

Table 5. Ivana's work - Top-down analysis - Episode 2

The Episode-1 bottom-up analysis shows that instrumental and semiotic genesis are activated to outline a particular quadrilateral according to elements of the referential such as the scale or the shape of the quadrilateral. In Episode-2, the quadrilateral construction in relation to its visualization may have suggested the use of the triangle area formula. This formula activates the instrumental genesis for the calculation and measurement of the length AC and the height from B in the ABC right triangle. This instrumental genesis is associated with a discourse on proof that justifies certain

calculations such as that of the length of the segment [AC]) with Pythagoras theorem in the right triangle ADC. It is also associated with the visualization for measuring and constructing the height from B. The formula of the triangle area is a symbolic artefact that guides the students' geometric work through the semiotic-instrumental (Actions 4 and 6) and instrumental-discursive (Action-5) planes.

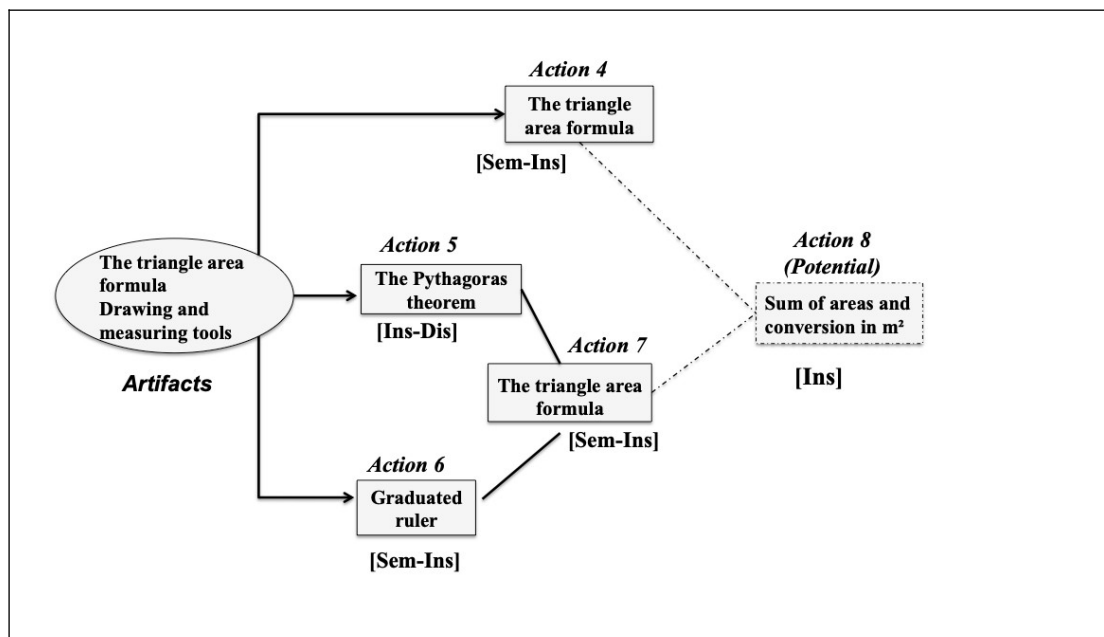


Table 6. Logical organization of Ivana's actions in episode-2.

In conclusion, surveyors' geometric work form relies on an instrumental construction process that promotes the use of drawing tools and formulas with measurement on drawings. The constructed figure is used as a support for reasoning and proof. It is therefore compliant with the requirements of Geometry-I paradigm. Within the MWS theory (sec 2.2), the mathematical work can be considered to be complete since all the MWS geneses are mobilized.

However, this geometrical work is not correct. It produces a faulty result since it is based on a particular figure derived from contrived data and a false theorem-in-action.

4.3 Explorers' geometric work

In this form of geometric work (produced by 4 out of 85 students), students are looking for figures that satisfy the required conditions, which leads them to construct several figures. The top-down analysis identifies a single-episode with two actions. Action 1 encompasses the construction to scale with ruler and compass (artifacts) of two quadrilaterals (representamen) that satisfied the conditions of the statement. In Action 2, students visualize and explore the figures. Some of them conclude that the area could take several values.

Clothilde: For me, the quadrilateral could have several areas depending on how it is built.

Other students focus on construction and conclude on the existence of several quadrilaterals satisfying the sides length conditions but do not refer to area.

Bottom-up analysis of the Explorers' work shows that the semiotic and instrumental geneses are mobilized to produce drawings that allow them to explore different configurations and, for some of

them, generate hypotheses about the areas of the figures. These drawings seem to have a heuristic role in the mathematical work produced. The exploration led students to contend that there may be several areas for the same figure, but they do not justify their assertion.

The geometric work is close to procedure P3b and is based on the semiotic-instrumental plane oriented by the discursive genesis. Compliant with Geometry-II paradigm, the work is complete even there is a lack of explanations and references to properties.

4.4 Constructors' geometric work

This work (13 students) was summarized as a single episode devoted to the sole construction to scale of a quadrilateral (representamen) with a graduated ruler and in most cases a compass (artifacts) (Action-1). The use of these tools makes it possible to adjust the construction of the fourth side guided by perception. The students then stop and abandon their pursuit (Action-2).

The work of the Constructors focuses on figure construction. It unfolds in the semiotic-instrumental [Sem-Ins] plane, and leans more forwards on the instrumental genesis (Action-1). In Action-2, students explain their blockage by their inability to provide an answer either because they do not recognize the particular shape of the constructed figure or because they do not remember the formula for calculating the area of a quadrilateral. This suggests that they have tried to mobilize the discursive dimension, without success.

In conclusion, it can be said that the work is compliant with Geometry-I paradigm and confined to the semiotic-instrumental plane because they think that all problem resolution need to be based on a specific figure or a formula.

4.5 Calculators' geometric work

In this work form that involved 24 participants students developed a formula or a calculation diagram and then applied it to estimate the area of the land. We describe below the procedure used by the twenty students who used a calculation diagram which enabled them to produce a result, albeit misrepresenting the real figure.

Top-down analysis shows that the students planned the work in a single episode. They started by constructing a trapezoid with two right angles (representamen) as union of a rectangle and a right-angled triangle (Action-1). They calculated the values for the rectangle and triangle areas by applying the appropriate formulas (artifact) and reported the two values on the diagram (Action-2). They deduced the area of the figure by summing the two values (Action-3). The diagram was used by the students as an artifact to perform calculations using known formulas.

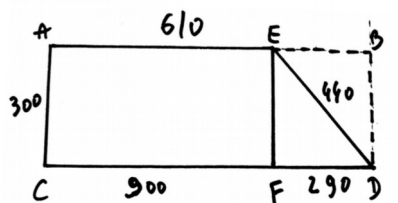
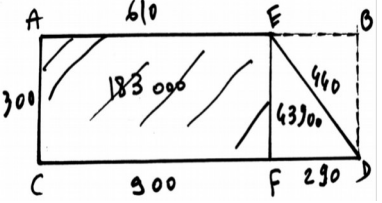
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| <p>Action-1. Drawing of the diagram seen as the union of a rectangle and a right-angled triangle</p>  | <p>Action-2. Calculation of the areas of each sub-figure and reporting of the values on the diagram.</p> $\mathcal{A}_{AECF} = 610 \times 300 = 183\,000 \text{ m}^2$ $FD = 900 - 610 = 290 \text{ m}$ $\mathcal{A}_{EFD} = \frac{290 \times 300}{2} = 43\,900$ $\mathcal{A} = 183\,000 + 43\,900 = 226\,500 \text{ m}^2$ | <p>Action-3. Calculation of the figure area as the sum of the rectangle and triangle areas</p>  |
|--|---|---|

Table 7. Calculators' work - Top-down analysis

Within the bottom-up analysis, we observe that through a semiotic genesis, the right trapezoid is considered as the union of a rectangle and a right-angled triangle (Table 7) which opens the path towards a conversion of the figure into a calculation artifact. The work is then pursued in the instrumental genesis with a calculation and submission of the result, all carried out without any reference to empirical or theoretical criteria. In summary, the work starts with the creation of artifact on semiotic genesis mode and is concluded with a calculation using that artifact in the instrumental genesis mode.

[Sem] → Artifact ----- > [Sem-Ins]

To justify their procedure, some students explained that they constructed that figure to overcome their blockage and answer the question. Indeed, the figure is a simple quadrilateral which comply with the conditions on lengths inequality and for which they know how to calculate the area. As a side comment, this work form tends to be more compliant with scholar expectations - producing a result, waiting for external validation by the teacher – rather than with the geometrical perspective. But, it is also possible to consider that it is confined in a the semiotic-instrumental plane where the instrumental genesis is totally oriented towards a calculation without any measure or use of construction using drawing tools. In that sense, it conforms to Geometry II paradigm without resorting to the discursive proof component. Such interpretation may find support in the output from the last four students who performed calculations without concern for empirical or theoretical validation. They availed themselves of a formula for the computation of the area of a rectangle for their calculation and transformed the initial figure into a rectangle to fit with the formula. They obtained the area of the quadrilateral by taking the product of the averages of the opposite sides.

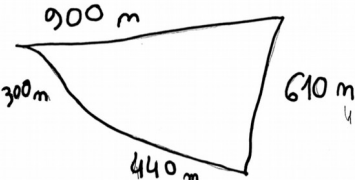
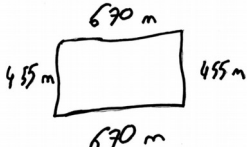
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| <p>Action-1. Freehand drawing of a quadrilateral</p>  | <p>Action-2. The Averages of the opposite sides</p> $\begin{array}{l} 900 - 230 = 670 \\ 490 + 230 = 670 \\ 610 - 155 = 455 \\ 300 + 155 = 455 \end{array}$ <p>Subtract from 900 and add to 440 to make 2 equal opposite sides</p> |
| <p>Action-3. Freehand drawing of the rectangle with the new lengths</p>  | <p>Action-4. The area of rectangle</p> $\text{Area} = 670 \times 455 = 304850 \text{ m}^2$ |

Table 8. Margaux's procedure

They thus developed procedures close to P4 by resorting to formulas applicable after a semiotic reconfiguration of the figure. We can also suppose an implicit use of the theorem in action TiA.

In conclusion, the work carried out could be seen as compliant to the Geometry II paradigm but it is confined to the semiotic-instrumental plane. Additionally, it exhibits the typical inclination of students to overcome blockages by falling back on calculations based on known formulas with the idea of being compliant with the traditional scholar paradigms.

4.6 Different geometric forms of work: A discussion

This study led us to identify five forms of geometric work, each characterized by their outcomes assessed along three criteria: compliance of the work carried out with a given geometric paradigm; degree of completeness; and correctness of the results achieved.

Using these three criteria, we are able to evaluate the work actually produced by the students and to propose a classification which we assume remains relevant beyond the particular task at hand. The geometric work forms of Surveyors and Constructors meet the expectations of the Geometry-I paradigm because the figure is constructed with drawing tools and serves as support for reasoning and proof. In Surveyors' case, work is carried out to completion through the mobilization of all the MWS geneses. For Constructor, students' work is mainly guided according to the instrumental genesis mode in which it remains confined for lack of a semiotic or discursive avenue that would enable them to push beyond the sole construction.

The Explorers', Dissectors' and Calculators' work forms are in compliance with the Geometry-II paradigm. The Explorers' work differs from Constructors by their heuristic use of constructions to survey the variety of possible figures according to the constraints imposed. That is why their work is viewed as compliant to the Geometry-II paradigm and complete. The Dissectors' form of work is confined to the semiotic-discursive plane and is based on exploratory work based on decomposition and rearrangement of figures but without the use of any drawing tools. In contrast, the working form of Calculators is confined to the semiotic-instrumental plane and oriented towards calculations with the creation of formulas and symbolic artifacts without any attempt to justify their statements.

Calculators' work form poses the problem of the exact nature of the paradigms involved in work which concerns most of the students. In general, pre-service students produced geometric work using various methods that appear to be compliant with a given paradigm, but they also introduced rules and practices that are not clearly part of any geometric paradigms but rather refer to expectations resulting from the academic practice of geometry. Almost all the teacher students think that all the geometric shapes they are asked to study must have some particular outlines and properties. And they purposefully assigned specific properties to the quadrilateral (right angle, parallel sides...). Consequently, their work cannot be considered to be valid. This suggests that there may be a sort of scholarly geometric paradigm which interferes with geometrical paradigms.

The geometric work forms characterization presented in this paper extends and complete the one we were able to give in Kuzniak and Rauscher (2011), which was essentially based on geometric paradigms. It also furthers the *a priori* classification of Duval (2005) which mainly takes into account the role of visualization in the type of iconic or non-iconic validation brought by junior high school students. In his classification, Duval introduces four forms and distinguishes too forms of Constructors and Surveyors. Our Dissectors' form is close to what he calls the Inventors-Tinkerers in that it relies on the search for properties with figure reconfigurations. The only form that we have not clearly found in our present study is that of Botanists (to Duval), which is based on the sole use of perception and is generally unproductive (Kuzniak and Rauscher, 2011). The present study also takes into account the artifacts and numbers used in the geometric work. This is how we have been able to highlight the categories of Explorers and Calculators. Moreover, unlike Duval's study, our study is based on the actual work done by the students, which leads us to consider their difficulties especially in relation to the discursive genesis as well as the possible bounces that allow

them to achieve the task. We have thus shown the importance of the school paradigm in understanding the students' results.

The influence of training on forms of geometric work needs also to be addressed. Kuzniak and Rauscher (2011) shown that student's use of the Geometry-I and Geometry-II paradigms depends on the level of training. Pre-service teachers generally work more in Geometry I at the beginning of their training and in Geometry II at the end. Our study took place at the end of the year after significant reminders of properties and operating rules of Geometry-II. For us, this does not alter the forms of geometric work themselves, which we consider stable, while the students' personal work is prone to change during the training process. One of the major teaching objectives is precisely to push this personal work towards forms of geometric work in affordance with the institutional expectations.

5. Conclusion and perspectives

This study identified and described five geometric work forms derived from MWS theory. It further outlined three criteria for evaluating the work actually produced in each of these forms.

Compliance by a work process to a certain paradigm indicates the accordance of its rules and practices, which in this study are specific to Geometry-I or Geometry-II. This is the case for the role and place of the figure, which may or may not serve as support for measurement or the exploration of the different solutions.

Completeness of the mathematical work requires mobilization of all forms of geneses in the MWS and the associated vertical planes to avoid certain blockages and facilitate a rebound of the work. For Dissectors, that would be achieved through the use of tools, such as drawing tools, to perform a heuristic exploration work.

Correctness of the work ensures the mathematical accuracy of results not based on false theorems in action or wrong constructions methods. However, some academic practices may occasionally give rise to a "classroom" paradigm that violates certain mathematics rules and makes the mathematical work incorrect.

Within the MWS theory, examination of these characteristics is based on a cognitive task analysis methodology divided into three steps. Top-down analysis step reveals the sequence of episodes and actions undertaken by students, while bottom-up analysis seeks to unveil the operating logic developed during the realization of the task. This analysis is based on the different tools of MWS theory, particularly its diagram. The third step start with the results from the top-down and bottom-up analyses and characterizes the work according the three evaluation criteria.

Beyond the field of geometry, we think that these three criteria, articulated with the tools of the MWS theory and our method of analysis, would be useful to evaluate work produced by students in other mathematical domains. Such studies would also be able to clarify the effect of the choice of particular tasks on mathematical work form in various institutional contexts.

Furthermore, we found that the students produce geometric work whose methods appear to be creative and various. Yet, their results are often wrong, which raises the question of the validity of this work from a mathematical perspective. This suggests that students have developed practices that conflict with the knowledge expected at this level of training. How do we ensure that the student are able to verify whether the work he or she produces is correct? Authors such as Balacheff

(2013) place the notion of control system at the core of a mastered approach to geometric reasoning based on semiotic tools, individual actions and problem classes to be solved. In the same vein, Arzarello and Sabena (2011), also introduce the idea of control to account for the fact that students make certain decisions about the selection and utilization of resources. According these authors, the control may be semiotic or theoretic according to the nature of the resources they select and implement. In our case, it is possible to envisage controls that would be semiotic, technological or theoretical depending on the type of tools of the epistemological plane used. These controls would concern the validity of statements and adequacy of the different tools selected.

In conclusion, it seems to us important to draw upon the results of this study and its description of the forms of geometric work and devise methods and developments that would unlock students' blockages and provide avenues for progress. Furthermore, it seems to us desirable to examine how the three criteria applied in the study would be useful in assisting future teachers in examining their work, reflecting on their mathematical knowledge, or becoming aware of their shortcomings, misconceptions, etc. This recommendation addresses the need to further provide students with the control means to master their geometric work forms and make them compliant, complete and correct. Furthermore, we surmise that, beyond their experience of geometry work while in training, these students will find these criteria very useful for their practice in teaching geometry by helping them to combine geometrical and pedagogical knowledge.

Acknowledgment

This is a post-peer-review, pre-copyedit version of an article published in *Educational Studies in Mathematics*. The final authenticated version is available online at: <http://dx.doi.org/DOI:10.1007/s10649-020-10011-2>.

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