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# From abstraction and indiscernibility to classification and types: revisiting Hermann Weyl's theory of ideal elements

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From abstraction and indiscernibility to  
classification and types:  
revisiting Hermann Weyl's theory of ideal  
elements

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## Contents

<b>1</b>	<b>Introduction. Essentialist and existentialist viewpoints on Types</b>	<b>2</b>
<b>2</b>	<b>Abstraction and Ideality: from Peano to Weyl</b>	<b>3</b>
2.1	<i>Peano's school's research programme on abstraction.</i>	3
2.2	<i>From Peano to Weyl: abstracta and ideal elements.</i>	5
2.3	<i>Monadic indiscernibility versus Relational indiscernibility.</i>	6
2.4	<i>Weyl's creative definitions: indiscernibility and ideal elements.</i>	7
2.5	<i>Which set-theoretic interpretation for Creative definitions ?</i>	7
<b>3</b>	<b>The notion of type</b>	<b>9</b>
3.1	<i>The orthogonality relation induced by a relation.</i>	9
3.2	<i>Orthogonality operators induced by a relation.</i>	9
3.3	<i>Types</i>	11
<b>4</b>	<b>Back to abstraction and classification</b>	<b>12</b>
4.1	<i>A type-oriented interpretation of Weyl's abstraction operators.</i>	13
4.2	<i>Reading abstraction through types: Classificatory and philosophical stakes</i>	14
<b>5</b>	<b>Conclusion</b>	<b>17</b>

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# 1 Introduction. Essentialist and existentialist viewpoints on Types

The notion of *Type* was introduced at the very beginning of 20th century by Bertrand Russell in [Russell, 1903] (Appendix B: “The Doctrine of Types”). The new fregean frame for Logic, in its set theoretical format, just happened to be proved inconsistent. Following Russell’s diagnosis, the fault originates in the consideration of mathematical objects as if they had an existence *per se*, a *pure* existence. Russell’s aim in introducing type theory was therefore to rebuild logic by rooting the existence of objects in *a kind of existence*, in other words *an essence* (their type); hence rejecting out of the scope of existence those individuals with no essence (meaningless, pseudo existing objects).

Four decades later, similar considerations led Alonzo Church to introduce types in his Lambda-calculus, a theory today considered as one of the first programming languages ever designed [Church, 1940]. In “Simply typed Lambda-calculus” (but also in the more powerful typing systems elaborated later on), it is the building of *programs* (lambda-terms: acting, behavioural individuals) that is tamed by the discipline of types. In that context, once again, the external norm introduced by types aims at excluding meaningless – untypable – individuals<sup>1</sup>.

It notably leaves aside the ones inducing infinite computations – the dynamical version of logical paradoxes –, hence “taming” the computational dynamic.

In the same field of Theoretical Computer Science, an alternative approach to the notion of type emerged however later on. Following the Propositions-as-Types point of view (i.e. the Brouwer-Heyting-Kolmogorov semantics of proofs revisited by the proof-as-programs viewpoint initiated by the Curry-Howard correspondence [Howard, 1980]) a given type is seen as some given set of programs, namely the set of all programs selected by the external norm of the typing discipline. With this point of view (types as sets of programs), the question naturally arises to characterise *intrinsically* which sets of programs are types, i.e. characterising them *from an existentialist viewpoint*, by considering *all pure programs* and their interactions in the computational process.

As existence appears rooted into essence in the first approach of the notion of type, this second point of view is naturally qualified as “essentialist” (as William Tait does, for instance in *Theory of Types and Natural Deduction*, the chapter 4 of [Tait, 1990], p. 65). For this alternative approach, existence precedes essence. Better even: essence is deduced from existence, deduced *a posteriori* from the behaviour (tell me how you act, I shall tell you who you are), a reason why it may be qualified as “non essentialist” or “existentialist” (as Jean-Yves Girard does, for instance in [Girard, 1990]).

In that latter existentialist point of view, a type describes a collective, common, behaviour so to speak: a set of acting individuals (programs) having, through the computational process, a similar, or comparable, behaviour (at

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<sup>1</sup>Actually, non typable terms are not always meaningless, especially when considering *simple types*. In Church’s times, the (blurred) delimitation between non-typable and paradoxal (in the logical sense mentioned further) was not clear (in particular, it’s only with Tait’s and Girard’s work that one could understand that lambda-terms of the shape  $(t)t$  may be non problematic – [Tait, 1966] and [Girard, 1971]).

least with some respect) in their interaction with their (pure) peers<sup>2</sup>.

Concretely, existentialist types are defined through a *closure by bi-orthogonality* operation (presented in section 3), where the involved notion of orthogonality<sup>3</sup> is relative to some given binary relation. In the particular case of theoretical computer science, the considered relation depends on the interactive dynamic<sup>4</sup> In this work, we will however consider the methodology from a more general point of view, i.e. for any given binary relation. From that set theoretical point of view, this operation of closure by bi-orthogonality provides a general tool for classification and the concept of type becomes a classifying notion.

The first classificatory notions emerged in late 19th century in the context of the logical investigations by Peano's school on *abstraction*. The historical and theoretical thread which links abstraction to classification (by means of the notions of *equivalence relation*, *equivalence classes* and *quotientation* of a set by an equivalence relation, drawn by Peano's school) is well known and well documented. Among the abundant literature on the history of Peano's "Definitions by abstraction" and Russell's "Abstraction principles" (the new name later on proposed and popularized by Russell for the former), one could refer in particular to [Consuegra, 1991] (whose first chapters underline the importance and the anteriority of the Peano's school investigations about abstraction and set theoretical interpretations of abstraction<sup>5</sup>) and [Mancosu, 2016] (whose Part 1 in entirety devoted to the history of Abstraction theory, from 19th century – and even earlier – to mid 20th century, in logic and in the mathematical practice).

In this work, we aim at showing that the same thread linking *abstraction* to the notion of *class* extends to the (more general) notion of *type*. Moreover, we advocate that such an extension is needed to understand the generalisation of Peano school's theory of abstraction that Hermann Weyl proposed in the first decade of 20th century and which roots ideals elements à la Hilbert into relational indiscernibility.

## 2 Abstraction and Ideality: from Peano to Weyl

### 2.1 Peano's school's research programme on abstraction.

Peano and his group launched around 1880 a research programme aiming at establishing a typology of definitions (as they actually occur in the practice of

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<sup>2</sup>The methodology is for example used in [Girard, 2001], [Krivine, 2001], [Highland-Schalk, 2003]. For a more theoretical/conceptual focus on the methodology itself : [Naibo-Petrolo-Seiller, 2016].

<sup>3</sup>The notion of *orthogonality* can be seen as a formulation of Garrett Birkhoff's *polarities*. In the sections 1., 5., 6. and 7. of chapter IV: "Complete lattices" of [Birkhoff, 1948], bridges are built between 1/ Closure operators (section 1), 2/ Orthogonality (Polarity in Birkhoff's terminology, section 5), 3/ Galois connections (section 6), starting from the Lattices viewpoint. Thanks to Alexandre Miquel for the reference and enthusiast discussions.

<sup>4</sup>Actually, the closure by bi-orthogonality methodology has in contemporary Logic, a large diffusion which exceeds the field of computational types. See for instance [Okada, 1998] where a completeness result for cut-free provability is reached by means of a notion of "fact", which rests on the same closure by bi-orthogonality methodology.

<sup>5</sup>In the sequel, we thus continuously refer to "the Peano's school approach", even if the philosophical literature on abstraction tends to promote Frege and Russell, because of the role they played in the logicist investigations about second order abstraction - see section 4

mathematicians along centuries), using the precise linguistic tools offered by the emerging formal logic. The thread of their investigations gradually led them to bring out (as candidates for a special kind of “definition”) statements of the form:

$$f_R(x) = f_R(x') \Leftrightarrow xRx',$$

where  $f_R$  is a newly introduced unary *function constant* and  $R$  is a binary predicate<sup>6</sup> satisfying three properties (for which they eventually coined the terms reflexivity, symmetry, and transitivity, after some terminological hesitations) characterising what they progressively called, as we still do: *equivalence relations*<sup>7</sup>. Peano’s group proposed to deem axioms of the form above as definitions and coined them “*definitions by abstraction*”<sup>8</sup>.

A simple, popular example of such definitions “by abstraction” is that of the *direction of a line*. One starts by considering the binary relation  $\parallel$  (“is parallel to”) over lines defined by  $x \parallel x' \text{ iff}_{\text{def}} \text{ the lines } x \text{ and } x' \text{ have all their points in common or no points in common}$ . One then introduces the new lexical element  $f_{\parallel}(x)$  (to be read: “the direction of line  $x$ ”) satisfying  $f_{\parallel}(x) = f_{\parallel}(x') \Leftrightarrow x \parallel x'$ ; i.e. the direction of the lines  $x$  and  $x'$  are the same if and only if  $x$  and  $x'$  are parallel.

The method described by the Peano’s school under the name “definition by abstraction” actually covered numerous examples which can be found previously in the history of mathematics<sup>9</sup>. Used as far back as in Euclid’s approach of rationals (abstracted from *ratios* comparison), the method happened however to gain a wide, increasing spread in the mathematical practice of 18th and overall 19th centuries. Among several mathematical pursuits, one may cite – again – *directions* abstracted from the relation of parallelism between lines; *shapes* abstracted from topological invariances; von Helmholtz’s *weights, brightness, pitch of tones*; and neither last, nor least, *cardinal numbers* abstracted from bijectibility between sets, a.k.a. *Hume’s principle*<sup>10</sup>.

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<sup>6</sup> To lighten the notation, we write  $xRx'$  to denote sentences  $R[x, x']$  having exactly two free variables  $x$  and  $x'$ .

<sup>7</sup> Following [Mancosu, 2016], p.22, “[...] it was in the Peano school that for the first time the three properties characterizing an equivalence relation were assigned a name. It was with Padoa 1908 that such relations acquired the characterizing name of ‘relazione egualiforme’ defined as a relation that satisfies reflexivity, symmetry, and transitivity”. Nevertheless (ibid. p. 88, footnote 57) “the explicit use of notions such as reflexivity, symmetry and transitivity in the Peano school seems to originate with Vailati (1892) and De Amicis (1892). Vailati in 1892 claims originality for introducing the word ‘reflexivity’. De Amicis also credits Vailati with the introduction of ‘reflexivity’ and both credit de Morgan with the introduction of ‘transitivity’. De Amicis coined ‘convertible’ [conversivo] for what we call ‘symmetric’ but his terminology did not catch on. Symmetric, in this sense, was introduced by Schröder in 1890”.

<sup>8</sup> Following [Mancosu, 2016], p. 2 and p. 13, the term *definitions by abstraction* appeared in print in 1894, but it was used earlier by members of the Peano school, at least from 1888. In the first decade of 20th century, Russell favoured the “abstraction principle” terminology to name those axioms (to the detriment of the “definition by abstraction” terminology).

<sup>9</sup> For a broad historical overview, see [Mancosu, 2016], Chapter 1: “The mathematical practice of definitions by abstraction from Euclid to Frege (and beyond)”.

<sup>10</sup>In §63 of his *Grundlagen*, Frege cites Hume as an ancestor of this idea, meanwhile developed by Cantor. The terminology *Hume’s principle* seems however to come from [Boolos, 1987]. For a stimulating detailed overview of the genesis of the treatment of numbers by Dedekind, Cantor and Frege, see [Tait, 1996].

A large part of the early philosophical debates about definitions by abstraction comes from the fact that, semantically, the codomain of  $f_R$  (whose elements are considered as the *abstracta*) is not determined by the new “definitional” axiom (favorizing interrogations about the ontological status of the thus potentially new entities : the *abstracta*). Once a minimum of set theory is assumed, a canonical solution is of course *to reduce abstraction to classification* by interpreting systematically  $f_R$  as the operator  $[\cdot]_R$  associating to any element its equivalence class<sup>11</sup> (thus choosing the quotient of the domain by  $R$  as the codomain of  $f_R$ )<sup>12</sup>. In some cases, an ontologically less costly variant of that classificatory solution (an Ockhamian solution to abstraction, so to speak) may consist in choosing a representative in each equivalence class (even if choosing is not at all an innocuous operation, in general...).

## 2.2 From Peano to Weyl: *abstracta and ideal elements*.

In 1910, H. Weyl generalizes Peano’s theory of “Definitions by abstraction”, by proposing the theory of what he coins “Creative definitions”, see [Weyl, 1910]. In [Weyl, 1927], he gives a more systematic presentation of the topic – a presentation that he will improve again (notably by giving complementary examples) in *Philosophy of Mathematics and Natural Sciences*, the augmented, revised, and translated in english edition of his 1927 book [Weyl, 1949]. His aim, with the concept of “creative definitions”, is not only to broaden the Peano’s school typology in order to cover examples overstepping definitions by abstraction<sup>13</sup>, but also to give a precise account of the Hilbertian process of introducing *ideal elements*. For sake of brevity and simplicity, we will not present the general form of “Creative definitions”. While Weyl considers creative definitions induced by  $k + 2$ -ary relations for any integer  $k$ , we will focus on the case where  $k = 0$ , i.e. creative definitions induced by binary relations<sup>14</sup>.

Even if Weyl himself does not conceptually justify his ideas in terms of *indiscernibility* and does rarely use that terminology (he contents himself with underlining how the notion of creative definitions fits with the definitional practices of mathematicians, through a list of specific instances), it is very clarifying to present his creative definitions by introducing the notion of *relational indiscernibility*.

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<sup>11</sup>The notion of *Equivalence class* was brought out by Mario Pieri, Cesare Burali-Forti and Alessandro Padoa, within their attempts to reformulate “definitions by abstraction” as “nominal definitions”, see [Consuegra, 1991]. Note that the “solution” interpreting  $f_R(x)$  as  $[x]_R$  is more appropriately described as introducing a binder – which in that particular case happens to be the binder corresponding to set formation:  $\{y; xRy\}$ , rather than a function constant  $f_R$ . About abstraction principles formulated with binders, see [Pollard, 1998] and [Tennant, 2017].

<sup>12</sup>Following [Mancosu, 2016], the first systematic and complete exposition of the partitioning/quotienting discipline in a mathematical textbook seems to occur only in the late 1920s (probably for the first time in van der Waerden’s *Abstrakte Algebra* [Van der Waerden, 1930], in the § 5, entitled “Klasseneinteilung. Äquivalenzrelation”). Nevertheless, “the technical details were already clear in the 1910s” for Russell, who, from 1902/1903, pleads for interpreting “abstracta” systematically by equivalence classes.

<sup>13</sup> The main examples presented or cited by Weyl are: *circles* in planar geometry, the notion of *function* over reals, *points at infinity*, *imaginary* elements in geometry, Kummer’s *ideal numbers*.

<sup>14</sup>We nevertheless believe this general case can be reduced to our binary setting or a simple generalisation thereof

### 2.3 *Monadic indiscernibility versus Relational indiscernibility.*

In the philosophical literature, the word *indiscernibility* is frequently used to qualify what would be better called *universal, absolute indiscernibility*, i.e. indiscernibility of  $x$  and  $x'$  from any possible viewpoints. That binary predicate (*absolute indiscernibility*) is usually paraphrased in second order monadic predicate logic as:

$$\forall P (Px \Leftrightarrow Px')$$

(see for instance the entry “Identity of Indiscernibles” of *The Stanford Encyclopedia of Philosophy*, [Forrest, 2016]).

In what follows, we will leave aside the idea of *absolute* indiscernibility, limiting thus ourselves to a more pedestrian notion, namely indiscernibility with respect to a given piece of first order language: *relative* (i.e. not absolute) indiscernibility. At first sight, this simply amounts to avoid the use of second order quantification in the definition of the indiscernibility binary predicate. However, if one does so (starting thus from the standard definition of absolute indiscernibility given above – namely  $\forall P (Px \Leftrightarrow Px')$ ), the resulting concept for, say, a given unary predicate  $P_0$  (indiscernibility of  $x$  and  $x'$  relatively to  $P_0$ , namely  $P_0x \Leftrightarrow P_0x'$ , that we could note  $x \sim_{P_0} x'$ ), happens to be particularly weak at least in terms of *classification*. If we consider a realization  $\mathfrak{m}$  of the language including  $P_0$ , then  $\sim_{P_0}$  is interpreted by an equivalence relation over the domain of  $\mathfrak{m}$  which partitions it into at most two classes. In other words, the indiscernibility predicate induced by  $P_0$  can create at best a bi-partition of the domain, i.e. it creates a classification à la Porphyry : the weakest kind of classification. Although the logical turn of the late 19th century overtook the notion of “property” in favour of the more general notion of  $n$ -ary predicate/relation, later philosophical investigations about the notion of *indiscernibility* essentially kept approaching that concept from the point of view of *monadic* predicate logic.

When one considers the indiscernibility induced by a *binary* relation  $R$  over a set  $X$ , a basic observation is that  $R$  induces over  $X$  *two* indiscernibility (binary) predicates. We will note them  $\overset{\text{td}}{\sim}_R$  and  $\overset{\text{tr}}{\sim}_R$ . They are defined by :

$$\begin{array}{ll} x \overset{\text{tr}}{\sim}_R x' \Leftrightarrow_{\text{def}} \forall y \in X (xRy \Leftrightarrow x'Ry) & x \overset{\text{td}}{\sim}_R x' \Leftrightarrow_{\text{def}} \forall y \in X (yRx \Leftrightarrow yRx') \\ \text{(we say: } x, x' \text{ are 'equi-targeters')} & \text{(we say: } x, x' \text{ are 'equi-targeted')} \end{array}$$

The *equi-targeted* and *equi-targeters* terminology evidently refers to graphical, sagittal representations of binary relations. The exponents  $\text{td}$  and  $\text{tr}$  conveniently recall **targeted** and **targeter**, respectively. The relations  $\overset{\text{td}}{\sim}_R$  and  $\overset{\text{tr}}{\sim}_R$  are *the two indiscernibility predicates* induced by the binary relation<sup>15</sup>  $R$ . We will write  $\overset{\text{t}}{\sim}_R$  to denote indifferently one of  $\overset{\text{tr}}{\sim}_R$  and  $\overset{\text{td}}{\sim}_R$ .

<sup>15</sup> Weyl considers the general case where  $R$  is a  $k+2$ -ary relation. *Indiscernibility predicates* are then given as the family of formulas:

$$\forall y_0 \dots \forall y_i \forall y_{i+2} \dots \forall y_{k+2} (Ry_0 \dots y_i x y_{i+2} \dots y_{k+2} \Leftrightarrow Ry_0 \dots y_i x' y_{i+2} \dots y_{k+2}).$$

## 2.4 Weyl's creative definitions: indiscernibility and ideal elements.

Weyl's view is that indiscernibility predicates are the true occasions for introducing *ideal elements* (Peano's *abstracta* being among them). Introducing a "creative definition" thus means to introduce an axiom of the form:

$$W_R(x) = W_R(x') \Leftrightarrow x \overset{\iota}{\sim}_R x'$$

where  $W_R$  ( $W$  for Weyl) is a newly introduced *function constant*<sup>16</sup> and  $R$  is any binary relation whatever.

As both indiscernibility predicates induced by  $R$  are equivalence relations over  $X$  (for any  $R$  whatever), those "ideal elements" could well be seen just as standard, usual *abstracta* à la Peano (abstracted from  $\overset{\text{td}}{\sim}_R$  – or from  $\overset{\text{tr}}{\sim}_R$  as well). Weyl actually defends a more accurate but dual view, according to which Peano's *abstracta* are but particular cases of *ideal elements* inducible from indiscernibility. Indeed, equivalence relations appear to be exactly the binary relations which *coincide* with the indiscernibility predicates they induce, i.e.  $R$  is an equivalence iff  $\overset{\iota}{\sim}_R = R$ . Hence, an *abstractum* introduced by a definition by abstraction is but an *ideality* introduced by a creative definition, in the very special case, proper to equivalences, where the indiscernibility predicate  $\overset{\iota}{\sim}_R$  collapses with the relation from which it was induced (namely  $R$  itself). Beyond the simple observation that definitions by abstraction are special cases of creative definitions, Weyl overall insists on the "finitist" specificity of definitions by abstraction among creative definitions: they tame the complexity inherent to indiscernibility predicates. Indeed, as an equivalence  $R$  allows to replace the indiscernibility predicates it induces by  $R$  itself (as  $x \overset{\iota}{\sim}_R x'$  is equivalent to  $xRx'$ , inasmuch  $R$  is an equivalence), equivalences allow to get rid of the universal quantifier present in the definition of indiscernibility predicates.

## 2.5 Which set-theoretic interpretation for Creative definitions ?

Similarly to what has already been observed for the case of a simple definition by abstraction à la Peano-Russell, the codomain of the operator introduced by a "creative definition" is left undetermined by the axiom. Exactly as Peano's school proposed a classificatory, set-theoretic interpretation of definitions by abstraction (the canonical interpretation of *abstracta* by equivalence classes), one may want to investigate classificatory set-theoretic accounts of  $W_R$  in "creative"

<sup>16</sup>As Weyl does compare Peano's Definitions by abstraction to his own Creative definitions (in order to defend that the latter do generalize the former), he presents them in a format analogous to Peano's one, namely as axioms introducing a fresh *function constant*, as Peano does. But in the examples that he develops, it is clear that Weyl is actually tempted to describe those axioms as introducing instead a definite descriptor and, thus, a binder and a "copula" (even if he does not explicitly introduce specific notations). This would lead to introduce a fresh binder  $\theta_R$ , say, and to write  $\theta_R x.x \overset{\iota}{\sim}_R x'$  instead of  $W_R(x')$  in the axiom and a copula that one would be tempted to note  $\in$ . See footnote 18.



definitions<sup>17</sup> i.e. in those axioms of the form:

$$W_R(x) = W_R(x') \Leftrightarrow x \overset{t}{\sim}_R x'$$

where  $W_R$  is a newly introduced *function constant* and  $R$  a binary relation whatever.

A first way could simply consist in keeping the “usual” canonical interpretation, i.e. interpreting Weyl through Peano. After all, indiscernibility predicates are themselves equivalences, and one may well choose to interpret Weyl’s *idealities* as equivalence classes for indiscernibility (i.e. sets made of indiscernible elements and maximal for that property)<sup>18</sup>.

Since it treats indiscernibility as if it were any equivalence whatever, this first way of reading creative definitions (thus reading them à la Peano) does not get the most out of the deep concept of indiscernibility. Notably, as soon as one devotes attention to the correlations between *the properties of R* on the one hand and *the properties of the classifying operations* induced by the indiscernibility on the other hand (as Weyl himself does when he relates, in the case of equivalence relations, the decrease in logical complexity of indiscernibility predicates to the properties of the relation inducing them), a finer understanding of the classificatory process at hand is wanted. Our thesis is that the notion of type (to be defined in the next section) is the relevant tool to that effect. As we will see in section 4, a bridge from the concept of indiscernibility to the notion of type may indeed be built from the observation that *individuals are indiscernible iff they belong to exactly the same types*.

That second way – that we will follow from now on – thus consists in interpreting  $W_R(x)$  not as the equivalence class of  $x$  for  $\overset{t}{\sim}_R$  anymore, but (at least to start with) as the set of types (induced by  $R$ ) to which  $x$  belongs (the set of types of  $x$  w.r.t.  $R$ , as we will say). This amounts to canonically interpret Weyl’s creative definitions (the generalized formulation of Peano’s theory of abstraction) along “Abstraction Principles” of the form:

$$x \overset{t}{\sim}_R x' \Leftrightarrow \text{Set-of-Types}_R(x) = \text{Set-of-Types}_R(x'),$$

where  $R$  is any binary relation.

We will come back to that alternative interpretation of Weyl’s Abstraction Principles in section 4, thus after having defined and presented the notion of type – a task to which the coming section is devoted.

<sup>17</sup> Weyl explicitly says that one may well choose to favour a set-theoretic interpretation of his ideal elements. As an epigone of Hilbert, he nevertheless underlines that, from his proof-theoretic point of view, such an interpretation is not compulsory, just a matter of taste.

<sup>18</sup> Let us notice that, in [Pollard, 1998], Stephen Pollard (who formulates Abstraction principles using definite descriptors, hence binders, instead of function symbols, and introduce a “copula” – see our footnote 16) claims that to do so actually amounts to *reduce* Weyl’s theory of abstraction/ideality to a weak set theory, made of: 1/ a weak comprehension scheme limited to “instantiable” properties, i.e. properties  $P[x]$  such that  $P[t/x]$  is provable for some term  $t$  (following an argument due to Dummett, such a weak comprehension scheme is derivable as soon as one accepts the small piece of comprehension enough to construct equivalence classes), 2/ together with a weak extensionality axiom equalizing only sets defined by provably equivalent concepts differing only w.r.t. complementary parameters (i.e.  $A[x, \vec{y}_i]$  and  $A[x, \vec{z}_i]$ , such that  $A[x, \vec{y}_i] \leftrightarrow A[x, \vec{z}_i]$  is provable in the current theory, where the  $y_i, z_i$  do not occur free); indeed this extensionality axiom is just a particular case of a creative definition à la Weyl (indiscernibility w.r.t. the copula  $\in$ ).

### 3 The notion of type

We will now present the notion of type induced by a binary relation  $R$ . That notion may be defined for any binary relations between arbitrary sets and there are no reasons to restrict ourselves to relations  $R$  over a single set here. We thus chose to provide more general definitions based on relations between arbitrary sets  $X$  and  $Y$ . Doing so we will be able to keep covering the particular case of classifications induced by a relation over a single set – e.g. quotientation by an equivalence relation –, but also to cover the more general case of classifications where the criterion for classifying the elements of a set depends on another set. Anyway, from now on,  $R$  will by default denote a subset of  $X \times Y$ , and we will follow the following notational conventions:  $A, A' \dots$  denote subsets of  $X$ ;  $B, B'$  denote subsets of  $Y$ ;  $x, x' \dots$  denote elements of  $X$ , and  $y, y' \dots$  denote elements of  $Y$ . The developments in this section are mainly technical, and we will discuss their consequences in the next section.

#### 3.1 The orthogonality relation induced by a relation.

The definition of types is based on a so-called *orthogonality relation* induced by  $R$ .

**Definition 1.** The right *orthogonality relation* induced by  $R$ , is the binary relation  $\perp_R \subseteq \mathcal{P}(X) \times \mathcal{P}(Y)$  defined by:  $A \perp_R B \Leftrightarrow_{\text{def}} A \times B \subseteq R$ . The relation  $A \perp_R B$  can be read both as “ $A$  is (left-)orthogonal to  $B$ ” and as “ $B$  is (right-)orthogonal to  $A$ ”.

Inasmuch only one relation  $R \subseteq X \times Y$  is involved, we frequently leave implicit the reference to  $R$  in those notation. In most situations, the context suffices to makes ambiguities disappear.

The following easy lemma characterises the orthogonality relation element-wise.

**Lemma 2.** Given two subsets  $A \subseteq X$  and  $B \subseteq Y$ ,

$$A \perp B \text{ if and only if } \forall x \in A, \forall y \in B, xRy.$$

#### 3.2 Orthogonality operators induced by a relation.

While the orthogonality relation defines a predicate over the product set  $\mathcal{P}(X) \times \mathcal{P}(Y)$ , it also induces two functions  $(\cdot)^\perp : \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$  and  ${}^\perp(\cdot) : \mathcal{P}(Y) \rightarrow \mathcal{P}(X)$  defined using the natural ordering of subsets induced by inclusion. Those functions will be called the orthogonality operators.

**Definition 3.** Given a subset  $A \subseteq X$ , we define the (right) orthogonal  $A^\perp$  of  $A$  as the largest subset of  $Y$  which is right-orthogonal to  $A$ , i.e.  $A^\perp =_{\text{def}} \max\{B \in \mathcal{P}(Y) \mid A \perp B\}$ . Similarly, the (left) orthogonal  ${}^\perp B$  of a subset  $B \subseteq Y$  is defined as the largest subset of  $X$  which is left-orthogonal to  $B$ , i.e.  ${}^\perp B =_{\text{def}} \max\{A \in \mathcal{P}(X) \mid A \perp B\}$ .

The fact that the notions of left- and right- orthogonal of a subset are well-defined is based on the following property, which is a direct consequence of the element-wise characterisation of the orthogonality relation.

**Lemma 4.** *We consider subsets  $A \subseteq X$ ,  $B \subseteq Y$  and  $B' \subseteq Y$ . If  $A \perp B$  and  $A \perp B'$ , then  $A \perp B \cup B'$ .*

Once again, the definition can be understood element-wise<sup>19</sup>.

**Lemma 5.** *Given subsets  $A \subseteq X$  and  $B \subseteq Y$ :*

$$\begin{aligned} A^\perp &= \{y \in Y ; \forall x \in A \ xRy\}, \\ {}^\perp B &= \{x \in X ; \forall y \in B \ xRy\}. \end{aligned}$$

When considering iterated applications of the orthogonality operators, and when the context will be clear, we will write abusively  $A^{\perp\perp}$  instead of  ${}^\perp(A^\perp)$  and  $B^{\perp\perp}$  instead of  $({}^\perp B)^\perp$ . Notice however that in the particular cases when  $X$  and  $Y$  overlap (and notably when  $X = Y$ ), the notation  $A^{\perp\perp}$  becomes ambiguous, as  ${}^\perp(A^\perp) \neq ({}^\perp A)^\perp$  in general.

*Remark 6.* We can check that left- and right- orthogonality operators are exchanged by considering the relation  $R^{-1} = \{(y, x) \in Y \times X \mid (x, y) \in R\}$  instead of  $R$ . As a consequence, it is enough to state and prove the properties of the right-orthogonality operator: by symmetry, the same property will hold for the left-operator. We thus now give statements about “orthogonal operators” by stating them for right-orthogonality.

**Proposition 7.** *Given  $A \subseteq X$ ,  $A \subseteq A^{\perp\perp}$ .*

*Proof.* This is a direct consequence of the fact that  $A \perp A^\perp$  and the definition of  ${}^\perp(A^\perp)$  as the maximal left-orthogonal to  $A^\perp$ .  $\square$

**Proposition 8 (Contravariance).** *The orthogonal operators are contravariant: given  $A \subseteq A' \subseteq X$ , one has  $A'^\perp \subseteq A^\perp$ .*

*Proof.* Let us pick  $y \in A'^\perp$ . By definition, for all  $x \in A$ ,  $x$  also belongs to  $A'$ , hence  $xRy$ . So  $A \perp A'^\perp$ , i.e.  $A'^\perp$  is right-orthogonal to  $A$ . Since  $A^\perp$  is defined as the maximal subset of  $Y$  which is right-orthogonal to  $A$ , this implies that  $A'^\perp \subseteq A^\perp$ .  $\square$

**Corollary 9.** *Given  $A \subseteq X$ ,  $A^{\perp\perp\perp} = A^\perp$ .*

*Proof.* We have that  $A \subseteq A^{\perp\perp}$  by Proposition 7, thus Proposition 8 allows us to conclude that  $A^{\perp\perp\perp} \subseteq A^\perp$ . The converse inclusion is given by Proposition 7 applied to the set  $A^\perp$ .  $\square$

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<sup>19</sup>So defined, the “orthogonality” and “orthogonal (operator)” terminology echoes and generalises the standard notion for vector spaces. In that latter case,  $A^\perp$  denotes “the orthogonal of a subset  $A$  of a vectorial space  $X$ ” defined as the sub-vectorial space whose elements are all the vectors orthogonal to all the vectors of  $A$ . In that particular case, vectors  $x, x'$  are said orthogonal w.r.t. to a given bilinear form  $(. | .)$  defined over  $X \times Y$  (notation  $x \perp x'$ ), when  $(x | y) = 0$ . So the notions we are considering are just generalising those ones, when  $x \perp x'$  means  $xRx'$  for any relation  $R \subseteq X \times Y$ , where  $X, Y$  also are arbitrary sets. Though the notion has a long genealogy in the history of mathematical practice (from Euclid), the theoretical focus on “orthogonality” in this broad sense seems to be notably due to Hilbert (and then Weyl).

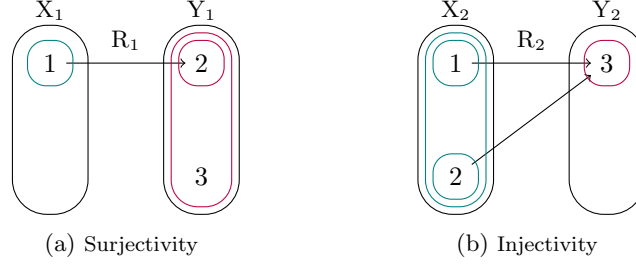


Figure 1: Illustration of the counter-examples from remark 11.

**Proposition 10.** Let  $I$  be a set, and  $\{A_i\}_{i \in I}$  an  $I$ -indexed family of subsets of  $X$ , i.e.  $\forall i \in I, A_i \in \mathcal{P}(X)$ . Then:

1.  $\left(\bigcup_{i \in I} A_i\right)^\perp = \bigcap_{i \in I} A_i^\perp$ ,
2.  $\bigcup_{i \in I} A_i^\perp \subseteq \left(\bigcap_{i \in I} A_i\right)^\perp$ .

However,  $\left(\bigcap_{i \in I} A_i\right)^\perp \not\subseteq \bigcup_{i \in I} A_i^\perp$  in general.

*Proof.* To prove the first item, we take  $y \in \bigcap_{i \in I} A_i^\perp$  and  $x \in \bigcup_{i \in I} A_i$  and show that  $xRy$ . By definition, there exists  $i_0 \in I$  such that  $x \in A_{i_0}$ . On the other hand,  $y \in A_i^\perp$  for all  $i \in I$ . Thus  $y \in A_{i_0}^\perp$ , and therefore  $xRy$  since  $x \in A_{i_0}$ . For the second item, as  $\bigcap_{i \in I} A_i \subseteq A_i$  for all  $i \in I$ , we use the contravariance to conclude that  $A_i^\perp \subseteq \left(\bigcap_{i \in I} A_i\right)^\perp$  for all  $i \in I$ . Consequently,  $\bigcup_{i \in I} A_i^\perp \subseteq \left(\bigcap_{i \in I} A_i\right)^\perp$ . Finally, for the last assertion, one observes that this inclusion does not hold in general, by considering  $X = Y = \{1, 2, 3\}$ ,  $R = \{(2, 2)\}$ ,  $A_1 = \{1, 2\}$ , and  $A_2 = \{2, 3\}$ . One then checks that  $(A_1 \cap A_2)^\perp = \{2\}^\perp = \{2\}$ , but  $A_1^\perp \cup A_2^\perp = \emptyset \cup \emptyset = \emptyset$ .  $\square$

*Remark 11.* In general, the operator  $(\cdot)^\perp$  is neither surjective nor injective. For surjectivity, let  $X_1 = \{1\}$ ,  $Y_1 = \{2, 3\}$  and let  $R_1 \subseteq X \times Y$  be the binary relation  $R_1 = \{(1, 2)\}$  (depicted below): we have  $\emptyset^\perp = \{2, 3\}$  and  $\{1\}^\perp = \{2\}$ , hence neither  $\{3\}$  nor  $\emptyset$  are reached by  $(\cdot)^\perp$  (even if  $\emptyset \perp \{3\}$  and  $\emptyset \perp \emptyset$ ). For injectivity, let  $X_2 = \{1, 2\}$ ,  $Y_2 = \{3\}$ , and  $R_2 \subseteq X \times Y$  be the binary relation  $R_2 = \{(1, 3), (2, 3)\}$ : we have  $\emptyset^\perp = \{1\}^\perp = \{1, 2\}^\perp = \{3\}$ . Both counter-examples are illustrated in Figure 1; types are shown in purple, generating sets of tests are shown in blue.

### 3.3 Types

**Definition 12.** A (left-)type is a subset  $A \subseteq X$  in the image of the left-orthogonality operator, i.e.  $A \subseteq X$  is a type if and only if there exists  $B \subseteq Y$  such that  $A = {}^\perp B$ . We call  $B$  a *generating set of tests* for the type  $A$ .

*Notation 13.* We denote  $\mathcal{T}_l(R)$  the set of left-types, i.e.  $\mathcal{T}_l(R) = \{A \in \mathcal{P}(X) \mid \exists B \subseteq Y, A = {}^\perp B\}$ . We denote  $\mathcal{T}_r(R)$  the set of right-types, i.e.  $\mathcal{T}_r(R) = \{B \in \mathcal{P}(Y) \mid \exists A \subseteq X, B = A^\perp\}$ .

Again, we will omit to mention  $R$  when it is not ambiguous. Moreover, we will also omit to mention the “direction” (right or left) any time it is not relevant (we thus states a proposition without mentioning the direction and prove it for one direction, indifferently).

*Remark 14.* Reformulated with the terminology just introduced, remark 11 may be rephrased as: in general, some subsets of  $Y$  are not types, and a given type may well have distinct generators.

We saw that any set is included in its double-orthogonal. We will now see that types are exactly those sets for which the converse inclusion holds, i.e. those one which are equal to their bi-orthogonal closure<sup>20</sup>.

**Proposition 15.** *A subset  $B \subseteq Y$  is a type if and only if  $B = B^{\perp\perp}$ .*

*Proof.* Evidently, if  $B = B^{\perp\perp}$ , then  $B$  has a generating set of tests, namely  $B^\perp$  and it is therefore a type. We will now prove the converse.

If  $B$  is a type, there exists  $A \in \mathcal{P}(X)$  such that  $B = A^\perp$ . We already know that  $B \subseteq B^{\perp\perp}$  by Corollary 7. Let us show the converse inclusion :

$$\begin{aligned}
B = A^\perp &\Rightarrow B \subseteq A^\perp \\
&\Rightarrow A^{\perp\perp} \subseteq B^\perp && \text{(by Proposition 8)} \\
&\Rightarrow B^{\perp\perp} \subseteq A^{\perp\perp\perp} && \text{(by Proposition 8)} \\
&\Rightarrow B^{\perp\perp} \subseteq A^\perp && \text{(by Corollary 9)} \\
&\Rightarrow B^{\perp\perp} \subseteq B && \text{(as we assumed } B = A^\perp\text{).} \quad \square
\end{aligned}$$

**Proposition 16.** *Let  $A \subseteq X$ . The type  $A^{\perp\perp}$  is the smallest type including  $A$ .*

*Proof.* Indeed,  $A^{\perp\perp}$  is a type (since it has  $A^\perp$  as its generator) which is contained in any type containing  $A$ . To see this, let us pick a type  $A'$  such that  $A \subseteq A'$ . As  $A \subseteq A'$ , we have  $A^{\perp\perp} \subseteq A'^{\perp\perp}$  (by Proposition 8 used twice). Since  $A'$  is a type, this gives  $A^{\perp\perp} \subseteq A'$  by Proposition 15.  $\square$

To close this fast presentation, let us mention that the inclusion order over types has lattice structure (the infimum is given by intersection, the supremum by the bi-orthogonal closure of union, which is indeed a supremum because of proposition 16).

## 4 Back to abstraction and classification

We now come back to the task left uncompleted in the end of section 2: working out a canonical alternative set-theoretic interpretation for the abstraction operators  $W_R(\cdot)$  introduced by Weyl’s Abstraction Principles (Creative definitions), i.e. by axioms of the form  $W_R(x) = W_R(x') \Leftrightarrow x \overset{t}{\sim}_R x'$ , where  $W_R$  is a fresh unary function constant and  $R$  any binary relation whatever. Our tool for that task will be the bridge between indiscernibility and types mentioned in the end of section 2, namely the proposition which states that to be  $R$ -indiscernible (following  $\overset{td}{\sim}_R$ , resp.  $\overset{tr}{\sim}_R$ , as well) means belonging to exactly the same (right, resp. left) types induced by  $R$ .

<sup>20</sup>The bi-orthogonality operator is an example of the notion of *closure operator*.

#### 4.1 A type-oriented interpretation of Weyl's abstraction operators.

*Notation 17.* Let  $R \subseteq X \times Y$ ,  $x \in X$  and  $y \in Y$ . We introduce the following notations – replacing our former informal notation “Set-of-Types $_R(x)$ ”:

$$\mathcal{T}_R(x) =_{\text{def}} \{A \in \mathcal{T}_l(R) ; x \in A\} \quad \text{and} \quad \mathcal{T}_R(y) =_{\text{def}} \{B \in \mathcal{T}_r(R) ; y \in B\}$$

The proposition below suggests a type-oriented canonical reading of the abstraction operator  $W_R(x)$  (in “Abstraction principles” à la Weyl, i.e. creative definitions) as  $\mathcal{T}_R(x)$ .

**Proposition 18.** *For any  $y, y' \in Y$ , one has:  $\mathcal{T}_R(y) = \mathcal{T}_R(y') \Leftrightarrow y \stackrel{\text{td}}{\sim}_R y'$ .*

*Proof.* Let  $y, y' \in Y$ . We need to show that:

$$\forall B \in \mathcal{T}_r(R) (y \in B \Leftrightarrow y' \in B) \Leftrightarrow y \stackrel{\text{td}}{\sim}_R y'.$$

- For the left-to-right implication, we let  $x \in X$  and verify:

$$xRy \underset{\text{by def of } (\cdot)^\perp}{\Leftrightarrow} y \in \{x\}^\perp \underset{\substack{\text{by the main hyp. of } (\Rightarrow) \\ \text{instanciated with type } \{x\}^\perp}}{\Leftrightarrow} y' \in \{x\}^\perp \underset{\text{by def of } (\cdot)^\perp}{\Leftrightarrow} xRy'.$$

This shows that  $\forall x \in X, (xRy \Leftrightarrow xRy')$ , i.e.  $y \stackrel{\text{td}}{\sim}_R y'$ .

- For the right-to-left implication, we let  $B$  be a right type for  $R$ . By definition of types,  $B = A^\perp$  for some  $A \subseteq X$ . Then:

$$\begin{aligned} y \in B \Leftrightarrow y \in A^\perp &\underset{\text{by def. of } A^\perp}{\Leftrightarrow} \forall x \in A, xRy \underset{\text{if } y \stackrel{\text{td}}{\sim}_R y'}{\Leftrightarrow} \forall x \in A, xRy' \\ &\underset{\text{by def. of } A^\perp}{\Leftrightarrow} y \in A^\perp \Leftrightarrow y' \in B. \end{aligned} \quad \square$$

Compared to the usual canonical reading of abstracta  $W_R(x)$  as being the  $[x]_{\stackrel{\text{t.}}{\sim}_R}$  (the equivalence classes for  $\stackrel{\text{t.}}{\sim}_R$ , which collapse to  $[x]_R$  when  $R$  is an equivalence), the logical “order” of the new interpretation proposed ( $W_R(x)$  as being  $\mathcal{T}_R(x)$ ) may seem a high price to pay for a refinement. Indeed, if  $R$  is defined over  $X$ , the codomain of the operator  $\mathcal{T}_R(\cdot)$  is the set  $\mathcal{P}(\mathcal{P}(X))$ . We will now see, however, that, notably because types are closed by intersection, we can canonically interpret the abstraction operator as the function associating to  $x$  (actually to  $\{x\}$ ) its minimal type, namely  $\{x\}^{\perp\perp}$ , by proposition 16.

**Proposition 19.** *Types are closed by intersection, i.e. let  $R \subseteq X \times Y$  and  $\{B_i\}_{i \in I}$ , a non empty family of right types. Then  $\bigcap_{i \in I} B_i$  is a right type.*

*Proof.* By definition of types, there is a family  $\{A_i\}_{i \in I}$  of subsets of  $X$ , such that  $\{B_i\}_{i \in I} = \{A_i^\perp\}_{i \in I}$ . Then  $\bigcap_{i \in I} B_i = \bigcap_{i \in I} A_i^\perp = \left(\bigcup_{i \in I} A_i\right)^\perp$  (by remark 10). By definition, being the orthogonal to some subset,  $\bigcap_{i \in I} B_i$  is thus a type.  $\square$

**Proposition 20.** *Let  $x, x' \in X$ .  $\mathcal{T}_R(x) = \mathcal{T}_R(x') \Leftrightarrow \{x\}^{\perp\perp} = \{x'\}^{\perp\perp}$ .*

*Proof.* We start by the left-to-right implication. If  $\mathcal{T}_R(x) = \mathcal{T}_R(x')$ , then  $\bigcap \mathcal{T}_R(x) = \bigcap \mathcal{T}_R(x')$ . So  $\{x\}^{\perp\perp} = \{x'\}^{\perp\perp}$ , since, by proposition 16,  $\{x\}^{\perp\perp}$  and  $\{x'\}^{\perp\perp}$  are the smallest types to which  $x$  and  $x'$  belongs, respectively.

For the right-to-left implication, if  $\{x\}^{\perp\perp} = \{x'\}^{\perp\perp}$ , then  $\{x\}^{\perp\perp\perp} = \{x'\}^{\perp\perp\perp}$ . Thus  $\{x\}^\perp = \{x'\}^\perp$  (by corollary 9). Hence  $x \overset{tr}{\sim}_R x'$ . So  $\mathcal{T}_R(x) = \mathcal{T}_R(x')$ , by proposition 18.  $\square$

**Proposition 21.**  $\{x\}^{\perp\perp} = \{x'\}^{\perp\perp} \Leftrightarrow x \overset{tr}{\sim}_R x'$

*Proof.* Corollary of propositions 18 and 20.  $\square$

Proposition 21 finally invites us to interpret the abstraction operator  $W_R(x)$  (in “Abstraction principles” à la Weyl, i.e. creative definitions) as  $\{x\}^{\perp\perp}$ , i.e. the minimal type of  $\{x\}$ .

## 4.2 *Reading abstraction through types: Classificatory and philosophical stakes*

That subsection is devoted to compare the canonical interpretation that we just proposed for Weyl’s abstraction operators  $W_R(\cdot)$  (following which  $W_R(x)$  is the minimal type of  $x$ ) with the old, traditional one (following which  $W_R(x)$  is  $x$ ’s equivalence class for the indiscernibility relation induced by  $R$  – which collapses with  $x$ ’s equivalence classe for  $R$ , when  $R$  is itself an equivalence relation). We aim not only to underline in which respects the classifications induced differ, but also to draw the consequences of the specificities of the new interpretation over the philosophical pursuits about abstraction initiated by Peano, Frege, Russell and their followers.

As a prefatory remark, we would like to observe first that, in order to compare both canonical interpretations, one has no particular reason to come back to the particular case where  $R$  is a relation over a single set  $X$ . Indeed, from a classificatory point of view, the case where  $R \subseteq X \times Y$  with  $X \neq Y$  corresponds to the general situation where the classifying criterion to be used is *external* to the classified set (i.e., for example, a type  $B$  included in  $Y$  is generated by a *set of tests* included in  $A$  – see Definition 12) and not by an internal one as, for example, in the case of the quotientation of a set by an equivalence relation. So, considering arbitrary sets  $X, Y$ , simply corresponds to the more general classificatory frame where the criterion is not necessarily internal. Focusing on relations over a single set is necessary only if one considers and investigates the types induced by a relation satisfying particular properties (like reflexivity, symmetry etc) whose definition requires that the relation is over a single and the same set.

From a classificatory point of view, the main feature which distinguishes the type-oriented interpretation from the class-oriented one is that, in the former case, generally, an induced classification does not generates a partition of the domain of  $R$ . Actually, in general, neither the set of types, nor the set of minimal types, nor the set of minimal types induced by singletons do induce partitions

(as shows the following example: Let  $X_1 = \{1, 2\}$ ,  $Y_1 = \{3, 4\}$ , and  $R_3 \subseteq X \times Y$  be the binary relation  $R_3 = \{(1, 3), (1, 4), (2, 4)\}$ ; we have  $\{1\}^{\perp\perp} = \{1\}$  and  $\{2\}^{\perp} = \{1, 2\}$ ). The classifications induced by the type-oriented interpretation thus are *multi*-classifying ones (i.e. a same element generally belongs to different types).

That ability, for types, to classify multiply a given object impacts the philosophy of abstraction in several respects. To see this, let us first recall that the original philosophical debates which troubled Peano's circle about abstracta, were partly centred around considerations on the methodology of science (Are definitions by abstraction reducible to nominal definitions ?), partly semantical or ontological (What kind of objects are abstracta ? Are they *additional* entities, completely new w.r.t. the ones from which they are induced ?).

The traditional set-theoretical canonical solution (abstracta are equivalence classes) brings a clear answer to both pursuits. Concerning the ontological one, the answer is however equivocal, in a sense. In the one hand, sets may be considered as new entities (some set theory is needed), moreover well separated ones, since the set of equivalence classes forms a *partition* of the original set. But, in the other hand, that same fact may be considered as supporting in a way an ontological parcimony (i.e. the idea that the new, fresh entities are fresh only in appearance). Indeed, when one partitions a set  $X$ , the cardinality of the resulting partition cannot exceed the cardinality of  $X$  – an observation which leaves open the door for a representation of the abstracta by the original objects, hence to parcimony<sup>21</sup>.

By the way, as soon as for ontological pursuits, one wishes to target only partitions (separated new individuals, but not more numerous than before), let us notice that the condition defining equivalence relations are sufficient, but not at all necessary. When  $R$  is a relation over a single set  $X$ , the conditions on  $R$  which *characterize the quotientation discipline* are the ones which define collusions (whose definition is recalled below). More precisely, if, for any relation  $R$  over  $X$  (be it an equivalence or not), we denote by  $[x]_R$ , the set  $\{x' \in X ; xRx'\}$  (“the class of  $x$  for  $R$ ”), then the set of all classes, namely  $\{[x]_R\}_{x \in X}$ , designs a partition of  $X$  iff  $R$  is a collusion (the condition “ $R$  equivalence” is sufficient, but not necessary). Let us give the definition of collusions in the more general case where  $R \subseteq X \times Y$ . Those ones are the relations  $R$  which are simultaneously:

- *collusive*:  $\forall x, x' \in X (\exists y \in Y (xRy \wedge x'Ry) \Rightarrow \forall y \in Y (xRy \Rightarrow x'Ry))$ <sup>22</sup>,
- *total*:  $\forall x \in X \exists y \in Y xRy$ ,
- *surjective*:  $\forall y \in Y \exists x \in X xRy$ .

<sup>21</sup>In case one requires, in order to recognize that cardinality could only decrease in case of partition, that there exists an injection from the partition to the original set, the result then rests upon the Partition Principle, which says that when there is a surjection  $s : X \rightarrow Y$ , then there exists an injection  $i : Y \rightarrow X$ . The Partition Principle is provable in ZFC. The question whether it is equivalent to the axiom of choice, seems to be still an open problem of set theory (in any case, it was still the case in 1995, see [Higasikawa, 1995]). It should be interesting, from an historical point of view, to determine whether the Peano's school members included that observation about cardinality into their ontological debates about definitions by abstraction. De facto, even if the Partition principle has been first established in 1902 by [Levi, 1902], Burali-Forti early proposed, in 1896, an approximation of it (actually a wrong formulation, later criticised by Russell, in 1906). For a precise historical and technical presentation of the Partition Principle, see [Banaschewski & Moore, 1990].

<sup>22</sup>The collusivity property may be reformulated in terms of indiscernibility predicates by:  $R$  is collusive iff  $\forall x, x' \in X (\exists y \in Y (xRy \wedge x'Ry) \Rightarrow x \overset{\text{tr}}{\sim}_R x')$



Note that a collusion from  $X$  to  $Y$  may also be seen as a bijection between the partition of  $X$  (whose elements are the left types for  $R$ ) and the partition of  $Y$  (whose elements are the right types for  $R$ )<sup>23</sup>.

In the particular case where  $X = Y$ , i.e. when  $R$  is a relation over a single set  $X$ , the fact that collusions *characterize the quotientation discipline* (formulated above in terms of classes), may be also formulated in terms of types: the set of the (right, resp. left) types induced by  $R$  forms a quasi-partition (of  $Y$ , resp. of  $X$ ) if and only if  $R$  is a collusion<sup>24</sup>. A rather thorough study of collusions over a single set can be found in [Joinet, 2019].

Actually, the ontological parcimony remark continues to prevail for the minimal types oriented interpretation that we designed, so that it cannot be invoked to require that the set of abstracta form a partition or the original set, i.e. to renounce to non partitioning, *multi-classifying* classifications. Indeed, in the minimal types interpretation, since the set of generated types is indexed by  $X$ , the cardinal of that set of types could at most decrease: the function which, to  $x$  associates  $x^{\perp\perp}$  is surjective.

By the way, the philosophical investigations on abstraction initiated by the work of the Peano’s school members, Frege and Russell have been early extended to non partitioning, *multi-classifying* classifications. In particular, several classificatory notions that may be seen somehow as “weakened” forms of equivalence classes did play a central role in 20th century’s philosophy of logic. This is typically the case of the notion of *clique* (a.k.a. *similarity classe*, following Carnap’s later terminology [Carnap, 1928]) and of *maximal cliques*, whose definitions are recalled below. As we will see, they can be directly defined in terms of orthogonal operators and in terms of the notion of type.

Those notions were first designed and studied from 1914 by Bertrand Russell and from 1915 by a post-doc of his department, Norbert Wiener. Russell was attempting to find a total ordering of instants “behind” any partial order of events (aiming, so to speak, at “quotienting” any partial order by the symmetric, non transitive binary relation: “ $x$  is incomparable to  $x'$  w.r.t. the partial order”). Wiener, following a thread more clearly linked with the theory of abstraction originated in Peano’s school, was attempting to study abstraction in case “equivalences” are replaced by “fuzzy equivalences”, namely similarities with threshold features (his ultimate aim was a reconstruction of concepts from the sensorial subjective experience). Later on, cliques will play a central role in Carnap’s *Quasi-analysis* research programme (see [Leitgeb, 2007] and [Gandon, 2016])<sup>25</sup>.

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<sup>23</sup>Indeed, we can prove that the set of collusions from  $X$  to  $Y$  may be bijectively mapped onto the set of bijections between partitions of  $X$  and partitions of  $Y$ . The article [Joinet-Seiller, 202?] should appear soon.

<sup>24</sup>We say a *quasi* partition because the non emptiness of classes condition is not satisfied. In the standard definition of a partition, a subset belonging to it, is required to be *non empty*. This cannot apply inasmuch we two disjoint types: since types are closed by intersection,  $\emptyset$  has then to be a type – the minimal type. But of course, if we focus on types whose generators are singletons, as we did to interpret Weyl’s Abstraction principles,  $\emptyset$  cannot be one of them.

<sup>25</sup>One may mention that, in almost recent times, the notion of clique happened to have a second life in Logic, namely in Girard’s Linear Logic. “Coherent spaces”, by which the proofs of Linear Logic are denotationally interpreted, are indeed spaces of cliques for “similarity relations”. See [Girard, 1987] and also [Girard, 2004] where the considered “Coherent spaces” are themselves defined through a bi-orthogonality construction.

*Maximal cliques* do play for *similarity relations* (i.e. reflexive, symmetric, but not necessary transitive ones), the same role than *equivalence classes* do for *equivalence relations*. If one look at a similarity relation as an equivalence to which transitivity is missing, one sees that the lost of transitivity entails that, contrary to equivalence classes, cliques do intersect. As we show below, maximale cliques may be characterised directly in terms of types. Let  $R$  be a binary relation over a single set  $X$ .

**Definition 22.** A subset  $C \subseteq X$  is a *clique* for  $R$  iff  $\forall x, x' \in C \ xRx'$ . And a clique  $C$  is a *maximal clique* for  $R$ , when, moreover: for all  $A \subseteq X$ , if  $A \supsetneq C$ , then  $A$  is not a clique.

**Proposition 23.** Let  $R \subseteq X \times X$ , a reflexive relation over  $X$  and  $C \subseteq X$ .  
 $C$  is a maximal clique  $\Leftrightarrow C^\perp \cap {}^\perp C = C$ . If  $R$  is moreover symmetric, then:  
 $C$  is a maximal clique  $\Leftrightarrow C = C^\perp$ .

Though Russell's, Wiener's and Carnap's investigations on cliques induced by "similarity relations" propagate the interpretation of abstraction based on classes toward *multi-classifying classification*, one may underline that the approach of classifications based on types goes much further. A first point is that the type-oriented classifications is completely general : it does not require from the relation  $R$  inducing the types any specific property whatever: it directly catches relational indiscernibility for any relation.

## 5 Conclusion

The results presented above show that the notion of type could play a central role to renew and improve the logical and philosophical investigations on abstraction. They draw an appropriate framework to interpret set-theoretically Weyl's creative definitions, as they build a set-theoretical counterpart for the objects that one may want to make emerge from relational indiscernibility.

The fact that, doing so, one reduces – again – abstraction to classification (again, since the members of the Peano's school and Russell did already perform such a reduction – even if only for the limited case of definitions by abstraction) leaves nevertheless open a series of non pedestrian logical and philosophical questions. Let us only briefly mention three lines of investigations that we hope to be able to follow in a next future.

The first one, would be to study systematically how the structure of the lattice of types depends on the properties of the relation inducing them (as we did, for instance, with collusions) and how, in case multiple relations are involved, the operations over the relations induce operations over the lattices of types.

A second one, would be to try to investigate *Second order abstraction principles* (which are central for logicism, including the contemporary "neo-logicist" trend<sup>26</sup>), but from the types point of view. In the case of traditional Abstraction Principles à la Peano-Russell, Second order ones have the ordinary shape of abstraction principles, but for the fact that the variables involved are second

<sup>26</sup>It was initiated by [Hale, 1987]. The literature about Second order abstraction principles is abundant. One can find good introductions to the topic in [Antonelli-May, 2005] and [Tennant, 2017].

order variables. So they are of the form :  $\forall X \forall Y (f_R(X) = f_R(Y) \Leftrightarrow R[X, Y])$ , where  $R[X, Y]$  satisfies the properties defining equivalences. In that case,  $f_R(X)$  is supposed to be a first order individual (an “object”). Two famous examples are Frege’s Axiom V and Hume principle. As soon as one considers second order abstraction schemes, things become hazardous: some of them leads to inconsistencies. At this stage, concerning that complex field, we are not able to develop a clear strategy: the concept of types will certainly add complexity. The *polymorphic* and *multi-scale* dimension of types could nevertheless constitute a relevant tool to face the second order dimension (quantification over the set of tests).

The last one, would be to investigate Weyl’s creative definitions again, but with the viewpoint and the tools of contemporary proof-theory. Apparently, only a few works did approach that way, the simpler case of definitions by abstraction à la Peano ([Tennant, 2017], does it partially). In a sense, such investigations would be faithful to Weyl’s dictum about *creative definitions*. In reality, knowing what are abstracta (and in particular whether they are sets) does not matter for us. What matters, with our creations, is to understand how we use them.

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