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François Digne, Jean Michel

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FORMULAE FOR TWO-VARIABLE GREEN FUNCTIONS

FRANÇOIS DIGNE AND JEAN MICHEL

Abstract. Based on results of Digne-Michel-Lehrer (2003) we give two formu-
lae for two-variable Green functions attached to Lusztig induction in a finite
reductive group. We present applications to explicit computation of these
Green functions, to conjectures of Malle and Rotilio, and to scalar products
between Lusztig inductions of Gelfand-Graev characters.

Let $G$ be a connected reductive group with Frobenius root $F$; that is, some
power $F^q$ is a Frobenius endomorphism attached to an $F_{q^d}$-structure on $G$, where
$q^d$ is a power of a prime $p$. Let $L$ be an $F$-stable Levi subgroup of a (non-necessarily
$F$-stable) parabolic subgroup $P$ of $G$. Let $U$ be the unipotent radical of $P$ and let
$X_U = \{ gU \in G/U \mid g^{-1}Fg \in U \cdot F^d U \}$ be the variety used to define the Lusztig
induction and restriction functors $R^G_L$ and $^*R^G_L$. For $u \in G^F$, $v \in L^F$ unipotent
elements, the two-variable Green function is defined as

$$Q^G_L(u, v) = \text{Trace}((u, v) \mid \sum_i (-1)^i H^i_c(X_U)).$$

In this paper, using the results of [5], we give two different formulae for two-
variable Green functions, and some consequences of these, including proving some
conjectures of [15].

The two-variables Green functions occur in the character formulae for Lusztig
induction and restriction. In particular, for unipotent elements these formulae read

**Proposition 1.** (See for example [8, 10.1.2])

- If $u$ is a unipotent element of $G^F$, and $\psi$ a class function on $L^F$, we have
  $$R^G_L(\psi)(u) = |L^F| \langle \psi, Q^G_L(u, -) \rangle_{L^F}.$$
- If $v$ is a unipotent element of $L^F$, and $\chi$ a class function on $G^F$, we have
  $$^*R^G_L(\chi)(v) = |L^F| \langle \chi, Q^G_L(-, v - 1) \rangle_{G^F}.$$

**Two formulae for two-variable Green functions**

For an element $u$ in a group $G$ we denote by $u^G$ the $G$-conjugacy class of $u$.

**Proposition 2.** Assume either the centre $ZG$ of $G$ is connected and $q > 2$, or $q$
is large enough (depending just on the Dynkin diagram of $G$). Then for $u$ regular,
$Q^G_L(u, -)$ vanishes outside a unique regular unipotent class of $L^F$. For $v$ in that
class, we have $Q^G_L(u, v) = |v^F|^{-1}$.

**Proof (Rotilio).** Let $\gamma^G_u$ be the normalized characteristic function of the $G^F$-conjugacy
class of $u$; that is, the function equal to 0 outside the class of $u$ and to $|C_G(u)^F|$ on that class. For $v' \in L^F$ unipotent, Proposition 1 gives

$$^*R^G_L(\gamma^G_u)(v') = |L^F| \langle \gamma^G_u, Q^G_L(-, v' - 1) \rangle_{G^F} = |L^F| Q^G_L(u, v' - 1).$$
Now, by [1, Theorem 15.2], there exists $v \in \mathbf{L}^F$ such that the left-hand side is equal to $\gamma_{[v]}(v')$. In [1, Theorem 15.2] $q$ is assumed large enough so that the Mackey formula holds and that $p$ and $q$ are such that the results of [10] hold (that is $p$ almost good and $q$ larger than some number depending on the Dynkin diagram of $G$). The Mackey formula is known to hold if $q > 2$ by [2] and the results of [10] hold without condition on $q$ if $ZG$ is connected by [17] or [18, Theorem 4.2] (the assumption that $p$ is almost good can be removed, see [13, section 89] and [12]). The proposition follows.

As in [5], we consider irreducible $G$-equivariant local systems on unipotent classes. These local systems are partitioned into “blocks” parametrised by cuspidal pairs formed by a Levi subgroup and a cuspidal local system supported on a unipotent class of that Levi subgroup. Let us call $I$ the unipotent $\iota$-conjugacy class which is the support of $\iota$ and if $(L_\iota, vF)$ is the cuspidal datum of $\iota$ (for a Levi $L$ of a reductive group $G$, we set $W_G(L) = N_G(L)/L$).

**Proposition 3.** Assume either $ZG$ is connected or $q$ is large enough (depending just on the Dynkin diagram of $G$). For $u$ a unipotent element of $G^F$ and $v$ a unipotent element of $\mathbf{L}^F$, we have

$$Q_{G}(u, v) = |L^F|^{-1} \sum_{\mathcal{I}} \sum_{wF \in W_L(L_{\mathcal{I}})} \frac{|Z^0(L_{\mathcal{I}})^{wF}|}{|W_L(L_{\mathcal{I}})|} Q_{G, \mathcal{I}}^G(u) Q_{wF}^L(v),$$

where $\mathcal{I}$ runs over the $F$-stable blocks of $L$ and where $\mathcal{I}_G$ is the block of $G$ with same cuspidal data as $\mathcal{I}$.

The part of the above sum for $\mathcal{I}$ the principal block is the same formula as [7, Corollaire 4.4].

**Proof.** Proposition 1 applied with $\psi = Q_{wF}^L$ gives, if we write $R_{L_F}^G$ instead of $R_{L}^G$ to keep track of the Frobenius,

$$\langle Q_{wF}^L, Q_{L}^G(u,-) \rangle_{L_F} = |L^F|^{-1} R_{L_F}^G(Q_{wF}^L)(u)$$

Now we have by [5, Proposition 3.2] $Q_{wF}^L = R_{L_F}^G \hat{\chi}_{1,wF}$ where $\hat{\chi}_{1,wF}$ is $q^{e_{\chi}}$ times the characteristic function of $(\iota, wF)$, a class function on $L_{wF}^F$. Here, as in [5, above Remark 2.1], for an irreducible $G$-equivariant local system $\iota$, we denote by $C_\iota$ the unipotent $G$-conjugacy class which is the support of $\iota$, and if $(L_\iota, vF)$ is the cuspidal datum of $\iota$ we set $e_\iota = \frac{1}{2}(\text{codim} C_{\iota} - \dim Z(L_\iota))$. In [5, Proposition 3.2] the assumptions on $p$ and $q$ come from [10] but by the same considerations than at the end of the proof of Proposition 2 it is sufficient to assume $ZG$ connected or $q$ large enough.

By the transitivity of Lusztig induction we get

$$\langle Q_{wF}^L, Q_{L}^G(u,-) \rangle_{L_F} = |L^F|^{-1} R_{L_F}^G(\hat{\chi}_{1,wF})(u) = |L^F|^{-1} Q_{wF}^G(u),$$

where $\mathcal{I}_G$ is the block of $G$ with same cuspidal data as $\mathcal{I}$. Using the orthogonality of the Green functions $Q_{wF}^L$, see [5, Corollary 3.5] (where the assumption that $p$ is almost good which comes from [9], so can be removed now by [12]) and the fact they form a basis of unipotently supported class functions on $L^F$, indexed by the $W_L(L_{\mathcal{I}})$-conjugacy classes of $W_L(L_{\mathcal{I}})F$, we get the proposition. □
Proposition 3 gives a convenient formula to compute automatically two-variable Green functions. Table 1 gives an example, computed with the package Chevie (see [16]).

We denote by \( Y_i \) the characteristic function of the \( F \)-stable local system \( i \), and by \( A(u) \) the group of components of the centralizer of a unipotent element \( u \).

**Proposition 4.** Assume either \( ZG \) is connected or \( q \) is large enough (depending just on the Dynkin diagram of \( G \)). Let \( R_{i,\gamma} \) be the polynomials which appear in [5, Lemma 6.9]. Then

\[
Q_{G}^{F}(u,v) = \left| v^{L_{F}} \right|^{-1} |A(v)|^{-1} \sum_{I} \sum_{i \in I_{G}, \gamma \in I^{F}} Y_i(u)Y_{\gamma}(v)R_{i,\gamma}q^{c_{i} - c_{\gamma}},
\]

where \( c_{i} = \frac{1}{2}(\text{codim} C_{i} - \dim Z(L_{I})) \).

**Proof.** For a block \( I \) of \( L \) and \( i \in I^{F} \), let \( \tilde{Q}_{i} \) be the function of [5, (4.1)]. Then by [5, (4.4)] applied respectively in \( G \) and \( L \) we have

\[
Q_{w_{F}}^{G,I_{G}}(u) = \sum_{i \in I_{G}} \tilde{Q}_{i}(w_{F}) Y_{i}(u) \quad \text{and} \quad Q_{w_{F}}^{L,I_{F}}(v) = \sum_{\kappa \in I^{F}} \tilde{Q}_{\kappa}(w_{F}) Y_{\kappa}(v),
\]

where \( \tilde{Y}_{i} = q^{c_{i}} Y_{i} \). Thus, using the notation \( Z_{L_{I}} \) as in [5, 3.3] to denote the function \( w_{F} \to |Z^{0}(L_{I})|^{w_{F}} \) on \( W_{G}(L_{I})F \), the term relative to a block \( I \) in the formula of Proposition 3 can be written

\[
|L_{F}|^{-1}(Z_{L_{I}} \sum_{\kappa \in I^{F}} \tilde{Q}_{\kappa} Y_{\kappa}(v), \sum_{i \in I_{G}} \tilde{Y}_{i}(u) \text{Res}_{W_{G}(L_{I})F}^{W_{G}(L_{I})F} \tilde{Q}_{i} Y_{i}^{(L_{I})F}).
\]

Applying now [5, Lemma 6.9] this is equal to

\[
|L_{F}|^{-1}(Z_{L_{I}} \sum_{\kappa \in I^{F}} \tilde{Q}_{\kappa} Y_{\kappa}(v), \sum_{i \in I_{G}} \tilde{Y}_{i}(u) R_{i,\gamma} Y_{\gamma}(v) W_{L_{I}}(L_{I})F),
\]

we use now [5, Corollary 5.2] which says that \( \langle \tilde{Q}_{\gamma}, Z_{L_{I}} \tilde{Q}_{\kappa} \rangle_{W_{L_{I}}(L_{I})F} = 0 \) unless \( C_{\gamma} = C_{\kappa} \) and in this last case is equal to

\[
|A(v)|^{-1} \sum_{a \in A(v)} |C_{L_{I}}^{0}(v_{a})F|q^{-2c_{\gamma}} Y_{\gamma}(v_{a}) Y_{\kappa}(v_{a}).
\]

Thus the previous sum becomes

\[
|L_{F}|^{-1} \sum_{i \in I_{G}, \gamma \in I^{F}} \tilde{Y}_{i}(u) R_{i,\gamma} |A(v)|^{-1} \sum_{a \in A(v)} |C_{L_{I}}^{0}(v_{a})F|q^{-2c_{\gamma}} Y_{\gamma}(v_{a}) \sum_{\kappa \in I^{F}} Y_{\kappa}(v_{a}) Y_{\kappa}(v).
\]

But by [5, (4.5)] we have \( \sum_{\kappa} Y_{\kappa}(v_{a}) Y_{\kappa}(v) = \begin{cases} q^{c_{\kappa}} |A(v)|^{F} & \text{if } v_{a} = v \\ 0 & \text{otherwise} \end{cases} \), where \( \kappa \) runs over all local systems. Thus, summing over all the blocks, we get the formula in the statement. \( \square \)

**Corollary 5.** Assume either \( ZG \) is connected or \( q \) is large enough (depending just on the Dynkin diagram of \( G \)). Then for any unipotent elements \( u \in G^{F} \) and \( v \in L_{F} \) we have:

(i) \( Q_{G}^{L}(u,v) \) vanishes unless \( v^{G} \subseteq u^{G} \subseteq \text{Ind}_{L}^{G}(v^{L}) \), where \( \text{Ind}_{L}^{G}(v^{L}) \) is the induced class in the sense of [14].

(ii) \( |v^{L_{F}}| |A(v)| Q_{G}^{L}(u,v) \) is an integer and is a polynomial in \( q \) with integral coefficients.
Proof. For (i), we use [5, Lemma 6.9(i)] which states that $R_{i, \gamma} = 0$ unless $C_\gamma \subseteq C_i \subseteq \mathrm{Ind}_L^G(C_\gamma)$. Since $\tilde{Y}_u(v)$ vanishes unless $C_\gamma \ni v$, the only non-zero terms in the formula of proposition 4 have $C_\gamma \ni v$, whence the result since $\tilde{Y}_v(u)$ vanishes unless $C_i \ni u$.

For (ii), we start with

**Lemma 6.** $q^{c_i - c_\gamma} R_{i, \gamma}$ is a polynomial in $q$ with integral coefficients.

Proof. The defining equation of the matrix $\tilde{R} = \{q^{c_i - c_\gamma} R_{i, \gamma}\}_{i, \gamma}$ reads (see the proof of [5, Lemma 6.9(ii)]):

$$\tilde{R} = P_G C_G IC_L^{-1} P_L^{-1}$$

where $C_G$ is the diagonal matrix with diagonal coefficients $q^{c_i}$ for $i \in I_G$, and $C_L$ is the similar matrix for $L$ and $I$, where $P_G$ is the matrix with coefficients $\{P_{i', i}\}_{i, i' \in I_G}$ where these polynomials are those defined in [11, 6.5], and $P_L$ is the similar matrix for $L$ and $I$, and finally $I$ is the matrix with coefficients $I_{i, \gamma} = \langle \mathrm{Ind}_{W_L(L_2^F)}^W(\tilde{\gamma}_\gamma, \tilde{\gamma}_i) \rangle_W G (L_2^F)$

where $\tilde{\gamma}_\gamma$ is the character of $W_L(L_2^F)$ which corresponds by the generalised Springer correspondence to $\gamma$ (and similarly for $\tilde{\gamma}_i$). Since $P_L$ and $P_G$ are unitriangular matrices with coefficients integral polynomials in $q$, thus $P_L^{-1}$ also, it suffices to prove that $C_G IC_L^{-1}$ has coefficients polynomial in $q$, or equivalently that

$$\text{if } \langle \mathrm{Ind}_{W_L(L_2^F)}^W(\tilde{\gamma}_\gamma, \tilde{\gamma}_i) \rangle_W G (L_2^F) \neq 0, \text{ then } c_i - c_\gamma \geq 0.$$ 

We now use [5, Proposition 2.3(ii)] which says that the non-vanishing above implies $C_\gamma \subseteq C_i \subseteq \mathrm{Ind}_L^G(C_\gamma)$. We now use that, according to the definitions, $c_i - c_\gamma = \dim B_u^G - \dim B_v^L$ where $B_u^G$ is the variety of Borel subgroups of $G$ containing an element $u$ of the support of $\iota$, and where $B_v^L$ is the variety of Borel subgroups of $L$ containing an element $v$ of the support of $\gamma$. Now the lemma follows from the fact that by [14, Theorem 1.3 (b)] we have $\dim B_u^G = \dim B_v^L$ if $u$ is an element of $\mathrm{Ind}_L^G(C_\gamma)$, and that $\dim B_u^G$ is greater for $u \in \mathrm{Ind}_L^G(C_\gamma) - \mathrm{Ind}_L^G(C_\gamma)$. 

Now (ii) results from the lemma: since the $\tilde{Y}$ have values algebraic integers, by Proposition 4 the expression in (ii) is a polynomial in $q$ with coefficients algebraic integers. But, since $[L^F]Q_\gamma^L(u, v)$ is a Lefschetz number (see for example [8, 8.1.3]), the expression in (ii) is a rational number; since this is true for an infinite number of integral values of $q$ the expression in (ii) is a polynomial with integral coefficients.

**Scalar products of induced Gelfand-Graev characters**

The pretext for this section is as follows: in [2, Remark 3.10] is pointed the problem of computing $(R_G^F \Gamma_{\iota}, R_L^F \Gamma_{\iota})_{G^F}$ when $(G, F)$ is simply connected of type $2E_6$, when $L$ is of type $A_2 \times A_2$, and when $\iota$ corresponds to a faithful character of $Z(L)/Z^0(L)$, and checking that the value is the same as given by the Mackey formula. We show now various ways to do this computation, where in this section we assume $p$ and $q$ large enough for all the results of [5] to hold (in particular, we assume $p$ good for $G$, thus not solving the problem of loc. cit. where we need $q = 2$).
Let \( Z = Z(G) \), and let \( \Gamma_z \) be the Gelfand-Graev character parameterized by \( z \in H^1(F, Z) \), see for instance [3, Definition 2.7]. Let \( u_z \) be a representative of the regular unipotent class parametrized by \( z \). As in [11, 7.5 (a)] for \( \ell \) an \( F \)-stable local system on the regular unipotent class we define \( \Gamma_\ell = c \sum_{z \in H^1(F, Z)} Y_\ell(u_z) \Gamma_z \) where \( c = \frac{|\mathbb{Z}/\mathbb{Z}||G^F|}{|H^1(F, Z)|} \).

Note that the cardinality \( |C_G(u_z)^F| \) is independent of \( z \); actually it is equal to \( |Z(G)^F|q^{\text{rk}G} \) (see [1, 15.5]). Thus we will denote this cardinality \( |C_G^F(u)| \) where \( u \in G^F \) is any regular unipotent element. There exists a character \( \zeta \) of \( H^1(F, Z) \) and a root of unity \( b_\ell \) (see [4, above 1.5]) such that \( Y_\ell(u_z) = b_\ell \zeta(z) \). With these notations, we have

**Proposition 7.** We have \( \Gamma_\ell = \eta_G \sigma_\zeta^{-1} c|C_G^F(u)|D\gamma \) where \( \eta_G \) and \( \sigma_\zeta \) are defined as in [4, 2.5].

**Proof.** This proposition could be deduced from [6, Theorem 2.8] using [4, Theorem 2.7]. We give here a more elementary proof.

With the notations of [3, (3.5')] we have \( D\gamma = \sum_{z \in H^1(F, Z)} c_{z,z} \zeta(z) \). By [4, lemma 2.3] we have \( c_{z,z'} = c_{z,z'}^{-1} \) and \( \sum_{z \in H^1(F, Z)} \zeta(z) c_{z,1} = \eta_G \sigma_\zeta^{-1} \). It follows that

\[
c^{-1}b_\ell^{-1}D\gamma = \sum_{z \in H^1(F, Z)} \zeta(z) D\gamma = \sum_{z,z' \in H^1(F, Z)} \zeta(z) c_{z,z'} \gamma_{z'}.
\]

\[
= \sum_{z,z' \in H^1(F, Z)} c_{z,z'}^{-1} \zeta(z) \gamma_{z'}
\]

\[
= \sum_{z' \in H^1(F, Z)} \zeta(z') \gamma_{z'} \sum_{z'' \in H^1(F, Z)} c_{z'',1} \zeta(z'')
\]

\[
= \eta_G \sigma_\zeta^{-1} \sum_{z' \in H^1(F, Z)} \zeta(z') \gamma_{z'} = \eta_G \sigma_\zeta^{-1} b_\ell^{-1} |C_G^F(u)| \gamma.
\]

\( \square \)

**Proposition 8.** If \( \ell \) is a local system supported on the regular unipotent class of \( L \) and \( I \) denotes its block, we have

\[
\langle R_{L}^{G} \Gamma_{\ell}^{L}, R_{L}^{G} \Gamma_{\ell}^{L} \rangle_{G^F} = \frac{|Z(L)|^2}{|Z_0(L)|^2} \sum_{w \in W_{L}(L_\mathbb{Z})} \frac{|Z_0(L_\mathbb{Z})^w| |W_G(L_\mathbb{Z})| |(wF)^{W_G(L_\mathbb{Z})} \cap W_{L}(L_\mathbb{Z})|}{|(wF)^{W_G(L_\mathbb{Z})}|}.
\]

Note that in a given block \( I \) there is at most one local system supported by the regular unipotent class (see [4, Corollary 1.10]).

**Proof.** When \( \ell \) is supported by the regular unipotent class we have \( Q_{\ell} \equiv 1 \), see the begining of section 7, bottom of page 130 in [5]. Using this in the last formula of the proof of [5, Proposition 6.1], we get that \( \Gamma_{\ell}^{L} \) is up to a root of unity equal to \( |A(C_i)| |W_{L}(L_\mathbb{Z})|^{-1} \sum_{w \in W_{L}(L_\mathbb{Z})} |Z_0(L_\mathbb{Z})^w| Q_{wF}^{L_\mathbb{Z}} \). Since \( R_{L}^{G} Q_{wF}^{L_\mathbb{Z}} = Q_{wF}^{G_\mathbb{Z}} \), we get

\[
\langle R_{L}^{G} \Gamma_{\ell}^{L}, R_{L}^{G} \Gamma_{\ell}^{L} \rangle_{G^F} = \langle Q_{wF}^{G_\mathbb{Z}}, Q_{wF}^{G_\mathbb{Z}} \rangle_{G^F}.
\]

\[
= |A(C_i)|^2 |W_{L}(L_\mathbb{Z})|^{-2} \sum_{w,w' \in W_{L}(L_\mathbb{Z})} |Z_0(L_\mathbb{Z})^w| |Z_0(L_\mathbb{Z})^{w'}| |Q_{wF}^{G_\mathbb{Z}}| |Q_{wF}^{G_\mathbb{Z}}|.
\]
By [5, 3.5] the last scalar product is zero unless \( wF \) and \( w'F \) are conjugate in \( W_G(L) \), and is equal to \( |C_{W_G(L)}(wF)|/|Z^0(L)wF| \) otherwise. We get
\[
\langle R^G_L(\Gamma_1), R^G_L(\Gamma_1) \rangle_{G^F} = |A(C_i)|^2 |W_L(L)|^{-2} \sum_{w \in W_L(L)} |Z^0(L)wF||C_{W_G(L)}(wF)||Z(L)wF|^2 \cap W_L(L),
\]
which gives the formula of the proposition since \( A(C_i) = Z(L)/Z(L)^0 \).

\[\Box\]

**Corollary 9.** Let \( \iota \) and \( \iota' \) be local systems supported on the regular unipotent class of \( G \), and \( L, L' \) be their respective blocks: then
\[
\begin{align*}
(i) \quad & \langle \Gamma^{G}_{L}, \Gamma^{G}_{L'} \rangle_{G^F} = \begin{cases} 
0 & \text{if } \iota \neq \iota', \\
|Z(G)|^2 |Z^0(L)\dim Z(L) - \dim Z(G)|^{-1} & \text{if } \iota = \iota'. 
\end{cases} \\
(ii) \quad & \langle \Upsilon^G_{\iota}, \Upsilon^G_{\iota'} \rangle_{G^F} = \begin{cases} 
0 & \text{if } \iota \neq \iota', \\
q^{-r_{\text{kes}}} |Z^0(G)^F|^{-1} & \text{if } \iota = \iota'. 
\end{cases}
\end{align*}
\]

**Proof.** The functions \( Q^{G,F}_{wF} \) and \( Q^{G,F}_{w'F} \) are orthogonal to each other when \( \mathcal{I}_G \neq \mathcal{I}'_G \) (see [9, V, 24.3.6] where the orthogonality is stated for the functions \( \chi_i \)). Since there is a unique \( \iota \) in a given block supported on the regular unipotent class, we get the orthogonality in (i). In the case \( \iota = \iota' \) in (i), the specialization \( L = G \) in Proposition 8 is \( \langle \Gamma^{G}_{\iota}, \Gamma^{G}_{\iota} \rangle_{G^F} = \frac{|Z(G)|^2 \sum_{w \in W_G(L)} |Z^0(L)wF|}{|W_G(L)|^2} \). By [5, Corollary 5.2], where we use that \( Q_{\iota} = 1 \) when \( \iota \) has regular support, we have \( \sum_{w \in W_G(L)} |Z^0(L)wF| = q^{-2c} |C_G(u)^0| \). Whence \( \langle \Gamma^{G}_{\iota}, \Gamma^{G}_{\iota} \rangle_{G^F} = \frac{|Z(G)|^2 q^{-r_{\text{kes}}} \dim Z(L) |C^0_G(u)^F|}{|W_G(L)|^2} \) using \( |C^0_G(u)^F| = q^{r_{\text{kes}}} |Z^0(G)^F| \), we get (i).

For (ii), we apply Proposition 7 in (i), using that \( D \) is an isometry and that \( \sigma_{L}=q^{r_{\text{kes}}}L \) by [4, proposition 2.5].

A particular case of Proposition 8 is

**Corollary 10.** If \( (L, \iota) \) is a cuspidal pair, that is \( L = L \), then
\[
\langle R^G_L(\Gamma_1), R^G_L(\Gamma_1) \rangle_{G^F} = \frac{|Z(L)|^2 |W_G(L)||Z(L)^0|^F}{|Z^0(L)|^2}.
\]

We remark that this coincides with the value predicted by the Mackey formula
\[
\langle R^G_L(\Gamma_1), R^G_L(\Gamma_1) \rangle_{G^F} = \sum_{x \in L^F \cap S(L,L)} \langle \ast R^G_{L \cap L}(\Gamma_1), \ast R^G_{L \cap L}x(\Gamma_1) \rangle_{L \cap L}.
\]

Indeed, since the block \( L \) which contains the local system \( \zeta \) is reduced to the unique cuspidal local system \( (C, \zeta) \) where \( C \) is the regular class of \( L \), all terms in the Mackey formula where \( L \cap \mathfrak{z} \neq L \) vanish. Thus the Mackey formula reduces to
\[
\langle R^G_L(\Gamma_1), R^G_L(\Gamma_1) \rangle_{G^F} = \sum_{x \in W_G(L)} \langle \Gamma^L_1, x \Gamma^L_1 \rangle_{L^F}
\]
and any \( x \) in \( W_L(L) \) acts trivially on \( H^1(F, Z(L)) \) since, the map \( h_L \) being surjective, any element of \( H^1(F, Z(L)) \) is represented by an element of \( H^1(F, Z(G)) \); thus all the terms in the sum are equal, and we get the same result as Corollary 10 by applying Corollary 10 in the case \( G = L \).

Another method for computing \( \langle R^G_L D(\Gamma_1), R^G_L D(\Gamma_1) \rangle_{G^F} \) would be to use Proposition 1 and the values of the two-variable Green functions.
We give these values in the following table in the particular case of $^2E_6$ for the $F$-stable standard Levi subgroup of type $A_2 \times A_2$, for $q \equiv -1 \pmod 3$, so that $F$ acts trivially on $Z(F)/Z(0)(G)$. This table has been computed in Chevie using Proposition 3. The method is to compute the one-variable Green functions which appear in the right-hand side sum by the Lusztig-Shoji algorithm; note that even though the characteristic function of cuspidal character sheaves are known only up to a root of unity, this ambiguity disappears when doing the sum, since such a scalar appears multiplied by its complex conjugate. However the Lusztig-Shoji algorithm depends also on the knowledge that when $\iota$, with support the class of the unipotent element $u$, is parameterized by $(u, \chi)$ where $\chi \in \text{Irr}(A(u))$ then for the unipotent element $u_\iota \in G^F$ parameterized by $a \in H^1(F, A(u))$ we have $\chi_i(u_\iota) = \chi(a)$. We assume that this hold. This is known when $\iota$ is in the principal block, but not for the two blocks with cuspidal datum supported on the Levi subgroup of type $A_2 \times A_2$.

Note that the table shows that the values of $|v^{L_F}|Q^G(u, v)$ are not in general polynomials with integral coefficients but may have denominators equal to $|A(v)|$.

**Table 1.** Values of $|v^{L_F}|Q^G(u, v)$ for $G = ^2E_6(q)$ simply connected and $L = A_2(q^2)/(q-1)^2$, for $q \equiv -1 \pmod 3$.

| $v'$ | $|E_6|$ $E_6(a_3)$ $E_6(a_2)$ $E_6(a_1)$ $E_6(a_1)_{(\zeta_3)}$ $E_6(a_1)_{(\zeta_2)}$ $D_5$ $E_6(a_3)$ |
|------|------|------|------|------|------|------|------|
| $111,111$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $21,21$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $1$ |
| $3,3$ | $(4q + 1)/3$ | $\Phi_2/3$ | $\Phi_2/3$ | $2q + 2q/3$ | $(7q^2 + 2q - 2)/q^2$ |
| $3,3_{(\zeta_3)}$ | $0$ | $0$ | $\Phi_2/3$ | $(4q + 1)/3$ | $\Phi_2/3$ | $2q + 2q/3$ | $(q - 2)/q^2/3$ |
| $3,3_{(\zeta_2)}$ | $0$ | $0$ | $\Phi_2/3$ | $(4q + 1)/3$ | $\Phi_2/3$ | $2q + 2q/3$ | $(q - 2)/q^2/3$ |

<table>
<thead>
<tr>
<th>$v'$</th>
<th>$E_6(a_3)_{(-\zeta_2)}$</th>
<th>$E_6(a_3)_{(\zeta_2)}$</th>
<th>$E_6(a_3)_{(-1)}$</th>
<th>$E_6(a_3)_{(\zeta_2)}$</th>
<th>$E_6(a_3)_{(-\zeta_2)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$111,111$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>$21,21$</td>
<td>$2q + 1$</td>
<td>$2q + 1$</td>
<td>$2q + 1$</td>
<td>$2q + 1$</td>
<td>$2q + 1$</td>
</tr>
<tr>
<td>$3,3$</td>
<td>$(q - 2)/q^2/3$</td>
<td>$\Phi_2/3$</td>
<td>$\Phi_2/3$</td>
<td>$\Phi_2/3$</td>
<td>$\Phi_2/3$</td>
</tr>
<tr>
<td>$3,3_{(\zeta_2)}$</td>
<td>$(3q^2 + 2q - 2)/q^2$</td>
<td>$\Phi_2/3$</td>
<td>$\Phi_2/3$</td>
<td>$\Phi_2/3$</td>
<td>$\Phi_2/3$</td>
</tr>
<tr>
<td>$3,3_{(\zeta_2)}$</td>
<td>$q^2\Phi_2/3$</td>
<td>$(7q^2 + 2q - 2)/q^2$</td>
<td>$\Phi_2/3$</td>
<td>$\Phi_2/3$</td>
<td>$\Phi_2/3$</td>
</tr>
</tbody>
</table>

| $v'$ | $A_5$ $A_3$ $A_3$ $A_3$ $A_3$ $A_3$ |
|------|------|------|------|------|------|
| $111,111$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $21,21$ | $(-2q - 1)/q^2$ | $(-2q - 1)/q^2$ | $(-2q - 1)/q^2$ | $(-2q - 1)/q^2$ | $3q + 3$ |
| $3,3$ | $q^2\Phi_2/3$ | $(-5q^2 - 2q + 1)/q^2/3$ | $(-5q^2 - 2q + 1)/q^2/3$ | $(-5q^2 - 2q + 1)/q^2/3$ | $3q + 3$ |
| $3,3_{(\zeta_3)}$ | $q^2\Phi_2\Phi_2/3$ | $(-5q^2 - 2q + 1)/q^2/3$ | $(-5q^2 - 2q + 1)/q^2/3$ | $(-5q^2 - 2q + 1)/q^2/3$ | $3q + 3$ |

| $v'$ | $A_4 + A_4$ $D_4$ $A_4$ $A_4$ $A_4$ |
|------|------|------|------|------|------|
| $111,111$ | $0$ | $1$ | $4q + 1$ | $4q + 1$ | $4q + 1$ |
| $21,21$ | $(2q + 1)/q^2\Phi_2/3$ | $q^2\Phi_2\Phi_2/3$ | $(3q^2 + q^2 + q + 1)/q^2$ | $(3q^2 + q^2 + q + 1)/q^2$ | $(4q + 1)/q^2\Phi_2/3$ |
| $3,3$ | $(2q + 1)/q^2\Phi_2/3$ | $q^2\Phi_2\Phi_2/3$ | $(3q^2 + q^2 + q + 1)/q^2$ | $(3q^2 + q^2 + q + 1)/q^2$ | $(4q + 1)/q^2\Phi_2/3$ |
| $3,3_{(\zeta_3)}$ | $(2q + 1)/q^2\Phi_2\Phi_2/3$ | $q^2\Phi_2\Phi_2\Phi_2/3$ | $(3q^2 + q^2 + q + 1)/q^2$ | $(3q^2 + q^2 + q + 1)/q^2$ | $(4q + 1)/q^2\Phi_2/3$ |
| $3,3_{(\zeta_3)}$ | $(2q + 1)/q^2\Phi_2\Phi_2/3$ | $q^2\Phi_2\Phi_2\Phi_2/3$ | $(3q^2 + q^2 + q + 1)/q^2$ | $(3q^2 + q^2 + q + 1)/q^2$ | $(4q + 1)/q^2\Phi_2/3$ |

| $v'$ | $D_4(a_1)_{(21)}$ $D_4(a_1)$ $A_3 + A_1$ $A_3$ |
|------|------|------|------|
| $111,111$ | $2q + 1$ | $(-2q - 1)/q^2$ | $(3q^2 + q^2 + q + 1)/q^2$ | $(3q^2 + q^2 + q + 1)/q^2$ |
| $21,21$ | $(8q^3 + 6q^2 + 2q + 2)/q^2$ | $(-4q^3 - 2q^2 + 1)/q^3$ | $(3q^2 + q^2 + q + 1)/q^2$ | $(3q^2 + q^2 + q + 1)/q^2$ |
| $3,3$ | $(2q + 1)/q^2\Phi_2\Phi_2/3$ | $(-2q - 1)/q^2\Phi_2/3$ | $(3q^2 + q^2 + q + 1)/q^2$ | $(3q^2 + q^2 + q + 1)/q^2$ |
| $3,3_{(\zeta_3)}$ | $(2q + 1)/q^2\Phi_2\Phi_2/3$ | $(-2q - 1)/q^2\Phi_2/3$ | $(3q^2 + q^2 + q + 1)/q^2$ | $(3q^2 + q^2 + q + 1)/q^2$ |
| $3,3_{(\zeta_3)}$ | $(2q + 1)/q^2\Phi_2\Phi_2/3$ | $(-2q - 1)/q^2\Phi_2/3$ | $(3q^2 + q^2 + q + 1)/q^2$ | $(3q^2 + q^2 + q + 1)/q^2$ |
\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
$v \backslash u$ & $2A_2+A_1$ & $2A_2+A_1(\zeta_3)$ & $2A_2+A_1(\zeta_2)$ & $2A_2$ \\
\hline
111, 111 & $\Phi_2 \Phi_3$ & $\Phi_2 \Phi_3$ & $\Phi_2 \Phi_3$ & $\Phi_2 \Phi_3$ \\
21, 21 & $(2q^3 + 2q^2 + 4q + 1)q^3 \Phi_2$ & $(2q^3 + 2q^2 + 4q + 1)q^3 \Phi_2$ & $(2q^3 + 2q^2 + 4q + 1)q^3 \Phi_2$ & $3q^3 \Phi_2 \Phi_3 \Phi_6$ \\
3, 3 & $q^6 \Phi_2 \Phi_3$ & $q^6 \Phi_2 \Phi_3$ & $q^6 \Phi_2 \Phi_3$ & $0$ \\
3, $\zeta_3(\zeta_3)$ & $0$ & $0$ & $0$ & $0$ \\
3, $\zeta_2(\zeta_2)$ & $0$ & $0$ & $0$ & $0$ \\
\hline
$v \backslash u$ & $2A_2(\zeta_3)$ & $2A_2(\zeta_2)$ & $A_2+2A_1$ & $A_2$ \\
\hline
111, 111 & $q^6 \Phi_2 \Phi_3 \Phi_6$ & $q^6 \Phi_2 \Phi_3 \Phi_6$ & $(2q^3 + 2q^2 + 4q + 1)q^3 \Phi_2$ & $q^6 \Phi_2 \Phi_3 \Phi_6$ \\
21, 21 & $(3q^4 + q^2 + q + 1)q^4 \Phi_2 \Phi_6$ & $(3q^4 + q^2 + q + 1)q^4 \Phi_2 \Phi_6$ & $(3q^4 + q^2 + q + 1)q^4 \Phi_2 \Phi_6$ & $(3q^4 + q^2 + q + 1)q^4 \Phi_2 \Phi_6$ \\
3, 3 & $0$ & $0$ & $0$ & $0$ \\
3, $\zeta_3(\zeta_3)$ & $0$ & $0$ & $0$ & $0$ \\
3, $\zeta_2(\zeta_2)$ & $0$ & $0$ & $0$ & $0$ \\
\hline
$v \backslash u$ & $3A_1$ & $2A_1$ & $A_2$ \\
\hline
111, 111 & $(-3q^3 - 3q^2 - 3q - q^3 - q^2 - q^2 - q - 1)\Phi_2$ & $(2q^3 + q^2 + q + 1)q^2 \Phi_2 \Phi_6$ & $q^6 \Phi_2 \Phi_3 \Phi_4 \Phi_6$ \\
21, 21 & $(-2q^2 - 1)q^3 \Phi_2 \Phi_6$ & $(2q^3 + q^2 + q + 1)q^2 \Phi_2 \Phi_6$ & $q^6 \Phi_2 \Phi_3 \Phi_4 \Phi_6$ \\
3, 3 & $0$ & $0$ & $0$ \\
3, $\zeta_3(\zeta_3)$ & $0$ & $0$ & $0$ \\
3, $\zeta_2(\zeta_2)$ & $0$ & $0$ & $0$ \\
\hline
$v \backslash u$ & $A_2$ & $A_3$ & $1$ \\
\hline
111, 111 & $(2q^{10} + q^8 + q^6 + q^4 + 2q^6 + q^6 + q^4 + q^2 + 1)\Phi_2^2 \Phi_6 \Phi_10 \Phi_2 \Phi_12 \Phi_18$ & $\Phi_2^4 \Phi_3 \Phi_2^2 \Phi_6 \Phi_10 \Phi_2 \Phi_12 \Phi_18$ & \\
21, 21 & $0$ & $0$ & \\
3, 3 & $0$ & $0$ & \\
3, $\zeta_3(\zeta_3)$ & $0$ & $0$ & \\
3, $\zeta_2(\zeta_2)$ & $0$ & $0$ & \\
\hline
\end{tabular}
\caption{}
\end{table}

References


(F. Digne) Laboratoire Amiénois de Mathématique Fondamentale et Appliquée, CNRS UMR 7352, Université de Picardie-Jules Verne, 80039 Amiens Cedex France.  

Email address: digne@u-picardie.fr  
URL: www.lamfa.u-picardie.fr/digne

(J. Michel) Institut Mathématique de Jussieu – Paris rive gauche, CNRS UMR 7586, Université de Paris, Bâtiment Sophie Germain, 75013, Paris France.  

Email address: jean.michel@imj-prg.fr  
URL: webusers.imj-prg.fr/~jean.michel