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FORMULAE FOR TWO-VARIABLE GREEN FUNCTIONS

FRANÇOIS DIGNE AND JEAN MICHEL

Abstract. Based on results of Digne-Michel-Lehrer (2003) we give two formulae for two-variable Green functions attached to Lusztig induction in a finite reductive group. We present applications to explicit computation of these Green functions, to conjectures of Malle and Rotilio, and to scalar products between Lusztig inductions of Gelfand-Graev characters.

Let $\mathbf{G}$ be a connected reductive group with Frobenius root F; that is, some power F^δ is a Frobenius endomorphism attached to an $\mathbb{F}_{q^\delta}$-structure on $\mathbf{G}$, where q^δ is a power of a prime p. Let $\mathbf{L}$ be an F-stable Levi subgroup of a (non-necessarily F-stable) parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$. Let $\mathbf{U}$ be the unipotent radical of $\mathbf{P}$ and let $\mathbf{X}_\mathbf{U} = \{g\mathbf{U} \in \mathbf{G}/\mathbf{U} \mid g^{-1\,F}g \in \mathbf{U} \cdot {}^F\mathbf{U}\}$ be the variety used to define the Lusztig induction and restriction functors $R_\mathbf{L}^\mathbf{G}$ and ${}^*R_\mathbf{L}^\mathbf{G}$. For $u \in \mathbf{G}^F, v \in \mathbf{L}^F$ unipotent elements, the two-variable Green function is defined as

$$Q_\mathbf{L}^\mathbf{G}(u,v) = \mathrm{Trace}((u,v) \mid \sum_i (-1)^i H_c^i(\mathbf{X}_\mathbf{U})).$$

In this paper, using the results of [5], we give two different formulae for two-variable Green functions, and some consequences of these, including proving some conjectures of [15].

The two-variables Green functions occur in the character formulae for Lusztig induction and restriction. In particular, for unipotent elements these formulae read

Proposition 1. *(See for example* [8, 10.1.2]*)*

- *If u is a unipotent element of $\mathbf{G}^F$, and ψ a class function on $\mathbf{L}^F$, we have*
$$R_\mathbf{L}^\mathbf{G}(\psi)(u) = |\mathbf{L}^F|\langle \psi, Q_\mathbf{L}^\mathbf{G}(u,-)\rangle_{\mathbf{L}^F}.$$
- *If v is a unipotent element of $\mathbf{L}^F$, and χ a class function on $\mathbf{G}^F$, we have*
$${}^*R_\mathbf{L}^\mathbf{G}(\chi)(v) = |\mathbf{L}^F|\langle \chi, Q_\mathbf{L}^\mathbf{G}(-,v^{-1})\rangle_{\mathbf{G}^F}.$$

Two formulae for two-variable Green functions

For an element u in a group G we denote by u^G the G-conjugacy class of u.

Proposition 2. *Assume either the centre $Z\mathbf{G}$ of $\mathbf{G}$ is connected and $q > 2$, or q is large enough (depending just on the Dynkin diagram of $\mathbf{G}$). Then for u regular, $Q_\mathbf{L}^\mathbf{G}(u,-)$ vanishes outside a unique regular unipotent class of $\mathbf{L}^F$. For v in that class, we have $Q_\mathbf{L}^\mathbf{G}(u,v) = |v^{\mathbf{L}^F}|^{-1}$.*

Proof (Rotilio). Let $\gamma_u^\mathbf{G}$ be the normalized characteristic function of the $\mathbf{G}^F$-conjugacy class of u; that is, the function equal to 0 outside the class of u and to $|C_\mathbf{G}(u)^F|$ on that class. For $v' \in \mathbf{L}^F$ unipotent, Proposition 1 gives

$${}^*R_\mathbf{L}^\mathbf{G}(\gamma_u^\mathbf{G})(v') = |\mathbf{L}^F|\langle \gamma_u^\mathbf{G}, Q_\mathbf{L}^\mathbf{G}(-,v'^{-1})\rangle_{\mathbf{G}^F} = |\mathbf{L}^F| Q_\mathbf{L}^\mathbf{G}(u,v'^{-1}).$$

Now, by [1, Theorem 15.2], there exists $v \in \mathbf{L}^F$ such that the left-hand side is equal to $\gamma_v^{\mathbf{L}}(v')$. In [1, Theorem 15.2] q is assumed large enough so that the Mackey formula holds and that p and q are such that the results of [10] hold (that is p almost good and q larger than some number depending on the Dynkin diagram of $\mathbf{G}$). The Mackey formula is known to hold if $q > 2$ by [2] and the results of [10] hold without condition on q if $Z\mathbf{G}$ is connected by [17] or [18, Theorem 4.2] (the assumption that p is almost good can be removed, see [13, section 89] and [12]). The proposition follows. □

As in [5], we consider irreducible $\mathbf{G}$-equivariant local systems on unipotent classes. These local systems are partitioned into "blocks" parametrised by cuspidal pairs formed by a Levi subgroup and a cuspidal local system supported on a unipotent class of that Levi subgroup. Let us call $Q_{wF}^{\mathbf{L},\mathcal{I}}$ the function defined in [5, 3.1(iii)] relative to an F-stable block $\mathcal{I}$ of unipotently supported local systems on $\mathbf{L}$ and to $wF \in W_{\mathbf{L}}(\mathbf{L}_{\mathcal{I}})F$ where $(\mathbf{L}_{\mathcal{I}}, \iota_{\mathcal{I}})$ is the cuspidal datum of $\mathcal{I}$ (for a Levi $\mathbf{L}$ of a reductive group $\mathbf{G}$, we set $W_{\mathbf{G}}(\mathbf{L}) = N_{\mathbf{G}}(\mathbf{L})/\mathbf{L}$).

Proposition 3. *Assume either $Z\mathbf{G}$ is connected or q is large enough (depending just on the Dynkin diagram of $\mathbf{G}$). For u a unipotent element of $\mathbf{G}^F$ and v a unipotent element of $\mathbf{L}^F$, we have*

$$Q_{\mathbf{L}}^{\mathbf{G}}(u,v) = |\mathbf{L}^F|^{-1} \sum_{\mathcal{I}} \sum_{w \in W_{\mathbf{L}}(\mathbf{L}_{\mathcal{I}})} \frac{|Z^0(\mathbf{L}_{\mathcal{I}})^{wF}|}{|W_{\mathbf{L}}(\mathbf{L}_{\mathcal{I}})|} \overline{Q_{wF}^{\mathbf{G},\mathcal{I}_{\mathbf{G}}}(u)} Q_{wF}^{\mathbf{L},\mathcal{I}}(v),$$

where $\mathcal{I}$ runs over the F-stable blocks of $\mathbf{L}$ and where $\mathcal{I}_{\mathbf{G}}$ is the block of $\mathbf{G}$ with same cuspidal data as $\mathcal{I}$.

The part of the above sum for $\mathcal{I}$ the principal block is the same formula as [7, Corollaire 4.4].

Proof. Proposition 1 applied with $\psi = Q_{wF}^{\mathbf{L},\mathcal{I}}$ gives, if we write $R_{\mathbf{L}^F}^{\mathbf{G}^F}$ instead of $R_{\mathbf{L}}^{\mathbf{G}}$ to keep track of the Frobenius,

$$\langle Q_{wF}^{\mathbf{L},\mathcal{I}}, Q_{\mathbf{L}}^{\mathbf{G}}(u,-)\rangle_{\mathbf{L}^F} = |\mathbf{L}^F|^{-1} R_{\mathbf{L}^F}^{\mathbf{G}^F}(Q_{wF}^{\mathbf{L},\mathcal{I}})(u)$$

Now we have by [5, Proposition 3.2] $Q_{wF}^{\mathbf{L},\mathcal{I}} = R_{\mathbf{L}_{\mathcal{I}}^{wF}}^{\mathbf{L}^F} \tilde{\mathcal{X}}_{\iota_{\mathcal{I}},wF}$ where $\tilde{\mathcal{X}}_{\iota_{\mathcal{I}},wF}$ is $q^{c_{\iota_{\mathcal{I}}}}$ times the characteristic function of $(\iota_{\mathcal{I}}, wF)$, a class function on $\mathbf{L}_{\mathcal{I}}^{wF}$. Here, as in [5, above Remark 2.1], for an irreducible $\mathbf{G}$-equivariant local system ι, we denote by C_ι the unipotent $\mathbf{G}$-conjugacy class which is the support of ι, and if $(\mathbf{L}_{\mathcal{I}}, \iota_{\mathcal{I}})$ is the cuspidal datum of ι we set $c_\iota = \frac{1}{2}(\operatorname{codim} C_\iota - \dim Z(\mathbf{L}_{\mathcal{I}}))$. In [5, Proposition 3.2] the assumptions on p and q come from [10] but by the same considerations than at the end of the proof of Proposition 2 it is sufficient to assume $Z\mathbf{G}$ connected or q large enough.

By the transitivity of Lusztig induction we get

$$\langle Q_{wF}^{\mathbf{L},\mathcal{I}}, Q_{\mathbf{L}}^{\mathbf{G}}(u,-)\rangle_{\mathbf{L}^F} = |\mathbf{L}^F|^{-1} R_{\mathbf{L}_{\mathcal{I}}^{wF}}^{\mathbf{G}^F}(\tilde{\mathcal{X}}_{\iota_{\mathcal{I}},wF})(u) = |\mathbf{L}^F|^{-1} Q_{wF}^{\mathbf{G},\mathcal{I}_{\mathbf{G}}}(u),$$

where $\mathcal{I}_{\mathbf{G}}$ is the block of $\mathbf{G}$ with same cuspidal data as $\mathcal{I}$. Using the orthogonality of the Green functions $Q_{wF}^{\mathbf{L},\mathcal{I}}$, see [5, Corollary 3.5] (where the assumption is that p is almost good which comes from [9], so can be removed now by [12]) and the fact they form a basis of unipotently supported class functions on $\mathbf{L}^F$, indexed by the $W_{\mathbf{L}}(\mathbf{L}_{\mathcal{I}})$-conjugacy classes of $W_{\mathbf{L}}(\mathbf{L}_{\mathcal{I}})F$, we get the proposition. □

Proposition 3 gives a convenient formula to compute automatically two-variable Green functions. Table 1 gives an example, computed with the package `Chevie` (see [16]).

We denote by $\mathcal{Y}_\iota$ the characteristic function of the F-stable local system ι, and by $A(u)$ the group of components of the centralizer of a unipotent element u.

Proposition 4. *Assume either $Z\mathbf{G}$ is connected or q is large enough (depending just on the Dynkin diagram of $\mathbf{G}$). Let $R_{\iota,\gamma}$ be the polynomials which appear in* [5, Lemma 6.9]. *Then*

$$Q_{\mathbf{L}}^{\mathbf{G}}(u,v) = |v^{\mathbf{L}^F}|^{-1}|A(v)|^{-1}\sum_{\mathcal{I}}\sum_{\iota\in\mathcal{I}_{\mathbf{G}}{}^F,\gamma\in\mathcal{I}^F}\overline{\mathcal{Y}_\iota(u)}\mathcal{Y}_\gamma(v)R_{\iota,\gamma}q^{c_\iota-c_\gamma},$$

where $c_\iota = \frac{1}{2}(\operatorname{codim} C_\iota - \dim Z(\mathbf{L}_{\mathcal{I}}))$.

Proof. For a block $\mathcal{I}$ of $\mathbf{L}$ and $\iota\in\mathcal{I}^F$, let $\tilde{Q}_\iota$ be the function of [5, (4.1)]. Then by [5, (4.4)] applied respectively in $\mathbf{G}$ and $\mathbf{L}$ we have

$$Q_{wF}^{\mathbf{G},\mathcal{I}_{\mathbf{G}}}(u) = \sum_{\iota\in\mathcal{I}_{\mathbf{G}}{}^F}\tilde{Q}_\iota(wF)\tilde{\mathcal{Y}}_\iota(u) \quad\text{and}\quad Q_{wF}^{\mathbf{L},\mathcal{I}}(v) = \sum_{\kappa\in\mathcal{I}^F}\tilde{Q}_\kappa(wF)\tilde{\mathcal{Y}}_\kappa(v)$$

where $\tilde{\mathcal{Y}}_\iota = q^{c_\iota}\mathcal{Y}_\iota$. Thus, using the notation $\mathcal{Z}_{\mathbf{L}_{\mathcal{I}}}$ as in [5, 3.3] to denote the function $wF\mapsto|Z^0(\mathbf{L}_{\mathcal{I}})^{wF}|$ on $W_{\mathbf{G}}(\mathbf{L}_{\mathcal{I}})F$, the term relative to a block $\mathcal{I}$ in the formula of Proposition 3 can be written

$$|\mathbf{L}^F|^{-1}\langle\mathcal{Z}_{\mathbf{L}_{\mathcal{I}}}\sum_{\kappa\in\mathcal{I}^F}\tilde{Q}_\kappa\tilde{\mathcal{Y}}_\kappa(v),\sum_{\iota\in\mathcal{I}_{\mathbf{G}}{}^F}\tilde{\mathcal{Y}}_\iota(u)\operatorname{Res}_{W_{\mathbf{L}}(\mathbf{L}_{\mathcal{I}})^F}^{W_{\mathbf{G}}(\mathbf{L}_{\mathcal{I}})^F}\tilde{Q}_\iota\rangle_{W_{\mathbf{L}}(\mathbf{L}_{\mathcal{I}})F}.$$

Applying now [5, Lemma 6.9] this is equal to

$$|\mathbf{L}^F|^{-1}\langle\mathcal{Z}_{\mathbf{L}_{\mathcal{I}}}\sum_{\kappa\in\mathcal{I}^F}\tilde{Q}_\kappa\tilde{\mathcal{Y}}_\kappa(v),\sum_{\iota\in\mathcal{I}_{\mathbf{G}}{}^F,\gamma\in\mathcal{I}^F}\tilde{\mathcal{Y}}_\iota(u)R_{\iota,\gamma}\tilde{Q}_\gamma\rangle_{W_{\mathbf{L}}(\mathbf{L}_{\mathcal{I}})F},$$

We use now [5, Corollary 5.2] which says that, $\langle\tilde{Q}_\gamma,\mathcal{Z}_{\mathbf{L}_{\mathcal{I}}}\tilde{Q}_\kappa\rangle_{W_{\mathbf{L}}(\mathbf{L}_{\mathcal{I}})F}=0$ unless $C_\gamma=C_\kappa$ and in this last case is equal to

$$|A(v)|^{-1}\sum_{a\in A(v)}|C_{\mathbf{L}}^0(v_a)^F|q^{-2c_\gamma}\mathcal{Y}_\gamma(v_a)\overline{\mathcal{Y}_\kappa(v_a)}$$

Thus the previous sum becomes

$$|\mathbf{L}^F|^{-1}\sum_{\iota\in\mathcal{I}_{\mathbf{G}}{}^F,\gamma\in\mathcal{I}^F}\overline{\tilde{\mathcal{Y}}_\iota(u)}R_{\iota,\gamma}|A(v)|^{-1}\sum_{a\in A(v)}|C_{\mathbf{L}}^0(v_a)^F|q^{-2c_\gamma}\mathcal{Y}_\gamma(v_a)\sum_{\kappa\in\mathcal{I}^F}\overline{\mathcal{Y}_\kappa(v_a)}\tilde{\mathcal{Y}}_\kappa(v).$$

But by [5, (4.5)] we have $\sum_\kappa\overline{\mathcal{Y}_\kappa(v_a)}\tilde{\mathcal{Y}}_\kappa(v)=\begin{cases}q^{c_\kappa}|A(v)^F| & \text{if } v_a=v\\ 0 & \text{otherwise}\end{cases}$, where κ runs over all local systems. Thus, summing over all the blocks, we get the formula in the statement. □

Corollary 5. *Assume either $Z\mathbf{G}$ is connected or q is large enough (depending just on the Dynkin diagram of $\mathbf{G}$). Then for any unipotent elements $u\in\mathbf{G}^F$ and $v\in\mathbf{L}^F$ we have:*

(i) *$Q_{\mathbf{L}}^{\mathbf{G}}(u,v)$ vanishes unless $v^{\mathbf{G}}\subseteq\overline{u^{\mathbf{G}}}\subseteq\operatorname{Ind}_{\mathbf{L}}^{\mathbf{G}}(v^{\mathbf{L}})$, where $\operatorname{Ind}_{\mathbf{L}}^{\mathbf{G}}(v^{\mathbf{L}})$ is the induced class in the sense of* [14].

(ii) *$|v^{\mathbf{L}^F}||A(v)|Q_{\mathbf{L}}^{\mathbf{G}}(u,v)$ is an integer and is a polynomial in q with integral coefficients.*

Proof. For (i), we use [5, Lemma 6.9(i)] which states that $R_{\iota,\gamma} = 0$ unless $C_\gamma \subseteq \overline{C_\iota} \subseteq \overline{\mathrm{Ind}_{\mathbf{L}}^{\mathbf{G}}(C_\gamma)}$. Since $\tilde{\mathcal{Y}}_\kappa(v)$ vanishes unless $C_\kappa \ni v$, the only non-zero terms in the formula of proposition 4 have $C_\gamma \ni v$, whence the result since $\tilde{\mathcal{Y}}_\iota(u)$ vanishes unless $C_\iota \ni u$.

For (ii), we start with

Lemma 6. *$q^{c_\iota - c_\gamma} R_{\iota,\gamma}$ is a polynomial in q with integral coefficients.*

Proof. The defining equation of the matrix $\tilde{R} = \{q^{c_\iota - c_\gamma} R_{\iota,\gamma}\}_{\iota,\gamma}$ reads (see the proof of [5, Lemma 6.9(i)]):

$$\tilde{R} = P_{\mathbf{G}} C_{\mathbf{G}} I C_{\mathbf{L}}^{-1} P_{\mathbf{L}}^{-1}$$

where $C_{\mathbf{G}}$ is the diagonal matrix with diagonal coefficients q^{c_ι} for $\iota \in \mathcal{I}_{\mathbf{G}}$, and $C_{\mathbf{L}}$ is the similar matrix for $\mathbf{L}$ and $\mathcal{I}$, where $P_{\mathbf{G}}$ is the matrix with coefficients $\{P_{\iota,\iota'}\}_{\iota,\iota' \in \mathcal{I}_{\mathbf{G}}}$ where these polynomials are those defined in [11, 6.5], and $P_{\mathbf{L}}$ is the similar matrix for $\mathbf{L}$ and $\mathcal{I}$, and finally I is the matrix with coefficients

$$I_{\iota,\gamma} = \langle \mathrm{Ind}_{W_{\mathbf{L}}(\mathbf{L}_{\mathcal{I}})F}^{W_{\mathbf{G}}(\mathbf{L}_{\mathcal{I}})F} \tilde{\varphi}_\gamma, \tilde{\varphi}_\iota \rangle_{W_{\mathbf{G}}(\mathbf{L}_{\mathcal{I}})F}$$

where $\tilde{\varphi}_\gamma$ is the character of $W_{\mathbf{L}}(\mathbf{L}_{\mathcal{I}})F$ which corresponds by the generalised Springer correspondence to γ (and similarly for $\tilde{\varphi}_\iota$). Since $P_{\mathbf{L}}$ and $P_{\mathbf{G}}$ are unitriangular matrices with coefficients integral polynomials in q, thus $P_{\mathbf{L}}^{-1}$ also, it suffices to prove that $C_{\mathbf{G}} I C_{\mathbf{L}}^{-1}$ has coefficients polynomial in q, or equivalently that

$$\text{if } \langle \mathrm{Ind}_{W_{\mathbf{L}}(\mathbf{L}_{\mathcal{I}})F}^{W_{\mathbf{G}}(\mathbf{L}_{\mathcal{I}})F} \tilde{\varphi}_\gamma, \tilde{\varphi}_\iota \rangle_{W_{\mathbf{G}}(\mathbf{L}_{\mathcal{I}})F} \neq 0, \text{ then } c_\iota - c_\gamma \geq 0.$$

We now use [5, Proposition 2.3(ii)] which says that the non-vanishing above implies $C_\gamma \subseteq \overline{C_\iota} \subseteq \overline{\mathrm{Ind}_{\mathbf{L}}^{\mathbf{G}}(C_\gamma)}$. We now use that, according to the definitions, $c_\iota - c_\gamma = \dim \mathcal{B}_u^{\mathbf{G}} - \dim \mathcal{B}_v^{\mathbf{L}}$ where $\mathcal{B}_u^{\mathbf{G}}$ is the variety of Borel subgroups of $\mathbf{G}$ containing an element u of the support of ι, and where $\mathcal{B}_v^{\mathbf{L}}$ is the variety of Borel subgroups of $\mathbf{L}$ containing an element v of the support of γ. Now the lemma follows from the fact that by [14, Theorem 1.3 (b)] we have $\dim \mathcal{B}_u^{\mathbf{G}} = \dim \mathcal{B}_v^{\mathbf{L}}$ if u is an element of $\mathrm{Ind}_{\mathbf{L}}^{\mathbf{G}}(C_\gamma)$, and that $\dim \mathcal{B}_u^{\mathbf{G}}$ is greater for $u \in \overline{\mathrm{Ind}_{\mathbf{L}}^{\mathbf{G}}(C_\gamma)} - \mathrm{Ind}_{\mathbf{L}}^{\mathbf{G}}(C_\gamma)$. □

Now (ii) results from the lemma: since the $\tilde{\mathcal{Y}}$ have values algebraic integers, by Proposition 4 the expression in (ii) is a polynomial in q with coefficients algebraic integers. But, since $|\mathbf{L}^F| Q_{\mathbf{L}}^{\mathbf{G}}(u,v)$ is a Lefschetz number (see for example [8, 8.1.3]), the expression in (ii) is a rational number; since this is true for an infinite number of integral values of q the expression in (ii) is a polynomial with integral coefficients. □

Scalar products of induced Gelfand-Graev characters

The pretext for this section is as follows: in [2, Remark 3.10] is pointed the problem of computing $\langle R_{\mathbf{L}}^{\mathbf{G}} \Gamma_\iota, R_{\mathbf{L}}^{\mathbf{G}} \Gamma_\iota \rangle_{\mathbf{G}^F}$ when $(\mathbf{G}, F)$ is simply connected of type ${}^2E_6$, when $\mathbf{L}$ is of type $A_2 \times A_2$, and when ι corresponds to a faithful character of $Z(\mathbf{L})/Z^0(\mathbf{L})$, and checking that the value is the same as given by the Mackey formula. We show now various ways to do this computation, where in this section we assume p and q large enough for all the results of [5] to hold (in particular, we assume p good for $\mathbf{G}$, thus not solving the problem of loc. cit. where we need $q = 2$).

Let $Z = Z(\mathbf{G})$, and let Γ_z be the Gelfand-Graev character parameterized by $z \in H^1(F, Z)$, see for instance [3, Definition 2.7]. Let u_z be a representative of the regular unipotent class parametrized by z. As in [11, 7.5 (a)] for ι an F-stable local system on the regular unipotent class we define $\Gamma_\iota = c\sum_{z\in H^1(F,Z)} \mathcal{Y}_\iota(u_z)\Gamma_z$ where $c = \frac{|Z/Z^0|}{|H^1(F,Z)|}$.

Note that the cardinality $|C_{\mathbf{G}}(u_z)^F|$ is independent of z; actually it is equal to $|Z(\mathbf{G})^F|q^{\mathrm{rkss}\mathbf{G}}$ (see [1, 15.5]). Thus we will denote this cardinality $|C_{\mathbf{G}^F}(u)|$ where $u \in \mathbf{G}^F$ is any regular unipotent element. There exists a character ζ of $H^1(F, Z)$ and a root of unity b_ι (see [4, above 1.5]) such that $\mathcal{Y}_\iota(u_z) = b_\iota\zeta(z)$. With these notations, we have

Proposition 7. *We have $\Gamma_\iota = \eta_{\mathbf{G}}\sigma_\zeta^{-1}c|C_{\mathbf{G}^F}(u)|D\mathcal{Y}_\iota$ where $\eta_{\mathbf{G}}$ and σ_ζ are defined as in* [4, 2.5],

Proof. This proposition could be deduced from [6, Theorem 2.8] using [4, Theorem 2.7]. We give here a more elementary proof.

With the notations of [3, (3.5')] we have $D\Gamma_z = \sum_{z'\in H^1(F,Z)} c_{z,z'}\gamma_{z'}$. By [4, lemma 2.3] we have $c_{z,z'} = c_{zz'^{-1},1}$ and $\sum_{z\in H^1(F,Z)} \zeta(z)c_{z,1} = \eta_{\mathbf{G}}\sigma_\zeta^{-1}$. It follows that

$$\begin{aligned} c^{-1}b_\iota^{-1}D\Gamma_\iota &= \sum_{z\in H^1(F,Z)} \zeta(z)D\Gamma_z = \sum_{z,z'\in H^1(F,Z)} \zeta(z)c_{z,z'}\gamma_{z'} \\ &= \sum_{z,z'\in H^1(F,Z)} c_{zz'^{-1},1}\zeta(z)\gamma_{z'} \\ &= \sum_{z'\in H^1(F,Z)} \zeta(z')\gamma_{z'} \sum_{z''\in H^1(F,Z)} c_{z'',1}\zeta(z'') \\ &= \eta_{\mathbf{G}}\sigma_\zeta^{-1} \sum_{z'\in H^1(F,Z)} \zeta(z')\gamma_{z'} = \eta_{\mathbf{G}}\sigma_\zeta^{-1}b_\iota^{-1}|C_{\mathbf{G}^F}(u)|\mathcal{Y}_\iota \end{aligned}$$

□

Proposition 8. *If ι is a local system supported on the regular unipotent class of $\mathbf{L}$ and $\mathcal{I}$ denotes its block, we have*

$$\langle R_{\mathbf{L}}^{\mathbf{G}}\Gamma_\iota^{\mathbf{L}}, R_{\mathbf{L}}^{\mathbf{G}}\Gamma_\iota^{\mathbf{L}}\rangle_{\mathbf{G}^F} = \\ |\frac{Z(\mathbf{L})}{Z^0(\mathbf{L})}|^2 \sum_{w\in W_{\mathbf{L}}(\mathbf{L}_{\mathcal{I}})} \frac{|Z^0(\mathbf{L}_{\mathcal{I}})^{wF}||W_{\mathbf{G}}(\mathbf{L}_{\mathcal{I}})|}{|W_{\mathbf{L}}(\mathbf{L}_{\mathcal{I}})|^2} \frac{|(wF)^{W_{\mathbf{G}}(\mathbf{L}_{\mathcal{I}})} \cap W_{\mathbf{L}}(\mathbf{L}_{\mathcal{I}})|}{|(wF)^{W_{\mathbf{G}}(\mathbf{L}_{\mathcal{I}})}|}.$$

Note that in a given block $\mathcal{I}$ there is at most one local system supported by the regular unipotent class (see [4, Corollary 1.10]).

Proof. When ι is supported by the regular unipotent class we have $\tilde{Q}_\iota = 1$, see the begining of section 7, bottom of page 130 in [5]. Using this in the last formula of the proof of [5, Proposition 6.1], we get that $\Gamma_\iota^{\mathbf{L}}$ is up to a root of unity equal to $|A(C_\iota))||W_{\mathbf{L}}(\mathbf{L}_{\mathcal{I}})|^{-1}\sum_{w\in W_{\mathbf{L}}(\mathbf{L}_{\mathcal{I}})} |Z^0(\mathbf{L}_{\mathcal{I}})^{wF}|Q_{wF}^{\mathbf{L},\mathcal{I}}$. Since $R_{\mathbf{L}}^{\mathbf{G}}Q_{wF}^{\mathbf{L},\mathcal{I}} = Q_{wF}^{\mathbf{G},\mathcal{I}_{\mathbf{G}}}$, we get

$$\langle R_{\mathbf{L}}^{\mathbf{G}}\Gamma_\iota, R_{\mathbf{L}}^{\mathbf{G}}\Gamma_\iota\rangle_{\mathbf{G}^F} = \\ |A(C_\iota))|^2|W_{\mathbf{L}}(\mathbf{L}_{\mathcal{I}})|^{-2} \sum_{w,w'\in W_{\mathbf{L}}(\mathbf{L}_{\mathcal{I}})} |Z^0(\mathbf{L}_{\mathcal{I}})^{wF}||Z^0(\mathbf{L}_{\mathcal{I}})^{w'F}|\langle Q_{wF}^{\mathbf{G},\mathcal{I}_{\mathbf{G}}}, Q_{w'F}^{\mathbf{G},\mathcal{I}_{\mathbf{G}}}\rangle_{\mathbf{G}^F}.$$

By [5, 3.5] the last scalar product is zero unless wF and $w'F$ are conjugate in $W_{\mathbf{G}}(\mathbf{L}_{\mathcal{I}})$, and is equal to $|C_{W_{\mathbf{G}}(\mathbf{L}_{\mathcal{I}})}(wF)|/|Z^0(\mathbf{L}_{\mathcal{I}})^{wF}|$ otherwise. We get

$$\langle R_{\mathbf{L}}^{\mathbf{G}}\Gamma_\iota, R_{\mathbf{L}}^{\mathbf{G}}\Gamma_\iota\rangle_{\mathbf{G}^F} = |A(C_\iota))|^2|W_{\mathbf{L}}(\mathbf{L}_{\mathcal{I}})|^{-2} \sum_{w\in W_{\mathbf{L}}(\mathbf{L}_{\mathcal{I}})} |Z^0(\mathbf{L}_{\mathcal{I}})^{wF}||C_{W_{\mathbf{G}}(\mathbf{L}_{\mathcal{I}})}(wF)||(wF)^{W_{\mathbf{G}}(\mathbf{L}_{\mathcal{I}})}\cap W_{\mathbf{L}}(\mathbf{L}_{\mathcal{I}})|,$$

which gives the formula of the proposition since $A(C_\iota) = Z(\mathbf{L})/Z(\mathbf{L})^0$. □

Corollary 9. *Let ι and ι' be local systems supported on the regular unipotent class of $\mathbf{G}$, and $\mathcal{I}$, $\mathcal{I}'$ be their respective blocks: then*

(i) $\langle \Gamma_\iota^{\mathbf{G}}, \Gamma_{\iota'}^{\mathbf{G}}\rangle_{\mathbf{G}^F} = \begin{cases} 0 & \text{if } \iota\neq\iota', \\ |\frac{Z(\mathbf{G})}{Z^0(\mathbf{G})}|^2|Z^0(\mathbf{G})^F|q^{\dim Z(\mathbf{L}_{\mathcal{I}})-\dim Z(\mathbf{G})} & \text{if } \iota=\iota'. \end{cases}$

(ii) $\langle \mathcal{Y}_\iota, \mathcal{Y}_{\iota'}\rangle_{\mathbf{G}^F} = \begin{cases} 0 & \text{if } \iota\neq\iota', \\ q^{-rkss\mathbf{G}}|Z^0(\mathbf{G})^F|^{-1} & \text{if } \iota=\iota'. \end{cases}$

Proof. The functions $Q_{wF}^{\mathbf{G},\mathcal{I}_{\mathbf{G}}}$ and $Q_{w'F}^{\mathbf{G},\mathcal{I}'_{\mathbf{G}}}$ are orthogonal to each other when $\mathcal{I}_{\mathbf{G}}\neq\mathcal{I}'_{\mathbf{G}}$ (see [9, V, 24.3.6] where the orthogonality is stated for the functions $\mathcal{X}_\iota$). Since there is a unique ι in a given block supported on the regular unipotent class, we get the orthogonality in (i). In the case $\iota=\iota'$ in (i), the specialization $\mathbf{L}=\mathbf{G}$ in Proposition 8 is $\langle \Gamma_\iota^{\mathbf{G}}, \Gamma_{\iota'}^{\mathbf{G}}\rangle_{\mathbf{G}^F} = |\frac{Z(\mathbf{G})}{Z^0(\mathbf{G})}|^2\sum_{w\in W_{\mathbf{G}}(\mathbf{L}_{\mathcal{I}})}\frac{|Z^0(\mathbf{L}_{\mathcal{I}})^{wF}|}{|W_{\mathbf{G}}(\mathbf{L}_{\mathcal{I}})|}$. By [5, Corollary 5.2], where we use that $\tilde{Q}_\iota=1$ when ι has regular support, we have $\sum_{w\in W_{\mathbf{G}}(\mathbf{L}_{\mathcal{I}})}\frac{|Z^0(\mathbf{L}_{\mathcal{I}})^{wF}|}{|W_{\mathbf{G}}(\mathbf{L}_{\mathcal{I}})|} = q^{-2c_\iota}|C_{\mathbf{G}}(u)^{0F}|$. Whence $\langle \Gamma_\iota^{\mathbf{G}}, \Gamma_{\iota'}^{\mathbf{G}}\rangle_{\mathbf{G}^F} = |\frac{Z(\mathbf{G})}{Z^0(\mathbf{G})}|^2 q^{-\mathrm{rk}\mathbf{G}+\dim Z(\mathbf{L}_{\mathcal{I}})}|C_{\mathbf{G}}^0(u)^F|$. Using $|C_{\mathbf{G}}^0(u)^F| = q^{\mathrm{rkss}\mathbf{G}}|Z^0(\mathbf{G})^F|$, we get (i).

For (ii), we apply Proposition 7 in (i), using that D is an isometry and that $\sigma_\zeta\overline{\sigma_\zeta} = q^{\mathrm{rkss}\mathbf{L}_{\mathcal{I}}}$ by [4, proposition 2.5]. □

A particular case of Proposition 8 is

Corollary 10. *If $(\mathbf{L},\iota)$ is a cuspidal pair, that is $\mathbf{L}=\mathbf{L}_{\mathcal{I}}$, then*

$$\langle R_{\mathbf{L}}^{\mathbf{G}}\Gamma_\iota^{\mathbf{L}}, R_{\mathbf{L}}^{\mathbf{G}}\Gamma_\iota^{\mathbf{L}}\rangle_{\mathbf{G}^F} = |\frac{Z(\mathbf{L})}{Z^0(\mathbf{L})}|^2|W_{\mathbf{G}}(\mathbf{L})||Z(\mathbf{L})^{0F}|.$$

We remark that this coincides with the value predicted by the Mackey formula

$$\langle R_{\mathbf{L}}^{\mathbf{G}}\Gamma_\iota^{\mathbf{L}}, R_{\mathbf{L}}^{\mathbf{G}}\Gamma_\iota^{\mathbf{L}}\rangle_{\mathbf{G}^F} = \sum_{x\in\mathbf{L}^F\backslash\mathcal{S}(\mathbf{L},\mathbf{L})/\mathbf{L}^F} \langle {}^*R_{\mathbf{L}\cap{}^x\mathbf{L}}^{\mathbf{L}}(\Gamma_\iota^{\mathbf{L}}), {}^*R_{\mathbf{L}\cap{}^x\mathbf{L}}^{{}^x\mathbf{L}}({}^x\Gamma_\iota^{\mathbf{L}})\rangle_{(\mathbf{L}\cap{}^x\mathbf{L})^F}$$

Indeed, since the block $\mathcal{I}$ which contains the local system ζ is reduced to the unique cuspidal local system (C,ζ) where C is the regular class of $\mathbf{L}$, all terms in the Mackey formula where $\mathbf{L}\cap{}^x\mathbf{L}\neq\mathbf{L}$ vanish. Thus the Mackey formula reduces to

$$\langle R_{\mathbf{L}}^{\mathbf{G}}\Gamma_\iota^{\mathbf{L}}, R_{\mathbf{L}}^{\mathbf{G}}\Gamma_\iota^{\mathbf{L}}\rangle_{\mathbf{G}^F} = \sum_{x\in W_{\mathbf{G}}(\mathbf{L})} \langle \Gamma_\iota^{\mathbf{L}}, {}^x\Gamma_\iota^{\mathbf{L}}\rangle_{\mathbf{L}^F}$$

and any x in $W_{\mathbf{L}}(\mathbf{L})$ acts trivially on $H^1(F,Z(\mathbf{L}))$ since, the map $h_{\mathbf{L}}$ being surjective, any element of $H^1(F,Z(\mathbf{L}))$ is represented by an element of $H^1(F,Z(\mathbf{G}))$; thus all the terms in the sum are equal, and we get the same result as Corollary 10 by applying Corollary 10 in the case $\mathbf{G}=\mathbf{L}$.

Another method for computing $\langle R_{\mathbf{L}}^{\mathbf{G}}D\Gamma_i, R_{\mathbf{L}}^{\mathbf{G}}D\Gamma_i\rangle_{\mathbf{G}^F}$ would be to use Proposition 1 and the values of the two-variable Green functions.

We give these values in the following table in the particular case of ${}^2E_6$ for the F-stable standard Levi subroup of type $A_2 \times A_2$, for $q \equiv -1 \pmod 3$, so that F acts trivially on $Z(\mathbf{G})/Z^0(\mathbf{G})$. This table has been computed in `Chevie` using Proposition 3. The method is to compute the one-variable Green functions which appear in the right-hand side sum by the Lusztig-Shoji algorithm; note that even though the characteristic function of cuspidal character sheaves are known only up to a root of unity, this ambiguity disappears when doing the sum, since such a scalar appears multiplied by its complex conjugate. However the Lusztig-Shoji algorithm depends also on the knowledge that when ι, with support the class of the unipotent element u, is parameterized by (u, χ) where $\chi \in \mathrm{Irr}(A(u))$ then for the unipotent element $u_a \in \mathbf{G}^F$ parameterized by $a \in H^1(F, A(u))$ we have $\mathcal{Y}_\iota(u_a) = \chi(a)$. We assume that this hold. This is known when ι is in the principal block, but not for the two blocks with cuspidal datum supported on the Levi subgroup of type $A_2 \times A_2$.

Note that the table shows that the values of $|v^{\mathbf{L}^F}|Q_{\mathbf{L}}^{\mathbf{G}}(u, v)$ are not in general polynomials with integral coefficients but may have denominators equal to $|A(v)|$.

TABLE 1. Values of $|v^{\mathbf{L}^F}|Q_{\mathbf{L}}^{\mathbf{G}}(u, v)$ for $\mathbf{G} = {}^2E_6(q)$ simply connected and $\mathbf{L} = A_2(q^2)(q-1)^2$, for $q \equiv -1 \pmod 3$.

$v\backslash u$	E_6	$E_{6(\zeta_3)}$	$E_{6(\zeta_3^2)}$	$E_6(a_1)$	$E_6(a_1)_{(\zeta_3)}$	$E_6(a_1)_{(\zeta_3^2)}$	D_5	$E_6(a_3)$
$111, 111$	0	0	0	0	0	0	0	0
$21, 21$	0	0	0	0	0	0	1	$4q+1$
$3, 3$	1	0	0	$(4q+1)/3$	$\Phi_2/3$	$\Phi_2/3$	$2q\Phi_2/3$	$(7q^2+2q-2)q/3$
$3, 3_{(\zeta_3)}$	0	1	0	$\Phi_2/3$	$(4q+1)/3$	$\Phi_2/3$	$2q\Phi_2/3$	$(q-2)q\Phi_2/3$
$3, 3_{(\zeta_3^2)}$	0	0	1	$\Phi_2/3$	$\Phi_2/3$	$(4q+1)/3$	$2q\Phi_2/3$	$(q-2)q\Phi_2/3$

$v\backslash u$	$E_6(a_3)_{(-\zeta_3^2)}$	$E_6(a_3)_{(\zeta_3)}$	$E_6(a_3)_{(-1)}$	$E_6(a_3)_{(\zeta_3^2)}$	$E_6(a_3)_{(-\zeta_3)}$
$111, 111$	0	0	0	0	0
$21, 21$	$2q+1$	$4q+1$	$2q+1$	$4q+1$	$2q+1$
$3, 3$	$q^2\Phi_2$	$(q-2)q\Phi_2/3$	$(3q+2)q^2$	$(q-2)q\Phi_2/3$	$q^2\Phi_2$
$3, 3_{(\zeta_3)}$	$(3q+2)q^2$	$(q-2)q\Phi_2/3$	$q^2\Phi_2$	$(7q^2+2q-2)q/3$	$q^2\Phi_2$
$3, 3_{(\zeta_3^2)}$	$q^2\Phi_2$	$(7q^2+2q-2)q/3$	$q^2\Phi_2$	$(q-2)q\Phi_2/3$	$(3q+2)q^2$

$v\backslash u$	A_5	$A_{5(\zeta_3)}$	$A_{5(\zeta_3^2)}$	$D_5(a_1)$
$111, 111$	0	0	0	0
$21, 21$	$(-2q-1)\Phi_2$	$(-2q-1)\Phi_2$	$(-2q-1)\Phi_2$	$3q+1$
$3, 3$	$q\Phi_2\Phi_3/3$	$(-5q^2-2q+1)q\Phi_2/3$	$(-5q^2-2q+1)q\Phi_2/3$	$q\Phi_1\Phi_2/3$
$3, 3_{(\zeta_3)}$	$(-5q^2-2q+1)q\Phi_2/3$	$q\Phi_2\Phi_3/3$	$(-5q^2-2q+1)q\Phi_2/3$	$q\Phi_1\Phi_2/3$
$3, 3_{(\zeta_3^2)}$	$(-5q^2-2q+1)q\Phi_2/3$	$(-5q^2-2q+1)q\Phi_2/3$	$q\Phi_2\Phi_3/3$	$q\Phi_1\Phi_2/3$

$v\backslash u$	$A_4{+}A_1$	D_4	A_4	$D_4(a_1)_{(111)}$
$111, 111$	0	1	0	$4q+1$
$21, 21$	$\Phi_2\Phi_3$	$3q\Phi_2\Phi_6$	$(3q^3+q^2+q+1)\Phi_2$	$(8q^3+2q^2+4q-2)q\Phi_2$
$3, 3$	$(2q+1)q^3\Phi_2/3$	$q\Phi_1\Phi_2^2\Phi_6/3$	$q^4\Phi_2^2$	$(4q+1)q^4\Phi_2^2/3$
$3, 3_{(\zeta_3)}$	$(2q+1)q^3\Phi_2/3$	$q\Phi_1\Phi_2^2\Phi_6/3$	$q^4\Phi_2^2$	$(4q+1)q^4\Phi_2^2/3$
$3, 3_{(\zeta_3^2)}$	$(2q+1)q^3\Phi_2/3$	$q\Phi_1\Phi_2^2\Phi_6/3$	$q^4\Phi_2^2$	$(4q+1)q^4\Phi_2^2/3$

$v\backslash u$	$D_4(a_1)_{(21)}$	$D_4(a_1)$	$A_3{+}A_1$	A_3
$111, 111$	$2q+1$	Φ_2	$(-2q-1)\Phi_2$	$(3q^3+2q+1)\Phi_2$
$21, 21$	$(8q^3+6q^2+2q+2)q^2$	$(2q+1)q\Phi_2\Phi_6$	$(-4q^3-q^2-2q+1)q\Phi_2^2$	$(3q^3-q^2+2q-1)q\Phi_2^2\Phi_4$
$3, 3$	$(2q+1)q^4\Phi_1\Phi_2/3$	$q^4\Phi_2\Phi_6/3$	$(-2q-1)q^5\Phi_2^2/3$	0
$3, 3_{(\zeta_3)}$	$(2q+1)q^4\Phi_1\Phi_2/3$	$q^4\Phi_2\Phi_6/3$	$(-2q-1)q^5\Phi_2^2/3$	0
$3, 3_{(\zeta_3^2)}$	$(2q+1)q^4\Phi_1\Phi_2/3$	$q^4\Phi_2\Phi_6/3$	$(-2q-1)q^5\Phi_2^2/3$	0

$v\backslash u$	$2A_2{+}A_1$	$2A_2{+}A_{1(\zeta_3)}$	$2A_2{+}A_{1(\zeta_3^2)}$	$2A_2$
$111,111$	$\Phi_2\Phi_3$	$\Phi_2\Phi_3$	$\Phi_2\Phi_3$	$\Phi_2^2\Phi_3\Phi_6$
$21,21$	$(2q^3+2q^2+4q+1)q^3\Phi_2$	$(2q^3+2q^2+4q+1)q^3\Phi_2$	$(2q^3+2q^2+4q+1)q^3\Phi_2$	$3q^4\Phi_2^2\Phi_3\Phi_6$
$3,3$	$q^6\Phi_2\Phi_3$	0	0	$q^6\Phi_2^2\Phi_3\Phi_6$
$3,3_{(\zeta_3)}$	0	$q^6\Phi_2\Phi_3$	0	0
$3,3_{(\zeta_3^2)}$	0	0	$q^6\Phi_2\Phi_3$	0

$v\backslash u$	$2A_{2(\zeta_3)}$	$2A_{2(\zeta_3^2)}$	$A_2{+}2A_1$	$A_2{+}A_1$
$111,111$	$\Phi_2^2\Phi_3\Phi_6$	$\Phi_2^2\Phi_3\Phi_6$	$(2q^4+q^3+q^2+q+1)\Phi_2$	$\Phi_2^2\Phi_3\Phi_6$
$21,21$	$3q^4\Phi_2^2\Phi_3\Phi_6$	$3q^4\Phi_2^2\Phi_3\Phi_6$	$(q^3+2q+1)q^5\Phi_2$	$(3q^2+2q+1)q^4\Phi_2^2\Phi_6$
$3,3$	0	0	0	0
$3,3_{(\zeta_3)}$	$q^6\Phi_2^2\Phi_3\Phi_6$	0	0	0
$3,3_{(\zeta_3^2)}$	0	$q^6\Phi_2^2\Phi_3\Phi_6$	0	0

$v\backslash u$	$A_{2(11)}$	A_2
$111,111$	$(3q^5+q^2+q+1)\Phi_2^2\Phi_6$	$5q^9+3q^8+4q^7+4q^6+5q^5+4q^4+2q^3+2q^2+2q+1$
$21,21$	$(4q^2+q+1)q^4\Phi_2^3\Phi_6^2$	$(2q+1)q^4\Phi_1\Phi_2\Phi_3\Phi_4\Phi_6$
$3,3$	0	0
$3,3_{(\zeta_3)}$	0	0
$3,3_{(\zeta_3^2)}$	0	0

$v\backslash u$	$3A_1$	$2A_1$
$111,111$	$(-3q^9-3q^8-3q^6-3q^5-2q^4-q^3-q^2-q-1)\Phi_2$	$(2q^8+q^6+q^5+q^4+1)\Phi_2^2\Phi_3\Phi_6$
$21,21$	$(-2q^2-1)q^7\Phi_2^3\Phi_6$	$q^9\Phi_2^3\Phi_3\Phi_4\Phi_6$
$3,3$	0	0
$3,3_{(\zeta_3)}$	0	0
$3,3_{(\zeta_3^2)}$	0	0

$v\backslash u$	A_1	1
$111,111$	$(2q^{10}+q^9+q^8+q^7+2q^6+q^5+q^4+q^2+q+1)\Phi_2^3\Phi_6\Phi_{10}$	$\Phi_2^4\Phi_3\Phi_4\Phi_6^2\Phi_8\Phi_{10}\Phi_{12}\Phi_{18}$
$21,21$	0	0
$3,3$	0	0
$3,3_{(\zeta_3)}$	0	0
$3,3_{(\zeta_3^2)}$	0	0

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