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HARDY'S INEQUALITY FOR FUNCTIONS VANISHING ON A PART OF THE BOUNDARY

MORITZ EGERT, ROBERT HALLER-DINTELMANN, AND JOACHIM REHBERG

ABSTRACT. We develop a geometric framework for Hardy's inequality on a bounded domain when the functions do vanish only on a closed portion of the boundary.

1. INTRODUCTION

Hardy's inequality is one of the classical items in analysis [27, 42]. Two milestones among many others in the development of the theory seem to be the result of Necas [41] that Hardy's inequality holds on strongly Lipschitz domains and the insight of Maz'ya [38], [39, Ch. 2.3] that its validity depends on measure theoretic conditions on the domain. Rather recently, the geometric framework in which Hardy's inequality remains valid was enlarged up to the frontiers of what is possible – as long as the boundary condition is purely Dirichlet, see [25, 28], compare also [3, 31, 48]. Moreover, over the last years it became manifest that Hardy's inequality plays an eminent role in modern PDE theory, see e.g. [7, 46, 43, 2, 13, 9, 16, 23, 32, 34].

What has not been treated systematically is the case where *only a part* D of the boundary of the underlying domain Ω is involved, reflecting the Dirichlet condition of the equation on this part – while on $\partial\Omega \setminus D$ other boundary conditions may be imposed, compare [11, 26, 2, 24, 8] including references therein. The aim of this paper is to set up a geometric framework for the domain Ω and the Dirichlet boundary part D that allow to deduce the corresponding Hardy inequality

$$\int_{\Omega} \left| \frac{u}{\text{dist}_D} \right|^p dx \leq c \int_{\Omega} |\nabla u|^p dx.$$

As in the well established case $D = \partial\Omega$ we in essence only require that D is l -thick in the sense of [28]. In our context this condition can be understood as an extremely weak compatibility condition between D and $\partial\Omega \setminus D$.

Our strategy of proof is first to reduce to the case $D = \partial\Omega$ by purely topological means, provided two major tools are applicable: An extension operator $\mathfrak{E} : W_D^{1,p}(\Omega) \rightarrow W_D^{1,p}(\mathbb{R}^d)$, the subscript D indicating the subspace of those Sobolev functions which vanish on D in an appropriate sense, and a Poincaré inequality on $W_D^{1,p}(\Omega)$. This abstract result is established in Section 5. In a second step in Sections 6 and 7 these partly implicit conditions are substantiated by more geometric assumptions that can be checked – more or less – by appearance. In particular, we prove that under the mere assumption that D is closed, every linear continuous extension operator $W_D^{1,p}(\Omega) \rightarrow W^{1,p}(\mathbb{R}^d)$ that is constructed by the usual procedure of gluing together local extension operators preserves the Dirichlet condition on D . This result even carries over to higher-order Sobolev spaces and sheds new light on some of the deep results on Sobolev extension operators obtained in [4].

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It is of course natural to ask, whether Hardy's inequality also characterizes the space $W_D^{1,p}(\Omega)$, i.e. whether the latter is precisely the space of those functions $u \in W^{1,p}(\Omega)$ for which u/dist_D belongs to $L^p(\Omega)$. Under very mild geometric assumptions we answer this question to the affirmative in Section 8.

Finally, in Section 9 we attend to the naive intuition that the part of $\partial\Omega$ that is far away from D should only be circumstantial for the validity of Hardy's inequality and in fact we succeed to weaken the previously discussed geometric assumptions considerably.

2. NOTATION

Throughout we work in Euclidean space \mathbb{R}^d , $d \geq 1$. We use \mathbf{x} , \mathbf{y} , etc. for vectors in \mathbb{R}^d and denote the open ball in \mathbb{R}^d around \mathbf{x} with radius r by $B(\mathbf{x}, r)$. The letter c is reserved for generic constants that may change their value from occurrence to occurrence. Given $F \subset \mathbb{R}^d$ we write dist_F for the function that measures the distance to F and $\text{diam}(F)$ for the diameter of F .

In our main results on Hardy's inequality we denote the underlying domain and its Dirichlet part by Ω and D . The various side results that are interesting in themselves and drop off on the way are identified by the use of Λ and E instead.

Next, let us introduce the common first-order Sobolev spaces of functions 'vanishing' on a part of the closure of the underlying domain that are most essential for the formulation of Hardy's inequality.

Definition 2.1. If Λ is an open subset of \mathbb{R}^d and E is a closed subset of $\overline{\Lambda}$, then for $p \in [1, \infty[$ the space $W_E^{1,p}(\Lambda)$ is defined as the completion of

$$C_E^\infty(\Lambda) := \{v|_\Lambda : v \in C_0^\infty(\mathbb{R}^d), \text{supp}(v) \cap E = \emptyset\}$$

with respect to the norm $v \mapsto (\int_\Lambda |\nabla v|^p + |v|^p \, dx)^{1/p}$. More generally, for $k \in \mathbb{N}$ we define $W_E^{k,p}(\Lambda)$ as the closure of $C_E^\infty(\Lambda)$ with respect to the norm $v \mapsto (\int_\Lambda \sum_{j=0}^k |D^j v|^p \, dx)^{1/p}$.

The situation we have in mind is of course when $\Lambda = \Omega$ and $E = D$ is the Dirichlet part D of the boundary $\partial\Omega$.

As usual, the Sobolev spaces $W^{k,p}(\Lambda)$ are defined as the space of those $L^p(\Lambda)$ functions whose distributional derivatives up to order k are in $L^p(\Lambda)$, equipped with the natural norm. Note that by definition $W_0^{k,p}(\Lambda) = W_{\partial\Lambda}^{k,p}(\Lambda)$ but in general $W_\emptyset^{k,p}(\Lambda) \subsetneq W^{k,p}(\Lambda)$, cf. [39, Sec. 1.1.6]

3. MAIN RESULTS

The following version of Hardy's inequality for functions vanishing on a part of the boundary is our main result. Readers not familiar with the measure theoretic concepts used to describe the regularity of the Dirichlet part D may refer to Section 4.1 beforehand.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, $D \subset \partial\Omega$ be a closed part of the boundary and $p \in]1, \infty[$. Suppose that the following three conditions are satisfied.*

- (i) *The set D is l -thick for some $l \in]d - p, d]$.*
- (ii) *The space $W_D^{1,p}(\Omega)$ can be equivalently normed by $\|\nabla \cdot\|_{L^p(\Omega)}$.*
- (iii) *There is a linear continuous extension operator $\mathfrak{E} : W_D^{1,p}(\Omega) \rightarrow W_D^{1,p}(\mathbb{R}^d)$.*

Then there is a constant $c > 0$ such that Hardy's inequality

$$(3.1) \quad \int_\Omega \left| \frac{u}{\text{dist}_D} \right|^p \, dx \leq c \int_\Omega |\nabla u|^p \, dx$$

holds for all $u \in W_D^{1,p}(\Omega)$.

Of course the conditions (ii) and (iii) in Theorem 3.1 are rather abstract and should be supported by more geometrical ones. This will be the content of Sections 6 and 7 where we shall give an extensive kit of such conditions. In particular, we will obtain the following version of Hardy's inequality.

Theorem 3.2 (A special Hardy inequality). *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and $p \in]1, \infty[$. Let $D \subset \partial\Omega$ be l -thick for some $l \in]d-p, d]$ and assume that for every $x \in \overline{\partial\Omega} \setminus D$ there is an open neighborhood U_x of x such that $\Omega \cap U_x$ is a $W^{1,p}$ -extension domain. Then there is a constant $c > 0$ such that*

$$\int_{\Omega} \left| \frac{u}{\text{dist}_D} \right|^p dx \leq c \int_{\Omega} |\nabla u|^p dx, \quad u \in W_D^{1,p}(\Omega).$$

Remark 3.3. The assumptions in the above theorem are met for all $p \in]1, \infty[$ if D is a $(d-1)$ -set and for every $x \in \overline{\partial\Omega} \setminus D$ there is an open neighborhood U_x of x such that $\Omega \cap U_x$ is a Lipschitz- or more generally an (ε, δ) -domain, see Subsections 4.1 and 6.4 for definitions.

Still, as we believe, the abstract framework traced out by the second and the third condition of Theorem 3.1 has the advantage that other sufficient geometric conditions for Hardy's inequality – tailor-suited for future applications – can be found much more easily. In fact the second condition is equivalent to the validity of Poincaré's inequality

$$\|u\|_{L^p} \leq c \|\nabla u\|_{L^p(\Omega)}, \quad u \in W_D^{1,p}(\Omega),$$

that is clearly *necessary* for Hardy's inequality (3.1). We give a detailed discussion of Poincaré's inequality within the present context in Section 7. For further reference the reader may consult [50, Ch. 4]. Concerning the third condition note carefully that we require the extension operator to preserve the Dirichlet boundary condition on D . Whereas extension of Sobolev functions is a well-established business, the preservation of traces is much more delicate and we devote Subsection 6.3 to this problem.

It is interesting to remark that under geometric assumptions very similar to those in Theorem 3.2 the space $W_D^{1,p}(\Omega)$ is the largest subspace of $W^{1,p}(\Omega)$ in which Hardy's inequality can hold. This is made precise by our third main result.

Theorem 3.4. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and $p \in]1, \infty[$. Let $D \subset \partial\Omega$ be porous and l -thick for some $l \in]d-p, d]$. Finally assume that for every $x \in \overline{\partial\Omega} \setminus D$ there is an open neighborhood U_x of x such that $\Omega \cap U_x$ is a $W^{1,p}$ -extension domain. If $u \in W^{1,p}(\Omega)$ is such that $u/\text{dist}_D \in L^p(\Omega)$, then already $u \in W_D^{1,p}(\Omega)$.*

Remark 3.5. (i) The assumption on D are met if D is an l -set for some $l \in]d-p, d]$, see Remark 4.4 below.
(ii) In the case $D = \partial\Omega$ the conclusion of Theorem 3.4 is classical [10, Thm. V.3.4] and remains true without any assumptions on $\partial\Omega$.

In Section 5 we give the proof of the general Hardy inequality from Theorem 3.1. The proofs of Theorem 3.2 and 3.4 are postponed to the end of Sections 6 and 8, respectively.

4. PRELIMINARIES

4.1. Regularity concepts for the Dirichlet part. For convenience we recall the notions from geometric measure theory that are used to describe the regularity of the Dirichlet part D in Hardy's inequality. For $l \in]0, \infty[$ the l -dimensional Hausdorff measure of $F \subset \mathbb{R}^d$ is

$$\mathcal{H}_l(F) := \liminf_{\delta \rightarrow 0} \left\{ \sum_{j=1}^{\infty} \text{diam}(F_j)^l : F_j \subset \mathbb{R}^d, \text{diam}(F_j) \leq \delta, F \subset \bigcup_{j=1}^{\infty} F_j \right\}$$

and its centered Hausdorff content is defined by

$$\mathcal{H}_l^\infty(F) := \inf \left\{ \sum_{j=1}^{\infty} r_j^l : x_j \in F, r_j > 0, F \subset \bigcup_{j=1}^{\infty} B(x_j, r_j) \right\}.$$

Definition 4.1. Let $l \in]0, \infty[$. A non-empty compact set $F \subset \mathbb{R}^d$ is called *l-thick* if there exist $R > 0$ and $\gamma > 0$ such that

$$(4.1) \quad \mathcal{H}_l^\infty(F \cap B(x, r)) \geq \gamma r^l$$

holds for all $x \in F$ and all $r \in]0, R]$. It is called *l-set* if there are two constants $c_0, c_1 > 0$ such that

$$c_0 r^l \leq \mathcal{H}_l(F \cap B(x, r)) \leq c_1 r^l$$

holds for all $x \in F$ and all $r \in]0, 1]$.

Remark 4.2. (i) If (4.1) holds for constants R, γ , then for all $S \geq R$ it also holds with R and γ replaced by S and $\gamma R^l S^{-l}$, respectively. For more information on this notion of *l-thick* sets the reader can refer to [28].

(ii) The notion of *l-sets* is due to [22, Sec. II.1]. It can be extended literally to arbitrary Borel sets F , see [22, Sec. VII.1.1].

Definition 4.3. A set $F \subset \mathbb{R}^d$ is *porous* if for some $\kappa \leq 1$ the following statement is true: For every ball $B(x, r)$ with $x \in \mathbb{R}^d$ and $0 < r \leq 1$ there is $y \in B(x, r)$ such that $B(y, \kappa r) \cap F = \emptyset$.

Remark 4.4. It is known that a set $F \subset \mathbb{R}^d$ is porous if and only if its so-called Assouad dimension is strictly less than the space dimension d , see [33, Thm. 5.2]. Recently it was shown in [30] that this notion of dimension coincides with the one introduced by Aikawa, that is the infimum of all $t > 0$ for which there exists $c_t > 0$ such that

$$\int_{B(x, r)} \text{dist}(x, F)^{t-d} dx \leq c_t r^t, \quad x \in F, r > 0.$$

In particular, each *l-set*, $l \in]0, d[$, has Aikawa dimension equal to l and thus is porous [29, Lem. 2.1].

For a later use we include a proof of the following two elementary facts. We remark that the first lemma is also implicit in [6, Lem. 2].

Lemma 4.5. *Let $l \in]0, \infty[$. If $F \subset \mathbb{R}^d$ is a compact *l-set*, then there are constants $c_0, c_1 > 0$ such that*

$$c_0 r^l \leq \mathcal{H}_l^\infty(F \cap B(x, r)) \leq c_1 r^l$$

*holds for all $r \in]0, 1[$ and all $x \in F$. In particular, F is *l-thick*.*

Proof. We prove $\mathcal{H}_l^\infty(A) \leq \mathcal{H}_l(A) \leq c \mathcal{H}_l^\infty(A)$ for all non-empty Borel subsets $A \subset F$.

First, fix $\varepsilon > 0$ and let $\{A_j\}_{j \in \mathbb{N}}$ be a covering of A by sets with diameter at most ε . If $A_j \cap A \neq \emptyset$, then A_j is contained in an open ball B_j centered in A and radius such that $r_j^l = \text{diam}(A_j)^l + \varepsilon 2^{-j}$. The so-obtained countable covering $\{B_j\}$ of A satisfies

$$\sum_{\substack{j \in \mathbb{N} \\ A_j \cap A \neq \emptyset}} \text{diam}(A_j)^l \geq \sum_{\substack{j \in \mathbb{N} \\ A_j \cap A \neq \emptyset}} (r_j^l - \varepsilon 2^{-j}) \geq \mathcal{H}_l^\infty(A) - \varepsilon.$$

Taking the infimum over all such coverings $\{A_j\}_{j \in \mathbb{N}}$ and passing to the limit $\varepsilon \rightarrow 0$ afterwards, $\mathcal{H}_l^\infty(A) \leq \mathcal{H}_l(A)$ follows. Conversely, let $\{B_j\}_{j \in \mathbb{N}}$ be a covering of A by open balls with radii r_j centered in A . If $r_j \leq 1$, then $\mathcal{H}_l(F \cap B_j) \leq c r_j^l$ since by assumption F is an *l-set*, and if $r_j > 1$,

then certainly $\mathcal{H}_l(F \cap B_j) \leq \mathcal{H}_l(F)r_j^l$. Note carefully that $0 < \mathcal{H}_l(F) < \infty$ holds for F can be covered by finitely many balls with radius 1 centered in F . Altogether,

$$\sum_{j=1}^{\infty} r_j^l \geq c \sum_{j=1}^{\infty} \mathcal{H}_l(F \cap B_j) \geq c\mathcal{H}_l\left(F \cap \bigcup_{j=1}^{\infty} B_j\right) \geq c\mathcal{H}_l(A).$$

Passing to the infimum, $\mathcal{H}_l^\infty(A) \geq c\mathcal{H}_l(A)$ follows. \square

Lemma 4.6. *If $F \subset \mathbb{R}^d$ is l -thick, then it is m -thick for every $m \in]0, l[$.*

Proof. Inspecting the definition of thick sets, the claim turns out to be a direct consequence of the inequality

$$\sum_{j=1}^N r_j^m \geq \left(\sum_{j=1}^N r_j^l\right)^{m/l}$$

for positive real numbers r_1, \dots, r_N . \square

4.2. Quasieverywhere defined functions. The results of Sections 6-8 rely on deep insights from potential theory and we shall recall the necessary notions beforehand. For further background we refer e.g. to [1].

Definition 4.7. Let $\alpha > 0$, $p \in]1, \infty[$ and let $F \subset \mathbb{R}^d$. Denote by $G_\alpha := \mathcal{F}^{-1}((1 + |\xi|^2)^{-\alpha/2})$ the Bessel kernel of order α . Then

$$C_{\alpha,p}(F) := \inf \left\{ \int_{\mathbb{R}^d} |f|^p : f \geq 0 \text{ on } \mathbb{R}^d \text{ and } G_\alpha * f \geq 1 \text{ on } F \right\}$$

is called (α, p) -capacity of F . The corresponding Bessel potential space is

$$H^{\alpha,p}(\mathbb{R}^d) := \{G_\alpha * f : f \in L^p(\mathbb{R}^d)\} \quad \text{with norm} \quad \|G_\alpha * f\|_{H^{\alpha,p}(\mathbb{R}^d)} = \|f\|_p.$$

It is well-known that for $k \in \mathbb{N}$ the spaces $H^{k,p}(\mathbb{R}^d)$ and $W^{k,p}(\mathbb{R}^d)$ coincide up to equivalent norms [45, Sec. 2.3.3]. The capacities $C_{\alpha,p}$ are outer measures on \mathbb{R}^d [1, Sec. 2.3]. A property that holds true for all x in some set $E \subset \mathbb{R}^d$ but those belonging to an exceptional set $F \subset E$ with $C_{\alpha,p}(F) = 0$ is said to be true (α, p) -quasieverywhere on E , abbreviated (α, p) -q.e. A property that holds true (α, p) -q.e. also holds true (β, p) -q.e. if $\beta < \alpha$. This is an easy consequence of [1, Prop. 2.3.13]. A more involved result in this direction is the following [1, Thm. 5.5.1]

Lemma 4.8. *Let $\alpha, \beta > 0$ and $1 < p, q < \infty$ be such that $\beta q < \alpha p < d$. Then each $C_{\alpha,p}$ -nullset also is a $C_{\beta,q}$ -nullset*

There is also a close connection between capacities and Hausdorff measures, cf. [1, Ch. 5.] for an exhaustive discussion. Most important for us is the following comparison theorem. In the case $p \in]1, d[$ this is proved in [1, Sec. 5] and if $p \in]d, \infty[$, then the result follows directly from [1, Prop. 2.6.1].

Theorem 4.9 (Comparison Theorem). *Let $1 < p < \infty$ and suppose $\alpha, l > 0$ are such that $d - l < \alpha p < \infty$. Then every $C_{\alpha,p}$ -nullset is also a \mathcal{H}_l - and thus a \mathcal{H}_l^∞ -nullset.*

Bessel capacities naturally occur when studying convergence of average integrals for Sobolev functions. In fact, if $\alpha > 0$, $p \in]1, \frac{d}{\alpha}]$ and $u \in H^{\alpha,p}(\mathbb{R}^d)$, then (α, p) -quasievery $y \in \mathbb{R}^d$ is a Lebesgue point for u in the L^p -sense, that is

$$(4.2) \quad \lim_{r \rightarrow 0} \frac{1}{|B(y, r)|} \int_{B(y, r)} u(x) \, dx =: u(y)$$

and

$$(4.3) \quad \lim_{r \rightarrow 0} \frac{1}{|B(y, r)|} \int_{B(y, r)} |u(x) - u(y)|^p \, dx = 0$$

hold [1, Thm. 6.2.1]. The (α, p) -quasieverywhere defined function \mathbf{u} reproduces u within its $H^{\alpha, p}$ -class. It gives rise to a meaningful (α, p) -quasieverywhere defined restriction $u|_E := \mathbf{u}|_E$ of u to E whenever E has non-vanishing (α, p) -capacity. For convenience we agree upon that $u|_E = 0$ is true for all $u \in H^{\alpha, p}(\mathbb{R}^d)$ if E has zero (α, p) -capacity. Note also that these results remain true if $p \in]\frac{d}{\alpha}, \infty[$, since in this case u has a Hölder continuous representative \mathbf{u} which then satisfies (4.2) and (4.3) for every $y \in \mathbb{R}^d$.

We obtain an alternate definition for Sobolev spaces with partially vanishing traces.

Definition 4.10. Let $k \in \mathbb{N}$, $p \in]1, \infty[$ and $E \subseteq \mathbb{R}^d$ be closed. Define

$$\mathcal{W}_E^{k, p}(\mathbb{R}^d) := \left\{ u \in W^{k, p}(\mathbb{R}^d) : D^\beta u|_E = 0 \text{ holds } (k - |\beta|, p)\text{-q.e. on } E \right. \\ \left. \text{for all multiindices } \beta, 0 \leq |\beta| \leq k - 1 \right\}$$

and equip it with the $W^{k, p}(\mathbb{R}^d)$ -norm.

The following theorem of Hedberg and Wolff is also known as the (k, p) -*synthesis*.

Theorem 4.11 ([1, Thm. 9.1.3]). *The spaces $\mathcal{W}_E^{k, p}(\mathbb{R}^d)$ and $\mathcal{W}_E^{k, p}(\mathbb{R}^d)$ coincide whenever $k \in \mathbb{N}$, $p \in]1, \infty[$ and $E \subset \mathbb{R}^d$ is closed.*

Hedberg and Wolff's theorem manifests the use of capacities in the study of traces of Sobolev functions. However, if one invests more on the geometry of E , e.g. if one assumes that it is an l -set, then by the subsequent recent result of Brewster, Mitrea, Mitrea and Mitrea capacities can be replaced by the l -dimensional Hausdorff measure at each occurrence.

Theorem 4.12 ([4, Thm. 4.4, Cor. 4.5]). *Let $k \in \mathbb{N}$, $p \in]1, \infty[$ and let $E \subset \mathbb{R}^d$ be closed and additionally an l -set for some $l \in]d - p, d]$. Then*

$$\mathcal{W}_E^{k, p}(\mathbb{R}^d) = \mathcal{W}_E^{k, p}(\mathbb{R}^d) = \left\{ u \in W^{k, p}(\mathbb{R}^d) : D^\beta u|_E = 0 \text{ holds } \mathcal{H}_{d-1}\text{-a.e. on } E \right. \\ \left. \text{for all multiindices } \beta, 0 \leq |\beta| \leq k - 1 \right\},$$

where on the right-hand side $D^\beta u|_E = 0$ means, as before, that for \mathcal{H}_{d-1} -almost every $y \in E$ the average integrals $\frac{1}{|B(y, r)|} \int_{B(y, r)} D^\beta u(x) \, dx$ vanish in the limit $r \rightarrow 0$.

5. PROOF OF THEOREM 3.1

We will deduce Theorem 3.1 from the following proposition that states the assertion in the case $D = \partial\Omega$.

Proposition 5.1 ([28], see also [25]). *Let $\Omega_\bullet \subseteq \mathbb{R}^d$ be a bounded domain and let $p \in]1, \infty[$. If $\partial\Omega_\bullet$ is l -thick for some $l \in]d - p, d]$, then Hardy's inequality is satisfied for all $u \in W_0^{1, p}(\Omega_\bullet)$, i.e. (3.1) holds with Ω replaced by Ω_\bullet and D by $\partial\Omega_\bullet$.*

Below we will reduce to the case $D = \partial\Omega$ by purely topological means, so that we can apply Proposition 5.1 afterwards. We will repeatedly use the following topological fact.

(■) Let $\{M_\lambda\}_\lambda$ be a family of connected subsets of a topological space. If $\bigcap_\lambda M_\lambda \neq \emptyset$, then $\bigcup_\lambda M_\lambda$ is again connected.

As required in Theorem 3.1 let now $\Omega \subseteq \mathbb{R}^d$ be a bounded domain and let D be a closed part of $\partial\Omega$. Then choose an open ball $B \supseteq \bar{\Omega}$ that, in what follows, will be considered as the relevant topological space. Consider

$$\mathcal{C} := \{M \subset B \setminus D : M \text{ open, connected and } \Omega \subset M\}$$

and for the rest of the proof put

$$\Omega_\bullet := \bigcup_{M \in \mathcal{C}} M.$$

In the subsequent lemma we collect some properties of Ω_\bullet . Our proof here is not the shortest possible, cf. [5, Lem. 6.4] but it has, however, the advantage to give a description of Ω_\bullet as the union of Ω , the boundary part $\partial\Omega \setminus D$ and those connected components of $B \setminus \bar{\Omega}$ whose boundary does not consist only of points from D . This completely reflects the naive geometric intuition.

Lemma 5.2. *It holds $\Omega \subseteq \Omega_\bullet \subseteq B$. Moreover, Ω_\bullet is open and connected and $\partial\Omega_\bullet = D$ in B .*

Proof. The first assertion is obvious. By construction Ω_\bullet is open. Since all elements from \mathcal{C} contain Ω the connectedness of Ω_\bullet follows by (■). It remains to show $\partial\Omega_\bullet = D$.

Let $x \in D$. Then x is an accumulation point of Ω and, since $\Omega \subseteq \Omega_\bullet$, also of Ω_\bullet . On the other hand, $x \notin \Omega_\bullet$ by construction. This implies $x \in \partial\Omega_\bullet$ and so $D \subseteq \partial\Omega_\bullet$.

In order to show the inverse inclusion, we first show that points from $\partial\Omega \setminus D$ cannot belong to $\partial\Omega_\bullet$. Indeed, since D is closed, for $x \in \partial\Omega \setminus D$ there is a ball $B_x \subseteq B$ around x that does not intersect D . Since x is a boundary point of Ω , we have $B_x \cap \Omega \neq \emptyset$. Both Ω and B_x are connected, so (■) yields that $\Omega \cup B_x$ is connected. Moreover, this set is open, contains Ω and avoids D , so it belongs to \mathcal{C} and we obtain $\Omega \cup B_x \subseteq \Omega_\bullet$. This in particular yields $x \in \Omega_\bullet$, so $x \notin \partial\Omega_\bullet$ since Ω_\bullet is open.

Summing up, we already know that $x \in \bar{\Omega}$ belongs to $\partial\Omega_\bullet$ if and only if $x \in D$. So, it remains to make sure that no point from $B \setminus \bar{\Omega}$ belongs to $\partial\Omega_\bullet$.

As $B \setminus \bar{\Omega}$ is open, it splits up into its open connected components Z_0, Z_1, Z_2, \dots . There are possibly only finitely many such components but at least one. We will show in a first step that for all these components it holds $\partial Z_j \subseteq \partial\Omega$. This allows to distinguish the two cases $\partial Z_j \subseteq D$ and $\partial Z_j \cap (\partial\Omega \setminus D) \neq \emptyset$. In Steps 2 and 3 we will then complete the proof by showing that in both cases Z_j does not intersect $\partial\Omega_\bullet$.

Step 1: $\partial Z_j \subseteq \partial\Omega$ for all j . First note that $\partial Z_j \cap \Omega = \emptyset$ for all j . Indeed, assuming this set to be non-empty and investing that Ω is open, we find that the set $Z_j \cap \Omega$ cannot be empty either and this contradicts the definition of Z_j .

Now, to prove the claim of Step 1, assume by contradiction that, for some j , there is a point $x \in \partial Z_j$ that does not belong to $\partial\Omega$. By the observation above we then have $x \notin \bar{\Omega}$ and consequently there is a ball B_x around x that does not intersect $\bar{\Omega}$. Now, the set $B_x \cup Z_j$ is connected thanks to (■), avoids $\bar{\Omega}$ and includes Z_j properly. However, this contradicts the property of Z_j to be a connected component of $B \setminus \bar{\Omega}$.

Step 2: If $\partial Z_j \subseteq D$, then $\bar{\Omega}_\bullet \cap Z_j = \emptyset$. We first note that it suffices to show $\Omega_\bullet \cap Z_j = \emptyset$. In fact, due to $\bar{\Omega}_\bullet = \partial\Omega_\bullet \cup \Omega_\bullet$ we then get $\bar{\Omega}_\bullet \cap Z_j = \emptyset$ since Z_j is open.

So, let us assume there is some $x \in \Omega_\bullet \cap Z_j$. Then $\Omega_\bullet \cup Z_j$ is connected due to (■). By assumption we have $\partial Z_j \subseteq D$ and by construction the sets Z_j and Ω_\bullet are both disjoint to D . So we can infer that $\partial Z_j \cap (\Omega_\bullet \cup Z_j) = \emptyset$ and this allows us to write

$$\Omega_\bullet \cup Z_j = (\Omega_\bullet \cup Z_j) \cap (Z_j \cup (B \setminus \bar{Z}_j)) = Z_j \cup (\Omega_\bullet \cap (B \setminus \bar{Z}_j)).$$

This is a decomposition of $\Omega_\bullet \cup Z_j$ into two open and mutually disjoint sets, so if we can show that both are nonempty then this yields a contradiction to the connectedness of $\Omega_\bullet \cup Z_j$ and the claim of Step 2 follows. Indeed, we even find

$$\Omega_\bullet \cap (B \setminus \bar{Z}_j) = \Omega_\bullet \setminus \bar{Z}_j = \Omega_\bullet \setminus (\partial Z_j \cup Z_j) \supset \Omega \setminus (D \cup Z_j) = \Omega \neq \emptyset,$$

since both D and Z_j do not intersect Ω .

Step 3: If $\partial Z_j \cap (\partial\Omega \setminus D) \neq \emptyset$, then $Z_j \subseteq \Omega_\bullet$. Let $x \in \partial Z_j \cap (\partial\Omega \setminus D)$, and let B_x be a ball around x that does not intersect D . The point x is a boundary point of Z_j , so $B_x \cap Z_j \neq \emptyset$ and we obtain that $B_x \cup Z_j$ is connected by (■). By the same argument, also the set $B_x \cup \Omega$ is connected and putting these two together a third reiteration of the argument yields that

$(B_x \cup \Omega) \cup (B_x \cup Z_j) = \Omega \cup B_x \cup Z_j$ is again connected. This last set is open and does not intersect D , so it belongs to \mathcal{C} and we end up with $\Omega \cup B_x \cup Z_j \subseteq \Omega_\bullet$. In particular we have $Z_j \subseteq \Omega_\bullet$. \square

Remark 5.3. Conversely, it can be shown that the asserted properties characterize Ω_\bullet uniquely in the sense that if an open, connected subset $\Xi \supset \Omega$ of B additionally satisfies $\partial\Xi = D$, then necessarily $\Xi = \Omega_\bullet$. In fact, since $\Xi \cap D = \emptyset$ one has $\Xi \subset \Omega_\bullet$, due to the definition of Ω_\bullet . In order to obtain the inverse inclusion we write

$$(5.1) \quad \Omega_\bullet = (\Omega_\bullet \cap \Xi) \cup (\Omega_\bullet \cap \partial\Xi) \cup (\Omega_\bullet \cap (B \setminus \bar{\Xi})) = \Xi \cup (\Omega_\bullet \cap (B \setminus \bar{\Xi})),$$

since $\Omega_\bullet \cap \partial\Xi = \Omega_\bullet \cap D = \emptyset$. Both $\Xi = \Xi \cap \Omega_\bullet$ and $\Omega_\bullet \cap (B \setminus \bar{\Xi})$ are open in Ω_\bullet , and $\Xi \supset \Omega$ is non-empty. Since Ω_\bullet is connected and $\Xi = \Xi \cap \Omega_\bullet$ is clearly disjoint to $\Omega_\bullet \cap (B \setminus \bar{\Xi})$, this latter set must be empty. Thus, (5.1) gives $\Xi = \Omega_\bullet$.

Corollary 5.4. *Consider Ω_\bullet as a subset of \mathbb{R}^d . Then Ω_\bullet is open and connected. Moreover, either $\partial\Omega_\bullet = D$ or $\partial\Omega_\bullet = D \cup \partial B$.*

Proof. It is clear that Ω_\bullet remains open. Assume that Ω_\bullet is not connected. Then there are disjoint open sets $U, V \subseteq \mathbb{R}^d$ such that $\Omega_\bullet = U \cup V$. However, the property $\Omega_\bullet \subseteq B$ then gives $\Omega_\bullet = \Omega_\bullet \cap B = (U \cap B) \cup (V \cap B)$, where $U \cap B$ and $V \cap B$ are open in B and disjoint to each other. This contradicts Lemma 5.2.

For the last assertion consider an annulus $A \subseteq B$ that is adjacent to ∂B and does not intersect $\bar{\Omega}$. Let Z_j be the connected component of $B \setminus \bar{\Omega}$ that contains A . We distinguish again the two cases of Step 2 and Step 3 in the proof of Lemma 5.2: If $\partial Z_j \subseteq D$, we have shown in Step 2 that Z_j is disjoint to Ω_\bullet and this implies $\partial\Omega_\bullet = \partial\Omega_\bullet \cap B = D$. In the second case, we infer from Step 3 in the above proof that $A \subseteq Z_j \subseteq \Omega_\bullet$ and this implies $\partial\Omega_\bullet = D \cup \partial B$. \square

Let us now conclude the proof of Theorem 3.1. We first observe that in both cases appearing in Corollary 5.4 the set $\partial\Omega_\bullet$ is m -thick for some $m \in]d - p, d - 1]$. In fact, D is l -thick for some $l \in]d - p, d]$ by assumption and using its local representation as the graph of a Lipschitz function, it can easily be checked that ∂B is a $(d - 1)$ -set, hence $(d - 1)$ -thick owing to Lemma 4.5. The claim follows from Lemma 4.6. Altogether, Proposition 5.1 applies to our special choice of Ω_\bullet .

Now, let \mathfrak{E} be the extension operator provided by Assumption (iii) of Theorem 3.1. In view of Corollary 5.4 we can define an extension operator $\mathfrak{E}_\bullet : W_D^{1,p}(\Omega) \rightarrow W_0^{1,p}(\Omega_\bullet)$ as follows: If $\partial\Omega_\bullet = D$, then we put $\mathfrak{E}_\bullet v := \mathfrak{E}v|_{\Omega_\bullet}$ and if $\partial\Omega_\bullet = D \cup \partial B$, then we choose $\eta \in C_0^\infty(B)$ with the property $\eta \equiv 1$ on $\bar{\Omega}$ and put $\mathfrak{E}_\bullet v := (\eta\mathfrak{E}v)|_{\Omega_\bullet}$. This allows us to apply Proposition 5.1 to the functions $\mathfrak{E}_\bullet u \in W_0^{1,p}(\Omega_\bullet)$, where u is taken from $W_D^{1,p}(\Omega)$. With a final help of Assumption (ii) in Theorem 3.1 this gives

$$\begin{aligned} \int_\Omega \left| \frac{u}{d_D} \right|^p dx &\leq \int_\Omega \left| \frac{u}{d_{\partial\Omega_\bullet}} \right|^p dx \leq \int_{\Omega_\bullet} \left| \frac{\mathfrak{E}_\bullet u}{d_{\partial\Omega_\bullet}} \right|^p dx \leq c \int_{\Omega_\bullet} |\nabla(\mathfrak{E}_\bullet u)|^p dx \\ &\leq c \|\mathfrak{E}_\bullet u\|_{W_0^{1,p}(\Omega_\bullet)}^p \leq c \|u\|_{W_D^{1,p}(\Omega)}^p \leq c \int_\Omega |\nabla u|^p dx \end{aligned}$$

for all $u \in W_D^{1,p}(\Omega)$ and the proof is complete.

Remark 5.5. (i) At the first glance one might think that Ω_\bullet could always be taken as $B \setminus D$. The point is that this set need not be connected, as the following example shows. Take $\Omega = \{x : 1 < |x| < 2\}$ and $D = \{x : |x| = 1\} \cup \{x : |x| = 2, x_1 \geq 0\}$. Obviously, if a ball B contains $\bar{\Omega}$, then $B \setminus D$ cannot be connected. In the spirit of Lemma 5.2, the set Ω_\bullet has here to be taken as $B \setminus (D \cup \{x : |x| < 1\})$. Thus, the somewhat subtle, topological considerations above cannot be avoided in general.

(ii) One might suggest that the procedure of this work is not limited to the proof of Hardy's inequality in the non-Dirichlet case. Possibly the combination of an application of the extension operator $\mathfrak{E}/\mathfrak{E}_\bullet$ and the construction of Ω_\bullet may serve for the reduction of other

problems on function spaces related to mixed boundary conditions to the pure Dirichlet case.

Finally, instead of its l -thickness we can also require that D is an l -set – a condition that promises to be more common to applications. One access to such a result is to prove that the l -property of $\partial\Omega$ implies the p -fatness of $\mathbb{R}^d \setminus \Omega$ – a result which was first obtained by Maz'ya [40]. Knowing this, Hardy's inequality may then be deduced from the results in [31] or [48]. Our approach is quite different and simply rests on Proposition 5.1 and Lemma 4.5. So we can record the following.

Corollary 5.6. *The assertion of Theorem 3.1 remains valid if instead of its l -thickness we require that D is an l -set.*

6. THE EXTENSION OPERATOR

In this section we discuss the second condition in our main result Theorem 3.1, that is the extendability for $W_D^{1,p}(\Omega)$ within the same class of Sobolev functions. We develop three abstract principles concerning Sobolev extension.

- Dirichlet cracks can be removed: We open the possibility of passing from Ω to another domain Ω_\star with a reduced Dirichlet boundary part, while $\Gamma = \partial\Omega \setminus D$ remains part of $\partial\Omega_\star$. In most cases this improves the boundary geometry in the sense of Sobolev extendability, see the example in the following Figure.

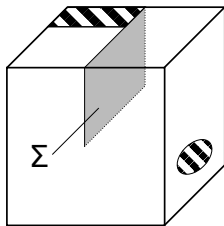


FIGURE 1. The set Σ does not belong to Ω , and carries – together with the striped parts – the Dirichlet condition.

- Sobolev extendability is a local property: We show that only the local geometry of the domain around the boundary part Γ plays a role for the existence of an extension operator.
- Preservation of traces: We prove under very general geometric assumptions that the extended functions do have the adequate trace behavior on D for *every* extension operator.

We believe that these results are of independent interest and therefore decided to directly present them for higher-order Sobolev spaces $W_E^{k,p}$. In the end we review some feasible commonly used geometric conditions which together with our abstract principles really imply the corresponding extendability.

6.1. Dirichlet cracks can be removed. As in Figure 1 there may be boundary parts which carry a Dirichlet condition and belong to the inner of the closure of the domain under consideration. Then one can extend the functions on Λ by 0 to such a boundary part, thereby enlarging the domain and simplifying the boundary geometry. In the following we make this precise.

Lemma 6.1. *Let $\Lambda \subset \mathbb{R}^d$ be a bounded domain and let $E \subset \partial\Lambda$ be closed. Define Λ_\star as the interior of the set $\Lambda \cup E$. Then the following hold true.*

- (i) The set Λ_\star is again a domain, $\Xi := \partial\Lambda \setminus E$ is a (relatively) open subset of $\partial\Lambda_\star$ and $\partial\Lambda_\star = \Xi \cup (E \cap \partial\Lambda_\star)$.
- (ii) Let $k \in \mathbb{N}$ and $p \in [1, \infty[$. Extending functions from $W_E^{k,p}(\Lambda)$ by 0 to Λ_\star , one obtains an isometric extension operator $\text{Ext}(\Lambda, \Lambda_\star)$ from $W_E^{k,p}(\Lambda)$ onto $W_E^{k,p}(\Lambda_\star)$.

Proof. (i) Due to the connectedness of Λ and the set inclusion $\Lambda \subset \Lambda_\star \subset \bar{\Lambda}$, the set Λ_\star is also connected, and, hence a domain. Obviously, one has $\bar{\Lambda}_\star = \bar{\Lambda}$. This, together with the inclusion $\Lambda \subset \Lambda_\star$ leads to $\partial\Lambda_\star \subset \partial\Lambda$. Since $\Xi \cap \Lambda_\star = \emptyset$, one gets $\Xi \subset \partial\Lambda_\star$. Furthermore, Ξ was relatively open in $\partial\Lambda$, so it is relatively open also in $\partial\Lambda_\star$.

The last asserted equality follows from $\partial\Lambda_\star = (\Xi \cap \partial\Lambda_\star) \cup (E \cap \partial\Lambda_\star)$ and $\Xi \subset \partial\Lambda_\star$.

- (ii) Consider any $\psi \in C_E^\infty(\mathbb{R}^d)$ and its restriction $\psi|_\Lambda$ to Λ . Since the support of ψ has a positive distance to E , one may extend $\psi|_\Lambda$ by 0 to the whole of Λ_\star without destroying the C^∞ -property. Thus, this extension operator provides a linear isometry from $C_E^\infty(\Lambda)$ onto $C_E^\infty(\Lambda_\star)$ (if both are equipped with the $W^{k,p}$ -norm). This extends to a linear extension operator $\text{Ext}(\Lambda, \Lambda_\star)$ from $W_E^{k,p}(\Lambda)$ onto $W_E^{k,p}(\Lambda_\star)$, see the two following commutative diagrams:

$$\begin{array}{ccc}
C_E^\infty(\mathbb{R}^d) & \xrightarrow{\text{restrict}_{\mathbb{R}^d \rightarrow \Lambda}} & C_E^\infty(\Lambda) \\
\text{restrict}_{\mathbb{R}^d \rightarrow \Lambda_\star} \downarrow & & \swarrow \text{extend}_{\Lambda \rightarrow \Lambda_\star} \\
C_E^\infty(\Lambda_\star) & &
\end{array}
\qquad
\begin{array}{ccc}
W_E^{k,p}(\mathbb{R}^d) & \xrightarrow{\text{restrict}_{\mathbb{R}^d \rightarrow \Lambda}} & W_E^{k,p}(\Lambda) \\
\text{restrict}_{\mathbb{R}^d \rightarrow \Lambda_\star} \downarrow & & \swarrow \text{extend}_{\Lambda \rightarrow \Lambda_\star} \\
W_E^{k,p}(\Lambda_\star) & &
\end{array}$$

□

Remark 6.2. (i) Note that no assumptions on E beside closedness are necessary.

- (ii) Having extended the functions from Λ to Λ_\star , the 'Dirichlet crack' Σ in Figure 1 has vanished, and one ends up with the whole cube. Here the problem of extending Sobolev functions is almost trivial. We suppose that this is the generic case – at least for problems arising in applications.

The above considerations suggest the following procedure: extend the functions from $W_E^{k,p}(\Lambda)$ first to Λ_\star , and afterwards to the whole of \mathbb{R}^d . The next lemma shows that this approach is universal.

Lemma 6.3. Let $k \in \mathbb{N}$ and $p \in [1, \infty[$. Let $\Lambda \subset \mathbb{R}^d$ be a bounded domain, let $E \subset \partial\Lambda$ be closed and as before define Λ_\star as the interior of the set $\Lambda \cup E$. Every linear, continuous extension operator $\mathfrak{F} : W_E^{k,p}(\Lambda) \rightarrow W_E^{k,p}(\mathbb{R}^d)$ factorizes as $\mathfrak{F} = \mathfrak{F}_\star \text{Ext}(\Lambda, \Lambda_\star)$ through a linear, continuous extension operator $\mathfrak{F}_\star : W_E^{k,p}(\Lambda_\star) \rightarrow W_E^{k,p}(\mathbb{R}^d)$.

Proof. Let \mathfrak{G} be the restriction operator from $W_E^{k,p}(\Lambda_\star)$ to $W_E^{k,p}(\Lambda)$. Then we define, for every $f \in W_E^{k,p}(\Lambda_\star)$, $\mathfrak{F}_\star f := \mathfrak{F} \mathfrak{G} f$. We obtain $\mathfrak{F}_\star \text{Ext}(\Lambda, \Lambda_\star) = \mathfrak{F} \mathfrak{G} \text{Ext}(\Lambda, \Lambda_\star) = \mathfrak{F}$. This shows that the factorization holds algebraically. However, one also has

$$\begin{aligned}
\|\mathfrak{F}_\star \text{Ext}(\Lambda, \Lambda_\star) f\|_{W_E^{k,p}(\mathbb{R}^d)} &= \|\mathfrak{F} f\|_{W_E^{k,p}(\mathbb{R}^d)} \leq \|\mathfrak{F}\|_{\mathcal{L}(W_E^{k,p}(\Lambda); W_E^{k,p}(\mathbb{R}^d))} \|f\|_{W_E^{k,p}(\Lambda)} \\
&= \|\mathfrak{F}\|_{\mathcal{L}(W_E^{k,p}(\Lambda); W_E^{k,p}(\mathbb{R}^d))} \|\text{Ext}(\Lambda, \Lambda_\star) f\|_{W_E^{k,p}(\Lambda_\star)}.
\end{aligned}$$

□

Having extended the functions already to Λ_\star one may proceed as follows: Since E is closed, so is $E_\star := E \cap \partial\Lambda_\star$. So, one can now consider the space $W_{E_\star}^{1,p}(\Lambda_\star)$ and has the task to establish an extension operator for this space – while afterwards one has to take into account that the original functions were 0 also on the set $E \cap \Lambda_\star$ and have not been altered by the extension operator thereon. However, note carefully that $E_\star := E \cap \partial\Lambda_\star$ may have a worse geometry than E . For

example, take Figure 2 and suppose that this time only Σ forms the whole Dirichlet part of the boundary. Then E is a $(d-1)$ -set whereas even $\mathcal{H}_{d-1}(E_\star) = 0$ holds.

To sum up, if one aims at an extension operator $\mathfrak{E} : W_E^{k,p}(\Lambda) \rightarrow W_E^{k,p}(\mathbb{R}^d)$, one is free to modify the domain Λ to Λ_\star . In most cases this improves the local geometry concerning Sobolev extensions and we do not have examples where the situation gets worse. Though we do not claim that this is, in a whatever precise sense, the generic case.

6.2. Sobolev extendability is a local property. Below, we make precise in which sense Sobolev extendability is a local property. We set up the following notation.

Definition 6.4. A domain $\Lambda \subset \mathbb{R}^d$ is a $W^{k,p}$ -extension domain for given $k \in \mathbb{N}$ and $p \in [1, \infty[$ if there exists a continuous extension operator $\mathfrak{E}_{k,p} : W^{k,p}(\Lambda) \rightarrow W^{k,p}(\mathbb{R}^d)$. If Λ is a $W^{k,p}$ -extension domain for all $k \in \mathbb{N}$ and all $p \in [1, \infty[$ in virtue of the *same* extension operator, then Λ is a *universal Sobolev extension domain*.

Proposition 6.5. *Let $k \in \mathbb{N}$ and $p \in [1, \infty[$. Let Λ be a bounded domain and let E be a closed part of its boundary. Assume that for every $\mathbf{x} \in \overline{\partial\Lambda} \setminus \overline{E}$ there is an open neighborhood $U_{\mathbf{x}}$ of \mathbf{x} such that $\Lambda \cap U_{\mathbf{x}}$ is a $W^{k,p}$ -extension domain. Then there is a continuous extension operator*

$$\mathfrak{E}_{k,p} : W_E^{k,p}(\Lambda) \rightarrow W^{k,p}(\mathbb{R}^d).$$

Moreover, if each local extension operator $\mathfrak{E}_{\mathbf{x}}$ maps the space $W_{E_{\mathbf{x}}}^{k,p}(\Lambda \cap U_{\mathbf{x}})$ into $W_{E_{\mathbf{x}}}^{k,p}(\mathbb{R}^d)$, where $E_{\mathbf{x}} := \overline{E} \cap \overline{U_{\mathbf{x}}} \subset \partial(\Lambda \cap U_{\mathbf{x}})$, then also

$$\mathfrak{E}_{k,p} : W_E^{k,p}(\Lambda) \rightarrow W_E^{k,p}(\mathbb{R}^d).$$

Proof. For the construction of the extension operator let for every $\mathbf{x} \in \overline{\partial\Lambda} \setminus \overline{E}$ denote $U_{\mathbf{x}}$ the open neighborhood of \mathbf{x} from the assumption. Let U_{x_1}, \dots, U_{x_n} be a finite subcovering of $\overline{\partial\Lambda} \setminus \overline{E}$. Since the compact set $\overline{\partial\Lambda} \setminus \overline{E}$ is contained in the open set $\bigcup_j U_{x_j}$, there is an $\varepsilon > 0$, such that the sets U_{x_1}, \dots, U_{x_n} , together with the open set $U := \{y \in \mathbb{R}^d : \text{dist}(y, \overline{\partial\Lambda} \setminus \overline{E}) > \varepsilon\}$, form an open covering of $\overline{\Lambda}$. Hence, on $\overline{\Lambda}$ there is a C_0^∞ -partition of unity $\eta, \eta_1, \dots, \eta_n$, with the properties $\text{supp}(\eta) \subset U$, $\text{supp}(\eta_j) \subset U_{x_j}$.

Assume $\psi \in C_E^\infty(\Lambda)$. Then $\eta\psi \in C_0^\infty(\Lambda)$. If one extends this function by 0 outside of Λ , then one obtains a function $\varphi \in C_{\partial\Lambda}^\infty(\mathbb{R}^d) \subset C_E^\infty(\mathbb{R}^d) \subset W_E^{k,p}(\mathbb{R}^d)$ with the property $\|\varphi\|_{W^{k,p}(\mathbb{R}^d)} = \|\eta\psi\|_{W^{k,p}(\Lambda)}$.

Now, for every fixed $j \in \{1, \dots, n\}$, consider the function $\psi_j := \eta_j\psi \in W^{k,p}(\Lambda \cap U_{x_j})$. Since $\Lambda \cap U_{x_j}$ is a $W^{k,p}$ -extension domain by assumption, there is an extension of ψ_j to a $W^{k,p}(\mathbb{R}^d)$ -function φ_j together with an estimate $\|\varphi_j\|_{W^{k,p}(\mathbb{R}^d)} \leq c\|\psi_j\|_{W^{k,p}(\Lambda \cap U_{x_j})}$, where c is independent from ψ . Clearly, one has a priori no control on the behavior of φ_j on the set $\Lambda \setminus U_{x_j}$. In particular φ_j may there be nonzero and, hence, cannot be expected to coincide with $\eta_j\psi$ on the whole of Λ . In order to correct this, let ζ_j be a $C_0^\infty(\mathbb{R}^d)$ -function which is identically 1 on $\text{supp}(\eta_j)$ and has its support in U_{x_j} . Then $\eta_j\psi$ equals $\zeta_j\varphi_j$ on all of Λ . Consequently, $\zeta_j\varphi_j$ really is an extension of $\eta_j\psi$ to the whole of \mathbb{R}^d which, additionally, satisfies the estimate

$$\|\zeta_j\varphi_j\|_{W^{k,p}(\mathbb{R}^d)} \leq c\|\varphi_j\|_{W^{k,p}(\mathbb{R}^d)} \leq c\|\eta_j\psi\|_{W^{k,p}(\Lambda \cap U_{x_j})} \leq c\|\psi\|_{W^{k,p}(\Lambda)},$$

where c is independent from ψ . Thus, defining $\mathfrak{E}_{k,p}(\psi) = \varphi + \sum_j \zeta_j\varphi_j$ one gets a linear, continuous extension operator from $C_E^\infty(\Lambda)$ into $W^{k,p}(\mathbb{R}^d)$. By density, $\mathfrak{E}_{k,p}$ uniquely extends to a linear, continuous operator

$$\mathfrak{E}_{k,p} : W_E^{k,p}(\Lambda) \rightarrow W^{k,p}(\mathbb{R}^d).$$

Finally, assume that the local extension operators map $W_{E_{x_j}}^{k,p}(\Lambda \cap U_{x_j})$ into $W_{E_{x_j}}^{k,p}(\mathbb{R}^d)$. Using the notation above, this means that φ_j can be approximated in $W^{k,p}(\mathbb{R}^d)$ by a sequence from

$C_{E_{x_j}}^\infty(\mathbb{R}^d)$. Since ζ_j is supported in U_{x_j} , multiplication by $\zeta_j \in C_0^\infty(\mathbb{R}^d)$ maps $C_{E_{x_j}}^\infty(\mathbb{R}^d)$ into $C_E^\infty(\mathbb{R}^d)$ boundedly with respect to the $W^{k,p}(\mathbb{R}^d)$ -topology. Hence, $\zeta_j \varphi_j \in W_E^{k,p}(\mathbb{R}^d)$. Since in any case $\varphi \in W_E^{k,p}(\mathbb{R}^d)$, the conclusion follows. \square

Remark 6.6. By construction one gets uniformity for \mathfrak{E} with respect to p and k if one invests the respective uniformity concerning the extension property for the local domains $\Lambda \cap U_x$. In particular, one obtains an extension operator that is bounded from $W_E^{k,p}(\Lambda)$ into $W^{k,p}(\mathbb{R}^d)$ for each $k \in \mathbb{N}$ and each $p \in [1, \infty[$ if the local domains are universal Sobolev extension domains.

6.3. Preservation of traces. Proposition 6.5 allows to construct Sobolev extension operators from $W_D^{k,p}(\Omega)$ into $W^{k,p}(\mathbb{R}^d)$ and gives a sufficient condition for preservation of the Dirichlet condition. In this section we prove that in fact *every* such extension operator has this feature. Recall that this is the crux of the matter in Assumption (iii) of Theorem 3.1. The key lemma is the following.

Lemma 6.7. *Let $k \in \mathbb{N}$ and $p \in]1, \infty[$. Let $\Lambda \subset \mathbb{R}^d$ be a domain, let $E \subset \partial\Lambda$ be closed and let $\mathfrak{E}_{k,p} : W_E^{k,p}(\Lambda) \rightarrow W^{k,p}(\mathbb{R}^d)$ be a bounded extension operator. Any of the following conditions guarantees that $\mathfrak{E}_{k,p}$ in fact maps into $W_E^{k,p}(\mathbb{R}^d)$.*

- (i) *For (k, p) -quasievery $y \in E$ balls around y in Λ have asymptotically nonvanishing relative volume, i.e.*

$$(6.1) \quad \liminf_{r \rightarrow 0} \frac{|B(y, r) \cap \Lambda|}{r^d} > 0.$$

- (ii) *The set E is an l -set for some $l \in]d - p, d]$ and (6.1) holds for \mathcal{H}_l -almost every $y \in E$.*
 (iii) *There exists $q > d$ such that $\mathfrak{E}_{k,p}$ maps $C_E^\infty(\Lambda)$ into $W^{k,q}(\mathbb{R}^d)$.*

Proof. As $C_E^\infty(\Omega)$ is dense in $W_E^{k,p}(\Lambda)$ and since $\mathfrak{E}_{k,p}$ is bounded, it suffices to prove that given $v \in C_E^\infty$ the function $u := \mathfrak{E}_{k,p}v$ belongs to $W_E^{k,p}(\mathbb{R}^d)$. The proof of (i) is inspired by [50, pp. 190-192]. Easy modifications of the argument will yield (ii) and (iii).

- (i) Fix an arbitrary multiindex β with $|\beta| \leq k - 1$. Let $\mathfrak{D}^\beta u$ be the representative of the distributional derivative $D^\beta u$ of u defined $(k - |\beta|, p)$ -q.e. on \mathbb{R}^d via

$$\mathfrak{D}^\beta u(y) := \lim_{r \rightarrow 0} \frac{1}{|B(y, r)|} \int_{B(y, r)} D^\beta u(x) \, dx.$$

Recall from (4.3) that then

$$(6.2) \quad \begin{aligned} & \lim_{r \rightarrow 0} \frac{1}{|B(y, r)|} \int_{B(y, r)} |\mathfrak{D}^\beta u(x) - \mathfrak{D}^\beta u(y)| \, dx \\ & \leq \lim_{r \rightarrow 0} \left(\frac{1}{|B(y, r)|} \int_{B(y, r)} |\mathfrak{D}^\beta u(x) - \mathfrak{D}^\beta u(y)|^p \, dx \right)^{1/p} = 0. \end{aligned}$$

holds for $(k - |\beta|, p)$ -q.e. $y \in \mathbb{R}^d$. Since (6.1) holds for (k, p) -quasievery $y \in E$, it a fortiori holds for $(k - |\beta|, p)$ -quasievery such y . Let now $N \subset \mathbb{R}^d$ be the exceptional set such that on $\mathbb{R}^d \setminus N$ the function $\mathfrak{D}^\beta u$ is defined and satisfies (6.2) and such that (6.1) holds for every $y \in E \setminus N$. Owing to Theorem 4.11 the claim follows once we have shown $\mathfrak{D}^\beta u(y) = 0$ for all $y \in E \setminus N$.

For the rest of the proof we fix $y \in E \setminus N$. For $r > 0$ we abbreviate $B(r) := B(y, r)$ and define

$$(6.3) \quad W_j := \{x \in \mathbb{R}^d \setminus N : |\mathfrak{D}^\beta u(x) - \mathfrak{D}^\beta u(y)| > 1/j\}.$$

Thanks to (6.2) for each $j \in \mathbb{N}$ we can choose some $r_j > 0$ such that $|B(r) \cap W_j| < 2^{-j}|B(r)|$ holds for all $r \in]0, r_j]$. Clearly, we can arrange that the sequence $\{r_j\}_j$ is decreasing. Now,

$$(6.4) \quad W := \bigcup_{j \in \mathbb{N}} \left\{ (B(r_j) \setminus B(r_{j+1})) \cap W_j \right\}$$

has vanishing Lebesgue density at y , i.e. $r^{-d}|B(r) \cap W|$ vanishes as r tends to 0: Indeed, if $r \in]r_{l+1}, r_l]$, then

$$\begin{aligned} |B(r) \cap W| &\leq \left| (B(r) \cap W_l) \cup \bigcup_{j \geq l+1} (B(r_j) \cap W_j) \right| \\ &\leq 2^{-l}|B(r)| + \sum_{j \geq l+1} 2^{-j}|B(r_j)| \leq 2^{-l+1}|B(r)|. \end{aligned}$$

Now, (6.1) allows to conclude

$$\liminf_{r \rightarrow 0} \frac{|B(r) \cap \Lambda \cap (\mathbb{R}^d \setminus W)|}{r^d} > 0.$$

Since u is an extension of $v \in C_E^\infty(\Lambda)$ and y is an element of E it holds $\mathfrak{D}^\beta u = 0$ a.e. on $B(r) \cap \Lambda$ with respect to the d -dimensional Lebesgue measure if $r > 0$ is small enough. The previous inequality gives $|B(r) \cap \Lambda \cap (\mathbb{R}^d \setminus W)| > 0$ if $r > 0$ is small enough. In particular, there exists a sequence $\{x_j\}_j$ in $\mathbb{R}^d \setminus W$ approximating y such that $\mathfrak{D}^\beta u(x_j) = 0$ for all $j \in \mathbb{N}$. Now, the upshot is that the restriction of $\mathfrak{D}^\beta u$ to $\mathbb{R}^d \setminus W$ is continuous at y since if $x \in \mathbb{R}^d \setminus W$ satisfies $|x - y| \leq r_j$ then by construction $|\mathfrak{D}^\beta u(x) - \mathfrak{D}^\beta u(y)| \leq 1/j$. Hence, $\mathfrak{D}^\beta u(y) = 0$ and the proof is complete.

- (ii) If E is an l -set for some $l \in]d - p, d]$, then we can appeal to Theorem 4.12 rather than Theorem 4.11 and the same argument as in (i) applies.
- (iii) By assumption $u \in W_E^{k,q}(\mathbb{R}^d)$, where $q > d$. By Sobolev embeddings each distributional derivative $D^\beta u$, $|\beta| \leq k - 1$, has a continuous representative $\mathfrak{D}^\alpha u$. As each $y \in E \subset \partial\Lambda$ is an accumulation point of $\Lambda \setminus E$ and since $D^\alpha u = D^\alpha v$ holds almost everywhere on Λ , the representative $\mathfrak{D}^\alpha u$ must vanish everywhere on E and Theorem 4.11 yields $u \in W_E^{k,p}(\mathbb{R}^d)$ as required. \square

Remark 6.8. If Λ is a d -set and E a $(d - 1)$ -set, then Lemma 6.7 is proved in [22, Sec. VIII.1].

We can now state and prove the remarkable result that every Sobolev extension operator that is constructed by localization techniques as in Proposition 6.5 preserves the Dirichlet condition.

Theorem 6.9. *Let $k \in \mathbb{N}$ and $p \in [1, \infty[$. Let Λ be a bounded domain and let E be a closed part of its boundary. Assume that for every $x \in \partial\Lambda \setminus E$ there is an open neighborhood U_x of x such that $\Lambda \cap U_x$ is a $W^{k,p}$ -extension domain. Then there exists a continuous extension operator*

$$\mathfrak{E}_{k,p} : W_E^{k,p}(\Lambda) \rightarrow W_E^{k,p}(\mathbb{R}^d).$$

For the proof we recall the following result of Hajlasz, Koskela and Tuominen.

Proposition 6.10 ([18, Thm. 2]). *If a domain $\Lambda \subset \mathbb{R}^d$ is a $W^{k,p}$ -extension domain for some $k \in \mathbb{N}$ and $p \in [1, \infty[$, then it is a d -set.*

Proof of Theorem 6.9. According to Proposition 6.5 it suffices to check that each local extension operator \mathfrak{E}_x maps $W_{E_x}^{k,p}(\Lambda \cap U_x)$ into $W_{E_x}^{k,p}(\mathbb{R}^d)$, where $E_x := \overline{E \cap U_x} \subset \partial(\Lambda \cap U_x)$. Owing to Proposition 6.10 the $W^{k,p}$ -extension domain $\Lambda \cap U_x$ is a d -set and as such satisfies (6.1) around every of its boundary points. So, Lemma 6.7.(i) yields the claim. \square

Remark 6.11. The extension operator in Theorem 6.9 is the same as in Proposition 6.5. Hence, the former result asserts that every Sobolev extension operator built by the common gluing-together of local extension operators automatically preserves the Dirichlet condition on E under the mere assumption that this set is closed. Moreover, all uniformity properties as in Remark 6.6 remain valid.

6.4. Geometric conditions. In this subsection we finally review common geometric conditions on the boundary part $\overline{\partial\Lambda \setminus E}$ such that the local sets $\Lambda \cap U_x$ really admit the Sobolev extension property required in Proposition 6.5.

A first condition, completely sufficient for the treatment of most real world problems, is the following Lipschitz condition.

Definition 6.12. A bounded domain $\Lambda \subset \mathbb{R}^d$ is called *bounded Lipschitz domain* if for each $x \in \partial\Lambda$ there is an open neighborhood U_x of x and a bi-Lipschitz mapping ϕ_x from U_x onto a cube, such that $\phi_x(\Lambda \cap U_x)$ is the (lower) half cube and $\partial\Lambda \cap U_x$ is mapped onto the top surface of this half cube.

It can be proved by elementary means that bounded Lipschitz domains are $W^{1,p}$ -extension domains for every $p \in [1, \infty[$, cf. e.g. [17] for the case $p = 2$. In fact, already the following (ε, δ) -condition of Jones [21] assures the existence of a universal Sobolev extension operator.

Definition 6.13. Let $\Lambda \subset \mathbb{R}^d$ be a domain and $\varepsilon, \delta > 0$. Assume that any two points $x, y \in \Lambda$, with distance not larger than δ , can be connected within Λ by a rectifiable arc γ with length $l(\gamma)$, such that the following two conditions are satisfied for all points z from the curve γ :

$$l(\gamma) \leq \frac{1}{\varepsilon} \|x - y\|, \quad \text{and} \quad \frac{\|x - z\| \|y - z\|}{\|x - y\|} \leq \frac{1}{\varepsilon} \text{dist}(z, \Lambda^c).$$

Then Λ is called (ε, δ) -domain.

Theorem 6.14 (Rogers). *Each (ε, δ) -domain is a universal Sobolev extension domain.*

- Remark 6.15.**
- (i) Theorem 6.14 is due to Rogers [44] and generalizes the celebrated result of Jones [21]. *Bounded (ε, δ) -domains* are known to be *uniform domains*, see [47, Ch. 4.2] and also [21, 36, 37, 35] for further information. In particular, every bounded Lipschitz domain is an (ε, δ) -domain, see e.g. [12, Rem. 5.11] for a sketch of proof.
 - (ii) Although the uniformity property is not necessary for a domain to be a Sobolev extension domain [49] it seems presently to be the broadest known class of domains for which this extension property holds – at least if one aims at all $p \in]1, \infty[$. For example Koch’s snowflake is an (ε, δ) -domain [21].

Plugging in Rogers extension operator into Theorem 6.9 lets us re-discover [4, Thm. 1.3] in case of bounded domains and p strictly between 1 and ∞ . We even obtain a universal extension operator that simultaneously acts on all $W_E^{k,p}$ -spaces and at the same time our argument reveals that the preservation of the trace is irrespective of the specific structure of Jones’ or Roger’s extension operators.

We believe that this sheds some more light also on [4, Thm. 1.3] though – of course – our argument cannot disclose the fundamental assertions on the support of the extended functions obtained in [4] by a careful analysis of Jones’ extension operator. We summarize our observations in the following theorem.

Theorem 6.16. *Let Λ be a bounded domain and let E be a closed part of its boundary. Assume that for every $x \in \overline{\partial\Lambda \setminus E}$ there is an open neighborhood U_x of x such that $\Lambda \cap U_x$ is a bounded Lipschitz or, more generally, an (ε, δ) -domain for some values $\varepsilon, \delta > 0$. Then there exists a universal operator \mathfrak{E} that restricts to a bounded extension operator $W_E^{k,p}(\Lambda) \rightarrow W_E^{k,p}(\mathbb{R}^d)$ for each $k \in \mathbb{N}$ and each $p \in]1, \infty[$.*

7. POINCARÉ'S INEQUALITY

In this section we will discuss sufficient conditions for Poincaré's inequality, thereby unwinding Assumption (ii) of Theorem 3.1. Our aim is not greatest generality as e.g. in [39] for functions defined on the whole of \mathbb{R}^d , but to include the aspect that our functions are only defined on a domain. Secondly, our interest is to give very general, but in some sense *geometric* conditions, which may be checked more or less 'by appearance' – at least for problems arising from applications.

The next proposition gives a condition that assures that a closed subspace of $W^{1,p}$ may be equivalently normed by the L^p -norm of the gradient of the corresponding functions only. We believe that this might also be of independent interest, compare also [50, Ch. 4]. Throughout $\mathbf{1}$ denotes the function that is identically one.

Proposition 7.1. *Let $\Lambda \subset \mathbb{R}^d$ be a bounded domain and suppose $p \in]1, \infty[$. Assume that X is a closed subspace of $W^{1,p}(\Lambda)$ that does not contain $\mathbf{1}$ and for which the restriction of the canonical embedding $W^{1,p}(\Lambda) \hookrightarrow L^p(\Lambda)$ to X is compact. Then X may be equivalently normed by $v \mapsto (\int_{\Lambda} |\nabla v|^p dx)^{1/p}$.*

Proof. First recall that both X and $L^p(\Lambda)$ are reflexive. In order to prove the proposition, assume to the contrary that there exists a sequence $\{v_k\}_k$ from X such that

$$\frac{1}{k} \|v_k\|_{L^p(\Lambda)} \geq \|\nabla v_k\|_{L^p(\Lambda)}.$$

After normalization we may assume $\|v_k\|_{L^p(\Lambda)} = 1$ for every $k \in \mathbb{N}$. Hence, $\{\nabla v_k\}_k$ converges to 0 strongly in $L^p(\Lambda)$. On the other hand, $\{v_k\}_k$ is a bounded sequence in X and hence contains a subsequence $\{v_{k_l}\}_l$ that converges weakly in X to an element $v \in X$. Since the gradient operator $\nabla : X \rightarrow L^p(\Lambda)$ is continuous, $\{\nabla v_{k_l}\}_l$ converges to ∇v weakly in $L^p(\Lambda)$. As the same sequence converges to 0 strongly in $L^p(\Lambda)$, the function ∇v must be zero and hence v is constant. But by assumption X does not contain constant functions except for $v = 0$. So, $\{v_{k_l}\}_l$ tends to 0 weakly in X . Owing to the compactness of the embedding $X \hookrightarrow L^p(\Lambda)$, a subsequence of $\{v_{k_l}\}_l$ tends to 0 strongly in $L^p(\Lambda)$. This contradicts the normalization condition $\|v_{k_l}\|_{L^p(\Lambda)} = 1$. \square

Remark 7.2. It is clear that in case $X = W_D^{1,p}(\Omega)$ the embedding $X \hookrightarrow L^p(\Omega)$ is compact, if there exists a continuous extension operator $\mathfrak{E} : W_D^{1,p}(\Omega) \rightarrow W^{1,p}(\mathbb{R}^d)$. Hence, the compactness of this embedding is no additional requirement in view of Theorem 3.1.

In the case that E is l -thick, the following lemma presents two conditions that are particularly easy to check and entail $\mathbf{1} \notin W_E^{1,p}(\Lambda)$. Loosely speaking, some knowledge on the common frontier of E and $\partial\Lambda \setminus E$ is required: Either not every point of E should lie thereon or $\partial\Lambda$ must not be too wild around this frontier.

Lemma 7.3. *Let $p \in]1, \infty[$, let Λ be a bounded domain and let $E \subset \partial\Lambda$ be l -thick for some $l \in]d - p, d]$. Both of the following conditions assure $\mathbf{1} \notin W_E^{1,p}(\Lambda)$.*

- (i) *The set E admits at least one relatively inner point x . Here, 'relatively inner' is with respect to $\partial\Lambda$ as ambient topological space.*
- (ii) *For every $x \in \overline{\partial\Lambda \setminus E}$ there is an open neighborhood U_x of x such that $\Lambda \cap U_x$ is a $W^{1,p}$ -extension domain.*

Proof. We treat both cases separately.

- (i) Assume the assertion was false and $\mathbf{1} \in W_E^{1,p}(\Lambda)$. Let x be the inner point of E from the hypotheses and let $B := B(x, r)$ be a ball that does not intersect $\partial\Lambda \setminus E$. Put $\frac{1}{2}B := B(x, \frac{r}{2})$ and let $\eta \in C_0^\infty(B)$ be such that $\eta \equiv 1$ on $\frac{1}{2}B$. We distinguish whether or not x is an interior point of $\overline{\Lambda}$.

First, assume it is not. For every $\psi \in C_E^\infty(\Lambda)$ the function $\eta\psi$ belongs to $W_0^{1,p}(\Lambda \cap B)$ and as such admits a $W^{1,p}$ -extension $\widehat{\eta\psi}$ by zero to the whole of \mathbb{R}^d . In particular,

$$\widehat{\eta\psi}(y) = \begin{cases} \psi(y), & \text{if } y \in \frac{1}{2}B \cap \Lambda \\ 0, & \text{if } y \in \frac{1}{2}B \setminus \Lambda \end{cases}$$

and consequently,

$$\|\nabla \widehat{\eta\psi}\|_{L^p(\frac{1}{2}B)} = \|\nabla \psi\|_{L^p(\frac{1}{2}B \cap \Lambda)}.$$

Since by assumption $\mathbf{1}$ is in the $W^{1,p}(\Lambda)$ -closure of $C_E^\infty(\Lambda)$ and since the mappings $W_E^{1,p}(\Lambda) \ni \psi \mapsto \nabla \widehat{\eta\psi} \in L^p(\frac{1}{2}B)$ and $W_E^{1,p}(\Lambda) \ni \psi \mapsto \nabla \psi \in L^p(\Lambda \cap \frac{1}{2}B)$ are continuous, the previous equation extends to $\psi = \mathbf{1}$:

$$\|\nabla \widehat{\eta\mathbf{1}}\|_{L^p(\frac{1}{2}B)} = \|\nabla \mathbf{1}\|_{L^p(\frac{1}{2}B \cap \Lambda)} = 0.$$

On the other hand x is not an inner point of $\bar{\Lambda}$ so that in particular $\frac{1}{2}B \setminus \bar{\Lambda}$ is non-empty. Since this set is open, $|\frac{1}{2}B \setminus \bar{\Lambda}| > 0$. Recall that by construction $\widehat{\eta\mathbf{1}} \in W^{1,p}(B)$ vanishes a.e. on $\frac{1}{2}B \setminus \bar{\Lambda}$. Hence, for some $c > 0$ the Poincaré inequality

$$\|\widehat{\eta\mathbf{1}}\|_{L^p(\frac{1}{2}B)} \leq c \|\nabla \widehat{\eta\mathbf{1}}\|_{L^p(\frac{1}{2}B)},$$

holds, cf. [50, Thm. 4.4.2]. However, we already know that the right hand side is zero, whereas the left hand side equals $|\frac{1}{2}B \cap \Lambda|^{1/p}$, which is nonzero since $\frac{1}{2}B \cap \Lambda$ is nonempty and open – a contradiction.

Now, assume x is contained in the interior of $\bar{\Lambda}$. Upon diminishing B we may assume $B \subset \bar{\Lambda}$. For every $\psi \in C_E^\infty(\mathbb{R}^d)$ we have $\eta\psi \in C_E^\infty(\mathbb{R}^d)$ with an estimate

$$\|\eta\psi\|_{W^{1,p}(\mathbb{R}^d)} \leq c \|\psi\|_{W^{1,p}(B)} = c \left(\int_B |\psi|^p + |\nabla \psi|^p \, dx \right)^{1/p}$$

for some constant $c > 0$ depending only on η and p . By our choice of B split

$$B = B \cap \bar{\Lambda} = (B \cap \Lambda) \cup (B \cap \partial\Lambda) = (B \cap \Lambda) \cup (B \cap E).$$

Since ψ vanishes in a neighborhood of E ,

$$(7.1) \quad \|\eta\psi\|_{W^{1,p}(\mathbb{R}^d)} \leq c \left(\int_{B \cap \Lambda} |\psi|^p + |\nabla \psi|^p \, dx \right)^{1/p} \leq c \|\psi\|_{W^{1,p}(\Lambda)}.$$

Taking into account $\eta \equiv 1$ on $\frac{1}{2}B$, the same reasoning gives

$$(7.2) \quad \int_{\frac{1}{2}B} |\nabla(\eta\psi)|^p \, dx = \int_{\frac{1}{2}B} |\nabla \psi|^p \, dx \leq \int_{\Lambda} |\nabla \psi|^p \, dx.$$

By assumption there is a sequence $\{\psi_j\}_j \subset C_E^\infty(\Lambda)$ tending to $\mathbf{1}$ in the $W^{1,p}(\Lambda)$ -topology. Due to (7.1) and the choice of η , the sequence $\{\eta\psi_j\}_j \subset C_E^\infty(\mathbb{R}^d)$ then tends to some $u \in W_E^{1,p}(\mathbb{R}^d)$ satisfying $u = 1$ a.e. on $\frac{1}{2}B \cap \Lambda$. Due to (7.2), $\nabla u = 0$ a.e. on $\frac{1}{2}B$, meaning that u is constant on this set. Since $\frac{1}{2}B \cap \Lambda$ as a non-empty open set has positive Lebesgue measure, all this can only happen if $u = 1$ a.e. on $\frac{1}{2}B$. Hence,

$$\lim_{r \rightarrow 0} \frac{1}{|B(y,r)|} \int_{B(y,r)} u \, dx = 1$$

for every $y \in \frac{1}{3}B \cap E$, which by Theorem 4.11 is only possible if $C_{1,p}(\frac{1}{3}B \cap E) = 0$. By Theorem 4.11 this in turn implies $\mathcal{H}_l^\infty(\frac{1}{3}B \cap E) = 0$ in contradiction to the l -thickness of E .

- (ii) Again assume the assertion was false. Then by (i) there exists some $x \in E$ that is not an inner point of E with respect to $\partial\Lambda$. Hence x is an accumulation point of $\partial\Lambda \setminus E$ and by assumption there is a neighborhood $U = U_x$ of x such that $\Lambda \cap U$ is a $W^{1,p}$ extension domain. We denote the corresponding extension operator by \mathfrak{E} . We shall localize the assumption $\mathbf{1} \in W_E^{1,p}(\Lambda)$ within U to arrive at a contradiction.

To this end, let $r_0 > 0$ be such that $\overline{B(x, r_0)} \subset U$ and let $\eta \in C_0^\infty(U)$ be such that $\eta \equiv 1$ on $B(x, r_0)$. Then also $\eta = \eta\mathbf{1} \in W_E^{1,p}(\Lambda)$ and in particular $\eta|_{\Lambda \cap U}$ belongs to $W_F^{1,p}(\Lambda \cap U)$, where $F := \overline{B(x, r_0/2)} \cap \overline{E} \subset \partial(\Lambda \cap U)$. Recall from Proposition 6.10 that the $W^{1,p}$ -extension domain $\Lambda \cap U$ satisfies in particular

$$\liminf_{r \rightarrow 0} \frac{|B(y, r) \cap \Lambda \cap U|}{r^d} > 0.$$

around every $y \in \partial(\Lambda \cap U)$. Thus, Lemma 6.7(i) yields $u := \mathfrak{E}(\eta|_{\Lambda \cap U}) \in W_F^{1,p}(\mathbb{R}^d)$.

On the other hand, similar to the proof of Lemma 6.7 let \mathbf{u} be the representative of u that is defined by limits of integral means on the complement of some exceptional set N with $C_{1,p}(N) = 0$ and fix $y \in F \setminus N$. Take W as in (6.3) and (6.4). Repeating the arguments in the proof of Lemma 6.7 reveals that the restriction of \mathbf{u} to $\mathbb{R}^d \setminus W$ is continuous at y and that $|B(y, r) \cap \Lambda \cap U \cap (\mathbb{R}^d \setminus W)| > 0$ if $r > 0$ is small enough. By construction $\mathbf{u} = 1$ a.e. on $B(y, r) \cap \Lambda \cap U \cap (\mathbb{R}^d \setminus W)$ if $r < r_0$. Hence, there is a sequence $\{x_j\}_j$ approximating y such that $\mathbf{u}(x_j) = 1$ for every $j \in \mathbb{N}$. By continuity $\mathbf{u}(y) = 1$ follows. This proves that $\mathbf{u} = 1$ holds $(1, p)$ -quasieverywhere on F .

By Theorem 4.11 this can only happen if $C_{1,p}(F) = 0$, which as in (i) contradicts the l -thickness of E . \square

- Remark 7.4.** (i) The proof of (i) shows that $\mathbf{1} \notin W_E^{1,p}(\Lambda)$ if E is merely closed and contains a relatively inner point that is not an inner point of $\overline{\Lambda}$.
- (ii) Of course the Poincaré inequality holds in the case $E = \partial\Lambda$ irrespective of any geometric considerations as long as Λ is bounded. This can be rediscovered by the results of this section. Indeed, E then only consists of relatively inner points and as $\emptyset \neq \partial\overline{\Lambda} \subset \partial\Lambda = E$ holds, it cannot be contained in the interior of $\overline{\Lambda}$. Hence $\mathbf{1} \notin W_0^{1,p}(\Lambda)$. The compactness of the embedding $W_0^{1,p}(\Lambda) \hookrightarrow L^p(\Lambda)$ is classical and Theorem 7.1 gives the claim.

Under the second assumption of Lemma 7.3 there exists a linear continuous Sobolev extension operator $\mathfrak{E} : W_E^{1,p}(\Lambda) \rightarrow W_E^{1,p}(\mathbb{R}^d)$, see Theorem 6.9. Then the compactness of the embedding $W_E^{1,p}(\Lambda) \hookrightarrow L^p(\Lambda)$ is classical and owing to Theorem 7.1 we can record the following special Poincaré inequality.

Proposition 7.5. *Let $p \in]1, \infty[$ and let Λ be a bounded domain. Suppose that $E \subset \partial\Lambda$ is l -thick for some $l \in]d - p, d]$ and that for each $x \in \overline{\partial\Lambda} \setminus \overline{E}$ there is an open neighborhood U_x of x such that $\Lambda \cap U_x$ is a $W^{1,p}$ -extension domain. Then $W_E^{1,p}(\Lambda)$ may equivalently be normed by $v \mapsto (\int_\Lambda |\nabla v|^p dx)^{1/p}$.*

Now, also Theorem 3.2 follows. In fact, this result is just the synthesis of the above proposition with Theorems 3.1 and 6.9.

8. PROOF OF THEOREM 3.4

The strategy of proof is to write u as the sum of $v \in W^{1,p}(\Omega)$ with $v/\text{dist}_{\partial\Omega} \in L^p(\Omega)$ and $w \in W^{1,p}$ with support within a neighborhood of $\overline{\partial\Omega} \setminus \overline{D}$. Then v can be handled by the following classical result.

Proposition 8.1 ([10, Thm. V.3.4]). *Let $\emptyset \subsetneq \Lambda \subsetneq \mathbb{R}^d$ be open and let $p \in]1, \infty[$. Then if $u \in W^{1,p}(\Lambda)$ and $u/\text{dist}_{\partial\Lambda} \in L^p(\Lambda)$, it follows $u \in W_0^{1,p}(\Lambda)$.*

For w we can – since local extension operators are available – rely on the techniques developed in Section 6. A key observation is an intrinsic relation between the property $\frac{u}{\text{dist}_D} \in L^p(\Omega)$ and Sobolev regularity of the function $\log(\text{dist}_D)$. In fact, a formal computation gives

$$\nabla(u \log(\text{dist}_D)) = \log(\text{dist}_D) \nabla u + \frac{u}{\text{dist}_D} \nabla \text{dist}_D.$$

Details are carried out in the following five consecutive steps.

Step 1: Splitting u and handling the easy term. As in the proof of Proposition 6.5 for every $x \in \overline{\partial\Omega} \setminus \overline{D}$, let U_x be the open neighborhood of x from the assumption, let U_{x_1}, \dots, U_{x_n} be a finite subcovering of $\overline{\partial\Omega} \setminus \overline{D}$ and let $\varepsilon > 0$ be such that the sets U_{x_1}, \dots, U_{x_n} , together with $U := \{y \in \mathbb{R}^d : \text{dist}(y, \overline{\partial\Omega} \setminus \overline{D}) > \varepsilon\}$, form an open covering of $\overline{\Omega}$. Finally, let $\eta, \eta_1, \dots, \eta_n$ be a subordinated C_0^∞ -partition of unity. The described splitting is $u = v + w$, where $v := \eta u$ and $w := \sum_{j=1}^n \eta_j u = (1 - \eta)u$. Since

$$\text{dist}_{\partial\Omega}(x) \geq \min\{\varepsilon, \text{dist}_D(x)\} \geq \min\{\varepsilon \text{diam}(\Omega)^{-1}, 1\} \cdot \text{dist}_D(x)$$

holds for every $x \in \text{supp}(\eta) \cap \Omega$, the function $v \in W^{1,p}(\Omega)$ satisfies

$$\int_{\Omega} \left| \frac{v}{\text{dist}_{\partial\Omega}} \right|^p dx \leq c \int_{\Omega} \left| \frac{v}{\text{dist}_D} \right|^p dx \leq c \int_{\Omega} \left| \frac{u}{\text{dist}_D} \right|^p dx < \infty$$

by assumption on u . Now, Proposition 8.1 yields $v \in W_0^{1,p}(\Omega) \subset W_D^{1,p}(\Omega)$.

Step 2: Extending w . By assumption the sets $\Omega \cap U_{x_j}$, $1 \leq j \leq n$, are $W^{1,p}$ -extension domains. Since $w = (1 - \eta)u$, where $(1 - \eta)$ has compact support in the union of these domains, an extension $\widehat{w} \in W^{1,p}(\mathbb{R}^d)$ of $w \in W^{1,p}(\Omega)$ with compact support within $\bigcup_{j=1}^n U_{x_j}$ can be constructed just as in the proof of Proposition 6.5. Now, if we can show $w \in W_D^{1,p}(\Omega)$, then by Step 1 also $u = v + w$ belongs to this space.

Step 3: Estimating the trace of \widehat{w} . To prove $\widehat{w} \in W_D^{1,p}(\mathbb{R}^d)$ we rely once more on the techniques used in the proof of Lemma 6.7. So, let $\widehat{\mathfrak{w}}$ be the representative of \widehat{w} defined on $\mathbb{R}^d \setminus N$ via

$$\widehat{\mathfrak{w}}(y) := \lim_{r \rightarrow 0} \frac{1}{|B(y, r)|} \int_{B(y, r)} \widehat{w} dx,$$

where the exceptional set N is of vanishing $(1, p)$ -capacity. Put

$$U_{\star} := \bigcup_{j=1}^n U_{x_j}, \quad \Omega_{\star} := \Omega \cap U_{\star}, \quad \text{and} \quad D_{\star} = \overline{D \cap U_{\star}} \subseteq \partial\Omega_{\star}.$$

Since \widehat{w} has support in U_{\star} it holds $\widehat{\mathfrak{w}}(y) = 0$ for every $y \in D \setminus D_{\star}$. For the rest of the step let $y \in D_{\star} \setminus N$.

By Proposition 6.10 each set $\Omega \cap U_{x_j}$ is a d -set and it can readily be checked that this property inherits to their union Ω_{\star} . Hence, Ω_{\star} satisfies the asymptotically nonvanishing relative volume condition (6.1) around y with a lower bound $c > 0$ on the limes inferior that is independent of y and – just as in the proof of Lemma 6.7 – a set $W \subset \mathbb{R}^d$ can be constructed such that the restriction of $\widehat{\mathfrak{w}}$ to $\mathbb{R}^d \setminus W$ is continuous at y and such that $|B(y, r) \cap \Omega_{\star} \cap (\mathbb{R}^d \setminus W)| \geq cr^d/2$ if

$r > 0$ is small enough. By these properties of W :

$$\begin{aligned} |\widehat{w}(y)| &= \left| \lim_{r \rightarrow 0} \frac{1}{|B(y, r) \cap \Omega_\star \cap (\mathbb{R}^d \setminus W)|} \int_{B(y, r) \cap \Omega_\star \cap (\mathbb{R}^d \setminus W)} \widehat{w} \, dx \right| \\ &\leq \limsup_{r \rightarrow 0} \frac{2}{cr^d} \int_{B(y, r) \cap \Omega_\star} |\widehat{w}| \, dx \\ &= \limsup_{r \rightarrow 0} \frac{2}{cr^d} \int_{B(y, r) \cap \Omega_\star} |w| \, dx. \end{aligned}$$

In order to force these mean-value integral to vanish in the limit $r \rightarrow 0$, introduce the function $\log(\text{dist}_D)^{-1}$, which is bounded above in absolute value by $|\log r|^{-1}$ on $B(y, r)$ if $r < 1$. It follows

$$(8.1) \quad |\widehat{w}(y)| \leq c \limsup_{r \rightarrow 0} |\log r|^{-1} \left(\frac{1}{r^d} \int_{B(y, r) \cap \Omega_\star} |w \log(\text{dist}_D)| \, dx \right).$$

So, since $|\log r|^{-1} \rightarrow 0$ as $r \rightarrow 0$ the function \widehat{w} vanishes at every $y \in D_\star \setminus N$ for which the mean value integrals on the right-hand side remain bounded as r tends to zero.

Step 4: Intermezzo on $w \log(\text{dist}_D)$. In this step we prove the following result.

Lemma 8.2. *Let $p \in]1, \infty[$, let $\Lambda \subset \mathbb{R}^d$ be a bounded d -set, and let $E \subset \partial\Lambda$ be closed and porous. Suppose $u \in W^{1,p}(\Lambda)$ has an extension $v \in W^{1,p}(\mathbb{R}^d)$ and satisfies $\frac{u}{\text{dist}_E} \in L^p(\Lambda)$. If $r \in]1, p[$ and $s \in]0, 1[$, then the function $|u \log(\text{dist}_E)|$ defined on Λ has an extension in the Bessel potential space $H^{s,r}(\mathbb{R}^d)$ that is positive almost everywhere.*

For the proof we need the following extension result of Jonsson and Wallin.

Proposition 8.3 ([22, Thm. V.1.1]). *Let $s \in]0, 1[$, $p \in]1, \infty[$ and let $\Lambda \subset \mathbb{R}^d$ be a d -set. Then there exists a linear operator \mathfrak{E} that extends every measurable function f on Λ that satisfies*

$$\|f\|_{L^p(\Lambda)} + \left(\iint_{\substack{x, y \in \Lambda \\ |x-y| < 1}} \frac{|f(x) - f(y)|^p}{|x-y|^{d+sp}} \, dx \, dy \right)^{1/p} < \infty$$

to a function $\mathfrak{E}f$ in the Besov space $B_s^{p,p}(\mathbb{R}^d)$ of all measurable functions g on \mathbb{R}^d such that

$$\|g\|_{L^p(\mathbb{R}^d)} + \left(\iint_{x, y \in \mathbb{R}^d} \frac{|g(x) - g(y)|^p}{|x-y|^{d+sp}} \, dx \, dy \right)^{1/p} < \infty.$$

Remark 8.4. The Besov spaces are nested with the Bessel potential spaces in the sense that $B_s^{p,p}(\mathbb{R}^d) \subset H^{s-\varepsilon,p}(\mathbb{R}^d)$ for each $s > 0$ and every $\varepsilon \in]0, s[$. Moreover, $W^{1,p}(\mathbb{R}^d) \subset B_s^{p,p}(\mathbb{R}^d)$. Proofs of these results can be found e.g. in [45, Sec. 2.3.2/2.5.1].

Proof of Lemma 8.2. Using Remark 8.4 it suffices to construct an extension in $B_s^{p,p}$ with the respective properties. Moreover, by the reverse triangle inequality it is enough to construct any extension $f \in B_s^{p,p}(\mathbb{R}^d)$ of $u \log \text{dist}_E$ – then $|f|$ can be used as the required extension of $|u \log \text{dist}_E|$. These considerations and Proposition 8.3 show that the claim follows provided

$$(8.2) \quad \|u \log(\text{dist}_D)\|_{L^r(\Lambda)} + \left(\iint_{\substack{x, y \in \Lambda \\ |x-y| < 1}} \frac{|u(x) \log(\text{dist}_E(x)) - u(y) \log(\text{dist}_E(y))|^r}{|x-y|^{d+sr}} \, dx \, dy \right)^{1/r}$$

is finite.

To bound the L^r norm on the left-hand side of (8.2) choose $q \in]1, \infty[$ such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ and apply Hölder's inequality

$$\|u \log(\text{dist}_E)\|_{L^r(\Lambda)} \leq \|u\|_{L^p(\Lambda)} \|\log(\text{dist}_D)\|_{L^q(\Lambda)}.$$

For the second term on the right-hand we use that the Aikawa dimension of the porous set E is strictly smaller than d , see Remark 4.4. This entails for some $\alpha < d$ and some $x \in E$ the estimate

$$\int_{\Lambda} \text{dist}_E(x)^{\alpha-d} dx \leq \int_{B(x, 2 \text{diam } \Lambda)} \text{dist}_E(x)^{\alpha-d} dx \leq c_{\alpha} (2 \text{diam } \Lambda)^{\alpha} < \infty.$$

Hence, some negative power of dist_E is integrable on Λ and by subordination of logarithmic growth $\log(\text{dist}_E) \in L^q(\Lambda)$ follows. Altogether, $u \log(\text{dist}_E) \in L^r(\Lambda)$ taking care of the first term in (8.2).

By symmetry the domain of integration for the second term on the left-hand side of (8.2) can be restricted to $\text{dist}_E(x) \geq \text{dist}_E(y)$. By adding and subtracting the term $u(y) \log(\text{dist}_E(x))$ it in fact suffices to prove that

$$(8.3) \quad \left(\int_{\Lambda} \int_{\Lambda} \frac{|u(x) - u(y)|^r}{|x - y|^{d+sr}} |\log(\text{dist}_E(x))|^r dx dy \right)^{1/r}$$

and

$$(8.4) \quad \left(\int_{\Lambda} |u(y)|^r \int_{\substack{x \in \Lambda \\ \text{dist}_E(x) \geq \text{dist}_E(y)}} \frac{|\log(\text{dist}_E(x)) - \log(\text{dist}_E(y))|^r}{|x - y|^{d+sr}} dx dy \right)^{1/r}$$

are finite. Fix $s < t < 1$, write (8.3) in the form

$$\left(\int_{\Lambda} \int_{\Lambda} \frac{|u(x) - u(y)|^r}{|x - y|^{dr/p+tr}} \frac{|\log(\text{dist}_E(x))|^r}{|x - y|^{dr/q+sr-tr}} dx dy \right)^{1/r}$$

and apply Hölder's inequality with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ to bound it by

$$\begin{aligned} &\leq \left(\int_{\Lambda} \int_{\Lambda} \frac{|u(x) - u(y)|^p}{|x - y|^{d+tp}} dx dy \right)^{1/p} \left(\int_{\Lambda} \int_{\Lambda} \frac{|\log(\text{dist}_E(x))|^q}{|x - y|^{d+(s-t)q}} dy dx \right)^{1/q} \\ &\leq \|\log(\text{dist}_E)\|_{L^q(\Lambda)} \left(\int_{\Lambda} \int_{\Lambda} \frac{|u(x) - u(y)|^p}{|x - y|^{d+tp}} dx dy \right)^{1/p} \left(\int_{|y| \leq \text{diam}(\Lambda)} \frac{1}{|y|^{d+(s-t)q}} dy \right)^{1/q} \end{aligned}$$

Now, $\log(\text{dist}_E) \in L^q(\Lambda)$ has been proved above and the third integral is absolutely convergent since $d + (s - t)q < d$. Finally note that by assumption u has an extension $v \in W^{1,p}(\mathbb{R}^d)$. Since $W^{1,p}(\mathbb{R}^d) \subset B_s^{p,p}(\mathbb{R}^d)$ the middle term above is finite as well, see Remark 8.4.

It remains to show that the most interesting term (8.4) is finite. Here, the additional assumptions on u , s and r enter the game. By the mean value theorem for the logarithm and since dist_E is a contraction, the r -th power of this term is bounded above by

$$\begin{aligned} &\int_{\Lambda} |u(y)|^r \int_{\substack{x \in \Lambda \\ \text{dist}_E(x) \geq \text{dist}_E(y)}} \frac{|\text{dist}_E(x) - \text{dist}_E(y)|^r}{\text{dist}_E(y)^r |x - y|^{d+sr}} dx dy \\ &\leq \int_{\Lambda} \left| \frac{u(y)}{\text{dist}_E(y)} \right|^r \int_{\Lambda} \frac{|x - y|^r}{|x - y|^{d+sr}} dx dy \\ &\leq \int_{\Lambda} \left| \frac{u(y)}{\text{dist}_E(y)} \right|^r dy \int_{|x| \leq \text{diam}(\Lambda)} \frac{1}{|x|^{d+r(s-1)}} dx. \end{aligned}$$

Now, the integral with respect to x is finite since $r(s - 1) < 0$. The integral with respect to y is finite since by assumption $\frac{u}{\text{dist}_E}$ is p -integrable on the bounded domain Λ and thus r -integrable for every $r < p$. \square

On noting that by Definition 4.3 a subset of a porous set is again porous, Lemma 8.2 applies to the bounded d -set Ω_{\star} and the porous set $D_{\star} \subset D$. Moreover, $w = (1 - \eta)u \in W^{1,p}(\Omega_{\star})$ has

the extension $\widehat{w} \in W^{1,p}(\mathbb{R}^d)$ and satisfies

$$\int_{\Omega_\star} \left| \frac{w(x)}{\text{dist}_{D_\star}(x)} \right|^p dx \leq \|1 - \eta\|_\infty \int_{\Omega} \left| \frac{u(x)}{\text{dist}_D(x)} \right|^p < \infty.$$

Hence we can record:

Corollary 8.5. *For every $r \in]1, p[$ and every $s \in]0, 1[$ the function $|w \log(\text{dist}_{D_\star})|$ defined on Ω_\star has an extension $f_{s,r} \in H^{s,r}(\mathbb{R}^d)$ that is positive almost everywhere.*

Step 5: Re-inspecting the right-hand side of (8.1). We return to (8.1). Given $r \in]1, p[$ and $s \in]0, 1[$ let $f_{s,r} \in H^{s,r}(\mathbb{R}^d)$ be as in Corollary 8.5. By (4.3) we can infer

$$\limsup_{r \rightarrow 0} \frac{1}{r^d} \int_{B(y,r) \cap \Omega_\star} |w \log(\text{dist}_D)| dx \leq \limsup_{r \rightarrow 0} \frac{1}{r^d} \int_{B(y,r)} f_{s,r} dx < \infty$$

for (s, r) -quasievery $y \in D_\star \setminus N$. By the conclusion of Step 3 this implies $\widehat{w}(y) = 0$ for (s, r) -quasievery $y \in D_\star \setminus N$. To proceed further, we distinguish two cases:

- (i) It holds $p \leq d$. Since the product $sr < p \leq d$ can get arbitrarily close to p , Lemma 4.8 yields for every $r \in]1, p[$ that $\widehat{w} = 0$ holds $(1, r)$ -quasieverywhere on $D_\star \setminus N$. Moreover, since $C_{1,p}(N) = 0$ by definition, $\widehat{w} = 0$ holds even $(1, r)$ -quasieverywhere on D_\star .
- (ii) It holds $p > d$. Then \widehat{w} is the continuous representative of $\widehat{w} \in W^{1,p}(\mathbb{R}^d)$ and N is empty, see the beginning of Step 3. Moreover, we can choose s and r such that $d - l < sr$ and conclude from the comparison theorem, Theorem 4.9, that \widehat{w} vanishes \mathcal{H}_l^∞ -a.e. on D_\star . Since D is l -thick and U_\star is open, for each $y \in D \cap U_\star$ the set $B(y, r) \cap D \cap U_\star$ coincides with $B(y, r) \cap D$ provided $r > 0$ is small enough and thus has strictly positive \mathcal{H}_l^∞ -measure. So, the continuous function \widehat{w} has to vanish *everywhere* on $D \cap U_\star$ as well as on the closure of the latter set – which by definition is D_\star .

Summing up, $\widehat{w} = 0$ has been shown to hold $(1, r)$ -quasieverywhere on D_\star for every $r \in]1, p[$. From the beginning of Step 3 we also know that \widehat{w} vanishes everywhere on $D \setminus D_\star$ and as $\widehat{w} \in W^{1,p}(\mathbb{R}^d)$ has compact support, Hölder's inequality yields $\widehat{w} \in W^{1,r}(\mathbb{R}^d)$. Combining these two observations with Theorem 4.11 we are eventually led to

$$(8.5) \quad \widehat{w} \in W^{1,p}(\mathbb{R}^d) \cap \bigcap_{1 < r < p} W_D^{1,r}(\mathbb{R}^d).$$

We continue by quoting the following result of Hedberg and Kilpeläinen.

Proposition 8.6 ([20, Cor. 3.5]). *Let $p \in]1, \infty[$ and let $\Lambda \subset \mathbb{R}^d$ be a bounded domain whose boundary is l -thick for some $l \in]d - p, d]$. Then*

$$W^{1,p}(\Lambda) \cap \bigcap_{1 < r < p} W_0^{1,r}(\Lambda) \subset W_0^{1,p}(\Lambda).$$

Remark 8.7. In [20] the requirement on Λ is that its complement is uniformly p -fat – a property that by the ingenious characterization in [28, Thm. 1] holds for every bounded set with l -thick boundary provided $l \in]d - p, d]$.

In order to apply this result to the case of mixed boundary conditions, we proceed similarly to the proof of Theorem 3.1: With $B \subset \mathbb{R}^d$ an open ball that contains the compact support of \widehat{w} define again

$$\mathcal{C} := \{M \subset B \setminus D : M \text{ open, connected and } \Omega \subset M\}$$

and

$$\Omega_\bullet := \bigcup_{M \in \mathcal{C}} M.$$

Then $\partial\Omega_\bullet \in \{D, D \cup \partial B\}$ by Corollary 5.4, subsequent to which it is also shown that $\partial\Omega_\bullet$ is m -thick for some $m \in]d - p, d]$. Finally, let $\eta \in C_0^\infty(B)$ be identically one on the support of \widehat{w} . As $\varphi \mapsto (\eta\varphi)|_{\Omega_\bullet}$ induces a bounded operator $W_D^{1,p}(\mathbb{R}^d) \rightarrow W_0^{1,p}(\Omega_\bullet)$, it follows from (8.5) that

$$\widehat{w}|_{\Omega_\bullet} = (\eta\widehat{w})|_{\Omega_\bullet} \in W^{1,p}(\Omega_\bullet) \cap \bigcap_{1 < r < p} W_0^{1,r}(\Omega_\bullet)$$

and thus $\widehat{w}|_{\Omega_\bullet} \in W_0^{1,p}(\Omega_\bullet)$ thanks to Proposition 8.6. Since by construction $\Omega \subset \Omega_\bullet$ and $D \subset \partial\Omega_\bullet$, we eventually conclude

$$w = \widehat{w}|_\Omega \in W_D^{1,p}(\Omega)$$

and the proof is complete. \square

9. A GENERALIZATION

If one asks: ‘What is the most restricting condition in Theorem 3.1?’, the answer doubtlessly is the assumption that a *global* extension operator shall exist. Certainly, this excludes all geometries that include cracks not belonging completely to the Dirichlet boundary part as in the subsequent Figure.

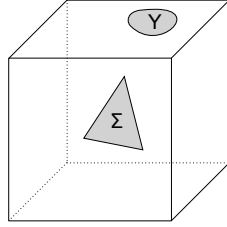


FIGURE 2. The domain Ω is the cube minus the triangle Σ . The Dirichlet boundary part D consists exactly of the six outer sides of the cube minus the droplet Υ on the cover plate.

Since the distance function dist_D measures only the distance to the Dirichlet boundary part D , points in $\partial\Omega$ that are far from D should not be of great relevance in view of the Hardy inequality (3.1). In the following considerations we intend to make this precise. Let $U, V \subset \mathbb{R}^d$ be two open, bounded sets with the properties

$$(9.1) \quad D \subset U, \quad \overline{V} \cap D = \emptyset, \quad \overline{\Omega} \subset U \cup V.$$

The philosophy behind this is to take U as a small neighborhood of D which – desirably – excludes the ‘nasty parts’ of $\partial\Omega \setminus D$. More properties of U, V will be specified below. Let $\eta_U \in C_0^\infty(U), \eta_V \in C_0^\infty(V)$ be two functions with $\eta_U + \eta_V = 1$ on $\overline{\Omega}$. Then one can estimate

$$\left(\int_{\Omega} |u|^p \text{dist}_D^{-p} \, dx \right)^{1/p} \leq \left(\int_{U \cap \Omega} |\eta_U u|^p \text{dist}_D^{-p} \, dx \right)^{1/p} + \left(\int_{V \cap \Omega} |\eta_V u|^p \text{dist}_D^{-p} \, dx \right)^{1/p}.$$

Since dist_D is larger than some $\varepsilon > 0$ on $\text{supp}(\eta_V) \subset V$, the second term can be estimated by $\frac{1}{\varepsilon} \left(\int_{\Omega} |u|^p \, dx \right)^{1/p}$. If one assumes, as above, Poincaré’s inequality, then this term may be estimated as required. In order to provide an adequate estimate also for the first term, we introduce the following assumption.

Assumption 9.1. *The set U from above can be chosen in such a way that $\Lambda := \Omega \cap U$ is again a domain and if one puts $\Gamma := (\partial\Omega \setminus D) \cap U$ and $E := \partial\Lambda \setminus \Gamma$, then there is a linear, continuous extension operator $\mathfrak{F} : W_E^{1,p}(\Lambda) \rightarrow W_E^{1,p}(\mathbb{R}^d)$.*

Clearly, this assumption is weaker than Condition (iii) in Theorem 3.1; in other words: Condition (iii) in Theorem 3.1 requires Assumption 9.1 to hold for an open set $U \supset \bar{\Omega}$.

Let us discuss the sense of Assumption 9.1 in extenso. Philosophically spoken, it allows to focus on the extension not of the functions u but the functions $\eta_U u$, which live on a set whose boundary does (possibly) not include the ‘nasty parts’ of $\partial\Omega \setminus D$ that are an obstruction against a global extension operator. In detail: one first observes that, for $\eta = \eta_U \in C_0^\infty(U)$ and $v \in W_D^{1,p}(\Omega)$, the function $\eta v|_\Lambda$ even belongs to $W_E^{1,p}(\Lambda)$, see [19, Thm. 5.8]. Secondly, we have by the definition of E

$$\partial U \cap \Omega = (\partial U \cap \Omega) \setminus \Gamma \subset \partial\Lambda \setminus \Gamma = E.$$

This shows that the ‘new’ boundary part $\partial U \cap \Omega$ of Λ belongs to E and is, therefore, uncritical in view of extension. Thirdly, one has $D = D \cap U \subseteq \partial\Omega \cap U \subset \partial\Lambda$, and it is clear that the ‘new Dirichlet boundary part’ E includes the ‘old’ one D . Hence, the extension operator \mathfrak{F} may be viewed also as a continuous one between $W_E^{1,p}(\Lambda)$ and $W_D^{1,p}(\mathbb{R}^d)$. Thus, concerning $v := \eta u = \eta_U u$ one is – mutatis mutandis – again in the situation of Theorem 3.1: $\eta u \in W_E^{1,p}(\Lambda) \subset W_D^{1,p}(\Lambda)$ admits an extension $\mathfrak{F}(\eta u) \in W_E^{1,p}(\mathbb{R}^d) \subseteq W_D^{1,p}(\mathbb{R}^d)$, which satisfies the estimate $\|\mathfrak{F}(\eta u)\|_{W_D^{1,p}(\mathbb{R}^d)} \leq c\|\eta u\|_{W_D^{1,p}(\Lambda)}$, the constant c being independent from u . This leads, as above, to a corresponding (continuous) extension operator $\mathfrak{F}_\bullet : W_E^{1,p}(\Lambda) \rightarrow W_0^{1,p}(\Lambda_\bullet)$. Here, of course, Λ_\bullet has again to be defined as the connected component of $B \setminus D$ that contains Λ . Thus one may proceed again as in the proof of Theorem 3.1, and gets, for $u \in W_D^{1,p}(\Omega)$,

$$\begin{aligned} \int_\Omega \left(\frac{|\eta u|}{\text{dist}_D} \right)^p dx &= \int_\Lambda \left(\frac{|\eta u|}{\text{dist}_D} \right)^p dx \leq \int_{\Lambda_\bullet} \left(\frac{|\mathfrak{F}_\bullet(\eta u)|}{\text{dist}_{\partial\Lambda_\bullet}} \right)^p dx \leq c\|\nabla(\mathfrak{F}_\bullet(\eta u))\|_{L^p(\Lambda_\bullet)}^p \\ &\leq c\|\mathfrak{F}_\bullet(\eta u)\|_{W^{1,p}(\Lambda_\bullet)}^p \leq c\|\eta u\|_{W^{1,p}(\Lambda)}^p \leq c(\|u\|_{L^p(\Omega)}^p + \|\nabla u\|_{L^p(\Omega)}^p), \end{aligned}$$

since ηu belongs to $W_E^{1,p}(\Lambda) \subset W_D^{1,p}(\Lambda)$. Exploiting a last time Poincaré’s inequality, whose validity will be discussed in a moment, one gets the desired estimate.

When aiming at Poincaré’s inequality, it seems convenient to follow again the argument in the proof of Proposition 7.1: as pointed out above, the property $\mathbb{1} \notin W_D^{1,p}(\Omega)$ has to do only with the local behavior of Ω around the points of D , cf. Lemma 7.3. Hence, this will not be discussed further here.

Concerning the compactness of the embedding $W_D^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$, one does not need the existence of a *global* extension operator $\mathfrak{E} : W_D^{1,p}(\Omega) \rightarrow W^{1,p}(\mathbb{R}^d)$. In fact, writing for every $v \in W_D^{1,p}(\Omega)$ again $v = \eta_U v + \eta_V v$ and supposing Assumption 9.1, one gets the following:

If $\{v_k\}_k$ is a bounded sequence in $W_D^{1,p}(\Omega)$, then the sequence $\{\eta_U v_k|_\Lambda\}_k$ is bounded in $W_E^{1,p}(\Lambda)$. Due to the extendability property, this sequence contains a subsequence $\{\eta_U v_{k_l}|_\Lambda\}_l$ that converges in $L^p(\Lambda)$ to an element v_U . Thus, $\{\eta_U v_{k_l}\}_l$ converges to the function on Ω that equals v_U on Λ and 0 on $\Omega \setminus \Lambda$. The elements $\eta_V v_k$ in fact live on the set $\Pi := \Omega \cap V$ and are zero on $\Omega \setminus V$. In particular they are zero in a neighborhood of D . Moreover, they form a bounded subset of $W^{1,p}(\Pi)$. Therefore it makes sense to require that Π is again a domain, and, secondly that Π meets one of the well-known compactness criteria $W^{1,p}(\Pi) \hookrightarrow L^p(\Pi)$, cf. [39, Ch. 1.4.6]. Keep in mind that such requirements are much weaker than the global $W^{1,p}$ -extendability, and in particular include the example in Figure 2, as long as the triangle Σ has a positive distance to the six outer sides of the cube. Resting on these criteria, one obtains again the convergence of a subsequence $\{\eta_V v_{k_l}|_\Pi\}_l$ that converges in $L^p(\Pi)$ towards a function v_V . The sequence $\{\eta_V v_{k_l}\}_l$ then converges in $L^p(\Omega)$ to a function that equals v_V on Π and zero on $\Omega \setminus V$.

Altogether, we have extracted a subsequence of $\{v_k\}_k$ that converges in $L^p(\Omega)$.

Remark 9.2. In fact one does not really need that Π is connected. By similar arguments as above it suffices to demand that it splits up in at most finitely many components Π_1, \dots, Π_n , such that each of these admits the compactness of the embedding $W^{1,p}(\Pi_j) \hookrightarrow L^p(\Pi_j)$.

We summarize these considerations in the following theorem.

Theorem 9.3. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and $D \subset \partial\Omega$ be a closed part of the boundary. Suppose that the following three conditions are satisfied:*

- (i) *The set D is l -thick for some $l \in]d - p, d]$.*
- (ii) *The space $W_D^{1,p}(\Omega)$ can be equivalently normed by $\|\nabla \cdot\|_{L^p(\Omega)}$.*
- (iii) *There are two open sets $U, V \subset \mathbb{R}^d$ that satisfy (9.1) and U satisfies Assumption 9.1.*

Then there is a constant $c > 0$ such that Hardy's inequality

$$\int_{\Omega} \left| \frac{u}{\text{dist}_D} \right|^p dx \leq c \int_{\Omega} |\nabla u|^p dx$$

holds for all $u \in W_D^{1,p}(\Omega)$.

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