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CONSTRUCTION OF THE MULTI-SOLITON TRAINS, MULTI KINK-SOLITON TRAINS OF THE DERIVATIVE NONLINEAR SCHRÖDINGER EQUATIONS BY THE FIXED POINT METHOD

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Abstract. We look for solutions to derivative nonlinear Schrodinger equations built upon ¨ solitons. We prove the existence of multi-soliton trains i.e. solutions behaving at large time as the sum of finite solitons. We also show that one can attach a kink at the begin of the train i.e multi kink-soliton trains. Our proofs proceed by fixed point arguments around the desired profile, using Strichartz estimates.

Contents

1. Introduction

We consider the derivative nonlinear Schrödinger equation:

$$iu_t + u_{xx} + i\alpha|u|^2u_x + i\mu u^2\overline{u_x} + b|u|^4u = 0, \tag{1.1}$$

where $\alpha, \mu, b \in \mathbb{R}$ and u is a complex value function of $(t, x) \in \mathbb{R} \times \mathbb{R}$.

In [22], the authors studied the Cauchy problem and gave a sufficient condition of global well-posedness for a general form of equation (1.1). Moreover, the author constructed modified wave operators on small and sufficiently regular asymptotic states. One can also cite the other works using the Gauge transformation as in [12, 13]. The Cauchy problem of (1.1) was also studied as in [23], where a gauge transformation and the Fourier restriction method is used to obtain local well-posedness in H^s, $s \geqslant 1/2$. Another approach was introduced as in [25, 26] using approximation method. The global well posedness of (1.1) in $H^1(\mathbb{R})$ is studied in many works (see e.g [27, 28, 11]) under small condition of initial data. In [4], the authors proved global result in $H^s(\mathbb{R})$, $s > \frac{2}{3}$ under small condition of initial data. Specially, in the case $b = 0$ and $\mu = 0$, (1.1) is a complete integral equation. In this case, the complete integrability structure of equation was used to prove the global existence of solution on $H^s(\mathbb{R})$, $s > \frac{1}{2}$ by the work as in [1].

Considering the case $\mu = 0$, the class of solitons of (1.1) has two parameters. In the case $b = 0$, a stability result of solitons was obtained as in [3] using variational technique. In [21], the author proved that for each $b > 0$ there exists unique $s^* = s^*(b) > 0 \in (0, 1)$ such that the soliton $u_{\omega,c}$ is

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orbitally stable if $-2\sqrt{\omega} < c < 2s^*\sqrt{\omega}$ and orbitally instable if $2s^*\sqrt{\omega} < c < 2\sqrt{\omega}$. The stability result is obtained in case $b < 0$ as in [10]. Specially, in the case $b = 0$, [14] proved a stability result of solitons in zero mass case. Removing the effect of scaling in stability result of this work is an open question.

Our main goal of this paper is to study the multi-solitons theory of (1.1).

1.1. **Multi-soliton trains.** We focus on studying the following special form of (1.1):

$$iu_t + u_{xx} + i|u|^2u_x + b|u|^4u = 0. \tag{1.2}$$

Our first goal in this paper is to study the long time behaviour of solution of (1.2). More precisely, we study the multi-solitons theory of (1.2). The existence of multi-solitons is a proof of the *soliton resolution conjecture* which state that all global solution of a dispersive equation behaves at large times as a sum of a radiative term and solitons. Theory of multi-solitons has attracted a lot of interest in mathematics. In [15, 16], the authors proved existence and uniqueness of finite and infinite soliton and kink-soliton trains of classical nonlinear Schrödinger equations, using fixed point arguments around of the desired profile. Another method was introduced in [18] for simple power nonlinear Schrödinger equation with L^2-subcritical nonlinearities. The proof was established by two ingredients: the uniform bounded and the compactness property. The arguments was later modified to obtain the results of L^2-supercritical equations [7] and of profile made with excited state [5]. One can also cite the works on the logarithmic Schrödinger equation (logNLS) in the focusing regime as in [9]. Specially, in [29], the inverse scattering transform method (IST) was used to construct multi-solitons in the one dimensional cubic focusing NLS. This method is powerful to show the problems of the complete integrable equations. We would like also to mention the works on the non-linear Klein-Gordon equation as in [8] and [6], and on the stability of multi-solitons for generalized Korteweg-de Vries equations and L^2-subcritical nonlinear Schrödinger equations from Martel, Merle and Tsai [19],[20]. In [17], the authors proved a stability result of multi-solitons of (1.2) in the case $b = 0$. As a sequence, there exists a solition of (1.1) which is close to a sum of solitons by a small constant as long as the initial data is enough close to a profile. In sense of multi-solitons theory, this solution can seem to be a multi-solitons but not really good. Our motivation is to prove the existence of a multi-solitons in similar sense as in [16, 15].

Consider equation (1.2). The soliton of equation (1.2) is solution of form $R_{\omega,c}(t,x) = e^{i\omega t}\phi_{\omega,c}(x-ct)$, where $\phi_{\omega,c} \in H^1(\mathbb{R})$ solves

$$-\phi_{xx} + \omega\phi + ic\phi_x - i|\phi|^2\phi_x - b|\phi|^4\phi = 0, \quad x \in \mathbb{R}. \tag{1.3}$$

Applying the following gauge transform to $\phi_{\omega,c}$

$$\phi_{\omega,c}(x) = \Phi_{\omega,c}(x)\exp\left(i\frac{c}{2}x - \frac{i}{4}\int_{-\infty}^{x}|\Phi_{\omega,c}(y)|^2\,dy\right),$$

it is easily verified that $\Phi_{\omega,c}$ satisfies the following equation.

$$-\Phi_{xx} + \left(\omega - \frac{c^2}{4}\right)\Phi + \frac{c}{2}|\Phi|^2\Phi - \frac{3}{16}\gamma|\Phi|^4\Phi = 0, \quad \gamma := 1 + \frac{16}{3}b. \tag{1.4}$$

The positive even solution of (1.4) is explicitly obtained by: if $\gamma > 0$ ($b > \frac{-3}{16}$),

$$\Phi^2_{\omega,c}(x) = \begin{cases} \frac{2(4\omega-c^2)}{\sqrt{c^2+\gamma(4\omega-c^2)}\cosh(\sqrt{4\omega-c^2}x)-c} & \text{if } -2\sqrt{\omega} < c < 2\sqrt{\omega}, \\ \frac{4c}{(cx)^2+\gamma} & \text{if } c = 2\sqrt{\omega}, \end{cases}$$

and if $\gamma \leqslant 0$ ($b \leqslant -\frac{3}{16}$),

$$\Phi^2_{\omega,c}(x) = \frac{2(4\omega-c^2)}{\sqrt{c^2+\gamma(4\omega-c^2)}\cosh(\sqrt{4\omega-c^2}x)-c} \text{ if } -2\sqrt{\omega} < c < -2s_*\sqrt{\omega},$$

where $s_* = s_*(\gamma) = \sqrt{\frac{-\gamma}{1-\gamma}}$. We note that the following condition of the parameters γ and (ω, c) is the necessary and sufficient condition for the existence of non-trivial solutions of (1.2) vanishing

at infinity (see [2]):

$$\text{if}\gamma > 0 \Leftrightarrow b > \frac{-3}{16}, -2\sqrt{\omega} < c \leqslant 2\sqrt{\omega},$$
$$\text{if}\gamma \leqslant 0 \Leftrightarrow b \leqslant \frac{-3}{16}, -2\sqrt{\omega} < c < -2s_*\sqrt{\omega}.$$

For each $j \in \{1, 2, .., K\}$, we set

$$R_j(t,x) = e^{i\theta_j} R_{\omega_j, c_j}(t, x - x_j).$$

The profile of an multi-solitons is a sum of the form:

$$R = \sum_{j=1}^{K} R_j. \tag{1.5}$$

A solution of (1.2) is called multi-solitons if, for the profile R:

$$\|u(t) - R(t)\|_{H^1} \to 0 \text{ as } t \to \infty,$$

For convenience, we set $h_j = \sqrt{4\omega_j - c_j^2}$. Let (c_j, ω_j) be satisfy the condition of existence of soliton, for each $1 \leqslant j \leqslant K$. We rewrite

$$\Phi_{\omega_j, c_j}(x) = \sqrt{2} h_j (\sqrt{c_j^2 + \gamma h_j^2}\cosh(h_j x) - c_j)^{\frac{-1}{2}}. \tag{1.6}$$

As each soliton is in $H^\infty(\mathbb{R})$, we have $R \in H^\infty(\mathbb{R})$. Our first main result is the following.

Theorem 1.1. *Let $K \in \mathbb{N}^*$ and for each $1 \leqslant j \leqslant K$, $(\theta_j, c_j, \omega_j, x_j)$ be sequence of parameters such that $x_j = 0$, $\theta_j \in \mathbb{R}$, $c_j \neq c_k \neq 0$, for $j \neq k$ and c_j such that $-2\sqrt{\omega_j} < c_j < 2\sqrt{\omega_j}$ if $\gamma > 0$ and $-2\sqrt{\omega_j} < c_j < -2s_*\sqrt{\omega_j}$ if $\gamma \leqslant 0$. The multi-solitons profile R is given as in (1.5). We assume that the parameters (ω_j, c_j) satisfy*

$$(1 + \|R_x\|_{L^\infty L^\infty})(1 + \|R\|_{L^\infty L^\infty}) + \|R\|^4_{L^\infty L^\infty} \ll v_* := \inf_{j \neq k} h_j |c_j - c_k|. \tag{1.7}$$

There exist $T_0 > 0$ and a solution u of (1.2) on $[T_0, \infty)$ such that

$$\|u - R\|_{H^1} \leqslant Ce^{-\lambda t}. \quad \forall t \geqslant T_0, \tag{1.8}$$

where $\lambda = \frac{v_*}{16}$ and C is a certain positive constant.

We observe that the formula of soliton in case $\gamma > 0$ and $\gamma \leqslant 0$ is similar. Thus, in the proof of Theorem 1.1, we only consider the case $\gamma > 0$. The case $\gamma \leqslant 0$ is treated by similar arguments.

Remark 1.2. We give a example of parameters satisfy (1.7) as follows. First, we chose $h_j \ll |c_j|$ and $c_j < 0$ for all j. We have

$$\|\Phi_{\omega_j, c_j}\|^2_{L^\infty} \leqslant \frac{2h_j^2}{\sqrt{c_j^2 + \gamma h_j^2} - c_j} \lesssim \frac{h_j^2}{|c_j|}.$$

Moreover,

$$\partial\Phi_{\omega_j, c_j} = \frac{-\sqrt{2}}{2} h_j^2 \sqrt{c_j^2 + \gamma h_j^2}\sinh(h_j x)(\sqrt{c_j^2 + \gamma h_j^2}\cosh(h_j x) - c_j)^{-\frac{3}{2}}.$$

Thus, choosing $c_j < 0$, for all j, we obtain

$$\begin{aligned}
|\partial\Phi_{\omega_j, c_j}| &\lesssim h_j^2 \sqrt{c_j^2 + \gamma h_j^2}|\sinh(h_j x)|(\sqrt{c_j^2 + \gamma h_j^2}\cosh(h_j x) - c_j)^{-\frac{3}{2}} \\
&\lesssim h_j^2 (\sqrt{c_j^2 + \gamma h_j^2}\cosh(h_j x) - c_j)^{-\frac{1}{2}} \\
&\approx h_j |\Phi_{\omega_j, c_j}| \lesssim \frac{h_j^2}{\sqrt{|c_j|}}.
\end{aligned}$$

In the addition, we have

$$\begin{aligned}\|\partial R_j\|_{L^\infty} &= \|\partial\phi_{\omega_j,c_j}\|_{L^\infty} \approx \|\partial\Phi_{\omega_j,c_j}\|_{L^\infty} + \|\frac{c_j}{2}\Phi_{\omega_j,c_j} - \Phi^3_{\omega_j,c_j}\|_{L^\infty}\\ &\leqslant \|\partial\Phi_{\omega_j,c_j}\|_{L^\infty} + \frac{|c_j|}{2}\|\Phi_{\omega_j,c_j}\|_{L^\infty} + \|\Phi_{\omega_j,c_j}\|^3_{L^\infty}\\ &\lesssim \frac{h_j^2}{\sqrt{|c_j|}} + h_j\sqrt{|c_j|} + \frac{h_j^3}{\sqrt{|c_j|^3}}.\end{aligned}$$

Thus, the left hand side of (1.7) is bounded by

$$\left(1+\sum_{1\leqslant j\leqslant K}\left(\frac{h_j^2}{\sqrt{|c_j|}} + h_j\sqrt{|c_j|} + \frac{h_j^3}{\sqrt{|c_j|^3}}\right)\right)\left(1+\sum_{1\leqslant j\leqslant K}\frac{h_j}{\sqrt{|c_j|}}\right) + \sum_{1\leqslant j\leqslant K}\frac{h_j^4}{c_j^2}. \tag{1.9}$$

Now, we give an example of parameters satisfying (1.7). Let $d_j < 0$, $h_j \in \mathbb{R}$ for all $j \in \{1, 2, ..., K\}$ such that $d_j \neq d_k$ for all $j \neq k$. Let $(c_j, \omega_j) = \big(Md_j, \frac{1}{4}(h_j^2 + M^2 d_j^2)\big)$. By this choosing, (1.9) is order M^0 and the right hand side of (1.7) is order M^1. Thus, (1.7) is satisfied (1.7) for M large enough.

1.2. **Multi kink-soliton trains.** Second, we consider another special case of (1.1) as follows

$$iu_t + u_{xx} + iu^2\overline{u_x} + b|u|^4u = 0. \tag{1.10}$$

Our goal is to construct the multi kink-soliton trains of (1.10). Motivation of this work is from [16], where the authors have been success to constructed an infinite multi kink-soliton trains for classical nonlinear Schrödinger equation by using fixed point arguments. However, in the case of (1.10), this method can not directly use by appearing of a derivative term. To overcome this difficult, we will use the similar method in the proof of Theorem 1.1. Specially, in the case $b = 0$, the same author of this paper [24] established local theory of (1.10) under nonvanishing boundary condition. To our knowledge, this is the first result in studying Cauchy problem of nonlinear Schrödinger equation with derivative term.

Consider the equation (1.10). First, we would like to define a kink solution of (1.10). Let $R_{\omega,c}$ be a smooth solution of (1.10) of form:

$$R_{\omega,c}(t,x) = e^{i\omega t}\phi_{\omega,c}(x - ct), \tag{1.11}$$

where $\phi_{\omega,c}$ is smooth and solves

$$-\phi_{xx} + \omega\phi + ic\phi_x - i\phi^2\overline{\phi_x} - b|\phi|^4\phi = 0, \quad x \in \mathbb{R}. \tag{1.12}$$

If $\phi_{\omega,c}\,|_{\mathbb{R}^+} \in H^1(\mathbb{R}^+)$ then the following Gauge transform is well defined:

$$\Phi_{\omega,c} = \exp\left(-i\frac{c}{2}x + \frac{i}{4}\int_\infty^x |\phi_{\omega,c}(y)|^2\,dy\right)\phi_{\omega,c}.$$

Since $\phi_{\omega,c}$ solves (1.12), $\Phi_{\omega,c}$ is smooth and solves

$$-\Phi_{xx} + \left(\omega - \frac{c^2}{4}\right)\Phi - \frac{3}{2}\mathcal{I}m(\overline{\Phi}\Phi_x)\Phi - \frac{c}{2}|\Phi|^2\Phi + \frac{3}{16}\gamma|\Phi|^4\Phi = 0, \quad \gamma := \frac{5}{3} - \frac{16}{3}b. \tag{1.13}$$

Since $\Phi_{\omega,c}\,|_{\mathbb{R}^+} \in H^2(\mathbb{R}^+)$, by similar arguments as in [3], we can prove that $\mathcal{I}m(\overline{\Phi_{\omega,c}}\partial_x\Phi_{\omega,c}) = 0$. Thus, $\Phi_{\omega,c}$ solves

$$-\Phi_{xx} + \left(\omega - \frac{c^2}{4}\right)\Phi - \frac{c}{2}|\Phi|^2\Phi + \frac{3}{16}\gamma|\Phi|^4\Phi = 0. \tag{1.14}$$

Now, we give the definition of a half-kink of (1.2).

Definition 1.3. The function $R_{\omega,c}$ is called a half-kink solution of (1.2) if $R_{\omega,c}$ is of form (1.11) and associated $\Phi_{\omega,c}$ is a real valued function solves (1.14) and satisfies:

$$\begin{cases} -\Phi'' + \tilde{\omega}\Phi - f(\Phi) = 0,\\ \lim\limits_{x\to\pm\infty}\Phi(x) \neq 0,\\ \lim\limits_{x\to\mp\infty}\Phi(x) = 0, \end{cases} \tag{1.15}$$

where $\tilde{\omega} = \omega - \frac{c^2}{4}$, $f : \mathbb{R} \to \mathbb{R}$ such that $f(s) = \frac{c}{2}s^3 - \frac{3}{16}\gamma s^5$.

The following result about the existence of a half-kink profile is stated in [15] as follows:

Proposition 1.4. *Let $f : \mathbb{R} \to \mathbb{R}$ be a C^1 function with $f(0) = 0$ and define $F(s) := \int_0^s f(t)\,dt$. For $\tilde{\omega} \in \mathbb{R}$, let*

$$\zeta(\tilde{\omega}) := \inf\left\{\zeta > 0, F(\zeta) - \frac{1}{2}\tilde{\omega}\zeta^2 = 0\right\}$$

and assume that there exists $\tilde{\omega}_1 \in \mathbb{R}$ such that

$$\zeta(\tilde{\omega}_1) > 0, \quad f'(0) - \tilde{\omega}_1 < 0, \quad f(\zeta(\tilde{\omega}_1)) - \tilde{\omega}_1\zeta(\tilde{\omega}_1) = 0. \tag{1.16}$$

Then, for $\tilde{\omega} = \tilde{\omega}_1$, there exists a half-kink profile $\Phi \in C^2(\mathbb{R})$ of (1.15) i.e Φ is unique (up to translation), positive and satisfies $\Phi > 0$, $\Phi' > 0$ on $\mathbb{R}$ and the boundary conditions

$$\lim_{x\to-\infty} \Phi(x) = 0, \quad \lim_{x\to\infty} \Phi(x) = \zeta(\tilde{\omega}_1) > 0. \tag{1.17}$$

If in addition,

$$f'(\zeta(\tilde{\omega}_1)) - \tilde{\omega}_1 < 0, \tag{1.18}$$

then for any $0 < a < \tilde{\omega}_1 - \max\{f'(0), f'(\zeta(\tilde{\omega}_1))\}$ there exists $D_a > 0$ such that

$$|\Phi'(x)| + |\Phi(x)1_{x<0}| + |(\zeta(\tilde{\omega}_1) - \Phi(x))1_{x>0}| \leqslant D_a e^{-a|x|}, \quad \forall x \in \mathbb{R}. \tag{1.19}$$

Remark 1.5.

(1) As [15, Remark 1.15], using the symmetry $x \to -x$ and Proposition 1.4 implies the existence and uniqueness of half-kink profile Φ satisfying

$$\lim_{x\to-\infty} \Phi(x) = \zeta(\tilde{\omega}_1) > 0. \quad \lim_{x\to\infty} \Phi(x) = 0.$$

(2) In our case, $f(s) = \frac{c}{2}s^3 - \frac{3}{16}\gamma s^5$. Let us see Proposition 1.4 under this nonlinear term. We may check that if $\gamma > 0$, $c > 0$ then there exists $\tilde{\omega}_1 = \frac{c^2}{4\gamma}$ and $\zeta(\tilde{\omega}_1) = \sqrt{\frac{2c}{\gamma}}$ satisfy the conditions (1.16), (1.18) and the definition of the function ζ. Thus, using Proposition 1.4, if $\gamma > 0$, $c > 0$ then there exists half-kink solution of (1.2) and the constant a in Proposition 1.4 satisfies

$$0 < a < \frac{c^2}{4\gamma}.$$

(3) Consider the half-kink profile Φ of Proposition 1.4. Since Φ solves (1.15) and satisfies (1.19), we have

$$|\Phi''(x)| + |\Phi'''(x)| \leqslant D_a e^{-a|x|}.$$

Now, we assume $\gamma > 0$. Let $K > 0$, $\theta_0, x_0, \omega_0, c_0 \in \mathbb{R}$ be such that $x_0 = 0$, $2\sqrt{\omega_0} > c_0 > \sqrt{2\gamma}$. For $1 \leqslant j \leqslant K$, let $(\theta_j, x_j, \omega_j, c_j) \in \mathbb{R}$ be such that $x_j = 0$, $c_j > c_0$, $c_j \neq c_k$ for $j \neq k$, $2\sqrt{\omega_j} > c_j > 2s_*\sqrt{\omega_j}$ for $s_* = \sqrt{\frac{\gamma}{1+\gamma}}$. Set $R_j = e^{i\theta_j}R_{\omega_j,c_j}$, where $R_{\omega_j,c_j} \in H^1(\mathbb{R})$ be soliton solution of (1.10) given associated to the profile:

$$\Phi^2_{\omega_j,c_j} = \frac{2h_j^2}{\sqrt{c_j^2 - \gamma h_j^2}\cosh(h_j x) + c_j}, \quad (h_j = \sqrt{4\omega_j - c_j^2}).$$

Let Φ_0 be the half-kink profile given as in Remark 1.5 (1) associated with parameters ω_0, c_0 and R_{ω_0,c_0} be the associated half-kink solution of (1.10). Set $R_0 = e^{i\theta}R_{\omega_0,c_0}$. The multi kink-soliton profile of (1.10) is defined as follows:

$$V = R_0 + \sum_{j=1}^{K} R_j. \tag{1.20}$$

Our second main result is the following:

Theorem 1.6. *Considering (1.10), we assume that $b < \frac{5}{16}$ ($\gamma > 0$). Let V be given as in (1.20). We assume that the parameters (ω_j, c_j) satisfy*

$$(1 + \|V_x\|_{L^\infty L^\infty})\,(1 + \|V\|_{L^\infty L^\infty}) + \|V\|^4_{L^\infty L^\infty} \ll v_* := \inf_{j\neq k} h_j|c_j - c_k|. \tag{1.21}$$

Then there exist a solution u to (1.10) such that

$$\|u - V\|_{H^1} \leqslant Ce^{-\lambda t}. \quad \forall t \geqslant T_0, \tag{1.22}$$

where $\lambda = \frac{v_*}{16}$ *and* C *is a certain positive constant.*

Remark 1.7.

(1) The condition $c_0^2 > 2\gamma$ in Theorem 1.6 is a technical condition and we can remove this. Under this condition, the constant a in Proposition 1.4 satisfies

$$0 < a < \frac{c_0^2}{4\gamma}.$$

Thus, we can choose $a = \frac{1}{2}$. This fact makes the proof is more easier and we have

$$|\Phi_0'''(x)| + |\Phi_0''(x)| + |\Phi_0'(x)| + |\Phi_0(x)1_{x>0}| + \left|\left(\sqrt{\frac{B}{2A}} - \Phi_0(x)\right)1_{x<0}\right| \lesssim e^{-\frac{1}{2}|x|}. \tag{1.23}$$

(2) By similar arguments as above, we can prove that there exists a half-kink solution of (1.2) which satisfies the definition 1.3. To our knowledge, there are no result about stability or instability of this kind of solution.

(3) Let $\gamma > 0$. We give an example of parameters satisfy the condition (1.21) of Theorem 1.6. As in remark 1.2, we have

$$\Phi_{\omega_j,c_j} = \sqrt{2}h_j\left(\sqrt{c_j^2 - \gamma h_j^2}\cosh(h_j x) + c_j\right)^{\frac{-1}{2}}, \quad \forall j.$$

Hence, choosing $h_j \ll c_j$, for all j, we have

$$\|\Phi_{\omega_j,c_j}\|_{L^\infty}^2 \leqslant \frac{2h_j^2}{\sqrt{c_j^2 - \gamma h_j^2} + c_j} \lesssim \frac{h_j^2}{c_j}.$$

By similar arguments in remark 1.2, for all $1 \leqslant j \leqslant K$, we have

$$\|\partial R_j\|_{L^\infty} \lesssim \frac{h_j^2}{\sqrt{c_j}} + h_j\sqrt{c_j} + \frac{h_j^3}{\sqrt{c_j^3}}.$$

Now, we treat to the case $j = 0$. Let Φ_0 be the profile given as in Proposition 1.4 associated to the parameters c_0, ω_0 and R_0 be associated half-kink solution of (1.2). Since (1.19), remark 1.5 and remark 1.7 we have

$$\begin{aligned}\|\Phi_0\|_{L^\infty} &\lesssim \sqrt{\frac{B}{2A}} \approx \sqrt{c_0},\\ \|\partial\Phi_0\|_{L^\infty} &\lesssim 1,\end{aligned}$$

Thus,

$$\begin{aligned}\|R_0\|_{L^\infty L^\infty} &\lesssim \sqrt{c_0},\\ \|\partial R_0\|_{L^\infty L^\infty} &\lesssim 1 + c_0^{\frac{3}{2}} \lesssim c_0^{\frac{3}{2}}.\end{aligned}$$

This implies that for $h_j \ll c_j$ $(j = 1, .., K)$ the left hand side of (1.21) is estimated by:

$$\left(1 + c_0^{\frac{3}{2}} + \sum_{j=1}^K\left(\frac{h_j^2}{\sqrt{c_j}} + h_j\sqrt{c_j} + \frac{h_j^3}{\sqrt{c_j^3}}\right)\right)\left(1 + \sqrt{c_0} + \sum_{j=1}^K\frac{h_j}{\sqrt{c_j}}\right).$$

Choosing $c_0 \approx 1$, the above expression is estimated by:

$$1 + \sum_{j=1}^K\left(\frac{h_j^2}{\sqrt{c_j}} + h_j\sqrt{c_j} + \frac{h_j^3}{\sqrt{c_j^3}}\right)\sum_{j=1}^K\frac{h_j}{\sqrt{c_j}}. \tag{1.24}$$

Let $h_j, d_j \in \mathbb{R}^+$, $d_j \neq d_k$ for all $j \neq k$, $1 \leqslant j, k \leqslant K$. Set $c_j = Md_j$, $\omega_j = \frac{1}{4}(h_j^2 + M^2d_j^2)$. We have (1.24) is of order M^0 and the right hand side of (1.21) is of order M^1. Thus, by these choices of parameters, when M is large enough, the condition (1.21) is satisfied.

The proof of Theorem 1.6 uses similar arguments as in the one of Theorem 1.1. To prove Theorem 1.1, our strategy is the following. Let R be the multi-solitons profile given as in Theorem 1.1. Our aim is to construct a solution of (1.2) which behaves as R at large times. Using the Gauge transform (2.1), we give a system of equations of (φ,ψ). Let h,k be the profile under the Gauge transform of R. Set $\tilde{\varphi}=\varphi-h$ and $\tilde{\psi}=\psi-k$. We see that if u solves (1.2) then $(\tilde{\varphi},\tilde{\psi})$ solves (2.10). By similar arguments as in [16], there exists a solution of this system which decays very fast at infinity. Using this property and combining with the condition (1.7), we may prove a relation of $\tilde{\varphi}$ and $\tilde{\psi}$. This relation allows us to obtain a solution of (1.2) satisfying the desired property.

We introduce the following notations using in this paper.

Notation.

(1) *For $t>0$, we note the Strichartz space $S([t,\infty))$ is defined via the norm*

$$\|u\|_{S([t,\infty))}=\sup_{(q,r)\ admissible}\|u\|_{L_t^qL_x^r([t,\infty)\times\mathbb{R})}$$

The dual space is denoted by $N([t,\infty))=S([t,\infty))^*$.

(2) *For $z=(a,b)\in\mathbb{C}^2$ is a vector, we denote $|z|=|a|+|b|$.*

(3) *We denote $a\lesssim b$, for $a,b>0$, if a is smaller than b up to multiply a positive constant. Moreover, we denote $a\approx b$ if a equal to b up to multiply a certain positive constant.*

(4) *We denote $a\lesssim_k b$ if there exists a constant $C(k)$ depend only on k such that $a\leqslant C(k)b$. Particularly, we denote $a\lesssim_p b$ if there exists a constant C depends only on parameters $\omega_1,...,\omega_K,c_1,...,c_K$ such that $a\leqslant Cb$.*

(5) *Let $f\in C^1(\mathbb{R})$. We use ∂f or f_x to denote the derivative in space of the function f.*

(6) *Let $f(x,y,z,..)$ be a function. We denote $|df|=|f_x|+|f_y|+|f_z|+....$*

2. Proof of Theorem 1.1

In this section, we give the proof of Theorem 1.1. We divide our proof into three steps.

Step 1. Preliminary analysis

Considering the following transform:

$$\begin{cases}\varphi(t,x)=\exp\left(\frac{i}{2}\int_{-\infty}^{x}|u(t,y)|^2\,dy\right)u(t,x),\\ \psi=\partial\varphi-\frac{i}{2}|\varphi|^2\varphi.\end{cases}\tag{2.1}$$

By similarly arguments as in [13] and [22], we see that if $u(t,x)$ solves (1.2) then (φ,ψ) solves the following system

$$\begin{cases}L\varphi=i\varphi^2\overline{\psi}-b|\varphi|^4\varphi,\\ L\psi=-i\psi^2\overline{\varphi}-3b|\varphi|^4\psi-2b|\varphi|^2\varphi^2\overline{\psi},\\ \varphi\mid_{t=0}=\varphi_0=\exp\left(\frac{i}{2}\int_{-\infty}^{x}|u_0(y)|^2\,dy\right)u_0,\\ \psi\mid_{t=0}=\psi_0=\partial\varphi_0-\frac{i}{2}|\varphi_0|^2\varphi_0,\end{cases}\tag{2.2}$$

where $L=\partial_t+\partial_{xx}$. For convenience, we define

$$P(\varphi,\psi)=i\varphi^2\overline{\psi}-b|\varphi|^4\varphi,$$
$$Q(\varphi,\psi)=-i\psi^2\overline{\varphi}-3b|\varphi|^4\psi-2b|\varphi|^2\varphi^2\overline{\psi}.$$

Let R be multi soliton profile given as in (1.5). Set $q=u-R$. Since R_j solves (1.2), for all j, by elementary calculation, we have

$$iR_t+R_{xx}+i|R|^2R_x+b|R|^4R=i(|R|^2R_x-\sum_{j=1}^{K}|R_j|^2R_{jx})+b(|R|^4R-\sum_{j=1}^{\infty}|R_j|^4R_j).\tag{2.3}$$

Since Lemma 4.1, we have

$$\||R|^2R_x-\sum_{j=1}^{K}|R_j|^2R_{jx}\|_{H^2}+\||R|^4R-\sum_{j=1}^{K}|R_j|^4R_j\|_{H^2}\leqslant e^{-\lambda t},\tag{2.4}$$

where $\lambda=\frac{1}{16}v_*$. Thus, we rewrite (2.3) as follows

$$iR_t+R_{xx}+i|R|^2R_x+b|R|^4R=e^{-\lambda t}v(t,x),\tag{2.5}$$

where $v(t)\in H^2(\mathbb{R})$ such that $\|v(t)\|_{H^2}$ uniformly bounded in t.

Define

$$h = \exp\left(\frac{i}{2}\int_{-\infty}^{x} |R|^2\,dy\right) R(t,x), \tag{2.6}$$

$$k = h_x - \frac{i}{2}|h|^2 h. \tag{2.7}$$

By elementary calculation as above, we have

$$Lh = ih^2\overline{k} - b|h|^4 h + e^{-t\lambda}m(t,x),$$
$$Lk = -ik^2\overline{h} - 3b|h|^4 k - 2b|h|^2 h^2\overline{k} + e^{-t\lambda}n(t,x),$$

where $m(t), n(t)$ satisfy

$$m = v\exp\left(\frac{i}{2}\int_{-\infty}^{x} |R|^2\,dy\right) - h\int_{-\infty}^{x} \mathcal{I}m(v\overline{R})\,dy, \tag{2.8}$$

$$n = m_x - i|h|^2 m + \frac{i}{2}h^2\overline{m}. \tag{2.9}$$

By Lemma 4.2 we have $\|m(t)\|_{H^1} + \|n(t)\|_{H^1}$ uniformly bounded in t. Let $\tilde{\varphi} = \varphi - h$ and $\tilde{\psi} = \psi - k$. Then $\tilde{\varphi}$, $\tilde{\psi}$ solve:

$$\begin{cases} L\tilde{\varphi} = P(\varphi,\psi) - P(h,k) - e^{-t\lambda}m(t,x), \\ L\tilde{\psi} = Q(\varphi,\psi) - Q(h,k) - e^{-t\lambda}n(t,x). \end{cases} \tag{2.10}$$

Set $\eta = (\tilde{\varphi}, \tilde{\psi})$, $W = (h,k)$ and $f(\varphi,\psi) = (P(\varphi,\psi), Q(\varphi,\psi))$. We find solution of (2.10) of following form:

$$\eta = -i\int_t^{\infty} S(t-s)[f(W+\eta) - f(W) + H](s)\,ds, \tag{2.11}$$

where $S(t-s)$ is Schrödinger group, $H = e^{-t\lambda}(m,n)$. Moreover, since $\psi = \partial\varphi - \frac{i}{2}|\varphi|^2\varphi$, we have

$$\tilde{\psi} = \partial\tilde{\varphi} - \frac{i}{2}(|\tilde{\varphi}+h|^2(\tilde{\varphi}+h) - |h|^2 h). \tag{2.12}$$

Step 2. Existence solution of (2.10)

Since Lemma 4.3, there exists $T_* \gg 1$ such that for $T_0 \geqslant T_*$ there exists unique solution η define on $[T_0,\infty)$ of (2.10) such that

$$e^{t\lambda}(\|\eta\|_{S([t,\infty))\times S([t,\infty))}) + e^{t\lambda}(\|\eta_x\|_{S([t,\infty))\times S([t,\infty))}) \leqslant 1, \quad \forall t \geqslant T_0, \tag{2.13}$$

for the constant $\lambda > 0$ defined as in **step 1**. Thus, for all $t \geqslant T_0$, we have

$$\|\tilde{\varphi}\|_{H^1} + \|\tilde{\psi}\|_{H^1} \lesssim e^{-\lambda t}, \tag{2.14}$$

Step 3. Existence of multi-solitons

We prove that the solution $\eta = (\tilde{\varphi}, \tilde{\psi})$ of (2.10) satisfying the relation (2.12). Indeed, let η be solution of (2.10) which we find in step 1. Set $\varphi = \tilde{\varphi} + h$, $\psi = \tilde{\psi} + k$ and $v = \partial\varphi - \frac{i}{2}|\varphi|^2\varphi$. Since h solves $Lh = P(h,k) + e^{-t\theta}m(t,x)$ and $\tilde{\varphi}$ solves $L\tilde{\varphi} = P(\varphi,\psi) - P(h,k) - e^{-t\theta}m(t,x)$, we have (φ,ψ) solves

$$\begin{cases} L\varphi = P(\varphi,\psi), \\ L\psi = Q(\varphi,\psi). \end{cases} \tag{2.15}$$

By similarly arguments as above, we have

$$\begin{aligned} L\psi - Lv &= Q(\varphi,\psi) - \left(\partial L\varphi - \frac{i}{2}L(|\varphi|^2\varphi)\right) \\ &= Q(\varphi,\psi) - \left(\partial L\varphi - \frac{i}{2}(L(\varphi^2)\overline{\varphi} + \varphi^2 L(\overline{\varphi}) + 2\partial(\varphi^2)\partial\overline{\varphi})\right) \\ &= Q(\varphi,\psi) - \left(\partial L\varphi - \frac{i}{2}(2L\varphi|\varphi|^2 + 2(\partial\varphi)^2\overline{\varphi} - \varphi^2\overline{L\varphi} + 2\varphi^2\partial_{xx}\overline{\varphi}) + 2\varphi|\partial\varphi|^2)\right). \end{aligned} \tag{2.16}$$

Moreover, we have

$$\begin{aligned} L\varphi = P(\varphi,\psi) &= i\varphi^2\overline{\psi} - b|\varphi|^4\varphi \\ &= i\varphi^2\overline{(\psi - v)} + i\varphi^2\overline{v} - b|\varphi|^4\varphi. \end{aligned} \tag{2.17}$$

Combining (2.17) and (2.16) and by elementary calculation, we obtain

$$\begin{aligned} L\psi - Lv &= Q(\varphi,\psi) - \partial(i\varphi^2\overline{(\psi-v)}) - |\varphi|^2\varphi^2\overline{(\psi-v)} - \frac{1}{2}|\varphi|^4(\psi-v) - Q(\varphi,v) \\ &= (Q(\varphi,\psi)-Q(\varphi,v)) - 2i\varphi\partial\varphi\overline{(\psi-v)} - i\varphi^2\partial\overline{(\psi-v)} - |\varphi|^2\varphi^2\overline{(\psi-v)} - \frac{1}{2}|\varphi|^4(\psi-v) \\ &= -i(\psi^2-v^2)\overline{\varphi} - 3b|\varphi|^4(\psi-v) - 2b|\varphi|^2\varphi^2\overline{(\psi-v)} - 2i\varphi\left(v+\frac{i}{2}|\varphi|^2\varphi\right)\overline{(\psi-v)} \\ &\quad - i\varphi^2\partial\overline{(\psi-v)} - |\varphi|^2\varphi^2\overline{(\psi-v)} - \frac{1}{2}|\varphi|^4(\psi-v). \end{aligned} \tag{2.18}$$

Define $\tilde{v} = v - k$. Since, $\tilde{\psi} - \tilde{v} = \psi - v$ and (2.18) we have

$$L\tilde{\psi} - L\tilde{v} = (\tilde{\psi}-\tilde{v})A(\tilde{\psi},\tilde{v},\tilde{\varphi},h,k) + \overline{(\tilde{\psi}-\tilde{v})}B(\tilde{\psi},\tilde{v},\tilde{\varphi},h,k) - i(\tilde{\varphi}+h)^2\partial\overline{(\tilde{\psi}-\tilde{v})}, \tag{2.19}$$

where

$$\begin{aligned} A &= -i(\tilde{\psi}+\tilde{v}+2k)(\overline{\tilde{\varphi}+h}) - 3b|\tilde{\varphi}+h|^4 - \frac{1}{2}|\tilde{\varphi}+h|^4, \\ B &= -2b|\tilde{\varphi}+h|^2(\tilde{\varphi}+h)^2 - 2i(\tilde{\varphi}+h)\left(\tilde{v}+k+\frac{i}{2}|\tilde{\varphi}+h|^2(\tilde{\varphi}+h)\right) - |\tilde{\varphi}+h|^2(\tilde{\varphi}+h)^2. \end{aligned}$$

We see that A, B are polynomials of degree at most 4 in $(\tilde{\psi},\tilde{v},\tilde{\varphi},h,k)$. Multiplying two sides of (2.19) with $\overline{\tilde{\psi}-\tilde{v}}$, take imaginary part and integral over space using integral by part , we obtain

$$\frac{1}{2}\partial_t\|\tilde{\psi}-\tilde{v}\|_{L^2}^2 = \mathcal{I}m\int_{\mathbb{R}}(\tilde{\psi}-\tilde{v})^2A(\tilde{\psi},\tilde{v},\tilde{\varphi},h,k) + \overline{(\tilde{\psi}-\tilde{v})}^2B(\tilde{\psi},\tilde{v},\tilde{\varphi},h,k) + \frac{i}{2}\partial(\tilde{\varphi}+h)^2\overline{(\tilde{\psi}-\tilde{v})}^2\,dx.$$

Thus,

$$\left|\frac{1}{2}\partial_t\|\tilde{\psi}-\tilde{v}\|_{L^2}^2\right| \lesssim \|\tilde{\psi}-\tilde{v}\|_{L^2}^2(\|A\|_{L^\infty} + \|B\|_{L^\infty} + \|\partial(\tilde{\varphi}+h)^2\|_{L^\infty}).$$

This implies that

$$\left|\frac{\partial_t\|\tilde{\psi}-\tilde{v}\|_{L^2}^2}{\|\tilde{\psi}-\tilde{v}\|_{L^2}^2}\right| \lesssim \|A\|_{L^\infty} + \|B\|_{L^\infty} + \|\partial(\tilde{\varphi}+h)^2\|_{L^\infty}).$$

Hence, for $N > t$, integrating from t to N both sides we obtain

$$\begin{aligned} \left|\int_t^N \frac{d}{d_t}\mathrm{Ln}(\|\tilde{\psi}-\tilde{v}\|_{L^2}^2)\,ds\right| &= \left|\int_t^N \frac{\partial_t\|\tilde{\psi}-\tilde{v}\|_{L^2}^2}{\|\tilde{\psi}-\tilde{v}\|_{L^2}^2}\,ds\right| \\ &\leqslant \int_t^N \left|\frac{\partial_t\|\tilde{\psi}-\tilde{v}\|_{L^2}^2}{\|\tilde{\psi}-\tilde{v}\|_{L^2}^2}\right| ds \lesssim \int_t^N (\|A\|_{L^\infty} + \|B\|_{L^\infty} + \|\partial(\tilde{\varphi}+h)^2\|_{L^\infty})\,ds, \end{aligned}$$

This implies that

$$\begin{aligned} \mathrm{Ln}(\|\tilde{\psi}(t)-\tilde{v}(t)\|_{L^2}^2) - \mathrm{Ln}(\|\tilde{\psi}(N)-\tilde{v}(N)\|_{L^2}^2) &\leqslant \left|\int_t^N \frac{d}{d_t}\mathrm{Ln}(\|\tilde{\psi}-\tilde{v}\|_{L^2}^2)\,ds\right| \\ &\lesssim \int_t^N (\|A\|_{L^\infty} + \|B\|_{L^\infty} + \|\partial(\tilde{\varphi}+h)^2\|_{L^\infty})\,ds \end{aligned}$$

We obtain

$$\|\tilde{\psi}(t)-\tilde{v}(t)\|_{L^2}^2 \lesssim \|\tilde{\psi}(N)-\tilde{v}(N)\|_{L^2}^2\exp\left(\int_t^N (\|A\|_{L^\infty} + \|B\|_{L^\infty} + \|\partial(\tilde{\varphi}+h)^2\|_{L^\infty}\,ds\right). \tag{2.20}$$

Since (2.13) and (2.14) and Sobolev embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(R)$ we have, for $t \geqslant T_0$:

$\|\tilde{\varphi}+h\|_{L^\infty} \lesssim 1 + \|h\|_{L^\infty}$,

$$\|\tilde{v}\|_{L^\infty} = \|\partial\tilde{\varphi} - \frac{i}{2}(|\tilde{\varphi}|^2(\tilde{\varphi}+h) - |h|^2h)\|_{L^\infty} \lesssim \|\partial\tilde{\varphi}\|_{L^\infty} + \|\tilde{\varphi}\|_{L^\infty}^3 + \|\tilde{\varphi}\|_{L^\infty}\|h\|_{L^\infty}^2 \lesssim 1 + \|\partial\tilde{\varphi}\|_{L^\infty} + \|h\|_{L^\infty}^2.$$

Thus, using $k = h_x - \frac{i}{2}|h|^2 h$ and $\tilde{v} = \partial\tilde{\varphi} - \frac{i}{2}(|\tilde{\varphi}+h|^2(\tilde{\varphi}+h) - |h|^2 h)$ and $|h| = |R|$, we have

$$
\begin{aligned}
&\int_t^N (\|A\|_{L^\infty} + \|B\|_{L^\infty} + \|\partial(\tilde{\varphi}+h)^2\|_{L^\infty})\,ds \\
&\lesssim \int_t^N (\|\tilde{\psi}\|_{L^\infty} + \|\tilde{v}\|_{L^\infty} + \|k\|_{L^\infty})\|\tilde{\varphi}+h\|_{L^\infty} + \|\tilde{\varphi}+h\|_{L^\infty}^4 + \|\tilde{\varphi}+h\|_{L^\infty}\|\tilde{v}+k\|_{L^\infty} \\
&\quad + (\|\tilde{\varphi}\|_{L^\infty} + \|h\|_{L^\infty})(\|\partial\tilde{\varphi}\|_{L^\infty} + \|h_x\|_{L^\infty})\,ds \\
&\lesssim \int_t^N (1 + \|\tilde{v}\|_{L^\infty} + \|k\|_{L^\infty})(1+\|h\|_{L^\infty}) + 1 + \|h\|_{L^\infty}^4 + (1+\|h\|_{L^\infty})(\|\tilde{v}\|_{L^\infty} + \|k\|_{L^\infty}) \\
&\quad + (1+\|h\|_{L^\infty})(\|\partial\tilde{\varphi}\|_{L^\infty} + \|h_x\|_{L^\infty})\,ds \\
&\lesssim \int_t^N 1 + \|h\|_{L^\infty}^4 + \|k\|_{L^\infty}(1+\|h\|_{L^\infty}) + \|\tilde{v}\|_{L^\infty}(1+\|h\|_{L^\infty}) + (1+\|h\|_{L^\infty})(\|\partial\tilde{\varphi}\|_{L^\infty} + \|h_x\|_{L^\infty})\,ds \\
&\lesssim \int_t^N 1 + \|h\|_{L^\infty}^4 + \|k\|_{L^\infty}(1+\|h\|_{L^\infty}) + \|\partial\tilde{\varphi}\|_{L^\infty}(1+\|h\|_{L^\infty}) + (1+\|h\|_{L^\infty})(\|\partial\tilde{\varphi}\|_{L^\infty} + \|k\|_{L^\infty} + \|h\|_{L^\infty}^3)\,ds \\
&\lesssim \int_t^N 1 + \|h\|_{L^\infty}^4 + \|k\|_{L^\infty}(1+\|h\|_{L^\infty}) + \|\partial\tilde{\varphi}\|_{L^\infty}(1+\|h\|_{L^\infty})\,ds \\
&\lesssim (N-t)(1 + \|h\|_{L^\infty L^\infty}^4 + \|k\|_{L^\infty L^\infty}(1+\|h\|_{L^\infty L^\infty})) + \|\partial\tilde{\varphi}\|_{L^4(t,N)L^\infty}(\|1\|_{L^{\frac{4}{3}}(t,N)} + \|h\|_{L^{\frac{4}{3}}(t,N)L^\infty}) \\
&\lesssim (N-t)(1 + \|R\|_{L^\infty L^\infty}^4 + (\|h_x\|_{L^\infty L^\infty} + \|R\|_{L^\infty L^\infty}^3)(1+\|R\|_{L^\infty L^\infty})) + (N-t)^{\frac{3}{4}}(1 + \|R\|_{L^\infty L^\infty}^{\frac{4}{3}}) \\
&\lesssim (N-t)(1 + \|R\|_{L^\infty L^\infty}^4 + \|R_x\|_{L^\infty L^\infty}(1+\|R\|_{L^\infty L^\infty})) + (N-t)^{\frac{3}{4}}(1 + \|R\|_{L^\infty L^\infty}^{\frac{4}{3}}).
\end{aligned}
$$

Thus, from the assumption (1.7), we have $(1+\|R_x\|_{L^\infty L^\infty})(1+\|R\|_{L^\infty L^\infty}) + \|R\|_{L^\infty L^\infty}^4 \ll \lambda$. Hence, fix t and let N large enough, we have

$$\int_t^N (\|A\|_{L^\infty} + \|B\|_{L^\infty} + \|\partial(\tilde{\varphi}+h)^2\|_{L^\infty})\,ds \leqslant (N-t)\lambda.$$

Combining with (2.14), we obtain, for N large enough:

$$\|\tilde{\psi}(t) - \tilde{v}(t)\|_{L^2}^2 \lesssim e^{-2\lambda N} e^{(N-t)\lambda} = e^{-\lambda N - t\lambda}.$$

Let $N \to \infty$, we obtain that

$$\|\tilde{\psi}(t) - \tilde{v}(t)\|_{L^2}^2 = 0.$$

This implies that $\tilde{\psi} = \tilde{v}$, hence,

$$\psi = v = \partial\varphi - \frac{i}{2}|\varphi|^2\varphi. \tag{2.21}$$

Let $u = \exp\left(-\frac{i}{2}\int_{-\infty}^x |\varphi(y)|^2\,dy\right)\varphi$. Combining (2.21) with the fact that (φ, ψ) solves

$$\begin{cases} L\varphi = P(\varphi,\psi), \\ L\psi = Q(\varphi,\psi), \end{cases}$$

we obtain u solves (1.2). Moreover,

$$
\begin{aligned}
\|u - R\|_{H^1} &= \|\exp\left(-\frac{i}{2}\int_{-\infty}^x |\varphi(y)|^2\,dy\right)\varphi - \exp\left(-\frac{i}{2}\int_{-\infty}^x |h(y)|^2\,dy\right)h\|_{H^1} \\
&\lesssim \|\varphi - h\|_{H^1} = \|\tilde{\varphi}\|_{H^1}
\end{aligned}
$$

Combining (2.14), for $t \geqslant T_0$, we have

$$\|u - R\|_{H^1} \leqslant e^{-\lambda t}.$$

This complete the proof of Theorem 1.1.

3. Proof of Theorem 1.6

In this section, we give the proof of Theorem 1.6. We divide our proof into three steps:

Step 1. Preliminary analysis

Set

$$v := u_x + \frac{i}{2}|u|^2 u.$$

By elementary calculation as in the proof of Theorem 1.1 we see that if u solves (1.2) then (u, v) solves the following system:

$$\begin{cases} Lu = -iu^2\overline{v} + \left(\frac{1}{2} - b\right)|u|^4 u, \\ Lv = iv^2\overline{u} + \left(\frac{3}{2} - 3b\right)|u|^4 v + (1-2b)|u|^2 u^2 \overline{v}, \\ u\,|_{t=0} = u_0, \\ v\,|_{t=0} = v_0 = \partial u_0 + \frac{i}{2}|u_0|^2 u_0. \end{cases} \tag{3.1}$$

Define

$$P(u,v) = -iu^2\overline{v} + \left(\frac{1}{2} - b\right)|u|^4 u,$$

$$Q(u,v) = iv^2\overline{u} + \left(\frac{3}{2} - 3b\right)|u|^4 v + (1-2b)|u|^2 u^2 \overline{v}.$$

Let V be multi kink-soliton profile given as in (1.20). Set $q = u - V$. Since R_j solves (1.2), for all j, by elementary calculation, we have

$$iV_t + V_{xx} + i|V|^2 V_x + b|V|^4 V = i(V^2\overline{V_x} - \sum_{j=0}^{K} R_j^2 \overline{R_{jx}}) + b(|V|^4 V - \sum_{j=0}^{K} |R_j|^4 R_j). \tag{3.2}$$

Since Lemma 4.4, we have

$$\|V^2\overline{V_x} - \sum_{j=0}^{K} R_j^2 \overline{R_{jx}}\|_{H^2} + \||V|^4 V - \sum_{j=0}^{K} |R_j|^4 R_j\|_{H^2} \leqslant e^{-\lambda t}, \tag{3.3}$$

for $\lambda = \frac{1}{16} v_*$. Thus, we rewrite (3.2) as follows

$$iV_t + V_{xx} + iV^2\overline{V_x} + b|V|^4 V = e^{-\lambda t} m(t,x), \tag{3.4}$$

where $m(t) \in H^2(\mathbb{R})$ such that $\|m(t)\|_{H^2}$ uniformly bounded in t. Define

$$h = V,$$

$$k = h_x + \frac{i}{2}|h|^2 h.$$

By elementary calculation as above, we have

$$Lh = -ih^2\overline{k} + \left(\frac{1}{2} - b\right)|h|^4 h + e^{-t\lambda} m = P(h,k) + e^{-t\lambda} m,$$

$$Lk = ik^2\overline{h} + \left(\frac{3}{2} - 3b\right)|h|^4 k + (1-2b)|h|^2 h^2 \overline{k} + e^{-t\lambda} n = Q(h,k) + e^{-t\lambda} n.$$

where $n = m_x + i|h|^2 m - \frac{i}{2} h^2 \overline{m}$ satisfy $\|n(t)\|_{H^1}$ is uniformly bounded in t. Let $\tilde{u} = u - h$ and $\tilde{v} = v - k$. Then $\tilde{u}$, $\tilde{v}$ solves:

$$\begin{cases} L\tilde{u} = P(u,v) - P(h,k) - e^{-t\lambda} m, \\ L\tilde{v} = Q(u,v) - Q(h,k) - e^{-t\lambda} n. \end{cases} \tag{3.5}$$

Set $\eta = (\tilde{u}, \tilde{v})$, $W = (h,k)$ and $f(u,v) = (P(u,v), Q(u,v))$. We find solution of (3.5) of following form:

$$\eta = -i \int_t^\infty S(t-s)[f(W+\eta) - f(W) + H](s)\, ds, \tag{3.6}$$

where $H = e^{-t\lambda}(m,n)$. Moreover, since $v = u_x + \frac{i}{2}|u|^2 u$, we have

$$\tilde{v} = \tilde{u}_x + \frac{i}{2}(|\tilde{u} + h|^2(\tilde{u} + h) - |h|^2 h). \tag{3.7}$$

Step 2. Existence solution of system equations

Since Lemma 4.3, there exists $T_* \gg 1$ such that for $T_0 \gg T_*$ there exists unique solution η define on $[T_0, \infty)$ of (3.5) such that

$$e^{t\lambda}\|\eta\|_{S([t,\infty))\times S([t,\infty))} + e^{t\lambda}\|\eta_x\|_{S([t,\infty))\times S([t,\infty))} \leqslant 1, \quad \forall t \geqslant T_0, \tag{3.8}$$

where the constant $\lambda > 0$ is defined as in **step 1**. Thus, for all $t \geqslant T_0$, we have

$$\|\tilde{u}\|_{H^1} + \|\tilde{v}\|_{H^1} \lesssim e^{-t\lambda}. \tag{3.9}$$

Step 3. Existence of multi kink-soliton for (1.2)

Using similar arguments in the proof of Theorem 1.1 we can prove that the solution $\eta = (\tilde{\varphi}, \tilde{\psi})$ of (3.5) satisfying the relation (3.7) under the condition (1.21). This implies that

$$\tilde{v} = \tilde{u}_x + \frac{i}{2}(|\tilde{u} + h|^2(\tilde{u} + h) - |h|^2 h).$$

Set $u = \tilde{u} + h$, $v = \tilde{v} + k$. We have

$$v = u_x + \frac{i}{2}|u|^2 u. \tag{3.10}$$

Since $(\tilde{u}, \tilde{v})$ solves (3.5), we have u, v solves

$$Lu = P(u, v),$$
$$Lv = Q(u, v).$$

Combining with (3.10) we have u solves (1.2). Moreover, for $t \geqslant T_0$, we have

$$\|u - V\|_{H^1} = \|\tilde{u}\|_{H^1} \lesssim e^{-\lambda t}.$$

This completes the proof of Theorem 1.6.

Remark 3.1. We do not have the proof for constructing the multi kink-soliton trains of equation (1.2) as (1.10). The reason is that if the profile R in the proof of Theorem 1.1 is not in $H^1(\mathbb{R})$ then the function h defined as in (2.6) is not in $H^1(\mathbb{R})$. Thus, the functions m, n defined as in (2.8) and (2.9) are not in $H^1(\mathbb{R})$ and we can not apply the Lemma 4.3 to construct solution of system (2.10).

4. Some technical lemmas

4.1. **Properties of solitons.** In this section, we prove some estimates of soliton profile using in the proof of Theorem 1.1.

Lemma 4.1. *There exist $T_0 > 0$ and a constant $\lambda > 0$ such that the estimate (2.4) is true uniformly for $t \geqslant T_0$.*

Proof. First, we need to some estimates on soliton profile. We have

$$|R_j(x,t)| = |\Phi_{\omega_j,c_j}(x - c_j t)| = \sqrt{2}h_j(\sqrt{c_j^2 + \gamma h_j^2}\cosh(h_j(x - c_j t)) - c_j)^{-\frac{1}{2}} \lesssim_{h_j,|c_j|} e^{\frac{-h_j}{2}|x - c_j t|}.$$

Moreover,

$$|\partial R_j(x,t)| = |\partial\phi_{\omega_j,c_j}(x - c_j t)| = \frac{-\sqrt{2}}{2}h_j^2\sqrt{c_j^2 + \gamma h_j^2}\sinh(h_j(x - c_j t))(\sqrt{c_j^2 + \gamma h_j^2}\cosh(h_j(x - c_j t)) - c_j)^{-\frac{3}{2}}$$
$$\lesssim_{h_j,|c_j|} e^{\frac{-h_j}{2}|x - c_j t|}.$$

By elementary calculation, we have

$$|\partial^2 R_j(x,t)| + |\partial^3 R_j(x,t)| \lesssim_{h_j,|c_j|} e^{\frac{-h_j}{2}|x - c - jt|}.$$

Now, let us comeback to prove Lemma 4.1. For convenience, we set

$$\chi_1 = i|R|^2 R_x - i\sum_{j=1}^{K}|R_j|^2 R_{jx},$$

$$\chi_2 = |R|^4 R - \sum_{j=1}^{K}|R_j|^4 R_j.$$

Fix $t > 0$. For each $x \in \mathbb{R}$, choose $m = m(x) \in \{1, 2, ..., K\}$ so that

$$|x - c_m t| = \min_j |x - c_j t|.$$

For $j \neq m$, we have

$$|x - c_j t| \geqslant \frac{1}{2}|c_j t - c_m t| = \frac{t}{2}|c_j - c_m|.$$

Thus, we have

$$\begin{aligned}
&|(R - R_m)(x,t)| + |(\partial R - \partial R_m(x,t))| + |\partial^2 R - \partial^2 R_m| + |\partial^3 R - \partial^3 R_m| \\
&\leqslant \sum_{j \neq m} (|R_j(x,t)| + |\partial R_j(x,t)| + |\partial^2 R_j(x,t)| + |\partial^3 R_j(x,t)|) \\
&\lesssim_{h_1,..,h_K,|c_1|,..,|c_K|} \delta_m(x,t) := \sum_{j \neq m} e^{\frac{-h_j}{2}|x - c_j t|}
\end{aligned}$$

Define

$$v_* = \inf_{j \neq k} h_j |c_j - c_k|.$$

We have

$$|(R - R_m)(x,t)| + |(\partial R - \partial R_m(x,t))| + |\partial^2 R - \partial^2 R_m| + |\partial^3 R - \partial^3 R_m| \lesssim \delta_m(x,t) \lesssim e^{\frac{-1}{4} v_* t}.$$

Let f_1, g_1, r_1 and f_2, g_2, r_2 be polynomials of u, u_x, u_{xx}, u_{xxx} and conjugates such that:

$$\begin{aligned}
i|u|^2 u_x = f_1(u, \overline{u}, u_x), \quad |u|^4 u = f_2(u, \overline{u}), \\
\partial(i|u|^2 u_x) = g_1(u, u_x, u_{xx}, \overline{u}, ..), \quad \partial(|u|^4 u) = g_2(u, u_x, \overline{u}, ..), \\
\partial^2(i|u|^2 u_x) = r_1(u, u_x, u_{xx}, u_{xxx}, \overline{u}, ..), \quad \partial^2(|u|^4 u) = r_2(u, u_x, u_{xx}, \overline{u}, ..).
\end{aligned}$$

Denote

$$A = \sup_{|z|+|z_x|+|z_{xx}|+|z_{xxx}| \leqslant \|R\|_{H^4}} (|df_1| + |df_2| + dg_1| + |dg_2| + |dr_1| + |dr_2|),$$

where we denote by $|df(x,y,z,...) = |f_x| + |f_y| + |f_z| + ...$ be absolute value of gradient of f at point $(x,y,z,...)$. We have

$$\begin{aligned}
&|\chi_1| + |\chi_2| + |\partial\chi_1| + |\partial\chi_2| + |\partial^2\chi_1| + |\partial^2\chi_2| \\
&\leqslant |f_1(R, R_x) - f_1(R_m, R_{mx})| + |f_2(R) - f_2(R_m)| + \sum_{j \neq m} (|f_1(R_j, R_{jx})| + |f_2(R_j)|) \\
&\quad + |g_1(R, R_x, R_{xx}, ..) - g_1(R_m, R_{mx}, R_{mxx}, ..)| + |g_2(R, R_x, ..) - g_2(R_m, R_{mx}, ..)| \\
&+ \sum_{j \neq m} (g_1(R_j, R_{jx}, R_{jxx}, ..) + g_2(R_j, R_{jx}), ..) \\
&\quad + |r_1(R, R_x, R_{xx}, R_{xxx}, ..) - r_1(R_m, R_{mx}, R_{mxx}, R_{mxxx}, ..)| + |r_2(R, R_x, R_{xx}, ..) - r_2(R_m, R_{mx}, R_{mxx}, ..)| \\
&\quad + \sum_{j \neq m} (r_1(R_j, R_{jx}, R_{jxx}, R_{jxxx}, ..) + r_2(R_j, R_{jx}, R_{jxx}, ..)) \\
&\leqslant A(|R - R_m| + |R_x - R_{mx}| + |R_{xx} - R_{mxx}| + |R_{xxx} - R_{mxxx}|) + \sum_{j \neq m} A(|R_j| + |R_{jx}| + |R_{jxx}| + |R_{jxxx}|) \\
&\leqslant 2A \sum_{j \neq m} (|R_j| + |R_{jx}| + |R_{jxx}| + |R_{jxxx}|) \\
&\leqslant 2A\delta_m(t,x).
\end{aligned}$$

In particular,

$$\|\chi_1\|_{W^{2,\infty}} + \|\chi_2\|_{W^{2,\infty}} \lesssim_p e^{-\frac{1}{4} v_* t}.$$

Moreover, we have

$$\begin{aligned}
&\|\chi_1\|_{W^{2,1}} + \|\chi_2\|_{W^{2,1}} \\
&\lesssim \sum_{j=1}^{K} (\||R_j|^2 R_{jx}\|_{L^1} + \|\partial(|R_j|^2 R_{jx})\|_{L^1} + \|\partial^2(|R_j|^2 R_{jx})\|_{L^1} + \|R_j^5\|_{L^1} + \|\partial(|R_j|^4 R_j)\|_{L^1} + \|\partial^2(|R_j|^4 R_j)\|_{L^1}) \\
&\lesssim \sum_{j=1}^{K} (\|R_j\|_{H^1}^3 + \|R_j\|_{H^2}^3 + \|R_j\|_{H^3}^3 + \|R_j\|_{H^1}^5 + \|R_j\|_{H^1}^5 + \|R_j\|_{H^2}^5) < C < \infty
\end{aligned}$$

By Holder inequality, for $1 < r < \infty$, we have

$$\|\chi_1\|_{W^{2,r}} + \|\chi_2\|_{W^{2,r}} \lesssim_p e^{-(1-\frac{1}{r})\frac{1}{4}v_* t}, \quad \forall r \in (s, \infty).$$

Choosing $r = 2$ we obtain:

$$\|\chi_1\|_{H^2} + \|\chi_2\|_{H^2} \lesssim_p e^{-\frac{v_*}{8}t},$$

Thus, for $t \geqslant T_0$, where T_0 large enough depend on the parameters $\omega_1, ..., \omega_K, c_1, ..., c_K$, we have

$$\|\chi_1\|_{H^2} + \|\chi_2\|_{H^2} \leqslant e^{-\frac{v_*}{16}t}, \quad \forall t \geqslant T_0.$$

Let $\lambda = \frac{-v_*}{16}$, we obtain the desired result. □

4.2. **Prove the boundedness of v, m, n.** Let v, m and n be given as in (2.3), (2.8) and (2.9) respectively. In this section, we prove the boundedness of $H^2(\mathbb{R})$-norm of v and $H^1(\mathbb{R})$-norm of m, n uniform in times. We have the following result:

Lemma 4.2. *The functions v, m, n satisfy*

$$\|v(t)\|_{H^2} + \|m(t)\|_{H^1} + \|n(t)\|_{H^1} \leqslant C,$$

uniformly on $t \geqslant T_0$, for some constant $C > 0$.

Proof. Let χ_1 be given function as in Lemma 4.1, we have

$$e^{-\lambda t} v = \chi_1$$

By Lemma 4.1, we have $\|v(t)\|_{H^2} \leqslant D$, for some constant $D > 0$. Since (2.8), we have

$$\|m\|_{H^2} \lesssim \|v\|_{H^2} + \|h\|_{H^2}\|v\|_{H^2}\|R\|_{H^2} \leqslant C_1,$$

for some constant $C_1 > 0$. Since, (2.9), we have

$$\|n\|_{L^2} \lesssim \|m_x\|_{L^2} + \|h\|^2_{H^1}\|m\|_{H^1} \leqslant \|m\|_{H^1}(1 + \|h\|^2_{H^1}) \leqslant C_2,$$

for some constant $C_2 > 0$. Moreover, we have

$$\|n_x\|_{L^2} \lesssim \|m_{xx}\|_{L^2} + \|h\|^2_{H^1}\|m\|_{H^1} \leqslant \|m\|_{H^2}(1 + \|h\|^2_{H^1}) \leqslant C_3,$$

for some constant $C_3 > 0$. Choosing $C = D + C_1 + C_2 + C_3$, we obtain the desired result. □

4.3. **Existence solution of system equation.** In this section, we prove the existence of solution of the following equation:

$$i\partial_t \eta + \partial_{xx}\eta = -[f(W + \eta) - f(W)] - H,$$

In Duhamel form,

$$\eta(t) = -i \int_t^\infty S(t - s)[f(W + \eta) - f(W) + H](s)\, ds, \tag{4.1}$$

where $f(\varphi, \psi) = (P(\varphi, \psi), Q(\varphi, \psi))$ and $P(\varphi, \psi) = i\varphi^2\overline{\psi} - b|\varphi|^4\varphi$, $Q(\varphi, \psi) = -i\psi^2\overline{\varphi} - 3b|\varphi|^4\psi - 2b|\varphi|^2\varphi^2\overline{\psi}$.

Lemma 4.3. *Let $H = H(t, x) : [0, \infty) \times \mathbb{R} \to \mathbb{C}^2$, $W = W(t, x) : [0, \infty) \times \mathbb{R} \to \mathbb{C}^2$ be given vector functions which satisfy for some $C_1 > 0$, $C_2 > 0$, $\lambda > 0$, $T_0 \geqslant 0$:*

$$\|W(t)\|_{L^\infty \times L^\infty} + e^{\lambda t}\|H(t)\|_{L^2 \times L^2} \leqslant C_1 \quad \forall t \geqslant T_0, \tag{4.2}$$

$$\|\partial W(t)\|_{L^2 \times L^2} + \|\partial W(t)\|_{L^\infty \times L^\infty} + e^{\lambda t}\|\partial H(t)\|_{L^2 \times L^2} \leqslant C_2, \quad \forall t \geqslant T_0. \tag{4.3}$$

Consider equation (4.1). There exists a constant λ_* independent of C_2 such that if $\lambda \geqslant \lambda_*$ then there exists a unique solution η to (4.1) on $[T_0, \infty) \times \mathbb{R}$ satisfying

$$e^{\lambda t}\|\eta\|_{S([t,\infty)) \times S([t,\infty))} + e^{\lambda t}\|\partial\eta\|_{S([t,\infty)) \times S([t,\infty))} \leqslant 1, \quad \forall t \geqslant T_0.$$

Proof. We use similarly arguments as in [15]. We write (4.1) as $\eta = \Phi\eta$. We shall show that, for λ sufficiently large, Φ is a contraction in the ball

$$B = \{\eta : \|\eta\|_X := e^{\lambda t}\|\eta\|_{S([t,\infty)) \times S([t,\infty))} + e^{\lambda t}\|\partial\eta\|_{S([t,\infty)) \times S([t,\infty))} \leqslant 1\}$$

Step 1. Prove Φ map B into B

Let $t \geqslant T_0$, $\eta = (\eta_1, \eta_2) \in B$, $W = (w_1, w_2)$ and $H = (h_1, h_2)$. By Strichartz estimates, we have

$$\|\Phi\eta\|_{S([t,\infty))\times S([t,\infty))} \lesssim \|f(W+\eta) - f(W)\|_{N([t,\infty))\times N([t,\infty))} \tag{4.4}$$
$$+ \|H\|_{L^1_\tau L^2_x([t,\infty))\times L^1_\tau L^2_x([t,\infty))}. \tag{4.5}$$

For (4.5), using (4.2), we have

$$\|H\|_{L^1_\tau L^2_x([t,\infty))\times L^1_\tau L^2_x([t,\infty))} = \|h_1\|_{L^1_\tau L^2_x([t,\infty))} + \|h_2\|_{L^1_\tau L^2_x([t,\infty))} \lesssim \int_t^\infty e^{-\lambda\tau}\, d\tau \leqslant \frac{1}{\lambda} e^{-\lambda t}. \tag{4.6}$$

For (4.4), we have

$$\begin{aligned}
|P(W+\eta) - P(W)| &= |P(w_1+\eta_1, w_2+\eta_2) - P(w_1, w_2)| \\
&\lesssim |(w_1+\eta_1)^2\overline{(w_2+\eta_2)} - w_1^2\overline{w_2}| + ||\eta_1+w_1|^4(\eta_1+w_1) - |w_1|^4 w_2| \\
&\lesssim |\eta_1| + |\eta_2| + |\eta_1|^5
\end{aligned}$$

Thus,

$$\begin{aligned}
\|P(W+\eta) - P(W)\|_{N([t,\infty))} &\lesssim \|\eta_1\|_{N([t,\infty))} + \|\eta_2\|_{N([t,\infty))} + \|\eta_1^5\|_{N([t,\infty))} \\
&\lesssim \|\eta_1\|_{L^1_\tau L^2_x(t,\infty)} + \|\eta_2\|_{L^1_\tau L^2_x(t,\infty)} + \|\eta_1^5\|_{L^1_\tau L^2_x(t,\infty)} \\
&\lesssim \int_t^\infty e^{-\lambda\tau}\, d\tau + \int_t^\infty \|\eta_1(\tau)\|_{L^10}^5\, d\tau \\
&\lesssim \frac{1}{\lambda} e^{-\lambda t} + \int_t^\infty \|\eta_1(\tau)\|_{L^2}^{\frac{7}{2}} \|\partial\eta_1(\tau)\|_{L^2}^{\frac{3}{2}} \\
&\lesssim \frac{1}{\lambda} e^{-\lambda t} + \int_t^\infty e^{-(7/2\lambda+3/2\lambda)\tau}\, d\tau \\
&\lesssim \frac{1}{\lambda} e^{-\lambda t} + \frac{1}{7/2\lambda+3/2\lambda} e^{-(7/2\lambda+3/2\lambda)t} \lesssim \frac{1}{\lambda} e^{-\lambda t}.
\end{aligned}$$

By similar arguments as above, we have

$$\|Q(W+\eta) - Q(W)\|_{N([t,\infty))} \lesssim \frac{1}{\lambda} e^{-\lambda t}.$$

Thus, for λ large enough, we have

$$\|\Phi\eta\|_{S([t,\infty)\times S([t,\infty)))} \leqslant \frac{1}{10} e^{-\lambda t}.$$

It remains to estimate $\|\partial\Phi\eta\|_{S([t,\infty)\times S([t,\infty)))}$. By Strichartz estimate we have

$$\|\partial\Phi\eta\|_{S([t,\infty)\times S([t,\infty)))} \lesssim \|\partial(f(W+\eta) - f(W))\|_{N([t,\infty))\times N([t,\infty))} \tag{4.7}$$
$$+ \|\partial H\|_{N([t,\infty))\times N([t,\infty))}. \tag{4.8}$$

For (4.8), using (4.3), we have

$$\begin{aligned}
\|\partial H\|_{N([t,\infty))\times N([t,\infty))} &\leqslant \|\partial h_1\|_{L^1_\tau L^2_x([t,\infty))} + \|\partial h_2\|_{L^1_\tau L^2_x([t,\infty))} \\
&\lesssim \int_t^\infty e^{-\lambda\tau}\, d\tau = \frac{1}{\lambda} e^{-\lambda t}.
\end{aligned} \tag{4.9}$$

For (4.7), we have

$$\|\partial(f(W+\eta) - f(W))\|_{N([t,\infty))\times N([t,\infty))} = \|\partial(P(W+\eta) - P(W))\|_{N([t,\infty))} + \|\partial(Q(W+\eta) - Q(W))\|_{N([t,\infty))}$$

Furthermore, using the notation (1.2) (3), we have

$$\begin{aligned}
|\partial(P(W+\eta) - P(W))| &\lesssim |\partial((w_1+\eta_1)^2\overline{(w_2+\eta_2)} - w_1^2\overline{w_2})| + |\partial(|w_1+\eta_1|^4(w_1+\eta_1) - |w_1|^4 w_1)| \\
&\lesssim |\partial\eta|(|\eta|^2 + |w|^2) + |\partial w|(|\eta|^2 + |w||\eta|) \\
&\quad + |\partial\eta|(|\eta|^4 + |w|^4) + |\partial w|(|\eta|^4 + |\eta||w|^3).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\|\partial(P(W+\eta) - P(W))\|_{N([t,\infty))} \\
&\lesssim \||\partial\eta|(|\eta|^2 + |w|^2)\|_{N([t,\infty))} + \||\partial w|(|\eta|^2 + |w||\eta|)\|_{N([t,\infty))} \qquad (4.10) \\
&\quad + \||\partial\eta|(|\eta|^4 + |w|^4)\|_{N([t,\infty))} + \||\partial w|(|\eta|^4 + |\eta||w|^3)\|_{N([t,\infty))}. \qquad (4.11)
\end{aligned}$$

For (4.10), using (4.2) and (4.3) and the assumption $\eta \in B$ we have

$$\begin{aligned}
&\||\partial\eta|(|\eta|^2+|w|^2)\|_{N([t,\infty))} + \||\partial w|(|\eta|^2+|w||\eta|)\|_{N([t,\infty))}\\
&\lesssim \||\partial\eta||\eta|^2\|_{L^1_\tau L^2_x([t,\infty))} + \||\partial\eta||w|^2\|_{L^1_\tau L^2_x([t,\infty))} + \||\partial w||\eta|^2\|_{L^1_\tau L^2_x([t,\infty))} + \||\partial w||w||\eta|\|_{L^1_\tau L^2_x([t,\infty))}\\
&\lesssim \|\|\partial\eta\|\|_{L^2_\tau L^2_x([t,\infty))}\|\|\eta\|\|^2_{L^4_\tau L^\infty} + \|\|\partial\eta\|\|_{L^1_\tau L^2_x([t,\infty))}\|\|w\|\|^2_{L^\infty L^\infty}\\
&\quad + \|\|\partial w\|\|_{L^\infty L^\infty}\|\|\eta\|\|_{L^4_\tau L^\infty_x([t,\infty))}\|\|\eta\|\|_{L^{4/3}_\tau L^2_x([t,\infty))} + \|\|w\|\|_{L^\infty L^\infty}\|\|\partial w\|\|_{L^\infty L^\infty}\|\|\eta\|\|_{L^1_\tau L^2_x([t,\infty))}\\
&\lesssim \frac{1}{\lambda}e^{-\lambda t}.
\end{aligned}$$

For (4.11), using (4.2) and (4.3) and the assumption $\eta \in B$ we have

$$\begin{aligned}
&\||\partial\eta|(|\eta|^4+|w|^4)\|_{N([t,\infty))} + \||\partial w|(|\eta|^4+|\eta||w|^3)\|_{N([t,\infty))}\\
&\lesssim \||\partial\eta|(|\eta|^4+|w|^4)\|_{L^1_\tau L^2_x([t,\infty))} + \||\partial w|(|\eta|^4+|\eta||w|^3)\|_{L^1_\tau L^2_x([t,\infty))}\\
&\lesssim \|\partial\eta\|_{L^\infty_\tau L^2_x([t,\infty))}\|\eta\|^4_{L^4_\tau L^\infty_x([t,\infty))} + \|w\|^4_{L^\infty L^\infty}\|\partial\eta\|_{L^1_\tau L^2_x([t,\infty))}\\
&\quad + \|\partial w\|_{L^\infty L^2}\|\eta\|^4_{L^4_\tau L^\infty_x([t,\infty))} + \|\partial w\|_{L^\infty L^\infty}\|w\|^3_{L^\infty L^\infty}\|\eta\|_{L^1_\tau L^2_x([t,\infty))}\\
&\lesssim \frac{1}{\lambda}e^{-\lambda t}.
\end{aligned}$$

Hence,

$$\|\partial(P(W+\eta)-P(W))\|_{N([t,\infty))} \lesssim \frac{1}{\lambda}e^{-\lambda t}. \tag{4.12}$$

By similarly arguments, we have

$$\|\partial(Q(W+\eta)-Q(W))\|_{N([t,\infty))} \lesssim \frac{1}{\lambda}e^{-\lambda t}. \tag{4.13}$$

Combining (4.12) and (4.13), we obtain

$$\|\partial(f(W+\eta)-f(W))\|_{N([t,\infty))\times N([t,\infty))} \lesssim \frac{1}{\lambda}e^{-\lambda t}. \tag{4.14}$$

Combining (4.9) and (4.14), we obtain

$$\|\partial\Phi\eta\|_{S([t,\infty))\times S([t,\infty))} \lesssim \frac{1}{\lambda}e^{-\lambda t} \leqslant \frac{1}{10}e^{-\lambda t},$$

if $\lambda > 0$ large enough. Thus, for $\lambda > 0$ large enough

$$\|\Phi\eta\|_X \leqslant 1. \tag{4.15}$$

Which implies that Φ map B onto B.

Step 2. Φ is contraction map on B

By using (4.2) and (4.3) and similarly estimate of (4.15), we can show that, for any $\eta \in B$, $\kappa \in B$,

$$\|\Phi\eta - \Phi\kappa\|_X \leqslant \frac{1}{2}\|\eta-\kappa\|_X.$$

By Banach fixed point theorem there exists unique solution on B of (4.1). □

4.4. **Properties of multi kink-soliton trains profile.** In this section, we prove some estimates of multi kink-soliton trains profile using in the proof of Theorem 1.6.

Lemma 4.4. *There exist $T_0 > 0$ and a constant $\lambda > 0$ such that the estimate (3.3) is true uniformly for $t \geqslant T_0$.*

Proof. For convenience, set

$$R = \sum_{j=1}^{K} R_j.$$

By similar arguments as in the proof of Lemma 4.1, we have

$$|R_j(x,t)| + |\partial R_j(x,t)| + |\partial^2 R_j(x,t)| + |\partial^3 R_j(x,t)| \lesssim_{h_j,|c_j|} e^{\frac{-h_j}{2}|x-c-jt|},$$

for all $1 \leqslant j \leqslant K$. Now, let us comeback to prove Lemma 4.1. For convenience, we set

$$\chi_1 = iV^2\overline{V_x} - i\sum_{j=0}^{K} R_j^2\overline{R_{jx}},$$

$$\chi_2 = |V|^4V - \sum_{j=0}^{K}|R_j|^4R_j.$$

As in the proof of Lemma 4.1, we fix $t > 0$, take any $x \in \mathbb{R}$ and choose $m = m(x) \in \mathbb{N}$ such that

$$|x - c_m t| = \min_{j\in\mathbb{N}} |x - c_j t|.$$

If $m \geqslant 1$ then by the assumption $c_0 < c_j$ for $j > 0$ we have $x > c_0 t$. Thus, by asymptotic behaviour of Φ_0 as in Remark 1.7, we can see R_0 as a soliton. More precise, we have

$$|R_0(t,x)| + |R_0'(t,x)| + |R_0''(t,x)| + |R_0'''(t,x)| \lesssim e^{-\frac{1}{2}|x-c_0t|} \lesssim e^{-\frac{1}{4}v_*t}.$$

Thus, by similar arguments as in the proof of Lemma 4.1, we have:

$$|(R - R_m)(x,t)| + |(\partial R - \partial R_m)(x,t)| + |(\partial^2 R - \partial^2 R_m)(x,t)| + |\partial^3 R - \partial^3 R_m| \lesssim e^{-\frac{1}{4}v_*t},$$

where

$$v_* = \inf_{j\neq k} h_j|c_j - c_k|.$$

Let f_1, g_1, r_1 and f_2, g_2, r_2 be polynomials of u, u_x, u_{xx}, u_{xxx} and conjugates such that for $u \in H^3(\mathbb{R})$:

$$\begin{aligned} iu^2\overline{u_x} &= f_1(u,\overline{u},u_x), \quad |u|^4u = f_2(u,\overline{u}),\\ \partial(iu^2\overline{u_x}) &= g_1(u,u_x,u_{xx},\overline{u},..), \quad \partial(|u|^4u) = g_2(u,u_x,\overline{u},..),\\ \partial^2(iu^2\overline{u_x}) &= r_1(u,u_x,u_{xx},u_{xxx},\overline{u},..), \quad \partial^2(|u|^4u) = r_2(u,u_x,u_{xx},\overline{u},..). \end{aligned}$$

Denote

$$A = \sup_{|z|+|z_x|+|z_{xx}|+|z_{xxx}|\leqslant\|R\|_{H^4(\mathbb{R})}} (|df_1| + |df_2| + dg_1| + |dg_2| + |dr_1| + |dr_2|),$$

where we denote by $|df(x,y,z,...)| = |f_x| + |f_y| + |f_z| + ...$ the absolute value of gradient of f at point $(x,y,z,...)$. Therefore, for $m \geqslant 1$ we have

$$\begin{aligned}
&|\chi_1| + |\chi_2| + |\partial\chi_1| + |\partial\chi_2| + |\partial^2\chi_1| + |\partial^2\chi_2|\\
&\lesssim |R_0|^2|R_{0x}| + |R_0|^5 + |f_1(V,..) - f_1(R,..)| + |f_2(V,...) - f_2(R,..)| + |g_1(V,..) - g_1(R,..)| + |g_2(V,..) - g_2(R,..)|\\
&\quad + |r_1(V,..) - r_1(R,..)| + |r_2(V,..) - r_2(R,..)| + |f_1(R,R_x,\overline{R}) - \sum_{j=1}^{K} f_1(R_j,R_{jx},\overline{R_j})|\\
&\quad + |f_2(R,\overline{R}) - \sum_{j=0}^{K} f_2(R_j,\overline{R_j})| + |g_1(R,R_x,..) - \sum_{j=1}^{K} g_1(R_j,R_{jx},..)| + |g_2(R,R_x,..) - \sum_{j=0}^{K} g_2(R_j,R_{jx},..)|\\
&\quad + |r_1(R,R_x,..) - \sum_{j=0}^{K} r_1(R_j,R_{jx},..)| + |r_2(R,R_x,..) - \sum_{j=0}^{K} r_2(R_j,R_{jx},..)|\\
&\lesssim |R_0|^2|R_{0x}| + |R_0|^5 + A|R_0|\\
&\quad + A(|(R - R_m)(x,t)| + |(\partial R - \partial R_m)(x,t)| + |(\partial^2 R - \partial^2 R_m)(x,t)| + |\partial^3 R - \partial^3 R_m|)\\
&\quad + A\sum_{j=1,j\neq m}^{K} (|R_j| + |\partial R_j| + |\partial^2 R_j| + |\partial^3 R_j|)\\
&\lesssim |R_0|^2|R_{0x}| + |R_0|^5 + A|R_0| + A\sum_{j=1,j\neq m}^{K} (|R_j| + |\partial R_j| + |\partial^2 R_j| + |\partial^3 R_j|)\\
&\lesssim_p e^{-\frac{1}{4}v_*t},
\end{aligned}$$

If $m = 0$ we have

$$\begin{aligned}
&|\chi_1| + |\chi_2| + |\partial\chi_1| + |\partial\chi_2| + |\partial^2\chi_1| + |\partial^2\chi_2| \\
&\lesssim \sum_{v=1,2}^{K} (|f_v(V, V_x, ..) - f_v(R_0, \partial R_0, ..)| + |g_v(V, V_x, ..) - g_v(R_0, \partial R_0, ..)| + |r_v(V, V_x, ..) - r_v(R_0, \partial R_0)|) \\
&\quad + \sum_{j=1,v=1,2}^{K} (|f_v(R_j, R_{jx}, ..)| + |g_v(R_j, R_{jx}, ..)| + |r_v(R_j, R_{jx}, ..)|) \\
&\lesssim A|R| + A\sum_{j=1}^{K}(|R_j| + |\partial R_j| + |\partial^2 R_j| + |\partial^3 R_j|) \\
&\lesssim_p e^{-\frac{1}{4}v_* t}.
\end{aligned}$$

This implies in all case we have

$$\|\chi_1(t)\|_{W^{2,\infty}} + \|\chi_2(t)\|_{W^{2,\infty}} \lesssim_p e^{-\frac{1}{4}v_* t}. \tag{4.16}$$

On one hand,

$$\begin{aligned}
&\|\chi_1(t)\|_{W^{2,1}} \\
&\lesssim \sum_{j=0}^{K}(\|R_j^2\overline{R_{jx}}\|_{L^1} + \|\partial(R_j^2\overline{R_{jx}})\|_{L^1} + \|\partial^2(R_j^2\overline{R_{jx}})\|_{L^1}) \\
&\lesssim \sum_{j=1}^{K}\|R_j\|_{H^3}^3 + \|\partial R_0\|_{W^{2,1}} < C < \infty.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\|\chi_2(t)\|_{W^{2,1}} \\
&\lesssim \||V|^4V - |R_0|^4R_0\|_{W^{2,1}} + \sum_{j=1}^{K}\||R_j|^4R_j\|_{W^{2,1}} \\
&\lesssim \||R_0|^4\sum_{j=1}^{K}|R_j| + |R_0|\sum_{j=1}^{K}|R_j|^4\|_{W^{2,1}} + \sum_{j=1}^{K}\|R_j\|_{H^3}^5 \\
&\lesssim \sum_{j=1}^{K}(\|R_j\|_{W^{2,1}}(\|R_0\|_{W^{2,\infty}} + 1) + \|R_j\|_{H^3}^5) < C < \infty.
\end{aligned}$$

Thus,

$$\|\chi_1(t)\|_{W^{2,1}} + \|\chi_1(t)\|_{W^{2,1}} < \infty. \tag{4.17}$$

Since (4.16) and (4.17), using Hölder inequality, we have

$$\|\chi_1(t)\|_{H^2} + \|\chi_2(t)\|_{H^2} \lesssim_p e^{-\frac{1}{8}v_* t}.$$

Let T_0 be large enough which depends on the parameters $\omega_1, ..., \omega_K, c_1, ..., c_K$, we have

$$\|\chi_1(t)\|_{H^2} + \|\chi_2(t)\|_{H^2} \leqslant e^{-\frac{1}{16}v_* t}, \quad \forall t \geqslant T_0.$$

Let $\lambda = \frac{1}{16}v_*$, we obtain the desired result. □

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