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CONSTRUCTION OF THE MULTI-SOLITON TRAINS, MULTI KINK-SOLITON TRAINS OF THE DERIVATIVE NONLINEAR SCHRÖDINGER EQUATIONS BY THE FIXED POINT METHOD

PHAN VAN TIN

Abstract. We look for solutions to derivative nonlinear Schrodinger equations built upon ¨ solitons. We prove the existence of multi-soliton trains i.e. solutions behaving at large time as the sum of finite solitons. We also show that one can attach a kink at the begin of the train i.e multi kink-soliton trains. Our proofs proceed by fixed point arguments around the desired profile, using Strichartz estimates.

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1. Introduction

We consider the derivative nonlinear Schrödinger equation:

$$iu_t + u_{xx} + i\alpha_1|u|^2u_x + i\alpha_2u^2\overline{u_x} + b|u|^4u = 0, \tag{1.1}$$

where α_1, α_2, b are given constants and $u : \mathbb{R}_t \times \mathbb{R}_x \to \mathbb{C}$.

In this paper, we are interested in two special forms of (1.1). First, we consider (1.1) in case $\alpha_1 = 1$, $\alpha_2 = 0$ as follows:

$$iu_t + u_{xx} + i|u|^2u_x + b|u|^4u = 0. \tag{1.2}$$

Second, we consider (1.1) in case $\alpha_1 = 0$, $\alpha_2 = 1$ as follows

$$iu_t + u_{xx} + iu^2\overline{u_x} + b|u|^4u = 0. \tag{1.3}$$

The derivative nonlinear Schrödinger equation was originally introduced in Plasma Physics as a simplified model for Alfvén wave propagation. Since then, it has attracted a lot of attention from the mathematical community (see e.g [4, 5, 7, 10, 11, 17, 18]).

The local well posedness of (1.1) in $H^1(\mathbb{R})$ is given by using Gauge transform as in [8, 9]. The global well posedness of (1.1) in $H^1(\mathbb{R})$ is given under small condition of the mass was considered in several papers (see [20] and the references therein). In [1], rely on complete integrability of equation, the authors proved the global well posedness of (1.1) in case $\alpha_1 = 2$, $\alpha_2 = 1$, $b = 0$ in $H^{\frac{1}{2}}(\mathbb{R})$ and furthermore proved that the $H^{\frac{1}{2}}$ norm of solution is globally bounded in time. To our

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knowledge, there are no many works on the Cauchy problem of (1.1) under non vanishing boundary condition. In [19], the author proved the local well posedness of (1.3) under $b = 0$ in some special spaces. However, the method use in [19] can not extend to general case (1.1). The main reason is that we can not find a proper transform to give a system without the derivative terms from (1.1).

The soliton of (1.2) is a profile of two parameters ω, c. In case $b = 0$, Colin and Ohta [3] proved that the soliton $u_{\omega,c}$ is orbitally stable when $\omega > \frac{c^2}{4}$ by variational arguments. In case $b > 0$ Ohta[15] proved that for each $b > 0$ there exists unique $s^* = s^*(b) > 0 \in (0, 1)$ such that the soliton $u_{\omega,c}$ is orbitally stable if $-2\sqrt{\omega} < c < 2s^*\sqrt{\omega}$ and orbitally instable if $2s^*\sqrt{\omega} < c < 2\sqrt{\omega}$. In [6], Hayashi investigate the structure of (1.2) from the viewpoint solitons.

Our goal in this paper is to construct the multi-soliton trains of (1.2) and multi kink-soliton trains of (1.3) i.e solutions which behave asymptotically as the sum of finitely solitons and as sum of a kink and finite solitons respectively. In [14], Le Coz and Wu proved stability of multi-soliton trains of (1.2) in case $b = 0$. For classical Schrödinger equation, in [12, 13] the authors constructed the infinite solitons train, infinite kink-soliton train using fixed point methods. In this paper, we prove the existence of multi-soliton trains and multi kink-soliton trains by using a fixed point method. By similar arguments in the proof of construct multi-soliton trains for (1.2), we can construct multi-soliton trains solution for (1.1). However, to construct the multi kink-soliton trains, our arguments do not work for the general case (1.1). Before state the main result, we give some preliminaries.

1.1. **Multi-soliton trains.** Consider equation (1.2). The soliton of equation (1.2) is solution of form $R_{\omega,c}(t, x) = e^{i\omega t}\phi_{\omega,c}(x - ct)$, where $\phi_{\omega,c} \in H^1(\mathbb{R})$ solves

$$-\phi_{xx} + \omega\phi + ic\phi_x - i|\phi|^2\phi_x - b|\phi|^4\phi = 0, \quad x \in \mathbb{R}. \tag{1.4}$$

Applying the following gauge transform to $\phi_{\omega,c}$

$$\phi_{\omega,c}(x) = \Phi_{\omega,c}(x) \exp\left(i\frac{c}{2}x - \frac{i}{4}\int_{-\infty}^{x} |\Phi_{\omega,c}(y)|^2\, dy\right),$$

it is easily verified that $\Phi_{\omega,c}$ satisfies the following equation.

$$-\Phi_{xx} + \left(\omega - \frac{c^2}{4}\right)\Phi + \frac{c}{2}|\Phi|^2\Phi - \frac{3}{16}\gamma|\Phi|^4\Phi = 0, \quad \gamma := 1 + \frac{16}{3}b. \tag{1.5}$$

The positive even solution of (1.5) is explicitly obtained as follows; if $\gamma > 0$ or equivalently $b > \frac{-3}{16}$,

$$\Phi^2_{\omega,c}(x) = \begin{cases} \frac{2(4\omega - c^2)}{\sqrt{c^2+\gamma(4\omega-c^2)}\cosh(\sqrt{4\omega-c^2}x)-c} & \text{if } -2\sqrt{\omega} < c < 2\sqrt{\omega}, \\ \frac{4c}{(cx)^2+\gamma} & \text{if } c = 2\sqrt{\omega}, \end{cases}$$

if $\gamma \leqslant 0$ or equivalently $b \leqslant -\frac{3}{16}$,

$$\Phi^2_{\omega,c}(x) = \frac{2(4\omega - c^2)}{\sqrt{c^2 + \gamma(4\omega - c^2)}\cosh(\sqrt{4\omega - c^2}x) - c} \text{ if } -2\sqrt{\omega} < c < -2s_*\sqrt{\omega},$$

where $s_* = s_*(\gamma) = \sqrt{\frac{-\gamma}{1-\gamma}}$. We note that the condition of two parameters γ and (ω, c)

$$\text{if}\gamma > 0 \Leftrightarrow b > \frac{-3}{16}, -2\sqrt{\omega} < c \leqslant 2\sqrt{\omega},$$
$$\text{if}\gamma \leqslant 0 \Leftrightarrow b \leqslant \frac{-3}{16}, -2\sqrt{\omega} < c < -2s_*\sqrt{\omega}.$$

is a necessary and sufficient condition for the existence of non-trivial solutions of (1.2) vanishing at infinity (see [2]). For each $j \in \{1, 2, .., K\}$, we set

$$R_j(t, x) = e^{i\theta_j} R_{\omega_j,c_j}(t, x - x_j).$$

The profile of an multisoliton is a sum of the form:

$$R = \sum_{j=1}^{K} R_j. \tag{1.6}$$

A solution of (1.2) is called multisoliton if, for some profile R:

$$u(t) - R(t) \to 0 \text{ as } t \to \infty,$$

in $H^1(\mathbb{R})$-norm. For convenience, we set $h_j = \sqrt{4\omega_j - c_j^2}$. Let (c_j, ω_j) be such that $-2\sqrt{\omega_j} < c_j < 2\sqrt{\omega_j}$ if $\gamma > 0$ or $-2\sqrt{\omega_j} < c_j < -2s_*\sqrt{\omega_j}$ if $\gamma \leqslant 0$, for all $1 \leqslant j \leqslant K$. We have

$$\Phi_{\omega_j,c_j}(x) = \sqrt{2}h_j(\sqrt{c_j^2 + \gamma h_j^2}\cosh(h_j x) - c_j)^{\frac{-1}{2}}. \tag{1.7}$$

As each soliton is in $H^\infty(\mathbb{R})$, we have $R \in H^\infty(\mathbb{R})$.

Our first main result is the following.

Theorem 1.1. *Let $K \in \mathbb{N}^*$ and for each $1 \leqslant j \leqslant K$, $(\theta_j, c_j, \omega_j, x_j)$ be sequence of parameters such that $x_j = 0$, $\theta_j \in \mathbb{R}$, $c_j \neq c_k \neq 0$, for $j \neq k$ and c_j such that $-2\sqrt{\omega_j} < c_j < 2\sqrt{\omega_j}$ if $\gamma > 0$ and $-2\sqrt{\omega_j} < c_j < -2s_*\sqrt{\omega_j}$ if $\gamma \leqslant 0$. The multisoliton profile R is given as (1.6). We assume that the parameters (ω_j, c_j) satisfy*

$$\max\{1; \|R_x\|_{L^\infty L^\infty}\|R\|_{L^\infty L^\infty} + \|R\|^4_{L^\infty L^\infty}\} \ll v_* := \inf_{j\neq k} h_j|c_j - c_k|. \tag{1.8}$$

There exist a solution u to (1.2) such that

$$\|u - R\|_{H^1} \leqslant Ce^{-\lambda t}. \quad \forall t \geqslant T_0, \tag{1.9}$$

for some constant $C > 0$, $\lambda = \frac{1}{8}v_*$.

The formula of soliton in case $\gamma > 0$ and $\gamma \leqslant 0$ is similar. Thus, from now on, we assume $\gamma > 0$. The case $\gamma \leqslant 0$ is treated by similar arguments.

Remark 1.2. In cases $\gamma > 0$, we give a example of parameters satisfy (1.8) as follows. First, chose $h_j \ll \min(|c_j|, 1)$ and $c_j < 0$ for all j. We have

$$\|\Phi_{\omega_j,c_j}\|^2_{L^\infty} \leqslant \frac{2h_j^2}{\sqrt{c_j^2 + \gamma h_j^2} - c_j} \lesssim \frac{h_j^2}{|c_j|}.$$

Moreover,

$$\partial\Phi_{\omega_j,c_j} = \frac{-\sqrt{2}}{2}h_j^2\sqrt{c_j^2 + \gamma h_j^2}\sinh(h_j x)(\sqrt{c_j^2 + \gamma h_j^2}\cosh(h_j x) - c_j)^{-\frac{3}{2}}.$$

Thus, choosing $c_j < 0$, for all j, we obtain

$$\begin{aligned}
|\partial\Phi_{\omega_j,c_j}| &\lesssim h_j^2\sqrt{c_j^2 + \gamma h_j^2}|\sinh(h_j x)|(\sqrt{c_j^2 + \gamma h_j^2}\cosh(h_j x) - c_j)^{-\frac{3}{2}} \\
&\lesssim h_j^2(\sqrt{c_j^2 + \gamma h_j^2}\cosh(h_j x) - c_j)^{-\frac{1}{2}} \\
&\approx h_j|\Phi_{\omega_j,c_j}| \lesssim \frac{h_j^2}{\sqrt{|c_j|}}.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
\|\partial R_j\|_{L^\infty} = \|\partial\phi_{\omega_j,c_j}\|_{L^\infty} &\approx \|\partial\Phi_{\omega_j,c_j}\|_{L^\infty} + \|\frac{c_j}{2}\Phi_{\omega_j,c_j} - \Phi^3_{\omega_j,c_j}\|_{L^\infty} \\
&\leqslant \|\partial\Phi_{\omega_j,c_j}\|_{L^\infty} + \frac{|c_j|}{2}\|\Phi_{\omega_j,c_j}\|_{L^\infty} + \|\Phi_{\omega_j,c_j}\|^3_{L^\infty} \\
&\lesssim \frac{h_j^2}{\sqrt{|c_j|}} + h_j\sqrt{|c_j|} + \frac{h_j^3}{\sqrt{|c_j|^3}}.
\end{aligned}$$

Thus, we only need to chose the parameters (c_j, ω_j) satisfy

$$\sum_{1\leqslant j\leqslant K}\left(\frac{h_j^2}{\sqrt{|c_j|}} + h_j\sqrt{|c_j|} + \frac{h_j^3}{\sqrt{|c_j|^3}}\right)\sum_{1\leqslant j\leqslant K}\frac{h_j}{\sqrt{|c_j|}} + \sum_{1\leqslant j\leqslant K}\frac{h_j^4}{c_j^2} \ll \inf_{j\neq k} h_j|c_j - c_k|, \tag{1.10}$$

then the assumption (1.8) is satisfied. Let $M \gg 1$ be large enough positive constant. Replace (c_j) by Mc_j, h_j bounded (hence, $\omega_j = \frac{1}{4}(h_j^2 + M^2c_j^2)$) for all j. As $M \to \infty$, the right hand sight of (1.10) is order M and the left side is order M^0. Thus, for M large enough we obtain the parameters (c_j, ω_j) satisfy (1.8).

1.2. **Multi kink-soliton trains.** Consider the equation (1.3). Let $R_{\omega,c}$ be a smooth solution of (1.2) of form:

$$R_{\omega,c}(t,x) = e^{i\omega t}\phi_{\omega,c}(x-ct), \tag{1.11}$$

where $\phi_{\omega,c}$ is smooth and solves

$$-\phi_{xx} + \omega\phi + ic\phi_x - i\phi^2\overline{\phi_x} - b|\phi|^4\phi = 0, \quad x \in \mathbb{R}. \tag{1.12}$$

If $\phi_{\omega,c}\,|_{\mathbb{R}^+} \in H^1(\mathbb{R}^+)$ then we can use the following Gauge transform:

$$\Phi_{\omega,c} = \exp\left(-i\frac{c}{2}x + \frac{i}{4}\int_{\infty}^{x} |\phi_{\omega,c}(y)|^2\,dy\right)\phi_{\omega,c}.$$

Since (1.12), $\Phi_{\omega,c}$ is smooth and solves

$$-\Phi_{xx} + \left(\omega - \frac{c^2}{4}\right)\Phi - \frac{3}{2}\mathcal{I}m(\overline{\Phi}\Phi_x)\Phi - \frac{c}{2}|\Phi|^2\Phi + \frac{3}{16}\gamma|\Phi|^4\Phi = 0, \quad \gamma := \frac{5}{3} - \frac{16}{3}b. \tag{1.13}$$

We note that signs of coefficients of the terms $|\phi|^2\Phi$ and $|\Phi|^4\Phi$ are not the same as (1.5). Since $\Phi_{\omega,c}\,|_{\mathbb{R}^+} \in H^2(\mathbb{R}^+)$, by similar arguments as in [3], we can prove that $\mathcal{I}m(\overline{\Phi_{\omega,c}}\partial_x\Phi_{\omega,c}) = 0$. Thus, $\Phi_{\omega,c}$ solves

$$-\Phi_{xx} + \left(\omega - \frac{c^2}{4}\right)\Phi - \frac{c}{2}|\Phi|^2\Phi + \frac{3}{16}\gamma|\Phi|^4\Phi = 0. \tag{1.14}$$

Now, we give the definition of kink solution of (1.2).

Definition 1.3. The function $R_{\omega,c}$ is called a half-kink solution of (1.2) if $R_{\omega,c}$ is of form (1.11) and associated $\Phi_{\omega,c}$ is a real valued function solves (1.14) and satisfies:

$$\begin{cases} -\Phi'' + \tilde{\omega}\Phi - f(\Phi) = 0, \\ \lim\limits_{x\to\pm\infty}\Phi(x) \neq 0, \\ \lim\limits_{x\to\mp\infty}\Phi(x) = 0, \end{cases} \tag{1.15}$$

where $\tilde{\omega} = \omega - \frac{c^2}{4}$, $f : \mathbb{R} \to \mathbb{R}$ such that $f(s) = \frac{c}{2}s^3 - \frac{3}{16}\gamma s^5$.

The following result about the existence of half-kink profile is stated in [12] as follows:

Proposition 1.4. *Let $f : \mathbb{R} \to \mathbb{R}$ be a C^1 function with $f(0) = 0$ and define $F(s) := \int_0^s f(t)\,dt$. For $\tilde{\omega} \in \mathbb{R}$, let*

$$\zeta(\tilde{\omega}) := \inf\left\{\zeta > 0, F(\zeta) - \frac{1}{2}\tilde{\omega}\zeta^2 = 0\right\}$$

and assume that there exists $\tilde{\omega}_1 \in \mathbb{R}$ such that

$$\zeta(\tilde{\omega}_1) > 0, \quad f'(0) - \tilde{\omega}_1 < 0, \quad f(\zeta(\tilde{\omega}_1)) - \tilde{\omega}_1\zeta(\tilde{\omega}_1) = 0. \tag{1.16}$$

Then, for $\tilde{\omega} = \tilde{\omega}_1$, there exists a half-kink profile $\Phi \in C^2(\mathbb{R})$ of (1.15) i.e Φ is unique (up to translation), positive and satisfies $\Phi > 0$, $\Phi' > 0$ on $\mathbb{R}$ and the boundary conditions

$$\lim_{x\to-\infty}\Phi(x) = 0, \quad \lim_{x\to\infty}\Phi(x) = \zeta(\tilde{\omega}_1) > 0. \tag{1.17}$$

If in addition,

$$f'(\zeta(\tilde{\omega}_1)) - \tilde{\omega}_1 < 0, \tag{1.18}$$

then for any $0 < a < \tilde{\omega}_1 - \max\{f'(0), f'(\zeta(\tilde{\omega}_1))\}$ there exists $D_a > 0$ such that

$$|\Phi'(x)| + |\Phi(x)1_{x<0}| + |(\zeta(\tilde{\omega}_1 - \Phi(x))1_{x>0}| \leqslant D_a e^{-a|x|}, \quad \forall x \in \mathbb{R}. \tag{1.19}$$

Remark 1.5.

(1) As [12, Remark 1.15], using the symmetry $x \to -x$ and Proposition 1.4 implies the existence and uniqueness of half-kink profile Φ satisfying

$$\lim_{x\to-\infty}\Phi(x) = \zeta(\tilde{\omega}_1) > 0. \quad \lim_{x\to\infty}\Phi(x) = 0.$$

(2) In this paper, we consider $f(s) = \frac{c}{2}s^3 - \frac{3}{16}\gamma s^5$. Let us see Proposition 1.4 under this nonlinear term. We can check if $\gamma > 0$, $c > 0$ then there exists $\tilde{\omega} = \frac{B^2}{2A}$ and $\zeta(\tilde{(\omega)}) = \sqrt{\frac{B}{2A}}$ satisfy the conditions (1.16), (1.18) and the definition of $\zeta(\tilde{\omega})$, where $A = \frac{3}{96}\gamma$, $B = \frac{c}{8}$. Thus, using Proposition 1.4, if $\gamma > 0$, $c > 0$ then there exists half-kink solution of (1.2). Moreover, by elementary calculation, we have the constant a in Proposition 1.4 satisfies

$$0 < a < \frac{c^2}{4\gamma}.$$

(3) Consider the half-kink profile Φ of Proposition 1.4. Since, Φ smooth solves (1.15) and satisfies (1.19) we have

$$|\Phi''(x)| + |\Phi'''(x)| \leqslant D_a e^{-a|x|}.$$

Now, we assume $\gamma > 0$. Let $K > 0$, $\theta_0, x_0, \omega_0, c_0 \in \mathbb{R}$ be such that $2\sqrt{\omega_0} > c_0 > 0$. For $1 \leqslant j \leqslant K$, let $(\theta_j, x_j, \omega_j, c_j) \in \mathbb{R}$ such that $2\sqrt{\omega_j} > c_j > 2s_*\sqrt{\omega_j}$ where $s_* = \sqrt{\frac{\gamma}{1+\gamma}}$ and $R_j \in H^1(\mathbb{R})$ be soliton solution of (1.3) given associated to the profile:

$$\Phi^2_{\omega_j,c_j} = \frac{2h_j^2}{\sqrt{c_j^2 - \gamma h_j^2}\cosh(h_j x) + c_j},$$

where $h_j = \sqrt{4\omega_j - c_j^2}$ and Φ_{ω_j,c_j} is localized solution of (1.14). Let Φ_0 be half-kink profile given as in Remark 1.5 (1) associated with parameters ω_0, c_0 such that

$$\lim_{x\to-\infty} \Phi_0 \neq 0, \text{ and } \lim_{x\to\infty} \Phi_0 = 0.$$

Let R_0 be the associated half-kink solution of (1.3). The multi kink-soliton profile of (1.3) is defined as follows:

$$V = R_0 + \sum_{j=1}^{K} R_j. \tag{1.20}$$

Our second main result is the following:

Theorem 1.6. *Considering (1.3), we assume that $b < \frac{5}{16}$ ($\gamma > 0$). Let $K \in \mathbb{N}^*$ and for each $1 \leqslant j \leqslant K$, $\theta_j, c_j, \omega_j, x_j$ be sequence of parameters such that $x_j = 0$, $\theta_j \in \mathbb{R}$, $c_j \neq c_k \neq 0$ for $j \neq k$ and c_j such that $2\sqrt{\omega_j} > c_j > 2s_*\sqrt{\omega_j}$. Let $(\theta_0, c_0, \omega_0, x_0)$ such that $x_0 = 0$, $2\sqrt{\omega_0} > c_0 > 0$, $c_0 < c_j$ for $K \geqslant j \geqslant 1$ and $c_0^2 > 2\gamma$ and R_0 be the associated half-kink solution given as above. The multi kink-soliton profile V is given as in (1.20). We assume that the parameters (ω_j, c_j) satisfy*

$$\max\{1; \|V_x\|_{L^\infty L^\infty}\|V\|_{L^\infty L^\infty} + \|V\|^4_{L^\infty L^\infty}\} \ll v_* := \inf_{j\neq k} h_j|c_j - c_k|. \tag{1.21}$$

Then there exist a solution u to (1.3) such that

$$\|u - V\|_{H^1} \leqslant Ce^{-\lambda t}. \quad \forall t \geqslant T_0, \tag{1.22}$$

for some constant $C > 0$, $\lambda = \frac{1}{8}v_*$.

Remark 1.7.

(1) The condition $c_0^2 > 2\gamma$ in Theorem 1.6 is a technical condition and we can remove this. Under this condition, the constant a in Proposition 1.4 satisfies

$$0 < a < \frac{c_0^2}{4\gamma}.$$

Thus, we can choose $a = \frac{1}{2}$. This fact makes the proof is more easier and we have

$$|\Phi_0'''(x)| + |\Phi_0''(x)| + |\Phi_0'(x)| + |\Phi_0(x)1_{x>0}| + \left|\left(\sqrt{\frac{B}{2A}} - \Phi_0(x)\right)1_{x<0}\right| \lesssim e^{-\frac{1}{2}|x|}. \tag{1.23}$$

(2) Let $\gamma > 0$. We give an example of parameters satisfy the condition (1.21) of Theorem 1.6. As in remark 1.2, we have

$$\Phi_{\omega_j,c_j} = \sqrt{2}h_j\left(\sqrt{c_j^2 - \gamma h_j^2}\cosh(h_j x) + c_j\right)^{\frac{-1}{2}}.$$

for all $1 \leqslant j \leqslant K$. Hence, choose $h_j \ll \min\{c_j, 1\}$ for all $1 \leqslant j \leqslant K$, we have

$$\|\Phi_{\omega_j,c_j}\|_{L^\infty}^2 \leqslant \frac{2h_j^2}{\sqrt{c_j^2 - \gamma h_j^2} + c_j} \lesssim \frac{h_j^2}{c_j}.$$

By similar arguments in remark 1.2, for all $1 \leqslant j \leqslant K$, we have

$$\|\partial R_j\|_{L^\infty} \lesssim \frac{h_j^2}{\sqrt{c_j}} + h_j\sqrt{c_j} + \frac{h_j^3}{\sqrt{c_j^3}}.$$

Now, we treat to the case $j = 0$. Let Φ_0 be profile given as in Proposition 1.4 with parameters c_0, ω_0 and R_0 be associated half-kink solution of (1.2). Since (1.19), remark 1.5 and remark 1.7 we have

$$\|\Phi_0\|_{L^\infty} \lesssim \tilde{\omega} = \frac{B^2}{2A} \approx c_0^2,$$
$$\|\partial\Phi_0\|_{L^\infty} \lesssim C,$$

for some constant $C > 0$. Thus,

$$\|R_0\|_{L^\infty} \lesssim c_0^2,$$
$$\|\partial R_0\|_{L^\infty} \lesssim C.$$

This implies that for $h_j \ll \min\{c_j, 1\}$ $(j = 1, .., K)$ we have

$$\begin{aligned}
&\|V_x\|_{L^\infty L^\infty}\|V\|_{L^\infty L^\infty} + \|V\|_{L^\infty L^\infty}^4 \\
&\lesssim \|V\|_{L^\infty L^\infty} + \|V\|_{L^\infty L^\infty}^4, \\
&\lesssim c_0^2 + \sum_{j=1}^K \frac{h_j}{\sqrt{c_j}} + c_0^8 + \sum_{j=1}^K \frac{h_j^4}{c_j^2} \lesssim c_0^2 + c_0^8 + \sum_{j=1}^K \frac{h_j}{\sqrt{c_j}}
\end{aligned}$$

We only need to choose the parameters (c_j, ω_j) satisfy

$$c_0^2 + c_0^8 + \sum_{j=1}^K \frac{h_j}{\sqrt{c_j}} \ll \inf_{j\neq k} h_j|c_j - c_k|, \tag{1.24}$$

then the assumption (1.21) is satisfied. Let $M \gg 1$. be large enough positive constant. Choosing $c_0 \approx 1$ and h_j uniformly bounded for all j. Replace c_j by Mc_j (hence, $\omega_j = \frac{1}{4}(h_j^2 + M^2c_j^2)$) for all $j \geqslant 1$. As $M \to \infty$, the right hand sight of (1.21) is order M and the left hand sight is order M^0. Thus, for M large enough we obtain the parameters (c_j, ω_j) satisfy (1.21).

To prove Theorem 1.1 and 1.6, our strategy is using a Gauge transform to give a system of two equations of φ, ψ from equation of u (1.2). Then, by fixed point method we prove that there exists a unique solution φ, ψ of this system that decay exponential in time when t is large enough. Using this property, we prove a relation between φ, ψ which allow us to obtain a solution u of (1.2). This solution satisfies the desired property. However, when we extend our result on construction of infinite soliton trains and infinite kink-soliton trains we meet some problems on selecting parameters. The reason is that in case of finite parameters the condition $\|R\|_{H^3}$ or $\|V - R_0\|_{H^3}$ bounded is automatically true but in case of infinite parameters, this condition requires some estimate on parameters. It is not easy to select the parameters which satisfy this estimate and the assumption (1.8) or (1.21). Moreover, the arguments in the proof of Theorem 1.1 can extend to construct the multi-soliton trains for general equation 1.1. However, we do not have prove to construct the multi kink-soliton trains for general equation as in 1.6.

We introduce the following notations using in this paper.

Notation.

(1) *For $t > 0$, we note the Strichartz space $S([t, \infty))$ is defined via the norm*

$$\|u\|_{S([t,\infty))} = \sup_{(q,r)\ admissible} \|u\|_{L_\tau^q L_x^r([t,\infty)\times\mathbb{R})}$$

The dual space is denoted by $N([t, \infty)) = S([t, \infty))^*$.

(2) *For $z = (a, b) \in \mathbb{C}^2$ is a vector, we denote $|z| = |a| + |b|$.*

(3) *We denote $a \lesssim b$, for $a, b > 0$, if a is smaller than b up to multiply a positive constant. Moreover, we denote $a \approx b$ if a equal to b up to multiply a certain positive constant.*
(4) *Let $f \in C^1(\mathbb{R})$. We use ∂f or f_x to denote the derivative in space of the function f.*

2. Proof of Theorem 1.1

In this section, we give the proof of Theorem 1.1. Our strategy is using the Banach fixed point theorem and Strichartz estimates. We divide our proof into steps.

Step 1. Preliminary analysis

Considering the following transform:

$$\varphi(t,x) = \exp\left(\frac{i}{2}\int_{-\infty}^{x} |u(t,y)|^2\,dy\right) u(t,x),$$
$$\psi = \partial\varphi - \frac{i}{2}|\varphi|^2\varphi.$$

By similarly arguments as in [9] and [16], we see that if $u(t,x)$ solves (1.2) then (φ,ψ) solves the following system

$$\begin{cases} L\varphi = i\varphi^2\overline{\psi} - b|\varphi|^4\varphi, \\ L\psi = -i\psi^2\overline{\varphi} - 3b|\varphi|^4\psi - 2b|\varphi|^2\varphi^2\overline{\psi}, \\ \varphi\,|_{t=0} = \varphi_0 = \exp\left(\frac{i}{2}\int_{-\infty}^{x}|u_0(y)|^2\,dy\right)u_0, \\ \psi\,|_{t=0} = \psi_0 = \partial\varphi_0 - \frac{i}{2}|\varphi_0|^2\varphi_0, \end{cases} \tag{2.1}$$

where $L = \partial_t + \partial_{xx}$. For convenience, we define

$$P(\varphi,\psi) = i\varphi^2\overline{\psi} - b|\varphi|^4\varphi,$$
$$Q(\varphi,\psi) = -i\psi^2\overline{\varphi} - 3b|\varphi|^4\psi - 2b|\varphi|^2\varphi^2\overline{\psi}.$$

Let R be multi soliton profile given as in (1.6). Set $q = u - R$. Since R_j solves (1.2), for all j, by elementary calculation, we have

$$iR_t + R_{xx} + i|R|^2R_x + b|R|^4R = i(|R|^2R_x - \sum_{j=1}^{K}|R_j|^2R_{jx}) + b(|R|^4R - \sum_{j=1}^{\infty}|R_j|^4R_j). \tag{2.2}$$

Since Lemma 4.1, we have

$$\||R|^2R_x - \sum_{j=1}^{K}|R_j|^2R_{jx}\|_{H^2} + \||R|^4R - \sum_{j=1}^{K}|R_j|^4R_j\|_{H^2} \leqslant e^{-\lambda t}, \tag{2.3}$$

for $\lambda = \frac{1}{8}v_*$. Thus, we rewrite (2.2) as follows

$$iR_t + R_{xx} + i|R|^2R_x + b|R|^4R = e^{-\lambda t}v(t,x), \tag{2.4}$$

where $v(t) \in H^2(\mathbb{R})$ such that $\|v(t)\|_{H^2}$ uniformly bounded in t.

Define

$$h = \exp\left(\frac{i}{2}\int_{-\infty}^{x}|R|^2\,dy\right)R(t,x),$$
$$k = h_x - \frac{i}{2}|h|^2h.$$

By elementary calculation as above, we have

$$Lh = ih^2\overline{k} - b|h|^4h + e^{-t\lambda}m(t,x),$$
$$Lk = -ik^2\overline{h} - 3b|h|^4k - 2b|h|^2h^2\overline{k} + e^{-t\lambda}n(t,x),$$

where $m(t), n(t)$ satisfy

$$m = v\exp\left(\frac{i}{2}\int_{-\infty}^{x}|R|^2\,dy\right) - h\int_{-\infty}^{x}\mathcal{I}m(v\overline{R})\,dy, \tag{2.5}$$

$$n = m_x - i|h|^2m + \frac{i}{2}h^2\overline{m}. \tag{2.6}$$

By Lemma 4.2 we have $\|m(t)\|_{H^1}+\|n(t)\|_{H^1}$ uniformly bounded in t. Let $\tilde{\varphi}=\varphi-h$ and $\tilde{\psi}=\psi-k$. Then $\tilde{\varphi}$, $\tilde{\psi}$ solve:

$$\begin{cases} L\tilde{\varphi}=P(\varphi,\psi)-P(h,k)-e^{-t\lambda}m(t,x),\\ L\tilde{\psi}=Q(\varphi,\psi)-Q(h,k)-e^{-t\lambda}n(t,x).\end{cases} \tag{2.7}$$

Set $\eta=(\tilde{\varphi},\tilde{\psi})$, $W=(h,k)$ and $f(\varphi,\psi)=(P(\varphi,\psi),Q(\varphi,\psi))$. We find solution of (2.7) of following form:

$$\eta=-i\int_t^\infty S(t-s)[f(W+\eta)-f(W)+H](s)\,ds, \tag{2.8}$$

where $S(t-s)$ is Schrödinger group, $H=e^{-t\lambda}(m,n)$. Moreover, since $\psi=\partial\varphi-\frac{i}{2}|\varphi|^2\varphi$, we have

$$\tilde{\psi}=\partial\tilde{\varphi}-\frac{i}{2}(|\tilde{\varphi}+h|^2(\tilde{\varphi}+h)-|h|^2h). \tag{2.9}$$

Step 2. Existence solution of system equations

Since Lemma 4.3, there exists $T_*\gg 1$ such that for $T_0\geqslant T_*$ there exists unique solution η define on $[T_0,\infty)$ of (2.7) such that

$$e^{t\lambda}(\|\eta\|_{S([t,\infty))\times S([t,\infty))})+e^{t\lambda}(\|\eta_x\|_{S([t,\infty))\times S([t,\infty))})\leqslant 1,\quad \forall t\geqslant T_0, \tag{2.10}$$

for the constant $\lambda>0$ defined as in **step 1**. Thus, for all $t\geqslant T_0$, we have

$$\|\tilde{\varphi}\|_{H^1}+\|\tilde{\psi}\|_{H^1}\lesssim e^{-\lambda t}, \tag{2.11}$$

Step 3. Existence of multisoliton for (1.2)

We prove that the solution $\eta=(\tilde{\varphi},\tilde{\psi})$ of (2.7) satisfying the relation (2.9). Indeed, let η be solution of (2.7) which we find in step 1. Set $\varphi=\tilde{\varphi}+h$, $\psi=\tilde{\psi}+k$ and $v=\partial\varphi-\frac{i}{2}|\varphi|^2\varphi$. Since h solves $Lh=P(h,k)+e^{-t\theta}m(t,x)$ and $\tilde{\varphi}$ solves $L\tilde{\varphi}=P(\varphi,\psi)-P(h,k)-e^{-t\theta}m(t,x)$, we have (φ,ψ) solves

$$\begin{cases} L\varphi=P(\varphi,\psi),\\ L\psi=Q(\varphi,\psi).\end{cases} \tag{2.12}$$

By similarly arguments as above, we have

$$\begin{aligned} L\psi-Lv&=Q(\varphi,\psi)-\left(\partial L\varphi-\frac{i}{2}L(|\varphi|^2\varphi)\right)\\ &=Q(\varphi,\psi)-\left(\partial L\varphi-\frac{i}{2}(L(\varphi^2)\overline{\varphi}+\varphi^2L(\overline{\varphi})+2\partial(\varphi^2)\partial\overline{\varphi})\right)\\ &=Q(\varphi,\psi)-\left(\partial L\varphi-\frac{i}{2}(2L\varphi|\varphi|^2+2(\partial\varphi)^2\overline{\varphi}-\varphi^2\overline{L\varphi}+2\varphi^2\partial_{xx}\overline{\varphi})+2\varphi|\partial\varphi|^2)\right). \end{aligned} \tag{2.13}$$

Moreover, we have

$$\begin{aligned} L\varphi=P(\varphi,\psi)&=i\varphi^2\overline{\psi}-b|\varphi|^4\varphi\\ &=i\varphi^2\overline{(\psi-v)}+i\varphi^2\overline{v}-b|\varphi|^4\varphi. \end{aligned} \tag{2.14}$$

Combining (2.14) and (2.13) and by elementary calculation, we obtain

$$\begin{aligned} L\psi-Lv&=Q(\varphi,\psi)-\partial(i\varphi^2\overline{(\psi-v)})-|\varphi|^2\varphi^2\overline{(\psi-v)}-\frac{1}{2}|\varphi|^4(\psi-v)-Q(\varphi,v)\\ &=(Q(\varphi,\psi)-Q(\varphi,v))-2i\varphi\partial\varphi\overline{(\psi-v)}-i\varphi^2\partial\overline{(\psi-v)}-|\varphi|^2\varphi^2\overline{(\psi-v)}-\frac{1}{2}|\varphi|^4(\psi-v)\\ &=-i(\psi^2-v^2)\overline{\varphi}-3b|\varphi|^4(\psi-v)-2b|\varphi|^2\varphi^2\overline{(\psi-v)}-2i\varphi\left(v+\frac{i}{2}|\varphi|^2\varphi\right)\overline{(\psi-v)}\\ &\quad-i\varphi^2\partial\overline{(\psi-v)}-|\varphi|^2\varphi^2\overline{(\psi-v)}-\frac{1}{2}|\varphi|^4(\psi-v). \end{aligned} \tag{2.15}$$

Define $\tilde{v}=v-k$. Since, $\tilde{\psi}-\tilde{v}=\psi-v$ and (2.15) we have

$$L\tilde{\psi}-L\tilde{v}=(\tilde{\psi}-\tilde{v})A(\tilde{\psi},\tilde{v},\tilde{\varphi},h,k)+\overline{(\tilde{\psi}-\tilde{v})}B(\tilde{\psi},\tilde{v},\tilde{\varphi},h,k)-i(\tilde{\varphi}+h)^2\partial\overline{(\tilde{\psi}-\tilde{v})}, \tag{2.16}$$

where

$$A = -i(\tilde{\psi} + \tilde{v} + 2k)(\overline{\tilde{\varphi} + h}) - 3b|\tilde{\varphi} + h|^4 - \frac{1}{2}|\tilde{\varphi} + h|^4,$$
$$B = -2b|\tilde{\varphi} + h|^2(\tilde{\varphi} + h)^2 - 2i(\tilde{\varphi} + h)\left(\tilde{v} + k + \frac{i}{2}|\tilde{\varphi} + h|^2(\tilde{\varphi} + h)\right) - |\tilde{\varphi} + h|^2(\tilde{\varphi} + h)^2.$$

We see that A, B are polynomials of degree at most 4 in $(\tilde{\psi}, \tilde{v}, \tilde{\varphi}, h, k)$. Multiplying two sides of (2.16) with $\overline{\tilde{\psi} - \tilde{v}}$, take imaginary part and integral over space using integral by part , we obtain

$$\begin{aligned}\frac{1}{2}\partial_t\|\tilde{\psi} - \tilde{v}\|_{L^2}^2 &= \mathcal{I}m\int_{\mathbb{R}}(\tilde{\psi} - \tilde{v})^2 A(\tilde{\psi}, \tilde{v}, \tilde{\varphi}, h, k) + \overline{(\tilde{\psi} - \tilde{v})}^2 B(\tilde{\psi}, \tilde{v}, \tilde{\varphi}, h, k) + \frac{i}{2}\partial(\tilde{\varphi} + h)^2\overline{(\tilde{\psi} - \tilde{v})}^2\, dx \\ &\lesssim \|\tilde{\psi} - \tilde{v}\|_{L^2}^2(\|A\|_{L^\infty} + \|B\|_{L^\infty} + \|\partial(\tilde{\varphi} + h)^2\|_{L^\infty}).\end{aligned}$$

By using Grönwall's inequality, we obtain

$$\|\tilde{\psi}(t) - \tilde{v}(t)\|_{L^2}^2 \lesssim \|\tilde{\psi}(N) - \tilde{v}(N)\|_{L^2}^2 \exp\left(\int_t^N (\|A\|_{L^\infty} + \|B\|_{L^\infty} + \|\partial(\tilde{\varphi} + h)^2\|_{L^\infty}\, ds\right). \quad (2.17)$$

Since (2.10) and (2.11) and bounded of $\|h\|_{H^4} + \|k\|_{H^4}$, for $t \geqslant T_0$, we have

$$\begin{aligned}&\int_t^N (\|A\|_{L^\infty} + \|B\|_{L^\infty} + \|\partial(\tilde{\varphi} + h)^2\|_{L^\infty})\, ds \\ &\lesssim \|\tilde{\varphi}\|_{L^1L^\infty((t,N\times\mathbb{R}))} + \|\tilde{\psi}\|_{L^1L^\infty((t,N)\times\mathbb{R})} + \|\partial\tilde{\varphi}\|_{L^1L^\infty((t,N)\times)\mathbb{R})} \\ &\quad + \|kh\|_{L^1L^\infty((t,N)\times\mathbb{R})} + \|h^4\|_{L^1L^\infty((t,N)\times\mathbb{R})} + \|\partial(h^2)\|_{L^1L^\infty((t,N)\times\mathbb{R})} \\ &\lesssim \|\tilde{\varphi}\|_{L^4L^\infty((t,N)\times\mathbb{R})}(N-t)^{\frac{3}{4}} + \|\tilde{\psi}\|_{L^4L^\infty((t,N)\times\mathbb{R})}(N-t)^{\frac{3}{4}} \\ &\quad + \|\partial\tilde{\varphi}\|_{L^4L^\infty((t,N)\times\mathbb{R})}(N-t)^{\frac{3}{4}} + (N-t)(\|h_xh\|_{L^\infty L^\infty} + \|h\|_{L^\infty L^\infty}^4) \\ &\lesssim (N-t)^{\frac{3}{4}}e^{-ct} + (N-t)(\|R_x\|_{L^\infty L^\infty}\|R\|_{L^\infty L^\infty} + \|R\|_{L^\infty L^\infty}^4).\end{aligned}$$

where we use $k = h_x - \frac{i}{2}|h|^2h$ and $\tilde{v} = \partial\tilde{\varphi} - \frac{i}{2}(|\tilde{\varphi} + h|^2(\tilde{\varphi} + h) - |h|^2h)$. Thus, from the assumption (1.8), we have $\|R_x\|_{L^\infty L^\infty}\|R\|_{L^\infty L^\infty} + \|R\|_{L^\infty L^\infty}^4 \ll \lambda$. Thus,

$$\begin{aligned}\|\tilde{\psi}(t) - \tilde{v}(t)\|_{L^2}^2 &\lesssim e^{-2\lambda N}\exp\left(C(N-t)^{\frac{3}{4}} + C(N-t)\|R\|_{W^{1,\infty}}\right) \\ &= \exp\left(C(N-t)^{\frac{3}{4}} + C(N-t)\|R\|_{W^{1,\infty}} - 2\lambda N\right).\end{aligned}$$

Let $N \to \infty$, we obtain that

$$\|\tilde{\psi}(t) - \tilde{v}(t)\|_{L^2}^2 = 0.$$

This implies that $\tilde{\psi} = \tilde{v}$, hence,

$$\psi = v = \partial\varphi - \frac{i}{2}|\varphi|^2\varphi. \quad (2.18)$$

Let $u = \exp\left(-\frac{i}{2}\int_{-\infty}^x |\varphi(y)|^2\, dy\right)\varphi$. Combining (2.18) with the fact that (φ, ψ) solves

$$\begin{cases} L\varphi = P(\varphi, \psi), \\ L\psi = Q(\varphi, \psi), \end{cases}$$

we obtain u solves (1.2). Moreover,

$$\begin{aligned}\|u - R\|_{H^1} &= \|\exp\left(-\frac{i}{2}\int_{-\infty}^x |\varphi(y)|^2\, dy\right)\varphi - \exp\left(-\frac{i}{2}\int_{-\infty}^x |h(y)|^2\, dy\right)h\|_{H^1} \\ &\lesssim \|\varphi - h\|_{H^1} = \|\tilde{\varphi}\|_{H^1}\end{aligned}$$

Combining (2.11), for $t \geqslant T_0$, we have

$$\|u - R\|_{H^1} \leqslant e^{-\lambda t}.$$

This complete the proof of Theorem 1.1.

Remark 2.1. We do not have the proof for constructing the multi kink-soliton trains of equation (1.2) as (1.3). The reason is that if the profile R in the proof of Theorem 1.1 is not in $H^1(\mathbb{R})$ then the function h is defined as above is not in $H^1(\mathbb{R})$. Thus, the functions m, n defined as in (2.5) and (2.6) are not in $H^1(\mathbb{R})$ and we can not apply the Lemma 4.3 to construct solution of system (2.7).

3. Proof of Theorem 1.6

In this section, we give the proof of Theorem 1.6. We use the similar arguments as in proof of Theorem 1.1. We divide our proof into three steps:

Step 1. Preliminary analysis

Set

$$v := u_x + \frac{i}{2}|u|^2 u.$$

By elementary calculation as in the proof of Theorem 1.1 we see that if u solves (1.2) then (u, v) solves the following system:

$$\begin{cases} Lu = -iu^2\overline{v} + \left(\frac{1}{2} - b\right)|u|^4 u, \\ Lv = iv^2\overline{u} + \left(\frac{3}{2} - 3b\right)|u|^4 v + (1-2b)|u|^2 u^2 \overline{v}, \\ u\,|_{t=0} = u_0, \\ v\,|_{t=0} = v_0 = \partial u_0 + \frac{i}{2}|u_0|^2 u_0. \end{cases} \tag{3.1}$$

Define

$$P(u,v) = -iu^2\overline{v} + \left(\frac{1}{2} - b\right)|u|^4 u,$$
$$Q(u,v) = iv^2\overline{u} + \left(\frac{3}{2} - 3b\right)|u|^4 v + (1-2b)|u|^2 u^2 \overline{v}.$$

Let V be multi kink-soliton profile given as in (1.20). Set $q = u - V$. Since R_j solves (1.2), for all j, by elementary calculation, we have

$$iV_t + V_{xx} + i|V|^2 V_x + b|V|^4 V = i(V^2\overline{V_x} - \sum_{j=0}^{K} R_j^2 \overline{R_{jx}}) + b(|V|^4 V - \sum_{j=0}^{K} |R_j|^4 R_j). \tag{3.2}$$

Since Lemma 4.4, we have

$$\|V^2\overline{V_x} - \sum_{j=0}^{K} R_j^2\overline{R_{jx}}\|_{H^2} + \||V|^4 V - \sum_{j=0}^{K} |R_j|^4 R_j\|_{H^2} \leqslant e^{-\lambda t}, \tag{3.3}$$

for $\lambda = \frac{1}{8}v_*$. Thus, we rewrite (3.2) as follows

$$iV_t + V_{xx} + iV^2\overline{V_x} + b|V|^4 V = e^{-\lambda t} m(t,x), \tag{3.4}$$

where $m(t) \in H^2(\mathbb{R})$ such that $\|m(t)\|_{H^2}$ uniformly bounded in t. Define

$$h = V,$$
$$k = h_x + \frac{i}{2}|h|^2 h.$$

By elementary calculation as above, we have

$$Lh = -ih^2\overline{k} + \left(\frac{1}{2} - b\right)|h|^4 h + e^{-t\lambda} m = P(h,k) + e^{-t\lambda} m,$$
$$Lk = ik^2\overline{h} + \left(\frac{3}{2} - 3b\right)|h|^4 k + (1-2b)|h|^2 h^2 \overline{k} + e^{-t\lambda} n = Q(h,k) + e^{-t\lambda} n.$$

where $n = m_x + i|h|^2 m - \frac{i}{2}h^2\overline{m}$ satisfy $\|m(t)\|_{H^1}$ is uniformly bounded in t, by similar arguments in the proof of Theorem 1.1. Let $\tilde{u} = u - h$ and $\tilde{v} = v - k$. Then $\tilde{u}$, $\tilde{v}$ solves:

$$\begin{cases} L\tilde{u} = P(u,v) - P(h,k) - e^{-t\lambda} m, \\ L\tilde{v} = Q(u,v) - Q(h,k) - e^{-t\lambda} n. \end{cases} \tag{3.5}$$

Set $\eta = (\tilde{u}, \tilde{v})$, $W = (h, k)$ and $f(u, v) = (P(u, v), Q(u, v))$. We find solution of (3.5) of following form:

$$\eta = -i \int_t^\infty S(t-s)[f(W+\eta) - f(W) + H](s)\, ds, \tag{3.6}$$

where $H = e^{-t\lambda}(m, n)$. Moreover, since $v = u_x + \frac{i}{2}|u|^2 u$, we have

$$\tilde{v} = \tilde{u}_x + \frac{i}{2}(|\tilde{u} + h|^2(\tilde{u} + h) - |h|^2 h). \tag{3.7}$$

Step 2. Existence solution of system equations

Since Lemma 4.3, there exists $T_* \gg 1$ such that for $T_0 \gg T_*$ there exists unique solution η define on $[T_0, \infty)$ of (3.5) such that

$$e^{t\lambda}\|\eta\|_{S([t,\infty))\times S([t,\infty))} + e^{t\lambda}\|\eta_x\|_{S([t,\infty))\times S([t,\infty))} \leqslant 1, \quad \forall t \geqslant T_0, \tag{3.8}$$

for the constant $\lambda > 0$ defined as in **step 1**. Thus, for all $t \geqslant T_0$, we have

$$\|\tilde{u}\|_{H^1} + \|\tilde{v}\|_{H^1} \lesssim e^{-t\lambda}. \tag{3.9}$$

Step 3. Existence of multi kink-soliton for (1.2)

Using similar arguments in the proof of Theorem 1.1 we can prove that the solution $\eta = (\tilde{\varphi}, \tilde{\psi})$ of (3.5) satisfying the relation (3.7) under the condition (1.21). This implies that

$$\tilde{v} = \tilde{u}_x + \frac{i}{2}(|\tilde{u} + h|^2(\tilde{u} + h) - |h|^2 h).$$

Set $u = \tilde{u} + h$, $v = \tilde{v} + k$. We have

$$v = u_x + \frac{i}{2}|u|^2 u. \tag{3.10}$$

Since $(\tilde{u}, \tilde{v})$ solves (3.5), we have u, v solves

$$Lu = P(u, v),$$
$$Lv = Q(u, v).$$

Combining with (3.10) we have u solves (1.2). Moreover, for $t \geqslant T_0$, we have

$$\|u - V\|_{H^1} = \|\tilde{u}\|_{H^1} \lesssim e^{-\lambda t}.$$

This completes the proof of Theorem 1.6.

4. Some technical lemmas

4.1. **Properties of solitons.** In this section, we prove some estimates of soliton profile using in the proof of Theorem 1.1.

Lemma 4.1. *There exist $C > 0$ and a constant $\lambda > 0$ such that for t large enough, the estimate (2.3) is true uniformly in t.*

Proof. First, we need to some estimates on soliton profile. We have

$$|R_j(x,t)| = |\Phi_{\omega_j, c_j}(x - c_j t)| = \sqrt{2} h_j (\sqrt{c_j^2 + \gamma h_j^2} \cosh(h_j(x - c_j t)) - c_j)^{-\frac{1}{2}} \lesssim_{h_j, |c_j|} e^{\frac{-h_j}{2}|x - c_j t|}.$$

Moreover,

$$|\partial R_j(x,t)| = |\partial \phi_{\omega_j, c_j}(x - c_j t)| = \frac{-\sqrt{2}}{2} h_j^2 \sqrt{c_j^2 + \gamma h_j^2} \sinh(h_j(x - c_j t))(\sqrt{c_j^2 + \gamma h_j^2} \cosh(h_j(x - c_j t)) - c_j)^{-\frac{3}{2}}$$
$$\lesssim_{h_j, |c_j|} e^{\frac{-h_j}{2}|x - c_j t|}.$$

By elementary calculation, we have

$$|\partial^2 R_j(x,t)| + |\partial^3 R_j(x,t)| \lesssim_{h_j, |c_j|} e^{\frac{-h_j}{2}|x - c - jt|}.$$

Now, let us comeback to prove Lemma 4.1. For convenience, we set

$$\chi_1 = i|R|^2 R_x - i \sum_{j=1}^{K} |R_j|^2 R_{jx},$$

$$\chi_2 = |R|^4 R - \sum_{j=1}^{K} |R_j|^4 R_j.$$

Fix $t>0$. For each $x\in\mathbb{R}$, choose $m=m(x)\in\{1,2,...,K\}$ so that

$$|x-c_mt|=\min_j|x-c_jt|.$$

For $j\neq m$, we have

$$|x-c_jt|\leqslant\frac{1}{2}|c_jt-c_mt|=\frac{t}{2}|c_j-c_m|.$$

Thus, we have

$$\begin{aligned}&|(R-R_m)(x,t)|+|(\partial R-\partial R_m(x,t))|+|\partial^2R-\partial^2R_m|+|\partial^3R-\partial^3R_m|\\&\leqslant\sum_{j\neq m}(|R_j(x,t)|+|\partial R_j(x,t)|+|\partial^2R_j(x,t)|+|\partial^3R_j(x,t)|)\\&\lesssim_{h_1,..,h_K,|c_1|,..,|c_K|}\delta_m(x,t):=\sum_{j\neq m}e^{\frac{-h_j}{2}|x-c_jt|}\end{aligned}$$

Define

$$v_*=\inf_{j\neq k}h_j|c_j-c_k|.$$

We have

$$|(R-R_m)(x,t)|+|(\partial R-\partial R_m(x,t))|+|\partial^2R-\partial^2R_m|+|\partial^3R-\partial^3R_m|\lesssim\delta_m(x,t)\lesssim e^{\frac{-1}{4}v_*t}.$$

Let f_1,g_1,r_1 and f_2,g_2,r_2 be polynomials of u,u_x,u_{xx},u_{xxx} and conjugates such that for $u\in H^3(\mathbb{R})$:

$$\begin{aligned}i|u|^2u_x=f_1(u,\overline{u},u_x),&\quad|u|^4u=f_2(u,\overline{u}),\\\partial(i|u|^2u_x)=g_1(u,u_x,u_{xx},\overline{u},..),&\quad\partial(|u|^4u)=g_2(u,u_x,\overline{u},..),\\\partial^2(i|u|^2u_x)=r_1(u,u_x,u_{xx},u_{xxx},\overline{u},..),&\quad\partial^2(|u|^4u)=r_2(u,u_x,u_{xx},\overline{u},..).\end{aligned}$$

Denote

$$A=\sup_{|z|+|z_x|+|z_{xx}|+|z_{xxx}|\leqslant\|R\|H^4(\mathbb{R})}(|df_1|+|df_2|+dg_1|+|dg_2|+|dr_1|+|dr_2|),$$

where we denote by df the differential of f). We have

$$\begin{aligned}&|\chi_1|+|\chi_2|+|\partial\chi_1|+|\partial\chi_2|+|\partial^2\chi_1|+|\partial^2\chi_2|\\&\leqslant|f_1(R,R_x)-f_1(R_m,R_{mx})|+|f_2(R)-f_2(R_m)|+\sum_{j\neq m}(|f_1(R_j,R_{jx})|+|f_2(R_j)|)\\&\quad+|g_1(R,R_x,R_{xx},..)-g_1(R_m,R_{mx},R_{mxx},..)|+|g_2(R,R_x,..)-g_2(R_m,R_{mx},..)|\\&+\sum_{j\neq m}(g_1(R_j,R_{jx},R_{jxx},..)+g_2(R_j,R_{jx}),..)\\&\quad+|r_1(R,R_x,R_{xx},R_{xxx},..)-r_1(R_m,R_{mx},R_{mxx},R_{mxxx},..)|+|r_2(R,R_x,R_{xx},..)-r_2(R_m,R_{mx},R_{mxx},..)|\\&\quad+\sum_{j\neq m}(r_1(R_j,R_{jx},R_{jxx},R_{jxxx},..)+r_2(R_j,R_{jx},R_{jxx},..))\\&\leqslant A(|R-R_m|+|R_x-R_{mx}|+|R_{xx}-R_{mxx}|+|R_{xxx}-R_{mxxx}|)+\sum_{j\neq m}A(|R_j|+|R_{jx}|+|R_{jxx}|+|R_{jxxx}|)\\&\leqslant 2A\sum_{j\neq m}(|R_j|+|R_{jx}|+|R_{jxx}|+|R_{jxxx}|)\\&\leqslant 2A\delta_m(t,x).\end{aligned}$$

In particular,

$$\|\chi_1\|_{W^{2,\infty}}+\|\chi_2\|_{W^{2,\infty}}\lesssim e^{-\frac{1}{4}v_*t}.$$

Moreover, we have

$$\|\chi_1\|_{W^{2,1}} + \|\chi_2\|_{W^{2,1}}$$
$$\lesssim \sum_{j=1}^{K} (\||R_j|^2 R_{jx}\|_{L^1} + \|\partial(|R_j|^2 R_{jx})\|_{L^1} + \|\partial^2(|R_j|^2 R_{jx})\|_{L^1} + \|R_j^5\|_{L^1} + \|\partial(|R_j|^4 R_j)\|_{L^1} + \|\partial^2(|R_j|^4 R_j)\|_{L^1})$$
$$\lesssim \sum_{j=1}^{K} (\|R_j\|_{H^1}^3 + \|R_j\|_{H^2}^3 + \|R_j\|_{H^3}^3 + \|R_j\|_{H^1}^5 + \|R_j\|_{H^1}^5 + \|R_j\|_{H^2}^5) < C < \infty$$

By Holder inequality, for $1 < r < \infty$, we have

$$\|\chi_1\|_{W^{2,r}} + \|\chi_2\|_{W^{2,r}} \leqslant C e^{-(1-\frac{1}{r})\frac{1}{4}v_* t}, \quad \forall r \in (s, \infty).$$

Choosing $r = 2$ and $\lambda = \frac{1}{8}v_*$ we obtain the desired result.

□

4.2. **Prove the boundedness of v, m, n.** Let v given as (2.2) and m, n be given as (2.5) and (2.6). In this section, we give a proof of boundedness for $H^2(\mathbb{R})$-norm of v and $H^1(\mathbb{R})$-norm of m and n. We have the following lemma

Lemma 4.2. *The functions v, m, n satisfy*

$$\|v(t)\|_{H^2} + \|m(t)\|_{H^1} + \|n(t)\|_{H^1} \leqslant C,$$

uniformly on t, for some constant $C > 0$.

Proof. Let χ_1 be given function as in Lemma 4.1, we have

$$e^{-\lambda t} v = \chi_1$$

By Lemma 4.1, we have $\|v(t)\|_{H^2} \leqslant D$, for some constant $D > 0$. Since (2.5), we have

$$\|m\|_{H^2} \lesssim \|v\|_{H^2} + \|h\|_{H^2}\|v\|_{H^2}\|R\|_{H^2} \leqslant C_1,$$

for some constant $C_1 > 0$. Since, (2.6), we have

$$\|n\|_{L^2} \lesssim \|m_x\|_{L^2} + \|h\|_{H^1}^2 \|m\|_{H^1} \leqslant \|m\|_{H^1}(1 + \|h\|_{H^1}^2) \leqslant C_2,$$

for some constant $C_2 > 0$. Moreover, we have

$$\|n_x\|_{L^2} \lesssim \|m_{xx}\|_{L^2} + \|h\|_{H^1}^2 \|m\|_{H^1} \leqslant \|m\|_{H^2}(1 + \|h\|_{H^1}^2) \leqslant C_3,$$

for some constant $C_3 > 0$. Choosing $C = D + C_1 + C_2 + C_3$, we obtain the desired result. □

4.3. **Existence solution of system equation.** In this section, we prove the existence of solution of the following equation:

$$i\partial_t \eta + \partial_{xx}\eta = -[f(W + \eta) - f(W)] - H,$$

In Duhamel form,

$$\eta(t) = -i \int_t^\infty S(t-s)[f(W+\eta) - f(W) + H](s)\, ds, \tag{4.1}$$

where $f(\varphi, \psi) = (P(\varphi, \psi), Q(\varphi, \psi))$ and $P(\varphi, \psi) = i\varphi^2\overline{\psi} - b|\varphi|^4\varphi$, $Q(\varphi, \psi) = -i\psi^2\overline{\varphi} - 3b|\varphi|^4\psi - 2b|\varphi|^2\varphi^2\overline{\psi}$.

Lemma 4.3. *Let $H = H(t, x) : [0, \infty) \times \mathbb{R} \to \mathbb{C}^2$, $W = W(t, x) : [0, \infty) \times \mathbb{R} \to \mathbb{C}^2$ be given vector functions which satisfy for some $C_1 > 0$, $C_2 > 0$, $\lambda > 0$, $T_0 \geqslant 0$:*

$$\|W(t)\|_{L^\infty \times L^\infty} + e^{\lambda t}\|H(t)\|_{L^2 \times L^2} \leqslant C_1 \quad \forall t \geqslant T_0, \tag{4.2}$$

$$\|\partial W(t)\|_{L^2 \times L^2} + \|\partial W(t)\|_{L^\infty \times L^\infty} + e^{\lambda t}\|\partial H(t)\|_{L^2 \times L^2} \leqslant C_2, \quad \forall t \geqslant T_0. \tag{4.3}$$

Consider equation (4.1). There exists a constant λ_* independent of C_2 such that if $\lambda \geqslant \lambda_*$ then there exists a unique solution η to (4.1) on $[T_0, \infty) \times \mathbb{R}$ satisfying

$$e^{\lambda t}\|\eta\|_{S([t,\infty)) \times S([t,\infty))} + e^{\lambda t}\|\partial\eta\|_{S([t,\infty)) \times S([t,\infty))} \leqslant 1, \quad \forall t \geqslant T_0.$$

Proof. We use similarly arguments as in [12]. We write (4.1) as $\eta = \Phi\eta$. We shall show that, for λ sufficiently large, Φ is a contraction in the ball

$$B = \big\{\eta : \|\eta\|_X := e^{\lambda t}\|\eta\|_{S([t,\infty))\times S([t,\infty))} + e^{\lambda t}\|\partial\eta\|_{S([t,\infty))\times S([t,\infty))} \leqslant 1\big\}$$

Step 1. Prove Φ map B into B

Let $t \geqslant T_0$, $\eta = (\eta_1, \eta_2) \in B$, $W = (w_1, w_2)$ and $H = (h_1, h_2)$. By Strichartz estimates, we have

$$\|\Phi\eta\|_{S([t,\infty))\times S([t,\infty))} \lesssim \|f(W+\eta) - f(W)\|_{N([t,\infty))\times N([t,\infty))} \tag{4.4}$$
$$+ \|H\|_{L^1_\tau L^2_x([t,\infty))\times L^1_\tau L^2_x([t,\infty))}. \tag{4.5}$$

For (4.5), using (4.2), we have

$$\|H\|_{L^1_\tau L^2_x([t,\infty))\times L^1_\tau L^2_x([t,\infty))} = \|h_1\|_{L^1_\tau L^2_x([t,\infty))} + \|h_2\|_{L^1_\tau L^2_x([t,\infty))} \lesssim \int_t^\infty e^{-\lambda\tau}\,d\tau \leqslant \frac{1}{\lambda}e^{-\lambda t}. \tag{4.6}$$

For (4.4), we have

$$\begin{aligned}
|P(W+\eta) - P(W)| &= |P(w_1+\eta_1, w_2+\eta_2) - P(w_1, w_2)| \\
&\lesssim |(w_1+\eta_1)^2\overline{(w_2+\eta_2)} - w_1^2\overline{w_2}| + ||\eta_1+w_1|^4(\eta_1+w_1) - |w_1|^4 w_2| \\
&\lesssim |\eta_1| + |\eta_2| + |\eta_1|^5
\end{aligned}$$

Thus,

$$\begin{aligned}
\|P(W+\eta) - P(W)\|_{N([t,\infty))} &\lesssim \|\eta_1\|_{N([t,\infty))} + \|\eta_2\|_{N([t,\infty))} + \|\eta_1^5\|_{N([t,\infty))} \\
&\lesssim \|\eta_1\|_{L^1_\tau L^2_x(t,\infty)} + \|\eta_2\|_{L^1_\tau L^2_x(t,\infty)} + \|\eta_1^5\|_{L^1_\tau L^2_x(t,\infty)} \\
&\lesssim \int_t^\infty e^{-\lambda\tau}\,d\tau + \int_t^\infty \|\eta_1(\tau)\|^5_{L^10}\,d\tau \\
&\lesssim \frac{1}{\lambda}e^{-\lambda t} + \int_t^\infty \|\eta_1(\tau)\|^{\frac{7}{2}}_{L^2}\|\partial\eta_1(\tau)\|^{\frac{3}{2}}_{L^2} \\
&\lesssim \frac{1}{\lambda}e^{-\lambda t} + \int_t^\infty e^{-(7/2\lambda+3/2\lambda)\tau}\,d\tau \\
&\lesssim \frac{1}{\lambda}e^{-\lambda t} + \frac{1}{7/2\lambda+3/2\lambda}e^{-(7/2\lambda+3/2\lambda)t} \lesssim \frac{1}{\lambda}e^{-\lambda t}.
\end{aligned}$$

By similar arguments as above, we have

$$\|Q(W+\eta) - Q(W)\|_{N([t,\infty))} \lesssim \frac{1}{\lambda}e^{-\lambda t}.$$

Thus, for λ large enough, we have

$$\|\Phi\eta\|_{S([t,\infty)\times S([t,\infty)))} \leqslant \frac{1}{10}e^{-\lambda t}.$$

It remains to estimate $\|\partial\Phi\eta\|_{S([t,\infty)\times S([t,\infty)))}$. By Strichartz estimate we have

$$\|\partial\Phi\eta\|_{S([t,\infty)\times S([t,\infty)))} \lesssim \|\partial(f(W+\eta) - f(W))\|_{N([t,\infty))\times N([t,\infty))} \tag{4.7}$$
$$+ \|\partial H\|_{N([t,\infty))\times N([t,\infty))}. \tag{4.8}$$

For (4.8), using (4.3), we have

$$\begin{aligned}
\|\partial H\|_{N([t,\infty))\times N([t,\infty))} &\leqslant \|\partial h_1\|_{L^1_\tau L^2_x([t,\infty))} + \|\partial h_2\|_{L^1_\tau L^2_x([t,\infty))} \\
&\lesssim \int_t^\infty e^{-\lambda\tau}\,d\tau = \frac{1}{\lambda}e^{-\lambda t}.
\end{aligned} \tag{4.9}$$

For (4.7), we have

$$\|\partial(f(W+\eta) - f(W))\|_{N([t,\infty))\times N([t,\infty))} = \|\partial(P(W+\eta) - P(W))\|_{N([t,\infty))} + \|\partial(Q(W+\eta) - Q(W))\|_{N([t,\infty))}$$

Furthermore, using the notation (1.2) (3), we have

$$\begin{aligned}
|\partial(P(W+\eta) - P(W))| &\lesssim |\partial((w_1+\eta_1)^2\overline{(w_2+\eta_2)} - w_1^2\overline{w_2})| + |\partial(|w_1+\eta_1|^4(w_1+\eta_1) - |w_1|^4 w_1)| \\
&\lesssim |\partial\eta|(|\eta|^2 + |w|^2) + |\partial w|(|\eta|^2 + |w||\eta|) \\
&\quad + |\partial\eta|(|\eta|^4 + |w|^4) + |\partial w|(|\eta|^4 + |\eta||w|^3).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\|\partial(P(W+\eta)-P(W))\|_{N([t,\infty))}\\
&\lesssim \||\partial\eta|(|\eta|^2+|w|^2)\|_{N([t,\infty))}+\||\partial w|(|\eta|^2+|w||\eta|)\|_{N([t,\infty))} \qquad (4.10)\\
&\quad+\||\partial\eta|(|\eta|^4+|w|^4)\|_{N([t,\infty))}+\||\partial w|(|\eta|^4+|\eta||w|^3)\|_{N([t,\infty))}. \qquad (4.11)
\end{aligned}$$

For (4.10), using (4.2) and (4.3) and the assumption $\eta\in B$ we have

$$\begin{aligned}
&\||\partial\eta|(|\eta|^2+|w|^2)\|_{N([t,\infty))}+\||\partial w|(|\eta|^2+|w||\eta|)\|_{N([t,\infty))}\\
&\lesssim \||\partial\eta||\eta|^2\|_{L^1_\tau L^2_x([t,\infty))}+\||\partial\eta||w|^2\|_{L^1_\tau L^2_x([t,\infty))}+\||\partial w||\eta|^2\|_{L^1_\tau L^2_x([t,\infty))}+\||\partial w||w||\eta|\|_{L^1_\tau L^2_x([t,\infty))}\\
&\lesssim \||\partial\eta|\|_{L^2_\tau L^2_x([t,\infty))}\||\eta|\|^2_{L^4_\tau L^\infty}+\||\partial\eta|\|_{L^1_\tau L^2_x([t,\infty))}\||w|\|^2_{L^\infty L^\infty}\\
&\quad+\||\partial w|\|_{L^\infty L^\infty}\||\eta|\|_{L^4_\tau L^\infty_x([t,\infty))}\||\eta|\|_{L^{4/3}_\tau L^2_x([t,\infty))}+\||w|\|_{L^\infty L^\infty}\||\partial w|\|_{L^\infty L^\infty}\||\eta|\|_{L^1_\tau L^2_x([t,\infty))}\\
&\lesssim \frac{1}{\lambda}e^{-\lambda t}.
\end{aligned}$$

For (4.11), using (4.2) and (4.3) and the assumption $\eta\in B$ we have

$$\begin{aligned}
&\||\partial\eta|(|\eta|^4+|w|^4)\|_{N([t,\infty))}+\||\partial w|(|\eta|^4+|\eta||w|^3)\|_{N([t,\infty))}\\
&\lesssim \||\partial\eta|(|\eta|^4+|w|^4)\|_{L^1_\tau L^2_x([t,\infty))}+\||\partial w|(|\eta|^4+|\eta||w|^3)\|_{L^1_\tau L^2_x([t,\infty))}\\
&\lesssim \|\partial\eta\|_{L^\infty_\tau L^2_x([t,\infty))}\|\eta\|^4_{L^4_\tau L^\infty_x([t,\infty))}+\|w\|^4_{L^\infty L^\infty}\|\partial\eta\|_{L^1_\tau L^2_x([t,\infty))}\\
&\quad+\|\partial w\|_{L^\infty L^2}\|\eta\|^4_{L^4_\tau L^\infty_x([t,\infty))}+\|\partial w\|_{L^\infty L^\infty}\|w\|^3_{L^\infty L^\infty}\|\eta\|_{L^1_\tau L^2_x([t,\infty))}\\
&\lesssim \frac{1}{\lambda}e^{-\lambda t}.
\end{aligned}$$

Hence,

$$\|\partial(P(W+\eta)-P(W))\|_{N([t,\infty))}\lesssim\frac{1}{\lambda}e^{-\lambda t}. \qquad (4.12)$$

By similarly arguments, we have

$$\|\partial(Q(W+\eta)-Q(W))\|_{N([t,\infty))}\lesssim\frac{1}{\lambda}e^{-\lambda t}. \qquad (4.13)$$

Combining (4.12) and (4.13), we obtain

$$\|\partial(f(W+\eta)-f(W))\|_{N([t,\infty))\times N([t,\infty))}\lesssim\frac{1}{\lambda}e^{-\lambda t}. \qquad (4.14)$$

Combining (4.9) and (4.14), we obtain

$$\|\partial\Phi\eta\|_{S([t,\infty))\times S([t,\infty))}\lesssim\frac{1}{\lambda}e^{-\lambda t}\leqslant\frac{1}{10}e^{-\lambda t},$$

if $\lambda>0$ large enough. Thus, for $\lambda>0$ large enough

$$\|\Phi\eta\|_X\leqslant 1. \qquad (4.15)$$

Which implies that Φ map B onto B.

Step 2. Φ is contraction map on B

By using (4.2) and (4.3) and similarly estimate of (4.15), we can show that, for any $\eta\in B$, $\kappa\in B$,

$$\|\Phi\eta-\Phi\kappa\|_X\leqslant\frac{1}{2}\|\eta-\kappa\|_X.$$

By Banach fixed point theorem there exists unique solution on B of (4.1). □

4.4. **Properties of multi kink-soliton trains profile.** In this section, we prove some estimates of multi kink-soliton trains profile using in the proof of Theorem 1.6.

Lemma 4.4. *There exist $C>0$ and a constant $\lambda>0$ such that for t large enough, the estimate (3.3) is true uniformly in t.*

Proof. For convenience, set

$$R = \sum_{j=1}^{K} R_j.$$

By similar arguments as in the proof of Lemma 4.1, we have

$$|R_j(x,t)| + |\partial R_j(x,t)| + |\partial^2 R_j(x,t)| + |\partial^3 R_j(x,t)| \lesssim_{h_j,|c_j|} e^{\frac{-h_j}{2}|x-c-jt|},$$

for all $1 \leqslant j \leqslant K$. Now, let us comeback to prove Lemma 4.1. For convenience, we set

$$\chi_1 = iV^2\overline{V_x} - i\sum_{j=0}^{K} R_j^2\overline{R_{jx}},$$

$$\chi_2 = |v|^4 V - \sum_{j=0}^{K} |R_j|^4 R_j.$$

As in the proof of Lemma 4.1, we fix $t > 0$, take any $x \in \mathbb{R}$ and choose $m = m(x) \in \mathbb{N}$ such that

$$|x - c_m t| = \min_{j\in\mathbb{N}} |x - c_j t|.$$

If $m \geqslant 1$ then by the assumption $c_0 < c_j$ for $j > 0$ we have $x > c_0 t$. Thus, by asymptotic behaviour of Φ_0 as in Remark 1.7, we can see R_0 as a soliton. More precise, we have

$$|R_0(t,x)| + |R_0'(t,x)| + |R_0''(t,x)| + |R_0'''(t,x)| \lesssim e^{-\frac{1}{2}|x-c_0 t|} \lesssim e^{-\frac{1}{4}v_* t}.$$

Thus, by similar arguments as in the proof of Lemma 4.1, we have:

$$|(R - R_m)(x,t)| + |(\partial R - \partial R_m)(x,t)| + |(\partial^2 R - \partial^2 R_m)(x,t)| + |\partial^3 R - \partial^3 R_m| \lesssim e^{-\frac{1}{4}v_* t},$$

where

$$v_* = \inf_{j\neq k} h_j|c_j - c_k|.$$

Let f_1, g_1, r_1 and f_2, g_2, r_2 be polynomials of u, u_x, u_{xx}, u_{xxx} and conjugates such that for $u \in H^3(\mathbb{R})$:

$$\begin{aligned} iu^2\overline{u_x} &= f_1(u,\overline{u},u_x), & |u|^4 u &= f_2(u,\overline{u}),\\ \partial(iu^2\overline{u_x}) &= g_1(u,u_x,u_{xx},\overline{u},..), & \partial(|u|^4 u) &= g_2(u,u_x,\overline{u},..),\\ \partial^2(iu^2\overline{u_x}) &= r_1(u,u_x,u_{xx},u_{xxx},\overline{u},..), & \partial^2(|u|^4 u) &= r_2(u,u_x,u_{xx},\overline{u},..). \end{aligned}$$

Denote

$$A = \sup_{|z|+|z_x|+|z_{xx}|+|z_{xxx}|\leqslant \|R\|H^4(\mathbb{R})} (|df_1| + |df_2| + dg_1| + |dg_2| + |dr_1| + |dr_2|),$$

where we denote by df the differential of f). Therefore, if $m \geqslant 1$ we have

$$\begin{aligned}
&|\chi_1| + |\chi_2| + |\partial\chi_1| + |\partial\chi_2| + |\partial^2\chi_1| + |\partial^2\chi_2|\\
&\lesssim |R_0|^2|R_{0x}| + |R_0|^5 + A|R_0| + |f_1(R,R_x,\overline{R}) - \sum_{j=1}^{K} f_1(R_j,R_{jx},\overline{R_j})| + |f_2(R,\overline{R}) - \sum_{j=0}^{K} f_2(R_j,\overline{R_j})|\\
&\quad + |g_1(R,R_x,..) - \sum_{j=1}^{K} g_1(R_j,R_{jx},..)| + |g_2(R,R_x,..) - \sum_{j=0}^{K} g_2(R_j,R_{jx},..)| + |r_1(R,R_x,..)\\
&\quad - \sum_{j=0}^{K} r_1(R_j,R_{jx},..)| + |r_2(R,R_x,..) - \sum_{j=0}^{K} r_2(R_j,R_{jx},..)|\\
&\lesssim |R_0|^2|R_{0x}| + |R_0|^5 + A|R_0|\\
&\quad + A\sum_{j=1}^{K}(|(R-R_m)(x,t)| + |(\partial R - \partial R_m)(x,t)| + |(\partial^2 R - \partial^2 R_m)(x,t)| + |\partial^3 R - \partial^3 R_m|)\\
&\lesssim e^{-\frac{1}{4}v_* t},
\end{aligned}$$

If $m = 0$ we have

$$\begin{aligned}&|\chi_1| + |\chi_2| + |\partial\chi_1| + |\partial\chi_2| + |\partial^2\chi_1| + |\partial^2\chi_2|\\ &\lesssim A|R| + \sum_{j=1,v=1,2}^{K} (|f_v(R_j, R_{jx}, ..)| + |g_v(R_j, R_{jx}, ..)| + |r_v(R_j, R_{jx}, ..)|) \lesssim e^{-\frac{1}{4}v_* t}.\end{aligned}$$

This implies in all case we have

$$\|\chi_1(t)\|_{W^{2,\infty}} + \|\chi_2(t)\|_{W^{2,\infty}} \lesssim e^{-\frac{1}{4}v_* t}. \tag{4.16}$$

On one hand,

$$\begin{aligned}&\|\chi_1(t)\|_{W^{2,1}}\\ &\lesssim \sum_{j=0}^{K} (\|R_j^2\overline{R_{jx}}\|_{L^1} + \|\partial(R_j^2\overline{R_{jx}})\|_{L^1} + \|\partial^2(R_j^2\overline{R_{jx}})\|_{L^1})\\ &\lesssim \sum_{j=1}^{K} \|R_j\|_{H^3}^3 + \|\partial R_0\|_{W^{2,1}} < C < \infty.\end{aligned}$$

On the other hand,

$$\begin{aligned}&\|\chi_2(t)\|_{W^{2,1}}\\ &\lesssim \||V|^4V - |R_0|^4R_0\|_{W^{2,1}} + \sum_{j=1}^{K} \||R_j|^4R_j\|_{W^{2,1}}\\ &\lesssim \||R_0|^4 \sum_{j=1}^{K} |R_j| + |R_0| \sum_{j=1}^{K} |R_j|^4\|_{W^{2,1}} + \sum_{j=1}^{K} \|R_j\|_{H^3}^5\\ &\lesssim \sum_{j=1}^{K} (\|R_j\|_{W^{2,1}}(\|R_0\|_{W^{2,\infty}} + 1) + \|R_j\|_{H^3}^5) < C < \infty.\end{aligned}$$

Thus,

$$\|\chi_1(t)\|_{W^{2,1}} + \|\chi_1(t)\|_{W^{2,1}} < \infty. \tag{4.17}$$

Since (4.16) and (4.17), using Hölder inequality, we have

$$\|\chi_1(t)\|_{W^{2,2}} + \|\chi_2(t)\|_{W^{2.2}} \lesssim e^{-\frac{1}{8}v_* t}.$$

Set $\lambda = \frac{1}{8}v_*$. This completes the proof. □

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