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# CONSTRUCTION OF THE MULTI-SOLITONS OF A DERIVATIVE NONLINEAR SCHRÖDINGER EQUATION BY A FIXED POINT METHOD

PHAN VAN TIN

Abstract. We look for solutions to derivative nonlinear Schrodinger equations built upon ¨ solitons. We prove the existence of multi-solitons, i.e. solutions behaving at large time as the sum of finite solitons. Our proofs proceed by fixed point arguments around the desired profile, using Strichartz estimates.

Contents



## 1. Introduction

We consider the derivative nonlinear Schrödinger equation:

$$iu_t + u_{xx} + i|u|^2 u_x + b|u|^4 u = 0, \tag{1.1}$$

where $u : \mathbb{R}_t \times \mathbb{R}_x \to \mathbb{C}$.

The derivative nonlinear Schrödinger equation was originally introduced in Plasma Physics as a simplified model for Alfvén wave propagation. Since then, it has attracted a lot of attention from the mathematical community (see e.g [3, 4, 6, 7, 9, 10, 16, 17]).

The soliton of (1.1) is a profile of two parameters $\omega, c$. In case $b = 0$, Colin and Ohta [2] proved that the soliton $u_{\omega,c}$ is orbitally stable when $\omega > \frac{c^2}{4}$ by variational arguments. In case $b > 0$, Ohta[14] proved that for each $b > 0$ there exists unique $s^* = s^*(b) > 0 \in (0, 1)$ such that the soliton $u_{\omega,c}$ is orbitally stable if $-2\sqrt{\omega} < c < 2s^*\sqrt{\omega}$ and orbitally instable if $2s^*\sqrt{\omega} < c < 2\sqrt{\omega}$. In [5], Hayashi investigate the structure of (1.1) from the viewpoint solitons.

Our goal in this paper is to constructing the multi-solitons of (1.1) i.e solutions which behave asymptotically as the sum of finitely solitons. In [13], Le Coz and Wu proved stability of multi-solitons of (1.1) in case $b = 0$. For classical Schrödinger equation, in [11, 12] the authors constructed the infinite solitons train, infinite kink-soliton train using fixed point methods. In this paper, we prove the existence of multi-solitons of (1.1), using a fixed point method. Before state the main result, we give some preliminaries.

Consider equation (1.1). The soliton of equation (1.1) is solution of form $R_{\omega,c}(t, x) = e^{i\omega t}\phi_{\omega,c}(x - ct)$, where $\phi_{\omega,c} \in H^1(\mathbb{R})$ solves

$$-\phi_{xx} + \omega\phi + ic\phi_x - i|\phi|^2\phi_x - b|\phi|^4\phi = 0, \quad x \in \mathbb{R}. \tag{1.2}$$

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Applying the following gauge transform to $\phi_{\omega,c}$

$$\phi_{\omega,c}(x) = \Phi_{\omega,c}(x) \exp\left(i\frac{c}{2}x - \frac{i}{4}\int_{-\infty}^{x} |\Phi_{\omega,c}(y)|^2\, dy\right),$$

it is easily verified that $\Phi_{\omega,c}$ satisfies the following equation.

$$-\Phi_{xx} + \left(\omega - \frac{c^2}{4}\right)\Phi + \frac{c}{2}|\Phi|^2\Phi - \frac{3}{16}\gamma|\Phi|^4\Phi = 0, \quad \gamma := 1 + \frac{16}{3}b. \tag{1.3}$$

The positive even solution of (1.3) is explicitly obtained as follows; if $\gamma > 0$ or equivalently $b > \frac{-3}{16}$,

$$\Phi^2_{\omega,c}(x) = \begin{cases} \frac{2(4\omega-c^2)}{\sqrt{c^2+\gamma(4\omega-c^2)}\cosh(\sqrt{4\omega-c^2}x)-c} & \text{if } -2\sqrt{\omega} < c < 2\sqrt{\omega}, \\ \frac{4c}{(cx)^2+\gamma} & \text{if } c = 2\sqrt{\omega}, \end{cases}$$

if $\gamma \leqslant 0$ or equivalently $b \leqslant -\frac{3}{16}$,

$$\Phi^2_{\omega,c}(x) = \frac{2(4\omega - c^2)}{\sqrt{c^2+\gamma(4\omega-c^2)}\cosh(\sqrt{4\omega-c^2}x) - c} \text{ if } -2\sqrt{\omega} < c < -2s_*\sqrt{\omega},$$

where $s_* = s_*(\gamma) = \sqrt{\frac{-\gamma}{1-\gamma}}$. We note that the condition of two parameters $\gamma$ and $(\omega, c)$

$$\begin{aligned} &\text{if} \gamma > 0 \Leftrightarrow b > \frac{-3}{16}, -2\sqrt{\omega} < c \leqslant 2\sqrt{\omega}, \\ &\text{if} \gamma \leqslant 0 \Leftrightarrow b \leqslant \frac{-3}{16}, -2\sqrt{\omega} < c < -2s_*\sqrt{\omega}. \end{aligned}$$

is a necessary and sufficient condition for the existence of non-trivial solutions of (1.1) vanishing at infinity (see [1]). For each $j \in \{1, 2, .., K\}$, we set

$$R_j(t,x) = e^{i\theta_j} R_{\omega_j,c_j}(t, x - x_j).$$

The profile of an multisoliton is a sum of the form:

$$R = \sum_{j=1}^{K} R_j. \tag{1.4}$$

A solution of (1.1) is called multisoliton if, for some profile $R$:

$$u(t) - R(t) \to 0 \text{ as } t \to \infty,$$

in $H^1(\mathbb{R})$-norm. For convenience, we set $h_j = \sqrt{4\omega_j - c_j^2}$. Let $(c_j, \omega_j)$ be such that $-2\sqrt{\omega_j} < c_j < 2\sqrt{\omega_j}$ if $\gamma > 0$ or $-2\sqrt{\omega_j} < c_j < -2s_*\sqrt{\omega_j}$ if $\gamma \leqslant 0$, for all $1 \leqslant j \leqslant K$. We have

$$\Phi_{\omega_j,c_j}(x) = \sqrt{2}h_j(\sqrt{c_j^2 + \gamma h_j^2}\cosh(h_j x) - c_j)^{\frac{-1}{2}}. \tag{1.5}$$

As each soliton is in $H^\infty(\mathbb{R})$, we have $R \in H^\infty(\mathbb{R})$.

Our main result is the following.

**Theorem 1.1.** *Let $K \in \mathbb{N}^*$ and for each $1 \leqslant j \leqslant K$, $(\theta_j, c_j, \omega_j, x_j)$ be sequence of parameters such that $x_j = 0$, $\theta_j \in \mathbb{R}$, $c_j \neq c_k \neq 0$, for $j \neq k$ and $c_j$ such that $-2\sqrt{\omega_j} < c_j < 2\sqrt{\omega_j}$ if $\gamma > 0$ and $-2\sqrt{\omega_j} < c_j < -2s_*\sqrt{\omega_j}$ if $\gamma \leqslant 0$. The multisoliton profile $R$ is given as (1.4). We assume that the parameters $(\omega_j, c_j)$ satisfy*

$$\min\{1; \|R_x\|_{L^\infty L^\infty}\|R\|_{L^\infty L^\infty} + \|R\|^4_{L^\infty L^\infty}\} \ll v_* := \inf_{j\neq k} h_j|c_j - c_k|. \tag{1.6}$$

*There exist a solution $u$ to (1.1) such that*

$$\|u - R\|_{H^1} \leqslant Ce^{-\lambda t}. \quad \forall t \geqslant T_0, \tag{1.7}$$

*for some constant $C > 0$, $\lambda = \frac{1}{8}v_*$.*

The formula of soliton in case $\gamma > 0$ and $\gamma \leqslant 0$ is similar. Thus, from now on, we assume $\gamma > 0$. The case $\gamma \leqslant 0$ is treated by similar arguments.

*Remark* 1.2. In cases $\gamma > 0$, we give a example of parameters satisfy (1.6) as follows. First, chose $h_j \ll \min(|c_j|, 1)$ and $c_j < 0$ for all $j$. We have

$$\|\Phi_{\omega_j,c_j}\|_{L^\infty}^2 \leqslant \frac{2h_j^2}{\sqrt{c_j^2+\gamma h_j^2}-c_j} \lesssim \frac{h_j^2}{|c_j|}.$$

Moreover,

$$\partial\Phi_{\omega_j,c_j} = \frac{-\sqrt{2}}{2}h_j^2\sqrt{c_j^2+\gamma h_j^2}\sinh(h_jx)(\sqrt{c_j^2+\gamma h_j^2}\cosh(h_jx)-c_j)^{-\frac{3}{2}}.$$

Thus, choosing $c_j < 0$, for all $j$, we obtain

$$\begin{aligned}|\partial\Phi_{\omega_j,c_j}| &\lesssim h_j^2\sqrt{c_j^2+\gamma h_j^2}|\sinh(h_jx)|(\sqrt{c_j^2+\gamma h_j^2}\cosh(h_jx)-c_j)^{-\frac{3}{2}}\\ &\lesssim h_j^2(\sqrt{c_j^2+\gamma h_j^2}\cosh(h_jx)-c_j)^{-\frac{1}{2}}\\ &\approx h_j|\Phi_{\omega_j,c_j}| \lesssim \frac{h_j^2}{\sqrt{|c_j|}}.\end{aligned}$$

Moreover, we have

$$\begin{aligned}\|\partial R_j\|_{L^\infty} = \|\partial\phi_{\omega_j,c_j}\|_{L^\infty} &\approx \|\partial\Phi_{\omega_j,c_j}\|_{L^\infty} + \|\frac{c_j}{2}\Phi_{\omega_j,c_j} - \Phi_{\omega_j,c_j}^3\|_{L^\infty}\\ &\leqslant \|\partial\Phi_{\omega_j,c_j}\|_{L^\infty} + \frac{|c_j|}{2}\|\Phi_{\omega_j,c_j}\|_{L^\infty} + \|\Phi_{\omega_j,c_j}\|_{L^\infty}^3\\ &\lesssim \frac{h_j^2}{\sqrt{|c_j|}} + h_j\sqrt{|c_j|} + \frac{h_j^3}{\sqrt{|c_j|^3}}.\end{aligned}$$

Thus, we only need to chose the parameters $(c_j, \omega_j)$ satisfy

$$\sum_{1\leqslant j\leqslant K}\left(\frac{h_j^2}{\sqrt{|c_j|}} + h_j\sqrt{|c_j|} + \frac{h_j^3}{\sqrt{|c_j|^3}}\right)\sum_{1\leqslant j\leqslant K}\frac{h_j}{\sqrt{|c_j|}} + \sum_{1\leqslant j\leqslant K}\frac{h_j^4}{c_j^2} \ll \inf_{j\neq k} h_j|c_j - c_k|, \tag{1.8}$$

then the assumption (1.6) is satisfied. Let $M \gg 1$ be large enough positive constant. Replace $(c_j)$ by $Mc_j$, $h_j$ bounded (hence, $\omega_j = \frac{1}{4}(h_j^2 + M^2c_j^2)$) for all $j$. As $M \to \infty$, the right hand sight of (1.8) is order $M$ and the left side is order $M^0$. Thus, for $M$ large enough we obtain the parameters $(c_j, \omega_j)$ satisfy (1.6).

Our strategy is using a Gauge transform to give a system of two equations of $\varphi, \psi$ from equation of $u$ (1.1). Then, by fixed point method we prove that there exists a unique solution $\varphi, \psi$ of this system that decay exponential in time when $t$ is large enough. Using this property, we prove a relation between $\varphi, \psi$ which allow us to obtain a solution $u$ of (1.1). This solution satisfies the desired property. However, when we extend our result on construction of infinite solitons, we meet some problems on selecting parameters. The reason is that in case of finite parameters the condition $\|R\|_{H^3}$ is automatically true but in case of infinite parameters, this condition requires some estimate on parameters. It is not easy to select the parameters which satisfy this estimate and the assumption (1.6).

We introduce the following notations using in this paper.

**Notation.**

(1) *For $t > 0$, we note the Strichartz space $S([t,\infty))$ is defined via the norm*

$$\|u\|_{S([t,\infty))} = \sup_{(q,r)\ admissible} \|u\|_{L_t^qL_x^r([t,\infty)\times\mathbb{R})}$$

*The dual space is denoted by $N([t,\infty)) = S([t,\infty))^*$.*

(2) *For $z = (a,b) \in \mathbb{C}^2$ is a vector, we denote $|z| = |a| + |b|$.*

(3) *We denote $a \lesssim b$, for $a, b > 0$, if $a$ is smaller than $b$ up to multiply a positive constant. Moreover, we denote $a \approx b$ if $a$ equal to $b$ up to multiply a certain positive constant.*

(4) *Let $f \in C^1(\mathbb{R})$. We use $\partial f$ or $f_x$ to denote the derivative in space of the function $f$.*

## 2. Proof of Theorem 1.1

In this section, we give the proof of Theorem 1.1. Our strategy is using the Banach fixed point theorem and Strichartz estimates. We divide our proof into steps.

**Step 1. Preliminary analysis**

Considering the following transform:

$$\varphi(t,x)=\exp\left(\frac{i}{2}\int_{-\infty}^{x}|u(t,y)|^2\,dy\right)u(t,x),$$
$$\psi=\partial\varphi-\frac{i}{2}|\varphi|^2\varphi.$$

By similarly arguments as in [8] and [15], we see that if $u(t,x)$ solves (1.1) then $(\varphi,\psi)$ solves the following system

$$\begin{cases} L\varphi=i\varphi^2\overline{\psi}-b|\varphi|^4\varphi,\\ L\psi=-i\psi^2\overline{\varphi}-3b|\varphi|^4\psi-2b|\varphi|^2\varphi^2\overline{\psi},\\ \varphi\mid_{t=0}=\varphi_0=\exp\left(\frac{i}{2}\int_{-\infty}^{x}|u_0(y)|^2\,dy\right)u_0,\\ \psi\mid_{t=0}=\psi_0=\partial\varphi_0-\frac{i}{2}|\varphi_0|^2\varphi_0, \end{cases}\tag{2.1}$$

where $L=\partial_t+\partial_{xx}$. For convenience, we define

$$P(\varphi,\psi)=i\varphi^2\overline{\psi}-b|\varphi|^4\varphi,$$
$$Q(\varphi,\psi)=-i\psi^2\overline{\varphi}-3b|\varphi|^4\psi-2b|\varphi|^2\varphi^2\overline{\psi}.$$

Let $R$ be infinite soliton profile given as in (1.4). Set $q=u-R$. Since $R_j$ solves (1.1), for all $j$, by elementary calculation, we have

$$iR_t+R_{xx}+i|R|^2R_x+b|R|^4R=i(|R|^2R_x-\sum_{j=0}^{\infty}|R_j|^2R_{jx})+b(|R|^4R-\sum_{j=0}^{\infty}|R_j|^4R_j).\tag{2.2}$$

Since Lemma 3.1, we have

$$\||R|^2R_x-\sum_{j=0}^{\infty}|R_j|^2R_{jx}\|_{H^2}+\||R|^4R-\sum_{j=0}^{\infty}|R_j|^4R_j\|_{H^2}\leqslant e^{-\lambda t},\tag{2.3}$$

for $\lambda=\frac{1}{8}v_*$. Thus, we rewrite (2.2) as follows

$$iR_t+R_{xx}+i|R|^2R_x+b|R|^4R=e^{-i\lambda t}v(t,x),\tag{2.4}$$

where $v(t)\in H^2(\mathbb{R})$ such that $\|v(t)\|_{H^2}$ uniformly bounded in $t$.

Define

$$h=\exp\left(\frac{i}{2}\int_{-\infty}^{x}|R|^2\,dy\right)R(t,x),$$
$$k=h_x-\frac{i}{2}|h|^2h.$$

By elementary calculation as above, we have

$$Lh=ih^2\overline{k}-b|h|^4h+e^{-t\lambda}m(t,x),$$
$$Lk=-ik^2\overline{h}-3b|h|^4k-2b|h|^2h^2\overline{k}+e^{-t\lambda}n(t,x),$$

where $m(t),n(t)$ satisfy

$$m=v-h\int_{-\infty}^{x}\mathcal{I}m(v\overline{R})\,dy,\tag{2.5}$$

$$n=m_x-i|h|^2m+\frac{i}{2}h^2\overline{m}.\tag{2.6}$$

By Lemma 3.2 we have $\|m(t)\|_{H^1}+\|n(t)\|_{H^1}$ uniformly bounded in $t$. Let $\tilde{\varphi}=\varphi-h$ and $\tilde{\psi}=\psi-k$. Then $\tilde{\varphi}$, $\tilde{\psi}$ solve:

$$\begin{cases} L\tilde{\varphi}=P(\varphi,\psi)-P(h,k)-e^{-t\lambda}m(t,x),\\ L\tilde{\psi}=Q(\varphi,\psi)-Q(h,k)-e^{-t\lambda}n(t,x). \end{cases}\tag{2.7}$$

Set $\eta = (\tilde{\varphi}, \tilde{\psi})$, $W = (h, k)$ and $f(\varphi, \psi) = (P(\varphi, \psi), Q(\varphi, \psi))$. We find solution of (2.7) of following form:

$$\eta = -i \int_t^\infty S(t-s)[f(W+\eta) - f(W) + H](s)\, ds, \tag{2.8}$$

where $S(t-s)$ is Schrödinger group, $H = e^{-t\lambda}(m, n)$. Moreover, since $\psi = \partial\varphi - \frac{i}{2}|\varphi|^2\varphi$, we have

$$\tilde{\psi} = \partial\tilde{\varphi} - \frac{i}{2}(|\tilde{\varphi} + h|^2(\tilde{\varphi} + h) - |h|^2 h). \tag{2.9}$$

**Step 2. Existence solution of system equations**

Since Lemma 3.3, there exists $T_* \gg 1$ such that for $T_0 \geqslant T_*$ there exists unique solution $\eta$ define on $[T_0, \infty)$ of (2.7) such that

$$e^{t\lambda}(\|\eta\|_{S([t,\infty))\times S([t,\infty))}) + e^{t\lambda}(\|\eta_x\|_{S([t,\infty))\times S([t,\infty))}) \leqslant 1, \quad \forall t \geqslant T_0, \tag{2.10}$$

for the constant $\lambda > 0$ defined as in **step 1**. Thus, for all $t \geqslant T_0$, we have

$$\|\tilde{\varphi}\|_{H^1} + \|\tilde{\psi}\|_{H^1} \lesssim e^{-\lambda t}, \tag{2.11}$$

**Step 3. Existence of multisoliton for (1.1)**

We prove that the solution $\eta = (\tilde{\varphi}, \tilde{\psi})$ of (2.7) satisfying the relation (2.9). Indeed, let $\eta$ be solution of (2.7) which we find in step 1. Set $\varphi = \tilde{\varphi} + h$, $\psi = \tilde{\psi} + k$ and $v = \partial\varphi - \frac{i}{2}|\varphi|^2\varphi$. Since $h$ solves $Lh = P(h, k) + e^{-t\theta}m(t, x)$ and $\tilde{\varphi}$ solves $L\tilde{\varphi} = P(\varphi, \psi) - P(h, k) - e^{-t\theta}m(t, x)$, we have $(\varphi, \psi)$ solves

$$\begin{cases} L\varphi = P(\varphi, \psi), \\ L\psi = Q(\varphi, \psi). \end{cases} \tag{2.12}$$

By similarly arguments as above, we have

$$\begin{aligned} L\psi - Lv &= Q(\varphi, \psi) - \left(\partial L\varphi - \frac{i}{2}L(|\varphi|^2\varphi)\right) \\ &= Q(\varphi, \psi) - \left(\partial L\varphi - \frac{i}{2}(L(\varphi^2)\overline{\varphi} + \varphi^2 L(\overline{\varphi}) + 2\partial(\varphi^2)\partial\overline{\varphi})\right) \\ &= Q(\varphi, \psi) - \left(\partial L\varphi - \frac{i}{2}(2L\varphi|\varphi|^2 + 2(\partial\varphi)^2\overline{\varphi} - \varphi^2\overline{L\varphi} + 2\varphi^2\partial_{xx}\overline{\varphi}) + 2\varphi|\partial\varphi|^2)\right). \end{aligned} \tag{2.13}$$

Moreover, we have

$$\begin{aligned} L\varphi = P(\varphi, \psi) &= i\varphi^2\overline{\psi} - b|\varphi|^4\varphi \\ &= i\varphi^2\overline{(\psi - v)} + i\varphi^2\overline{v} - b|\varphi|^4\varphi. \end{aligned} \tag{2.14}$$

Combining (2.14) and (2.13) and by elementary calculation, we obtain

$$\begin{aligned} L\psi - Lv &= Q(\varphi, \psi) - \partial(i\varphi^2\overline{(\psi - v)}) - |\varphi|^2\varphi^2\overline{(\psi - v)} - \frac{1}{2}|\varphi|^4(\psi - v) - Q(\varphi, v) \\ &= (Q(\varphi, \psi) - Q(\varphi, v)) - 2i\varphi\partial\varphi\overline{(\psi - v)} - i\varphi^2\partial\overline{(\psi - v)} - |\varphi|^2\varphi^2\overline{(\psi - v)} - \frac{1}{2}|\varphi|^4(\psi - v) \\ &= -i(\psi^2 - v^2)\overline{\varphi} - 3b|\varphi|^4(\psi - v) - 2b|\varphi|^2\varphi^2\overline{(\psi - v)} - 2i\varphi\left(v + \frac{i}{2}|\varphi|^2\varphi\right)\overline{(\psi - v)} \\ &\quad - i\varphi^2\partial\overline{(\psi - v)} - |\varphi|^2\varphi^2\overline{(\psi - v)} - \frac{1}{2}|\varphi|^4(\psi - v). \end{aligned} \tag{2.15}$$

Define $\tilde{v} = v - k$. Since, $\tilde{\psi} - \tilde{v} = \psi - v$ and (2.15) we have

$$L\tilde{\psi} - L\tilde{v} = (\tilde{\psi} - \tilde{v})A(\tilde{\psi}, \tilde{v}, \tilde{\varphi}, h, k) + \overline{(\tilde{\psi} - \tilde{v})}B(\tilde{\psi}, \tilde{v}, \tilde{\varphi}, h, k) - i(\tilde{\varphi} + h)^2\partial\overline{(\tilde{\psi} - \tilde{v})}, \tag{2.16}$$

where

$$\begin{aligned} A &= -i(\tilde{\psi} + \tilde{v} + 2k)(\overline{\tilde{\varphi} + h}) - 3b|\tilde{\varphi} + h|^4 - \frac{1}{2}|\tilde{\varphi} + h|^4, \\ B &= -2b|\tilde{\varphi} + h|^2(\tilde{\varphi} + h)^2 - 2i(\tilde{\varphi} + h)\left(\tilde{v} + k + \frac{i}{2}|\tilde{\varphi} + h|^2(\tilde{\varphi} + h)\right) - |\tilde{\varphi} + h|^2(\tilde{\varphi} + h)^2. \end{aligned}$$

We see that $A, B$ are polynomials of degree at most 4 in $(\tilde{\psi}, \tilde{v}, \tilde{\varphi}, h, k)$. Multiplying two sides of (2.16) with $\overline{\tilde{\psi} - \tilde{v}}$, take imaginary part and integral over space using integral by part , we obtain

$$\begin{aligned}\frac{1}{2}\partial_t\|\tilde{\psi} - \tilde{v}\|_{L^2}^2 &= \mathcal{I}m\int_{\mathbb{R}}(\tilde{\psi} - \tilde{v})^2 A(\tilde{\psi}, \tilde{v}, \tilde{\varphi}, h, k) + \overline{(\tilde{\psi} - \tilde{v})}^2 B(\tilde{\psi}, \tilde{v}, \tilde{\varphi}, h, k) + \frac{i}{2}\partial(\tilde{\varphi} + h)^2\overline{(\tilde{\psi} - \tilde{v})}^2\, dx\\ &\lesssim \|\tilde{\psi} - \tilde{v}\|_{L^2}^2(\|A\|_{L^\infty} + \|B\|_{L^\infty} + \|\partial(\tilde{\varphi} + h)^2\|_{L^\infty}).\end{aligned}$$

By using Grönwall's inequality, we obtain

$$\|\tilde{\psi}(t) - \tilde{v}(t)\|_{L^2}^2 \lesssim \|\tilde{\psi}(N) - \tilde{v}(N)\|_{L^2}^2 \exp\left(\int_t^N (\|A\|_{L^\infty} + \|B\|_{L^\infty} + \|\partial(\tilde{\varphi} + h)^2\|_{L^\infty}\, ds\right). \quad (2.17)$$

Since (2.10) and (2.11) and bounded of $\|h\|_{H^4} + \|k\|_{H^4}$, for $t \geqslant T_0$, we have

$$\begin{aligned}&\int_t^N (\|A\|_{L^\infty} + \|B\|_{L^\infty} + \|\partial(\tilde{\varphi} + h)^2\|_{L^\infty})\, ds\\ &\lesssim \|\tilde{\varphi}\|_{L^1L^\infty((t,N\times\mathbb{R}))} + \|\tilde{\psi}\|_{L^1L^\infty((t,N)\times\mathbb{R})} + \|\partial\tilde{\varphi}\|_{L^1L^\infty((t,N)\times)\mathbb{R})}\\ &\quad + \|kh\|_{L^1L^\infty((t,N)\times\mathbb{R})} + \|h^4\|_{L^1L^\infty((t,N)\times\mathbb{R})} + \|\partial(h^2)\|_{L^1L^\infty((t,N)\times\mathbb{R})}\\ &\lesssim \|\tilde{\varphi}\|_{L^4L^\infty((t,N)\times\mathbb{R})}(N-t)^{\frac{3}{4}} + \|\tilde{\psi}\|_{L^4L^\infty((t,N)\times\mathbb{R})}(N-t)^{\frac{3}{4}}\\ &\quad + \|\partial\tilde{\varphi}\|_{L^4L^\infty((t,N)\times\mathbb{R})}(N-t)^{\frac{3}{4}} + (N-t)(\|h_xh\|_{L^\infty L^\infty} + \|h\|_{L^\infty L^\infty}^4)\\ &\lesssim (N-t)^{\frac{3}{4}}e^{-ct} + (N-t)(\|R_x\|_{L^\infty L^\infty}\|R\|_{L^\infty L^\infty} + \|R\|_{L^\infty L^\infty}^4).\end{aligned}$$

where we use $k = h_x - \frac{i}{2}|h|^2h$ and $\tilde{v} = \partial\tilde{\varphi} - \frac{i}{2}(|\tilde{\varphi}+h|^2(\tilde{\varphi}+h) - |h|^2h)$. Thus, from the assumption (1.6), we have $\|R_x\|_{L^\infty L^\infty}\|R\|_{L^\infty L^\infty} + \|R\|_{L^\infty L^\infty}^4 \ll \lambda$. Thus,

$$\begin{aligned}\|\tilde{\psi}(t) - \tilde{v}(t)\|_{L^2}^2 &\lesssim e^{-2\lambda N}\exp\left(C(N-t)^{\frac{3}{4}} + C(N-t)\|R\|_{W^{1,\infty}}\right)\\ &= \exp\left(C(N-t)^{\frac{3}{4}} + C(N-t)\|R\|_{W^{1,\infty}} - 2\lambda N\right).\end{aligned}$$

Let $N \to \infty$, we obtain that

$$\|\tilde{\psi}(t) - \tilde{v}(t)\|_{L^2}^2 = 0.$$

This implies that $\tilde{\psi} = \tilde{v}$, hence,

$$\psi = v = \partial\varphi - \frac{i}{2}|\varphi|^2\varphi. \quad (2.18)$$

Let $u = \exp\left(-\frac{i}{2}\int_{-\infty}^x |\varphi(y)|^2\, dy\right)\varphi$. Combining (2.18) with the fact that $(\varphi, \psi)$ solves

$$\begin{cases} L\varphi = P(\varphi, \psi),\\ L\psi = Q(\varphi, \psi),\end{cases}$$

we obtain $u$ solves (1.1). Moreover,

$$\begin{aligned}\|u - R\|_{H^1} &= \|\exp\left(-\frac{i}{2}\int_{-\infty}^x |\varphi(y)|^2\, dy\right)\varphi - \exp\left(-\frac{i}{2}\int_{-\infty}^x |h(y)|^2\, dy\right)h\|_{H^1}\\ &\lesssim \|\varphi - h\|_{H^1} = \|\tilde{\varphi}\|_{H^1}\end{aligned}$$

Combining (2.11), for $t \geqslant T_0$, we have

$$\|u - R\|_{H^1} \leqslant e^{-\lambda t}.$$

This complete the proof of Theorem 1.1.

## 3. Some technical lemmas

3.1. **Properties of solitons.** In this section, we prove some estimates of soliton profile using in the proof of the main theorem.

**Lemma 3.1.** *There exist $C > 0$ and a constant $\lambda > 0$ such that for $t$ large enough, the estimate (2.3) is true uniformly in $t$.*

*Proof.* First, we need to some estimates on soliton profile. We have

$$|R_j(x,t)| = |\Phi_{\omega_j,c_j}(x-c_jt)| = \sqrt{2}h_j(\sqrt{c_j^2+\gamma h_j^2}\cosh(h_j(x-c_jt))-c_j)^{-\frac{1}{2}} \lesssim_{h_j,|c_j|} e^{\frac{-h_j}{2}|x-c_jt|}.$$

Moreover,

$$|\partial R_j(x,t)| = |\partial\phi_{\omega_j,c_j}(x-c_jt)| = \frac{-\sqrt{2}}{2}h_j^2\sqrt{c_j^2+\gamma h_j^2}\sinh(h_j(x-c_jt))(\sqrt{c_j^2+\gamma h_j^2}\cosh(h_j(x-c_jt))-c_j)^{-\frac{3}{2}}$$
$$\lesssim_{h_j,|c_j|} e^{\frac{-h_j}{2}|x-c_jt|}.$$

By elementary calculation, we have

$$|\partial^2 R_j(x,t)| + |\partial^3 R_j(x,t)| \lesssim_{h_j,|c_j|} e^{\frac{-h_j}{2}|x-c-jt|}.$$

Now, let us comeback to prove Lemma 3.1. For convenience, we set

$$\chi_1 = i|R|^2R_x - i\sum_{j=0}^{K}|R_j|^2R_{jx},$$
$$\chi_2 = |R|^4R - \sum_{j=0}^{K}|R_j|^4R_j.$$

Fix $t>0$. For each $x\in\mathbb{R}$, choose $m=m(x)\in\{1,2,...,K\}$ so that

$$|x-c_mt| = \min_j|x-c_jt|.$$

For $j\neq m$, we have

$$|x-c_jt| \leqslant \frac{1}{2}|c_jt-c_mt| = \frac{t}{2}|c_j-c_m|.$$

Thus, we have

$$\begin{aligned}&|(R-R_m)(x,t)| + |(\partial R-\partial R_m(x,t))| + |\partial^2R-\partial^2R_m| + |\partial^3R-\partial^3R_m|\\ &\leqslant \sum_{j\neq m}(|R_j(x,t)| + |\partial R_j(x,t)| + |\partial^2R_j(x,t)| + |\partial^3R_j(x,t)|)\\ &\lesssim_{h_1,..,h_K,|c_1|,..,|c_K|} \delta_m(x,t) := \sum_{j\neq m} e^{\frac{-h_j}{2}|x-c_jt|}\end{aligned}$$

Define

$$v_* = \inf_{j\neq k} h_j|c_j-c_k|.$$

We have

$$|(R-R_m)(x,t)| + |(\partial R-\partial R_m(x,t))| + |\partial^2R-\partial^2R_m| + |\partial^3R-\partial^3R_m| \lesssim \delta_m(x,t) \lesssim e^{\frac{-1}{4}v_*t}.$$

Let $f_1,g_1,r_1$ and $f_2,g_2,r_2$ be polynomials of $u,u_x,u_{xx},u_{xxx}$ and conjugates such that for $u\in H^3(\mathbb{R})$:

$$\begin{aligned} i|u|^2u_x = f_1(u,\overline{u},u_x), &\quad |u|^4u = f_2(u,\overline{u}),\\ \partial(i|u|^2u_x) = g_1(u,u_x,u_{xx},\overline{u},..), &\quad \partial(|u|^4u) = g_2(u,u_x,\overline{u},..),\\ \partial^2(i|u|^2u_x) = r_1(u,u_x,u_{xx},u_{xxx},\overline{u},..), &\quad \partial^2(|u|^4u) = r_2(u,u_x,u_{xx},\overline{u},..).\end{aligned}$$

Denote

$$A = \sup_{|z|+|z_x|+|z_{xx}|+|z_{xxx}|\leqslant\|R\|H^4(\mathbb{R})} (|df_1|+|df_2|+dg_1|+|dg_2|+|dr_1|+|dr_2|),$$

where we denote by $df$ the differential of f). We have

$$
\begin{aligned}
&|\chi_1| + |\chi_2| + |\partial\chi_1| + |\partial\chi_2| + |\partial^2\chi_1| + |\partial^2\chi_2| \\
&\leqslant |f_1(R, R_x) - f_1(R_m, R_{mx})| + |f_2(R) - f_2(R_m)| + \sum_{j\neq m}(|f_1(R_j, R_{jx})| + |f_2(R_j)|) \\
&\quad + |g_1(R, R_x, R_{xx}, ..) - g_1(R_m, R_{mx}, R_{mxx}, ..)| + |g_2(R, R_x, ..) - g_2(R_m, R_{mx}, ..)| \\
&+ \sum_{j\neq m}(g_1(R_j, R_{jx}, R_{jxx}, ..) + g_2(R_j, R_{jx}), ..) \\
&\quad + |r_1(R, R_x, R_{xx}, R_{xxx}, ..) - r_1(R_m, R_{mx}, R_{mxx}, R_{mxxx}, ..)| + |r_2(R, R_x, R_{xx}, ..) - r_2(R_m, R_{mx}, R_{mxx}, ..)| \\
&\quad + \sum_{j\neq m}(r_1(R_j, R_{jx}, R_{jxx}, R_{jxxx}, ..) + r_2(R_j, R_{jx}, R_{jxx}, ..)) \\
&\leqslant A(|R - R_m| + |R_x - R_{mx}| + |R_{xx} - R_{mxx}| + |R_{xxx} - R_{mxxx}|) + \sum_{j\neq m} A(|R_j| + |R_{jx}| + |R_{jxx}| + |R_{jxxx}|) \\
&\leqslant 2A\sum_{j\neq m}(|R_j| + |R_{jx}| + |R_{jxx}| + |R_{jxxx}|) \\
&\leqslant 2A\delta_m(t, x).
\end{aligned}
$$

In particular,

$$\|\chi_1\|_{W^{2,\infty}} + \|\chi_2\|_{W^{2,\infty}} \lesssim e^{-\frac{1}{4}v_* t}.$$

Moreover, we have

$$
\begin{aligned}
&\sum_{j\in\mathbb{N}}(\|\chi_1\|_{W^{2,1}} + \|\chi_2\|_{W^{2,1}}) \\
&= \sum_{j\in\mathbb{N}}(\||R_j|^2 R_{jx}\|_{L^1} + \|\partial(|R_j|^2 R_{jx})\|_{L^1} + \|\partial^2(|R_j|^2 R_{jx})\|_{L^1} + \|R_j^5\|_{L^1} + \|\partial(|R_j|^4 R_j)\|_{L^1} + \|\partial^2(|R_j|^4 R_j)\|_{L^1}) \\
&\lesssim \sum_{j\in\mathbb{N}}(\|R_j\|_{H^1}^3 + \|R_j\|_{H^2}^3 + \|R_j\|_{H^3}^3 + \|R_j\|_{H^1}^5 + \|R_j\|_{H^1}^5 + \|R_j\|_{H^2}^5) < C < \infty
\end{aligned}
$$

By Holder inequality, for $1 < r < \infty$, we have

$$\|\chi_1\|_{W^{2,r}} + \|\chi_2\|_{W^{2,r}} \leqslant Ce^{-(1-\frac{1}{r})\frac{1}{4}v_* t}, \quad \forall r \in (s, \infty).$$

Choosing $r = 2$ and $\lambda = \frac{1}{8}v_*$ we obtain the desired result.

□

3.2. **Prove the boundedness of $v, m, n$.** Let $v$ given as (2.2) and $m, n$ be given as (2.5) and (2.6). In this section, we give a proof of boundedness for $H^2(\mathbb{R})$-norm of $v$ and $H^1(\mathbb{R})$-norm of $m$ and $n$. We have the following lemma

**Lemma 3.2.** *The functions $v, m, n$ satisfy*

$$\|v(t)\|_{H^2} + \|m(t)\|_{H^1} + \|n(t)\|_{H^1} \leqslant C,$$

*uniformly on $t$, for some constant $C > 0$.*

*Proof.* Let $\chi_1$ be given function as in Lemma 3.1, we have

$$e^{-\lambda t}v = \chi_1$$

By Lemma 3.1, we have $\|v(t)\|_{H^2} \leqslant D$, for some constant $D > 0$. Since (2.5), we have

$$\|m\|_{H^2} \lesssim \|v\|_{H^2} + \|h\|_{H^2}\|v\|_{H^2}\|R\|_{H^2} \leqslant C_1,$$

for some constant $C_1 > 0$. Since, (2.6), we have

$$\|n\|_{L^2} \lesssim \|m_x\|_{L^2} + \|h\|_{H^1}^2\|m\|_{H^1} \leqslant \|m\|_{H^1}(1 + \|h\|_{H^1}^2) \leqslant C_2,$$

for some constant $C_2 > 0$. Moreover, we have

$$\|n_x\|_{L^2} \lesssim \|m_{xx}\|_{L^2} + \|h\|_{H^1}^2\|m\|_{H^1} \leqslant \|m\|_{H^2}(1 + \|h\|_{H^1}^2) \leqslant C_3,$$

for some constant $C_3 > 0$. Choosing $C = D + C_1 + C_2 + C_3$, we obtain the desired result. □

3.3. **Existence solution of system equation.** In this section, we prove the existence of solution of the following equation:

$$i\partial_t\eta + \partial_{xx}\eta = -[f(W+\eta) - f(W)] - H,$$

In Duhamel form,

$$\eta(t) = -i\int_t^\infty S(t-s)[f(W+\eta) - f(W) + H](s)\,ds, \tag{3.1}$$

where $f(\varphi,\psi) = (P(\varphi,\psi), Q(\varphi,\psi))$ and $P(\varphi,\psi) = i\varphi^2\overline{\psi} - b|\varphi|^4\varphi$, $Q(\varphi,\psi) = -i\psi^2\overline{\varphi} - 3b|\varphi|^4\psi - 2b|\varphi|^2\varphi^2\overline{\psi}$.

**Lemma 3.3.** *Let $H = H(t,x) : [0,\infty)\times\mathbb{R}\to\mathbb{C}^2$, $W = W(t,x) : [0,\infty)\times\mathbb{R}\to\mathbb{C}^2$ be given vector functions which satisfy for some $C_1 > 0$, $C_2 > 0$, $\lambda > 0$, $T_0 \geqslant 0$:*

$$\|W(t)\|_{L^\infty\times L^\infty} + e^{\lambda t}\|H(t)\|_{L^2\times L^2} \leqslant C_1 \quad \forall t \geqslant T_0, \tag{3.2}$$

$$\|\partial W(t)\|_{L^2\times L^2} + \|\partial W(t)\|_{L^\infty\times L^\infty} + e^{\lambda t}\|\partial H(t)\|_{L^2\times L^2} \leqslant C_2, \quad \forall t \geqslant T_0. \tag{3.3}$$

*Consider equation (3.1). There exists a constant $\lambda_*$ independent of $C_2$ such that if $\lambda \geqslant \lambda_*$ then there exists a unique solution $\eta$ to (3.1) on $[T_0,\infty)\times\mathbb{R}$ satisfying*

$$e^{\lambda t}\|\eta\|_{S([t,\infty))\times S([t,\infty))} + e^{\lambda t}\|\partial\eta\|_{S([t,\infty))\times S([t,\infty))} \leqslant 1, \quad \forall t \geqslant T_0.$$

*Proof.* We use similarly arguments as in [11]. We write (3.1) as $\eta = \Phi\eta$. We shall show that, for $\lambda$ sufficiently large, $\Phi$ is a contraction in the ball

$$B = \left\{\eta : \|\eta\|_X := e^{\lambda t}\|\eta\|_{S([t,\infty))\times S([t,\infty))} + e^{\lambda t}\|\partial\eta\|_{S([t,\infty))\times S([t,\infty))} \leqslant 1\right\}$$

**Step 1. Prove $\Phi$ map $B$ into $B$**

Let $t \geqslant T_0$, $\eta = (\eta_1,\eta_2) \in B$, $W = (w_1,w_2)$ and $H = (h_1,h_2)$. By Strichartz estimates, we have

$$\|\Phi\eta\|_{S([t,\infty))\times S([t,\infty))} \lesssim \|f(W+\eta) - f(W)\|_{N([t,\infty))\times N([t,\infty))} \tag{3.4}$$

$$+ \|H\|_{L^1_\tau L^2_x([t,\infty))\times L^1_\tau L^2_x([t,\infty))}. \tag{3.5}$$

For (3.5), using (3.2), we have

$$\|H\|_{L^1_\tau L^2_x([t,\infty))\times L^1_\tau L^2_x([t,\infty))} = \|h_1\|_{L^1_\tau L^2_x([t,\infty))} + \|h_2\|_{L^1_\tau L^2_x([t,\infty))} \lesssim \int_t^\infty e^{-\lambda\tau}\,d\tau \leqslant \frac{1}{\lambda}e^{-\lambda t}. \tag{3.6}$$

For (3.4), we have

$$\begin{aligned}|P(W+\eta) - P(W)| &= |P(w_1+\eta_1, w_2+\eta_2) - P(w_1,w_2)|\\ &\lesssim |(w_1+\eta_1)^2\overline{(w_2+\eta_2)} - w_1^2\overline{w_2}| + ||\eta_1+w_1|^4(\eta_1+w_1) - |w_1|^4w_2|\\ &\lesssim |\eta_1| + |\eta_2| + |\eta_1|^5\end{aligned}$$

Thus,

$$\begin{aligned}\|P(W+\eta) - P(W)\|_{N([t,\infty))} &\lesssim \|\eta_1\|_{N([t,\infty))} + \|\eta_2\|_{N([t,\infty))} + \|\eta_1^5\|_{N([t,\infty))}\\ &\lesssim \|\eta_1\|_{L^1_\tau L^2_x(t,\infty)} + \|\eta_2\|_{L^1_\tau L^2_x(t,\infty)} + \|\eta_1^5\|_{L^1_\tau L^2_x(t,\infty)}\\ &\lesssim \int_t^\infty e^{-\lambda\tau}\,d\tau + \int_t^\infty \|\eta_1(\tau)\|_{L^1}^5\,d\tau\\ &\lesssim \frac{1}{\lambda}e^{-\lambda t} + \int_t^\infty \|\eta_1(\tau)\|_{L^2}^{\frac{7}{2}}\|\partial\eta_1(\tau)\|_{L^2}^{\frac{3}{2}}\\ &\lesssim \frac{1}{\lambda}e^{-\lambda t} + \int_t^\infty e^{-(7/2\lambda+3/2\lambda)\tau}\,d\tau\\ &\lesssim \frac{1}{\lambda}e^{-\lambda t} + \frac{1}{7/2\lambda+3/2\lambda}e^{-(7/2\lambda+3/2\lambda)t} \lesssim \frac{1}{\lambda}e^{-\lambda t}.\end{aligned}$$

By similar arguments as above, we have

$$\|Q(W+\eta) - Q(W)\|_{N([t,\infty))} \lesssim \frac{1}{\lambda}e^{-\lambda t}.$$

Thus, for $\lambda$ large enough, we have

$$\|\Phi\eta\|_{S([t,\infty)\times S([t,\infty)))} \leqslant \frac{1}{10}e^{-\lambda t}.$$

It remains to estimate $\|\partial\Phi\eta\|_{S([t,\infty)\times S([t,\infty)))}$. By Strichartz estimate we have

$$\|\partial\Phi\eta\|_{S([t,\infty)\times S([t,\infty)))} \lesssim \|\partial(f(W+\eta)-f(W))\|_{N([t,\infty))\times N([t,\infty))} \tag{3.7}$$
$$+ \|\partial H\|_{N([t,\infty))\times N([t,\infty))}. \tag{3.8}$$

For (3.8), using (3.3), we have

$$\|\partial H\|_{N([t,\infty))\times N([t,\infty))} \leqslant \|\partial h_1\|_{L^1_\tau L^2_x([t,\infty))} + \|\partial h_2\|_{L^1_\tau L^2_x([t,\infty))}$$
$$\lesssim \int_t^\infty e^{-\lambda\tau}\,d\tau = \frac{1}{\lambda}e^{-\lambda t}. \tag{3.9}$$

For (3.7), we have

$$\|\partial(f(W+\eta)-f(W))\|_{N([t,\infty))\times N([t,\infty))} = \|\partial(P(W+\eta)-P(W))\|_{N([t,\infty))} + \|\partial(Q(W+\eta)-Q(W))\|_{N([t,\infty))}$$

Furthermore, using the notation (1) (3), we have

$$\begin{aligned}|\partial(P(W+\eta)-P(W))| &\lesssim |\partial((w_1+\eta_1)^2\overline{(w_2+\eta_2)} - w_1^2\overline{w_2})| + |\partial(|w_1+\eta_1|^4(w_1+\eta_1)-|w_1|^4w_1)|\\ &\lesssim |\partial\eta|(|\eta|^2+|w|^2) + |\partial w|(|\eta|^2+|w||\eta|)\\ &\quad + |\partial\eta|(|\eta|^4+|w|^4) + |\partial w|(|\eta|^4+|\eta||w|^3).\end{aligned}$$

Thus, we have

$$\begin{aligned}&\|\partial(P(W+\eta)-P(W))\|_{N([t,\infty))}\\ &\lesssim \||\partial\eta|(|\eta|^2+|w|^2)\|_{N([t,\infty))} + \||\partial w|(|\eta|^2+|w||\eta|)\|_{N([t,\infty))} \qquad (3.10)\\ &\quad + \||\partial\eta|(|\eta|^4+|w|^4)\|_{N([t,\infty))} + \||\partial w|(|\eta|^4+|\eta||w|^3)\|_{N([t,\infty))}. \qquad (3.11)\end{aligned}$$

For (3.10), using (3.2) and (3.3) and the assumption $\eta\in B$ we have

$$\begin{aligned}&\||\partial\eta|(|\eta|^2+|w|^2)\|_{N([t,\infty))} + \||\partial w|(|\eta|^2+|w||\eta|)\|_{N([t,\infty))}\\ &\lesssim \||\partial\eta||\eta|^2\|_{L^1_\tau L^2_x([t,\infty))} + \||\partial\eta||w|^2\|_{L^1_\tau L^2_x([t,\infty))} + \||\partial w||\eta|^2\|_{L^1_\tau L^2_x([t,\infty))} + \||\partial w||w||\eta|\|_{L^1_\tau L^2_x([t,\infty))}\\ &\lesssim \|\|\partial\eta\|_{L^2_\tau L^2_x([t,\infty))}\|\eta\|^2_{L^4_\tau L^\infty} + \|\|\partial\eta\|_{L^1_\tau L^2_x([t,\infty))}\|w\|^2_{L^\infty L^\infty}\\ &\quad + \|\|\partial w\|\|_{L^\infty L^\infty}\|\|\eta\|\|_{L^4_\tau L^\infty_x([t,\infty))}\|\|\eta\|\|_{L^{4/3}_\tau L^2_x([t,\infty))} + \|\|w\|\|_{L^\infty L^\infty}\|\|\partial w\|\|_{L^\infty L^\infty}\|\|\eta\|\|_{L^1_\tau L^2_x([t,\infty))}\\ &\lesssim \frac{1}{\lambda}e^{-\lambda t}.\end{aligned}$$

For (3.11), using (3.2) and (3.3) and the assumption $\eta\in B$ we have

$$\begin{aligned}&\||\partial\eta|(|\eta|^4+|w|^4)\|_{N([t,\infty))} + \||\partial w|(|\eta|^4+|\eta||w|^3)\|_{N([t,\infty))}\\ &\lesssim \||\partial\eta|(|\eta|^4+|w|^4)\|_{L^1_\tau L^2_x([t,\infty))} + \||\partial w|(|\eta|^4+|\eta||w|^3)\|_{L^1_\tau L^2_x([t,\infty))}\\ &\lesssim \|\partial\eta\|_{L^\infty_\tau L^2_x([t,\infty))}\|\eta\|^4_{L^4_\tau L^\infty_x([t,\infty))} + \|w\|^4_{L^\infty L^\infty}\|\partial\eta\|_{L^1_\tau L^2_x([t,\infty))}\\ &\quad + \|\partial w\|_{L^\infty L^2}\|\eta\|^4_{L^4_\tau L^\infty_x([t,\infty))} + \|\partial w\|_{L^\infty L^\infty}\|w\|^3_{L^\infty L^\infty}\|\eta\|_{L^1_\tau L^2_x([t,\infty))}\\ &\lesssim \frac{1}{\lambda}e^{-\lambda t}.\end{aligned}$$

Hence,

$$\|\partial(P(W+\eta)-P(W))\|_{N([t,\infty))} \lesssim \frac{1}{\lambda}e^{-\lambda t}. \tag{3.12}$$

By similarly arguments, we have

$$\|\partial(Q(W+\eta)-Q(W))\|_{N([t,\infty))} \lesssim \frac{1}{\lambda}e^{-\lambda t}. \tag{3.13}$$

Combining (3.12) and (3.13), we obtain

$$\|\partial(f(W+\eta)-f(W))\|_{N([t,\infty))\times N([t,\infty))} \lesssim \frac{1}{\lambda}e^{-\lambda t}. \tag{3.14}$$

Combining (3.9) and (3.14), we obtain

$$\|\partial\Phi\eta\|_{S([t,\infty))\times S([t,\infty))} \lesssim \frac{1}{\lambda}e^{-\lambda t} \leqslant \frac{1}{10}e^{-\lambda t},$$

if $\lambda > 0$ large enough. Thus, for $\lambda > 0$ large enough

$$\|\Phi\eta\|_X \leqslant 1. \tag{3.15}$$

Which implies that $\Phi$ map $B$ onto $B$.

**Step 2.** *Phi* **is contraction map on** $B$

By using (3.2) and (3.3) and similarly estimate of (3.15), we can show that, for any $\eta \in B$, $\kappa \in B$,

$$\|\Phi\eta - \Phi\kappa\|_X \leqslant \frac{1}{2}\|\eta - \kappa\|_X.$$

By Banach fixed point theorem there exists unique solution on $B$ of (3.1). □

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