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## ► To cite this version:

Frédéric Bonnans, Guillaume Bonnet, Jean-Marie Mirebeau. A linear finite-difference scheme for approximating Randers distances on Cartesian grids. 2021. hal-03125879v1

**HAL Id: hal-03125879**

**<https://hal.science/hal-03125879v1>**

Preprint submitted on 29 Jan 2021 (v1), last revised 21 Mar 2022 (v3)

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# A linear finite-difference scheme for approximating Randers distances on Cartesian grids

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## Abstract

Using an extension of Varadhan’s formula to Randers manifolds, we notice that Randers distances may be approximated by a logarithmic transformation of a linear second-order partial differential equation. Following an idea introduced by Crane, Weischedel, and Wardetzky in the case of Riemannian distances, we study a numerical method for approximating Randers distances which involves a discretization of this linear equation. We propose to use Selling’s formula, which originates from the theory of low-dimensional lattice geometry, to build a monotone and linear finite-difference scheme. By injecting the logarithmic transformation in this linear scheme, we are able to prove convergence of this numerical method to the Randers distance, as well as consistency to the order two thirds far from the boundary of the considered domain. We explain how this method may be used to approximate optimal transport distances, how has been previously done in the Riemannian case.

## 1 Introduction

One variant of Varadhan’s formula [12] states that if  $x$  and  $y$  are two points of a Riemannian manifold  $\mathcal{M}$ , whose distance and Laplace-Beltrami operator we denote respectively by  $\text{dist}_{\mathcal{M}}$  and  $\Delta_{\mathcal{M}}$ , then

$$\text{dist}_{\mathcal{M}}(x, y) = -\lim_{t \rightarrow 0} \sqrt{t} \log(\mathcal{P}_t(x, y)),$$

where the kernel  $\mathcal{P}_t$  is such that  $g = \int_{\mathcal{M}} \mathcal{P}_t(\cdot, y) f(y) dy$  whenever  $f, g: \mathcal{M} \rightarrow \mathbb{R}$  satisfy  $(Id - t\Delta_{\mathcal{M}})g = f$ . Authors of [7] proposed to use Varadhan’s formula to approximate the gradient field of the distance function to a point on  $\mathcal{M}$ , and to recover the distance function itself as a postprocessing step, for better accuracy. In [11], Varadhan’s formula is used, together with Sinkhorn’s matrix scaling algorithm [8], to solve the optimal transport problem

$$\inf_{\gamma \in \Pi(\mu, \nu)} \int_{\mathcal{M} \times \mathcal{M}} \text{dist}_{\mathcal{M}}(x, y) d\gamma(x, y),$$

where  $\mu$  and  $\nu$  are probability measures on  $\mathcal{M}$  and  $\Pi(\mu, \nu)$  is the set of probability measures on  $\mathcal{M} \times \mathcal{M}$  whose first and second marginals coincide respectively with  $\mu$  and  $\nu$ . We extend those works to the case of Randers manifolds.

In section 2, we recall the definition of Randers distances and introduce an extension of Varadhan’s formula to Randers manifolds. We show that the logarithm of the solution of a particular linear second-order partial differential equation is approximately proportional to the corresponding Randers distance.

In section 3, we define a linear finite-difference scheme for this linear equation. We show that the result of applying the logarithmic transformation of section 2 to the solution of this scheme may be seen as the solution to another, nonlinear, scheme, whose we prove convergence and consistency with the

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Frédéric Bonnans acknowledges support from the Chair Finance & Sustainable Development and of the FiME Lab (Institut Europlace de Finance).

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This work was partly supported by ANR research grant MAGA, ANR-16-CE40-0014.

Hamilton-Jacobi equation of the Randers distance. We also explain how the techniques introduced in [7] to improve numerical results may apply to our case.

In section 4, following [11], we explain how the numerical scheme proposed in section 3 may be used in the numerical resolution of optimal transport problems.

In section 5, we illustrate our results by numerical experiments in which we approximate Randers distances to finite sets, the associated minimal paths, and the 1-Wasserstein distance on some Randers manifolds.

## 2 Setting

We denote by  $|\cdot|$  the Euclidean norm on  $\mathbb{R}^d$ , by  $B_d(0,1)$  the unit ball of  $\mathbb{R}^d$ , and, respectively, by  $\mathcal{S}_d$ ,  $\mathcal{S}_d^+$ , and  $\mathcal{S}_d^{++}$  the sets of symmetric, symmetric positive semidefinite, and symmetric positive definite matrices of size  $d$ . For any  $A \in \mathcal{S}_d^{++}$  and  $b \in \mathbb{R}^d$ , we define

$$|A| := \sup_{x \in B_d(0,1)} |Ax|, \quad |b|_A := \langle b, Ab \rangle^{1/2}.$$

For any bounded matrix field  $A: \mathbb{R}^d \rightarrow \mathcal{S}_d^{++}$  and bounded vector field  $b: \mathbb{R}^d \rightarrow \mathbb{R}^d$ , we define

$$\|A\|_\infty := \sup_{x \in \mathbb{R}^d} |A(x)|, \quad \|b\|_\infty := \sup_{x \in \mathbb{R}^d} |b(x)|.$$

From now on, we consider an open, bounded, connected, and nonempty domain  $\Omega \subset \mathbb{R}^d$  with a  $W^{3,\infty}$  boundary, a matrix field  $A: \mathbb{R}^d \rightarrow \mathcal{S}_d^{++}$ , and a vector field  $b: \mathbb{R}^d \rightarrow \mathbb{R}^d$ . We assume  $A$  and  $b$  are bounded, Lipschitz continuous, and that there is  $0 < \delta < 1$  such that for any  $x \in \mathbb{R}^d$ ,

$$|b(x)|_{A^{-1}(x)} \leq \delta. \quad (1)$$

We also define the matrix field  $A_b: \mathbb{R}^d \rightarrow \mathcal{S}_d$  by

$$A_b(x) := A(x) - b(x) \otimes b(x).$$

The following proposition applies to  $A(x)$ ,  $b(x)$ , and  $A_b(x)$  for any  $x \in \mathbb{R}^d$ , and in particular shows that  $A_b(x) \in \mathcal{S}_d^{++}$ :

**Proposition 2.1.** *Let  $A \in \mathcal{S}_d^{++}$  and  $b \in \mathbb{R}^d$  be such that  $|b|_{A^{-1}} < 1$ , and let  $A_b := A - b \otimes b$ . Then  $A_b \in \mathcal{S}_d^{++}$  and for any  $p \in \mathbb{R}^d$ ,*

$$\begin{aligned} |p|_A \leq 1 + \langle b, p \rangle &\iff |p|_{A_b}^2 \leq 1 + 2\langle b, p \rangle, \\ |p|_A \geq 1 + \langle b, p \rangle &\iff |p|_{A_b}^2 \geq 1 + 2\langle b, p \rangle. \end{aligned}$$

*Proof.* For any  $p \in \mathbb{R}^d \setminus \{0\}$ ,

$$\langle p, A_b p \rangle = |p|_A^2 - \langle b, p \rangle^2 = |p|_A^2 - \langle A^{-1/2} b, A^{1/2} p \rangle^2 \geq (1 - |b|_{A^{-1}}^2) |p|_A^2 > 0.$$

Thus  $A_b \in \mathcal{S}_d^{++}$  and for any  $p \in \mathbb{R}^d$ ,  $|p|_{A_b}^2 = |p|_A^2 - \langle b, p \rangle^2 \geq 0$ , hence  $|p|_A \geq |\langle b, p \rangle|$ . Then, still for any  $p \in \mathbb{R}^d$ ,

$$|p|_A \leq |1 + \langle b, p \rangle| \implies |\langle b, p \rangle| \leq |1 + \langle b, p \rangle| \implies 1 + \langle b, p \rangle > 0,$$

hence

$$|p|_A \leq |1 + \langle b, p \rangle| \implies |p|_A \leq 1 + \langle b, p \rangle, \quad (2)$$

$$|p|_A < |1 + \langle b, p \rangle| \implies |p|_A < 1 + \langle b, p \rangle. \quad (3)$$

Therefore, using (2) for the first equivalence,

$$\begin{aligned} |p|_A \leq 1 + \langle b, p \rangle &\iff |p|_A^2 \leq |1 + \langle b, p \rangle|^2 \\ &\iff |p|_A^2 \leq 1 + \langle b, p \rangle^2 + 2\langle b, p \rangle \\ &\iff |p|_{A_b}^2 \leq 1 + 2\langle b, p \rangle. \end{aligned}$$

This chain of equivalences is still true for the nonstrict converse inequality, using the contrapositive of (3).  $\square$

## 2.1 The Randers distance

For any  $x \in \overline{\Omega}$  and integrable function  $\alpha: \mathbb{R}_+ \rightarrow \mathbb{R}^d$ , we define the path  $\gamma^{x,\alpha}: \mathbb{R}_+ \rightarrow \mathbb{R}^d$  and the exit time  $\tau^{x,\alpha} \in \mathbb{R}_+$  by

$$\begin{aligned}\gamma^{x,\alpha}(t) &:= x + \int_0^t A^{1/2}(\gamma^{x,\alpha}(s))\alpha(s) + b(\gamma^{x,\alpha}(s)) ds, \\ \tau^{x,\alpha} &:= \inf \{t \geq 0 \mid \gamma^{x,\alpha}(t) \notin \overline{\Omega}\}.\end{aligned}$$

We denote by  $\mathcal{A}$  the set of measurable functions  $\alpha: \mathbb{R}_+ \rightarrow B_d(0,1)$ . Formally, the definition of  $\mathcal{A}$  imposes that for any  $x \in \overline{\Omega}$ ,  $\alpha \in \mathcal{A}$ , and  $t \in \mathbb{R}_+$ ,

$$|\dot{\gamma}^{x,\alpha}(t) - b(\gamma^{x,\alpha}(t))|_{A^{-1}(\gamma^{x,\alpha}(t))} \leq 1.$$

The *Randers distance*  $\text{dist}_{A,b}: \overline{\Omega} \times \overline{\Omega} \rightarrow \mathbb{R}$  associated to  $A$  and  $b$  on  $\overline{\Omega}$  is defined by

$$\text{dist}_{A,b}(x, y) := \inf \{t \geq 0 \mid \exists \alpha \in \mathcal{A}, \tau^{x,\alpha} \geq t \text{ and } \gamma^{x,\alpha}(t) = y\}. \quad (4)$$

The function  $\text{dist}_{A,b}$  is not symmetric, but it is nevertheless a quasidistance on  $\overline{\Omega}$ : for any  $x, y, z \in \overline{\Omega}$ ,

$$\text{dist}_{A,b}(x, y) = 0 \iff x = y, \quad \text{dist}_{A,b}(x, z) \leq \text{dist}_{A,b}(x, y) + \text{dist}_{A,b}(y, z).$$

Symmetry is replaced by the following property:

**Proposition 2.2.** *Let  $x, y \in \overline{\Omega}$ . Then  $\text{dist}_{A,b}(x, y) = \text{dist}_{A,-b}(y, x)$ .*

*Proof.* For any integrable function  $\hat{\alpha}: \mathbb{R}_+ \rightarrow \mathbb{R}^d$ , let us define  $\hat{\gamma}^{y,\hat{\alpha}}: \mathbb{R}_+ \rightarrow \mathbb{R}^d$  and  $\hat{\tau}^{y,\hat{\alpha}}$  by

$$\begin{aligned}\hat{\gamma}^{y,\hat{\alpha}}(t) &:= y + \int_0^t A^{1/2}(\hat{\gamma}^{y,\hat{\alpha}}(s))\hat{\alpha}(s) - b(\hat{\gamma}^{y,\hat{\alpha}}(s)) ds, \\ \hat{\tau}^{y,\hat{\alpha}} &:= \inf \{t \geq 0 \mid \hat{\gamma}^{y,\hat{\alpha}}(t) \notin \overline{\Omega}\}.\end{aligned}$$

Let  $t \geq 0$  and  $\alpha, \hat{\alpha} \in \mathcal{A}$  be such that  $\hat{\alpha}(s) = -\alpha(t-s)$  for any  $s \in [0, t]$ . If  $\gamma^{x,\alpha}(t) = y$ , then  $\gamma^{x,\alpha}(s) = \hat{\gamma}^{y,\hat{\alpha}}(t-s)$  for any  $s \in [0, t]$ , and if moreover  $\tau^{x,\alpha} \geq t$ , then  $\hat{\tau}^{y,\hat{\alpha}} \geq t$ . Reciprocally, if  $\hat{\gamma}^{y,\hat{\alpha}}(t) = x$ , then  $\hat{\gamma}^{y,\hat{\alpha}}(s) = \gamma^{x,\alpha}(t-s)$  for any  $s \in [0, t]$ , and if moreover  $\hat{\tau}^{y,\hat{\alpha}} \geq t$ , then  $\tau^{x,\alpha} \geq t$ . Therefore,

$$\begin{aligned}\text{dist}_{A,b}(x, y) &= \inf \{t \geq 0 \mid \exists \alpha \in \mathcal{A}, \tau^{x,\alpha} \geq t \text{ and } \gamma^{x,\alpha}(t) = y\} \\ &= \inf \{t \geq 0 \mid \exists \hat{\alpha} \in \mathcal{A}, \hat{\tau}^{y,\hat{\alpha}} \geq t \text{ and } \hat{\gamma}^{y,\hat{\alpha}}(t) = y\} \\ &= \text{dist}_{A,-b}(y, x).\end{aligned}$$

□

The following proposition provides an upper bound on  $\text{dist}_{A,b}$  on convex subdomains of  $\overline{\Omega}$ .

**Proposition 2.3.** *Let  $x, y \in \overline{\Omega}$ , and assume that  $(1-t)x + ty \in \overline{\Omega}$  for any  $t \in [0, 1]$ . Then*

$$\text{dist}_{A,b}(x, y) \leq (1-\delta)^{-1} \|A^{-1}\|_\infty^{1/2} |x - y|.$$

*Proof.* We define the measurable function  $\alpha: \mathbb{R}_+ \rightarrow \mathbb{R}^d$  by

$$\alpha(t) := A^{-1/2}(\gamma^{x,\alpha}(t)) \left( \frac{(1-\delta)(y-x)}{\|A^{-1}\|_\infty^{1/2} |x-y|} - b(\gamma^{x,\alpha}(t)) \right).$$

For any  $t \in \mathbb{R}_+$ , using (1) for the last inequality,

$$|\alpha(t)| = \left| \frac{(1-\delta)(y-x)}{\|A^{-1}\|_\infty^{1/2} |x-y|} - b(\gamma^{x,\alpha}(t)) \right|_{A^{-1}(\gamma^{x,\alpha}(t))}$$

$$\leq \frac{(1-\delta)}{\|A^{-1}\|_\infty^{1/2}} |x-y|_{A^{-1}(\gamma^{x,\alpha}(t))} + |b(\gamma^{x,\alpha}(t))|_{A^{-1}(\gamma^{x,\alpha}(t))} \leq 1.$$

Thus  $\alpha \in \mathcal{A}$ . We conclude by noticing that for any  $t \in \mathbb{R}_+$ ,

$$\gamma^{x,\alpha}(t(1-\delta)^{-1}\|A^{-1}\|_\infty^{1/2}|x-y|) = (1-t)x + ty.$$

□

Let  $g \in C(\partial\Omega)$ . In the following, it will be convenient to define

$$g_{\min} := \min_{x \in \partial\Omega} g(x), \quad g_{\max} := \max_{x \in \partial\Omega} g(x).$$

We aim to compute a numerical approximation of the function  $v: \overline{\Omega} \rightarrow \mathbb{R}$  defined by

$$v(x) := \inf_{y \in \partial\Omega} \text{dist}_{A,b}(x, y) + g(y). \quad (5)$$

In some applications, we consider domains  $\Omega$  whose boundary is the union of two disjoint close components  $\Gamma^1$  and  $\Gamma^2$  and functions  $g$  such that  $g(x) = 0$  on  $\Gamma^1$  and  $g(x) \geq \sup_{y \in \Gamma^1} \text{dist}_{A,b}(x, y)$  on  $\Gamma^2$ . In this case,  $v$  is the distance function to  $\Gamma^1$  on  $\overline{\Omega}$  according to the Randers metric.

Note that for any  $x \in \overline{\Omega}$  and  $y \in \partial\Omega$ ,

$$\text{dist}_{A,b}(x, y) = \inf \{ \tau^{x,\alpha} \mid \alpha \in \mathcal{A}, \gamma^{x,\alpha}(\tau^{x,\alpha}) = y \}.$$

Thus the definition of  $v$  may be rewritten as the optimal control problem

$$v(x) = \inf_{\alpha \in \mathcal{A}} \tau^{x,\alpha} + g(\gamma^{x,\alpha}(\tau^{x,\alpha})). \quad (6)$$

Using that, for any  $p \in \mathbb{R}^d$  and  $x \in \overline{\Omega}$ ,

$$\sup_{\alpha \in B_d(0,1)} -\langle A^{1/2}(x)\alpha + b(x), p \rangle = |p|_{A(x)} - \langle b(x), p \rangle,$$

we deduce — see Theorem 2.7 below — that this optimal control problem is associated to the following Hamilton-Jacobi equation:

$$\begin{cases} |Dv(x)|_{A(x)} - \langle b(x), Dv(x) \rangle - 1 = 0 & \text{in } \Omega, \\ v(x) - g(x) = 0 & \text{on } \partial\Omega. \end{cases} \quad (7)$$

## 2.2 Varadhan's formula on Randers manifolds

We propose to use the following extension of Varadhan's formula to Randers manifolds to approximate the function  $v: \overline{\Omega} \rightarrow \mathbb{R}$  defined by (5): for  $x \in \Omega$ ,

$$v(x) = -\lim_{\varepsilon \rightarrow 0} \varepsilon \log \left( \mathbb{E} \left[ \exp \left( -\tilde{\tau}^{x,\varepsilon} - \frac{g(\tilde{X}_{\tilde{\tau}^{x,\varepsilon}}^{x,\varepsilon})}{\varepsilon} \right) \right] \right), \quad (8)$$

where for  $\varepsilon > 0$ , the stochastic process  $(\tilde{X}_t^{x,\varepsilon})_{t \geq 0}$  and exit time  $\tilde{\tau}^{x,\varepsilon}$  are defined by

$$\begin{aligned} d\tilde{X}_t^{x,\varepsilon} &= 2\varepsilon b(\tilde{X}_t^{x,\varepsilon}) dt + \sqrt{2\varepsilon} A_b^{1/2}(\tilde{X}_t^{x,\varepsilon}) dW_t, \quad \tilde{X}_0^{x,\varepsilon} = x, \\ \tilde{\tau}^{x,\varepsilon} &:= \inf \{ t \geq 0 \mid \tilde{X}_t^{x,\varepsilon} \notin \Omega \}, \end{aligned}$$

and  $(W_t)_{t \geq 0}$  is a  $d$ -dimensional Wiener process. We prove formula (8) in Theorem 2.9 below.

For  $\varepsilon > 0$ , we define  $u_\varepsilon: \overline{\Omega} \rightarrow \mathbb{R}_-$  by

$$u_\varepsilon(x) = -\mathbb{E} \left[ \exp \left( -\tilde{\tau}^{x,\varepsilon} - \frac{g(\tilde{X}_{\tilde{\tau}^{x,\varepsilon}}^{x,\varepsilon})}{\varepsilon} \right) \right]. \quad (9)$$

According to the Feynman-Kac formula — see Theorem 2.8 below — the function  $u_\varepsilon$  is solution to the linear equation

$$\begin{cases} u_\varepsilon(x) - 2\varepsilon \langle b(x), Du_\varepsilon(x) \rangle - \varepsilon^2 \langle A_b(x), D^2 u_\varepsilon(x) \rangle = 0 & \text{in } \Omega, \\ u_\varepsilon(x) + \exp(-g(x)/\varepsilon) = 0 & \text{on } \partial\Omega. \end{cases} \quad (10)$$

### 2.3 Viscosity solutions

To study equations (7) and (17), we need to recall the definition of *viscosity solutions* to first- and second-order degenerate elliptic equations. An operator  $F: \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{S}_d \rightarrow \mathbb{R}$  is called *degenerate elliptic* if it is nonincreasing with respect to its last variable for the Loewner order (meaning that  $F(x, r, p, X_1) \leq F(x, r, p, X_2)$  whenever  $X_1 - X_2$  is positive semidefinite). The Dirichlet problem for a degenerate elliptic equation writes as

$$\begin{cases} F(x, u(x), Du(x), D^2u(x)) = 0 & \text{in } \Omega, \\ u(x) - \psi(x) = 0 & \text{on } \partial\Omega, \end{cases} \quad (11)$$

where  $\psi: \partial\Omega \rightarrow \mathbb{R}$ . For example when considering equation (7), one should choose

$$F(x, t, p, X) = |p|_{A(x)} - \langle b(x), p \rangle - 1, \quad \psi(x) = g(x).$$

For any bounded function  $u: \overline{\Omega} \rightarrow \mathbb{R}^d$ , we denote respectively by  $u^*: \overline{\Omega} \rightarrow \mathbb{R}$  and  $u_*: \overline{\Omega} \rightarrow \mathbb{R}$  its upper semicontinuous and lower semicontinuous envelopes, defined by

$$u^*(x) := \limsup_{y \in \overline{\Omega}, y \rightarrow x} u(y), \quad u_*(x) := \liminf_{y \in \overline{\Omega}, y \rightarrow x} u(y).$$

**Definition 2.4.** Let  $F: \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{S}_d \rightarrow \mathbb{R}$  be a continuous degenerate elliptic operator and  $\psi \in C(\partial\Omega)$ . A bounded function  $u: \overline{\Omega} \rightarrow \mathbb{R}$  is a *viscosity subsolution* to (11) if for any  $\varphi \in C^2(\overline{\Omega})$  and local maximum  $x \in \overline{\Omega}$  of  $u^* - \varphi$ ,

$$\begin{cases} F(x, u^*(x), D\varphi(x), D^2\varphi(x)) \leq 0 & \text{if } x \in \Omega, \\ (u^*(x) - \psi(x)) \wedge F(x, u^*(x), D\varphi(x), D^2\varphi(x)) \leq 0 & \text{if } x \in \partial\Omega. \end{cases}$$

It is a *viscosity supersolution* if for any  $\varphi \in C^2(\overline{\Omega})$  and local minimum  $x \in \overline{\Omega}$  of  $u_* - \varphi$ ,

$$\begin{cases} F(x, u_*(x), D\varphi(x), D^2\varphi(x)) \geq 0 & \text{if } x \in \Omega, \\ (u_*(x) - \psi(x)) \vee F(x, u_*(x), D\varphi(x), D^2\varphi(x)) \geq 0 & \text{if } x \in \partial\Omega. \end{cases}$$

It is a *viscosity solution* if it is both a viscosity subsolution and supersolution.

A well-known property of viscosity solutions is their stability under monotone changes of variables.

**Proposition 2.5.** Let  $F: \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{S}_d \rightarrow \mathbb{R}$  be a continuous degenerate elliptic operator,  $\psi \in C(\partial\Omega)$ ,  $I, J \subset \mathbb{R}$  be open intervals,  $\eta: I \rightarrow J$  be a strictly increasing  $C^2$ -diffeomorphism, and  $v: \overline{\Omega} \rightarrow I$  be bounded away from  $\partial I$ . Define the continuous degenerate elliptic operator  $G: \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{S}_d \rightarrow \mathbb{R}$  by

$$G(x, t, p, X) := F(x, \eta(t), \eta'(t)p, \eta''(t)p \otimes p + \eta'(t)X).$$

Then  $u := \eta \circ v$  is a viscosity subsolution (respectively supersolution) to (11) if and only if  $v$  is a viscosity subsolution (respectively supersolution) to

$$\begin{cases} G(x, v(x), Dv(x), D^2v(x)) = 0 & \text{in } \Omega, \\ v(x) - \eta^{-1}(\psi(x)) = 0 & \text{on } \partial\Omega. \end{cases} \quad (12)$$

*Proof.* We only show the result for subsolutions, since the case of supersolutions is similar. We assume that  $v$  is a subsolution to (12) and prove that  $u$  is a subsolution to (11). The proof of the converse is the same, using that

$$F(x, t, p, X) = G(x, \eta^{-1}(t), (\eta^{-1})'(t)p, (\eta^{-1})''(t)p \otimes p + (\eta^{-1})'(t)X).$$

Let  $\varphi \in C^2(\overline{\Omega})$  and  $x \in \overline{\Omega}$  be a local maximum of  $u^* - \varphi$ . Without loss of generality, we may assume that  $\varphi(\overline{\Omega}) \subset J$ . Let  $\tilde{\varphi} := \eta^{-1} \circ \varphi$ . Using that  $\eta$  is strictly increasing,  $u^* = (\eta \circ v)^* = \eta \circ v^*$ , and  $\varphi = \eta \circ \tilde{\varphi}$ , we deduce that  $x$  is a local maximum of  $v^* - \tilde{\varphi}$ . We conclude the proof by noticing that

$$F(x, u^*(x), D\varphi(x), D^2\varphi(x)) = G(x, v^*(x), D\tilde{\varphi}(x), D^2\tilde{\varphi}(x))$$

and, if  $x \in \partial\Omega$ , that  $u^*(x) - \psi(x)$  and  $v^*(x) - \eta^{-1}(\psi(x))$  have the same sign.  $\square$

Problems (7) and (10) admit the following *strong comparison principle*, which in particular implies that their viscosity solutions are uniquely determined on  $\Omega$  — though not on  $\partial\Omega$ .

**Theorem 2.6.** *Let  $\bar{v}, \underline{v}: \bar{\Omega} \rightarrow \mathbb{R}$  be respectively either a subsolution and a supersolution of (7), or a subsolution and a supersolution to (10), for  $\varepsilon > 0$ . Then*

$$\bar{v}^* \leq \underline{v}_* \quad \text{in } \Omega.$$

*Proof.* Note that since applications  $A$  and  $A_b$  are Lipschitz continuous and their lower eigenvalues are bounded away from zero,  $A^{1/2}$  and  $A_b^{1/2}$  are Lipschitz continuous too.

For problem (10), the result is a direct application of [3, Theorem 2.1], using that  $A_b^{1/2}: \mathbb{R}^d \rightarrow \mathcal{S}_d^{++}$  and  $b: \mathbb{R}^d \rightarrow \mathbb{R}^d$  are Lipschitz continuous,  $\partial\Omega$  is of class  $W^{3,\infty}$ , and  $g \in C(\partial\Omega)$ .

For problem (7), whose equation on  $\Omega$  has no zeroth order term, we perform the Kruzhkov change of variables. If  $\bar{v}$  and  $\underline{v}$  are respectively a viscosity subsolution and supersolution to (7), then by Proposition 2.5,  $\bar{u} := -\exp(-\bar{v})$  and  $\underline{u} := -\exp(-\underline{v})$  are respectively a viscosity subsolution and supersolution to

$$\begin{cases} |Du(x)|_{A(x)} - \langle b(x), Du(x) \rangle + u(x) = 0 & \text{in } \Omega, \\ u(x) + \exp(-g(x)) = 0 & \text{on } \partial\Omega. \end{cases}$$

Note that the equation in  $\Omega$  may be rewritten as

$$\sup_{\alpha \in B_d(0,1)} -\langle A^{1/2}(x)\alpha + b(x), Du(x) \rangle + u(x) = 0 \quad \text{in } \Omega.$$

Since  $\partial\Omega$  is of class  $W^{3,\infty}$ ,  $g \in C(\partial\Omega)$ , the function

$$\mathbb{R}^d \times B_d(0,1) \rightarrow \mathbb{R}^d, \quad (x, \alpha) \mapsto A^{1/2}(x)\alpha + b(x)$$

is continuous in  $\alpha$  and Lipschitz continuous in  $x$ , uniformly over  $\alpha$ , and for any  $x \in \mathbb{R}^d$  and  $p \in \mathbb{R}^d \setminus \{0\}$ , there are  $\alpha_1, \alpha_2 \in B_d(0,1)$  such that

$$\langle A^{1/2}(x)\alpha_1 + b(x), p \rangle < 0, \quad \langle A^{1/2}(x)\alpha_2 + b(x), p \rangle > 0,$$

(choose  $\{\alpha_1, \alpha_2\} = \{A^{-1/2}(x)(\pm(1-\delta)p/|p|_{A^{-1}(x)} - b(x))\}$ ), then by [3, Theorem 2.1],  $\bar{u}^* \leq \underline{u}_*$  in  $\Omega$ . Therefore  $\bar{v}^* \leq \underline{v}_*$  in  $\Omega$ .  $\square$

Viscosity solutions to (7) and (10) may be explicitly defined by (5) and (9), as proven by the following results:

**Theorem 2.7.** *The function  $v: \bar{\Omega} \rightarrow \mathbb{R}$  defined by (5) is a bounded viscosity solution to (7).*

*Proof.* The boundedness follows from Proposition 2.3: for any  $x \in \bar{\Omega}$ ,

$$v(x) \leq (1-\delta)^{-1} \|A^{-1}\|_\infty^{1/2} \text{diam}(\Omega) + \sup_{y \in \partial\Omega} g(y).$$

Since  $g \in C(\partial\Omega)$  and the function

$$\mathbb{R}^d \times B_d(0,1) \rightarrow \mathbb{R}^d, \quad (x, \alpha) \mapsto A^{1/2}(x)\alpha + b(x)$$

is continuous in  $\alpha$  and Lipschitz continuous in  $x$ , uniformly over  $\alpha$ , [1, Theorem V.4.13 and Remark V.4.14] implies that  $v$  is a viscosity solution to (7).  $\square$

**Theorem 2.8.** *For any  $\varepsilon > 0$ , the function  $u_\varepsilon: \bar{\Omega} \rightarrow \mathbb{R}_-$  defined by (9) is a viscosity solution to (10) and is bounded away from zero and  $-\infty$ .*

*Proof.* Since  $A_b^{1/2}: \mathbb{R}^d \rightarrow \mathcal{S}_d^{++}$  and  $b: \mathbb{R}^d \rightarrow \mathbb{R}^d$  are Lipschitz continuous,  $\partial\Omega$  is of class  $W^{3,\infty}$ , and  $g \in C(\partial\Omega)$ , [3, Theorem 3.1] implies that  $u_\varepsilon$  is a viscosity solution to (10).

We next prove the boundedness properties. For any  $p \in \mathbb{R}^d$ ,

$$\begin{aligned} & 1 + 2\langle b(x), p \rangle - |p|_{A_b(x)}^2 \\ &= 1 + 2\langle b(x), p \rangle + \langle b(x), p \rangle^2 - |p|_{A(x)}^2 \\ &\leq 1 + 2|b(x)|_{A^{-1}(x)}|p|_{A(x)} + (|b(x)|_{A^{-1}(x)}^2 - 1)|p|_{A(x)}^2 \\ &\leq 1 + 2\delta\|A\|_\infty^{1/2}|p| + (\delta^2 - 1)\|A^{-1}\|_\infty^{-1}|p|^2. \end{aligned} \tag{13}$$

Choose  $|p|$  large enough so that the right hand side of the above is nonpositive. For any  $\varepsilon > 0$ , define  $\bar{u}_\varepsilon, \underline{u}_\varepsilon: \bar{\Omega} \rightarrow \mathbb{R}_-$  by

$$\bar{u}_\varepsilon(x) := -\exp\left(-\frac{g_{\min}}{\varepsilon}\right), \quad \underline{u}_\varepsilon(x) := \sup_{y \in \partial\Omega} -\exp\left(\frac{\langle p, y - x \rangle - g(y)}{\varepsilon}\right).$$

Then  $\bar{u}_\varepsilon$  is a subsolution to (10). The definition of  $\underline{u}_\varepsilon$  may be rewritten as

$$\underline{u}_\varepsilon(x) = \left( \sup_{y \in \partial\Omega} -\exp\left(\frac{\langle p, y \rangle - g(y)}{\varepsilon}\right) \right) \exp\left(-\frac{\langle p, x \rangle}{\varepsilon}\right),$$

and  $\underline{u}_\varepsilon$  is a supersolution to (10), since for any  $x \in \Omega$ , using (13),

$$\begin{aligned} & \underline{u}_\varepsilon(x) - 2\varepsilon\langle b(x), D\underline{u}_\varepsilon(x) \rangle - \varepsilon^2\langle A_b(x), D^2\underline{u}_\varepsilon(x) \rangle \\ &= \underline{u}_\varepsilon(x) \left( 1 + 2\langle b(x), p \rangle - |p|_{A_b(x)}^2 \right) \geq 0. \end{aligned}$$

Thus by Theorem 2.6,

$$\begin{aligned} -\exp\left(-\frac{g_{\min}}{\varepsilon}\right) &= \bar{u}_\varepsilon \leq (u_\varepsilon)_* \leq (u_\varepsilon)^* \leq \underline{u}_\varepsilon \\ &\leq -\exp\left(-\frac{g_{\max} + |p|\operatorname{diam}(\Omega)}{\varepsilon}\right) \quad \text{on } \Omega. \end{aligned} \tag{14}$$

□

## 2.4 Asymptotic analysis

Formula (8) states that if the function  $u_\varepsilon: \bar{\Omega} \rightarrow \mathbb{R}_-$  is defined by (9) — and thus is a viscosity solution to (10) — for small  $\varepsilon > 0$ , then  $v_\varepsilon := -\varepsilon \log(-u_\varepsilon)$  approaches the function  $v: \bar{\Omega} \rightarrow \mathbb{R}$  defined by (5). According to Proposition 2.5,  $u_\varepsilon$  is a viscosity subsolution (respectively supersolution) to (10), bounded away from zero and  $-\infty$ , if and only if  $v_\varepsilon$  is a bounded viscosity subsolution (respectively supersolution) to

$$\begin{cases} |Dv_\varepsilon(x)|_{A_b(x)}^2 - 2\langle b(x), Dv_\varepsilon(x) \rangle - \varepsilon\langle A_b(x), D^2v_\varepsilon(x) \rangle - 1 = 0 & \text{in } \Omega, \\ v_\varepsilon(x) - g(x) = 0 & \text{on } \partial\Omega. \end{cases} \tag{15}$$

This suggests studying the limit equation

$$\begin{cases} |Dv(x)|_{A_b(x)}^2 - 2\langle b(x), Dv(x) \rangle - 1 = 0 & \text{in } \Omega, \\ v(x) - g(x) = 0 & \text{on } \partial\Omega. \end{cases} \tag{16}$$

But, by Proposition 2.1, problems (16) and (7) are equivalent, in the sense that they admit the same viscosity subsolutions and supersolutions. We are able to complete the proof of formula (8) by making rigorous the passing to the limit between problems (15) and (16). Note that we follow a standard sketch of proof, already used in [2, Proposition II.6] for example.

**Theorem 2.9.** *As  $\varepsilon > 0$  approaches zero, if  $u_\varepsilon: \bar{\Omega} \rightarrow \mathbb{R}_-$  is defined by (9), then  $v_\varepsilon := -\varepsilon \log(-u_\varepsilon)$  converges uniformly on compact subsets of  $\Omega$  to the viscosity solution  $v$  to (7) defined by (5).*



*Proof.* For any  $\varepsilon > 0$ , the function  $v_\varepsilon$  is a viscosity solution to (15). By (14), choosing  $p$  as in the proof of Theorem 2.8,

$$g_{\min} \leq (v_\varepsilon)_* \leq (v_\varepsilon)^* \leq g_{\max} + |p| \operatorname{diam}(\Omega) \quad \text{on } \Omega.$$

Thus the functions  $\bar{v}, \underline{v}: \bar{\Omega} \rightarrow \mathbb{R}$  defined by

$$\bar{v}(x) := \limsup_{\varepsilon \rightarrow 0, y \rightarrow x} v_\varepsilon(y), \quad \underline{v}(x) := \liminf_{\varepsilon \rightarrow 0, y \rightarrow x} v_\varepsilon(y)$$

are bounded uniformly over  $x$  and  $\varepsilon$ . Thus we can apply [6, Lemma 6.1 and Remark 6.3] to functions  $(v_\varepsilon)_*$  and  $(v_\varepsilon)^*$ , and deduce that  $\bar{v}$  and  $\underline{v}$  are respectively a viscosity subsolution and supersolution to (16), or equivalently to (7). Hence by Theorem 2.6,  $\bar{v} \leq v_* \leq v^* \leq \underline{v}$  on  $\Omega$ . By definition,  $\bar{v} \geq \underline{v}$  on  $\bar{\Omega}$ . Therefore  $\bar{v} = v = \underline{v}$  on  $\Omega$ . The locally uniform convergence of  $v_\varepsilon$  to  $v$  on  $\Omega$  follows from the definitions of  $\bar{v}$  and  $\underline{v}$ .  $\square$

### 3 The numerical scheme

For  $\varepsilon > 0$  and some discretization step  $h > 0$ , we propose to approximate the solution  $u_\varepsilon: \bar{\Omega} \rightarrow \mathbb{R}_-$  to (10) defined by (9) on the Cartesian grid  $\mathcal{G}_h := \Omega \cap h\mathbb{Z}^d$ . We define  $\bar{\mathcal{G}}_h := \mathcal{G}_h \cup \partial\Omega$  and for any  $x \in \mathcal{G}_h$  and  $e \in \mathbb{Z}^d$ , we let

$$\tilde{h}_{x,h}^e := \sup \{t \leq h \mid x + t'e \in \Omega, \forall t' \in [0, t]\},$$

so that  $x + \tilde{h}_{x,h}^e \in \bar{\mathcal{G}}_h$ . Notice that for any  $x \in \mathcal{G}_h$  and  $e \in \mathbb{Z}^d$ , there is  $h_0$  small enough so that  $\tilde{h}_{x,h}^e = h$  whenever  $h \leq h_0$ . For any  $x \in \mathcal{G}_h$ ,  $e \in \mathbb{Z}^d$ , and function  $u: \bar{\mathcal{G}}_h \rightarrow \mathbb{R}$ , we define the difference

$$\delta_{x,h}^e u := u(x + \tilde{h}_{x,h}^e e) - u(x).$$

We approximate  $u_\varepsilon$  by a solution  $u^h: \bar{\mathcal{G}}_h \rightarrow \mathbb{R}$  to a numerical scheme of the form

$$\begin{cases} u^h(x) - \sum_{e \in \mathbb{Z}^d} \eta_{x,h}^e \delta_{x,h}^e u^h = 0 & \text{in } \mathcal{G}_h, \\ u^h(x) + \exp(-g(x)/\varepsilon) = 0 & \text{on } \partial\Omega, \end{cases} \quad (17)$$

where  $(\eta_{x,h}^e)_{x \in \Omega, h > 0, e \in \mathbb{Z}^d}$  are nonnegative coefficients. Note that there is an implicit dependence on the parameter  $\varepsilon > 0$  in coefficients  $\eta_{x,h}^e$  and the solution  $u^h$  to the scheme. We drop the notation for this dependence since for now  $\varepsilon$  is fixed, and later we will choose  $\varepsilon$  as a function of  $h$ . We explain in the next section how to choose coefficients  $\eta_{x,h}^e$  so that the scheme (17) is consistent with equation (10), but let us first show that it is well-posed.

**Theorem 3.1.** *For any  $h > 0$  and  $\varepsilon > 0$ , there is a unique solution  $u^h: \bar{\mathcal{G}}_h \rightarrow \mathbb{R}$  to (17), and  $u^h$  takes nonpositive values.*

*Proof.* Let  $u^h: \bar{\mathcal{G}}_h \rightarrow \mathbb{R}$  and define  $U^h \in \mathbb{R}^{\mathcal{G}_h}$  by  $U_x^h := u^h(x)$ . Then  $u^h$  is a solution to (17) if and only if  $u^h = -\exp(-g/\varepsilon)$  on  $\partial\Omega$  and

$$M^h U^h = G^h, \quad (18)$$

where  $M^h \in \mathbb{R}^{\mathcal{G}_h \times \mathcal{G}_h}$  and  $G^h \in \mathbb{R}^{\mathcal{G}_h}$  are defined by

$$M_{x,y}^h := \begin{cases} 1 + \sum_{e \in \mathbb{Z}^d} \eta_{x,h}^e & \text{if } y = x, \\ -\eta_{x,h}^e & \text{if } x + \tilde{h}_{x,h}^e e = y \text{ for some } e \in \mathbb{Z}^d \setminus \{0\}, \\ 0 & \text{else.} \end{cases}$$

$$G_x^h := - \sum_{\substack{e \in \mathbb{Z}^d \\ x + \tilde{h}_{x,h}^e e \in \partial\Omega}} \eta_{x,h}^e \exp(-g(x + \tilde{h}_{x,h}^e e)/\varepsilon_h).$$

Since the matrix  $M^h$  has nonpositive extradiagonal elements and is strictly diagonally dominant, it is a nonsingular  $M$ -matrix. Therefore there is a unique solution  $U^h \in \mathbb{R}^{\mathcal{G}_h}$  to (18) and, since  $M^h U^h = G^h \leq 0 = M^h 0$ , where inequalities between vectors are taken elementwise, it follows that  $U^h \leq 0$ .  $\square$

### 3.1 Choice of coefficients

We will see that an appropriate choice of coefficients  $\eta_{x,h}^e$  is given by the formula

$$\eta_{x,h}^e := \frac{4\varepsilon^2 a_x^e}{\tilde{h}_{x,h}^e(\tilde{h}_{x,h}^e + \tilde{h}_{x,h}^{-e})} + \frac{2\varepsilon b_x^e}{\tilde{h}_{x,h}^e}, \quad (19)$$

where  $(a_x^e)_{x \in \Omega, e \in \mathbb{Z}^d}$  are nonnegative coefficients satisfying

$$a_x^e = a_x^{-e}, \quad A_b(x) = \sum_{e \in \mathbb{Z}^d} a_x^e e \otimes e, \quad (20)$$

and  $(b_x^e)_{x \in \Omega, e \in \mathbb{Z}^d}$  are real coefficients satisfying

$$b(x) = \sum_{e \in \mathbb{Z}^d} b_x^e e, \quad (21)$$

chosen so that  $\eta_{x,h}^e$  remain nonnegative. In this case, the first equation in (17) rewrites as

$$u^h(x) - 2\varepsilon \sum_{e \in \mathbb{Z}^d} b_x^e \frac{\delta_{x,h}^e u^h}{\tilde{h}_{x,h}^e} - \varepsilon^2 \sum_{e \in \mathbb{Z}^d} a_x^e \frac{2}{\tilde{h}_{x,h}^e + \tilde{h}_{x,h}^{-e}} \left( \frac{\delta_{x,h}^e u^h}{\tilde{h}_{x,h}^e} + \frac{\delta_{x,h}^{-e} u^h}{\tilde{h}_{x,h}^{-e}} \right) = 0 \quad \text{in } \mathcal{G}_h.$$

Replacing values of  $u^h$  by fourth-order Taylor expansions of  $u_\varepsilon$  in the above scheme, we see that it is first-order consistent with the first equation in (10), and that at points  $x \in \Omega$  such that  $\tilde{h}_{x,h}^e = \tilde{h}_{x,h}^{-e}$  whenever  $a_x^e \neq 0$  or  $b_x^e \neq 0$ , second-order consistency is achieved if for any  $e \in \mathbb{Z}^d$ ,

$$b_x^e = -b_x^{-e}. \quad (22)$$

For the scheme to be useable in practice, we need that at any point  $x \in \mathcal{G}_h$ , only finitely many coefficients  $a_x^e$  and  $b_x^e$  be nonzero, which may be written as

$$\sup \{|e| \mid e \in \mathbb{Z}^d, a_e(x) > 0 \text{ or } b_e(x) \neq 0\} < \infty. \quad (23)$$

We gather all requested properties of coefficients of the scheme, excluding (22), in the following assumption:

**Assumption 3.2.** *Nonnegative coefficients  $(\eta_{x,h}^e)_{x \in \Omega, h > 0, e \in \mathbb{Z}^d}$  are defined by (19), where coefficients  $a_x^e$  and  $b_x^e$  satisfy (20), (21), and (23).*

In dimensions two and three and for any  $x \in \mathcal{G}_h$ , we compute the nonnegative coefficients  $a_x^e$  satisfying properties (20) and (23) using *Selling's algorithm*, which we describe below.

**Theorem 3.3.** *Assume that  $d \in \{2, 3\}$  and let  $A \in \mathcal{S}_d^{++}$ . Then there exist algorithmically computable nonnegative coefficients  $(a^e)_{e \in \mathbb{Z}^d}$  satisfying*

$$a^e = a^{-e}, \quad A = \sum_{e \in \mathbb{Z}^d} a^e e \otimes e, \quad (24)$$

and

$$\max \{|e| \mid e \in \mathbb{Z}^d, a^e > 0\} \leq C_d |A|^{1/2} |A^{-1}|^{1/2}, \quad (25)$$

where  $C_2 := 2$  and  $C_3 := 2\sqrt{3}$ .

*Proof.* Let us recall the notion of *superbase* of  $\mathbb{Z}^d$ . A superbase of  $\mathbb{Z}^d$ , by definition, is a family  $v = (v_1, \dots, v_{d+1}) \in (\mathbb{Z}^d)^{d+1}$  satisfying

$$v_1 + \dots + v_{d+1} = 0, \quad \det(v_1, \dots, v_d) = \pm 1.$$

It is said to be *A-obtuse* if  $\langle v_i, Av_j \rangle \leq 0$ , for any  $1 \leq i < j \leq d+1$ .

For any superbase  $v$  of  $\mathbb{Z}^d$ , we define vectors  $(e_{ij})_{1 \leq i < j \leq d+1}$  by

$$\begin{aligned} e_{ij} &:= \pm v_k^\perp & \text{if } d = 2 \text{ and } \{i, j, k\} = \{1, 2, 3\}, \\ e_{ij} &:= \pm v_k \wedge v_l & \text{if } d = 3 \text{ and } \{i, j, k, l\} = \{1, 2, 3, 4\}, \end{aligned}$$

choosing the signs arbitrarily. Then  $(e_{ij} \otimes e_{ij})_{1 \leq i < j \leq d+1}$  is the dual basis in  $\mathcal{S}_d$  to  $((-v_i \otimes v_j - v_j \otimes v_i)/2)_{1 \leq i < j \leq d+1}$ , and *Selling's formula* holds:

$$A = - \sum_{1 \leq i < j \leq d+1} \langle v_i, Av_j \rangle e_{ij} \otimes e_{ij}.$$

Therefore coefficients  $(a^e)_{e \in \mathbb{Z}^d}$  defined by

$$a^e := \begin{cases} -\frac{1}{2} \langle v_i, Av_j \rangle & \text{if } e = \pm e_{ij}, 1 \leq i < j \leq d+1, \\ 0 & \text{else,} \end{cases}$$

satisfy (24) and are nonnegative as soon as the superbase  $v$  is  $A$ -obtuse.

We define coefficients  $(a^e)_{e \in \mathbb{Z}^d}$  using the  $A$ -obtuse superbase  $v$  returned by *Selling's algorithm*, which is initialized with an arbitrary superbase  $v^0$  of  $\mathbb{Z}^d$ , and at each step  $k \in \mathbb{N}$  of which the following operations are performed:

- If there is a permutation  $\sigma \in \mathfrak{S}_{d+1}$  such that  $\langle v_{\sigma(1)}^k, Av_{\sigma(2)}^k \rangle > 0$ , then choose

$$\begin{aligned} v^{k+1} &:= (-v_{\sigma(1)}^k, v_{\sigma(2)}^k, v_{\sigma(1)}^k - v_{\sigma(2)}^k) & \text{if } d = 2, \\ v^{k+1} &:= (-v_{\sigma(1)}^k, v_{\sigma(2)}^k, v_{\sigma(3)}^k + v_{\sigma(1)}^k, v_{\sigma(4)}^k + v_{\sigma(1)}^k) & \text{if } d = 3, \end{aligned}$$

and proceed with next step.

- Else, stop and return  $v^k$ , which is  $A$ -obtuse.

We refer to [10] or [5] for a proof that Selling's algorithm stops and to [9, Proposition 4.8] for a proof of (25).  $\square$

We study two versions of (10), corresponding to two ways of choosing coefficient  $b_x^e$  satisfying (21) and (23):

*The upwind scheme.* We denote by  $(e_1, \dots, e_d)$  the canonical basis of  $\mathbb{R}^d$ . For any  $x \in \mathcal{G}_h$  and  $e \in \mathbb{Z}^d$ , we choose

$$b_x^e := \begin{cases} \langle b(x), e \rangle^+ & \text{if } e = \pm e_i, 1 \leq i \leq d \\ 0 & \text{else.} \end{cases}$$

Then all coefficients  $b_x^e$  are nonnegative, and so are  $\eta_{x,h}^e$ .

*The centered scheme.* We use that for any  $x \in \mathcal{G}_h$ ,

$$b(x) = b(x)A_b^{-1}(x)A_b(x) = \sum_{e \in \mathbb{Z}^d} a_x^e \langle b(x), A_b^{-1}(x)e \rangle e$$

and choose  $b_x^e := a_x^e \langle b(x), A_b^{-1}(x)e \rangle$ . Since  $b_x^e$  may be negative, we have to check whether coefficients  $\eta_{x,h}^e$  remain nonnegative. By definition,

$$\begin{aligned} \eta_{x,h}^e &= \frac{4\varepsilon^2 a_x^e}{\tilde{h}_{x,h}^e (\tilde{h}_{x,h}^e + \tilde{h}_{x,h}^{-e})} + \frac{2\varepsilon b_x^e}{\tilde{h}_{x,h}^e} \\ &= \frac{2\varepsilon a_x^e}{\tilde{h}_{x,h}^e} \left( \frac{2\varepsilon}{\tilde{h}_{x,h}^e + \tilde{h}_{x,h}^{-e}} + \langle e, A_b^{-1}(x)b(x) \rangle \right). \end{aligned}$$

Let  $C_d$  be as in Theorem 3.3. Let us assume that  $|e| \leq C_d \|A_b\|_\infty^{1/2} \|A_b^{-1}\|_\infty^{1/2}$ , since otherwise  $a_x^e = 0$  and thus  $\eta_{x,h}^e = 0$ . Then

$$\frac{2\varepsilon}{\tilde{h}_{x,h}^e + \tilde{h}_{x,h}^{-e}} + \langle e, A_b^{-1}(x)b(x) \rangle \geq \frac{\varepsilon}{h} - C_d \|A_b\|_\infty^{1/2} \|A_b^{-1}\|_\infty^{3/2} \|b\|_\infty.$$

Therefore  $\eta_{x,h}^e$  is nonnegative as soon as

$$\varepsilon \geq C_d \|A_b\|_\infty^{1/2} \|A_b^{-1}\|_\infty^{3/2} \|b\|_\infty h.$$

We advocate to use the centered scheme, since coefficients  $b_x^e$  choosen this way satisfy (22), yielding a scheme (17) which is second-order consistent with (10) far from  $\partial\Omega$ .

### 3.2 Convergence

For  $\varepsilon > 0$ , let  $u_\varepsilon: \bar{\Omega} \rightarrow \mathbb{R}_-$  be defined by (9). Since by Theorem 2.9, the function  $v_\varepsilon := -\varepsilon \log(-u_\varepsilon)$  approaches the function  $v: \bar{\Omega} \rightarrow \mathbb{R}$  defined by (5) on  $\Omega$  as  $\varepsilon$  approaches 0, we propose to approximate  $v$  by  $v^h := -\varepsilon \log(-u^h)$ , where  $u^h: \bar{\mathcal{G}}_h \rightarrow \mathbb{R}_-$  is the solution to (17), for small values of the parameters  $h$  and  $\varepsilon$ . We study properties of  $v^h$  when  $h$  and  $\varepsilon$  simultaneously approach zero, so for any  $h > 0$ , we let  $\varepsilon_h > 0$  be the associated value of the  $\varepsilon$ . We consider the following choice of parameters  $\varepsilon_h$ :

**Assumption 3.4.** *There are  $C > 0$  and  $0 < r < 1$  such that for any  $h > 0$ ,  $\varepsilon_h = Ch^r$ .*

Let Assumption 3.2 hold and let  $h > 0$ ,  $u^h: \bar{\mathcal{G}}_h \rightarrow \mathbb{R}_*$ , and  $v^h := -\varepsilon_h \log(-u^h)$ . Then by replacing  $u^h$  by  $-\exp(-v^h/\varepsilon_h)$  in (17) and multiplying the equation in  $\mathcal{G}_h$  by  $\exp(v^h/\varepsilon_h)$ , we see that  $u^h$  is solution to (17) with  $\varepsilon = \varepsilon_h$  if and only if  $v^h$  is solution to

$$S_h(x, v^h(x), v^h) = 0 \quad \text{in } \bar{\mathcal{G}}_h, \quad (26)$$

where for  $h > 0$ ,  $x \in \bar{\mathcal{G}}_h$ ,  $t \in \mathbb{R}$ , and  $v: \bar{\mathcal{G}}_h \rightarrow \mathbb{R}$ ,

$$S_h(x, t, v) := \begin{cases} \sum_{e \in \mathbb{Z}^d} \eta_{x,h}^e (\exp((t - v(x + \tilde{h}_{x,h}^e e))/\varepsilon_h) - 1) - 1 & \text{if } x \in \mathcal{G}_h, \\ t - g(x) & \text{if } x \in \partial\Omega. \end{cases}$$

This equivalence is still true when replacing equalities by inequalities in (17) and (26). We see (26) as a nonlinear numerical scheme solved by  $v^h$ , and show that it is consistent with equation (16) when Assumption 3.4 holds.

**Theorem 3.5.** *Let Assumption 3.2 and Assumption 3.4 hold. Let  $\varphi \in C^\infty(\bar{\Omega})$ . Then for small  $h > 0$ ,*

$$S_h(x, \varphi(x), \varphi) = |D\varphi(x)|_{A_b(x)}^2 - 2\langle b(x), D\varphi(x) \rangle - 1 + O(h^{r \wedge 1 - r})$$

*uniformly over  $x \in \mathcal{G}_h$ . If moreover (22) holds, then on any compact subset  $X$  of  $\Omega$ ,*

$$S_h(x, \varphi(x), \varphi) = |D\varphi(x)|_{A_b(x)}^2 - 2\langle b(x), D\varphi(x) \rangle - 1 + O(h^{r \wedge 2 - 2r})$$

*uniformly over  $x \in X \cap \mathcal{G}_h$  for  $h$  small enough.*

*Proof.* For  $x \in \mathcal{G}_h$ ,

$$\begin{aligned} & S_h(x, \varphi(x), \varphi) \\ &= \sum_{e \in \mathbb{Z}^d} \eta_{x,h}^e (\exp(-\delta_{x,h}^e \varphi / \varepsilon_h) - 1) - 1 \\ &= \sum_{e \in \mathbb{Z}^d} \eta_{x,h}^e \left( \exp \left( -\frac{\tilde{h}_{x,h}^e}{\varepsilon_h} \langle e, D\varphi(x) \rangle + O \left( \frac{(\tilde{h}_{x,h}^e)^2}{\varepsilon_h} \right) \right) - 1 \right) - 1 \\ &= \sum_{e \in \mathbb{Z}^d} \eta_{x,h}^e \left( \sum_{i=1}^{\infty} (-1)^i \frac{(\tilde{h}_{x,h}^e)^i}{i! \varepsilon_h^i} \langle e, D\varphi(x) \rangle^i + O \left( \frac{(\tilde{h}_{x,h}^e)^2}{\varepsilon_h} \right) \right) - 1 \end{aligned}$$

$$\begin{aligned}
&= \sum_{e \in \mathbb{Z}^d} a_x^e \left( \sum_{i=1}^{\infty} (-1)^i \frac{4(\tilde{h}_{x,h}^e)^{i-1}}{i! \varepsilon_h^{i-2} (\tilde{h}_{x,h}^e + \tilde{h}_{x,h}^{-e})} \langle e, D\varphi(x) \rangle^i + O(\varepsilon_h) \right) \\
&\quad + 2 \sum_{e \in \mathbb{Z}^d} b_x^e \left( \sum_{i=1}^{\infty} (-1)^i \frac{(\tilde{h}_{x,h}^e)^{i-1}}{i! \varepsilon_h^{i-1}} \langle e, D\varphi(x) \rangle^i + O(h) \right) - 1 \\
&= \sum_{e \in \mathbb{Z}^d} a_x^e \left( \langle e, D\varphi(x) \rangle^2 + \frac{\tilde{h}_{x,h}^e - \tilde{h}_{x,h}^{-e}}{3\varepsilon_h} \langle e, D\varphi(x) \rangle^3 + O(h^{r \wedge 2 - 2r}) \right) \\
&\quad - 2 \sum_{e \in \mathbb{Z}^d} b_x^e \left( \langle e, D\varphi(x) \rangle - \frac{\tilde{h}_{x,h}^e}{2\varepsilon_h} \langle e, D\varphi(x) \rangle^2 + O(h^{1 \wedge 2 - 2r}) \right) - 1.
\end{aligned}$$

If (22) holds, we choose  $h$  small enough such that for any  $x \in X$ ,  $\tilde{h}_{x,h}^e = \tilde{h}_{x,h}^{-e}$  whenever  $a_x^e \neq 0$  or  $b_x^e \neq 0$ .  $\square$

Note that  $\max_{0 < r < 1} r \wedge 1 - r = 1/2$  and  $\max_{0 < r < 1} r \wedge 2 - 2r = 2/3$ , and those maxima are attained respectively at  $r = 1/2$  and  $r = 2/3$ .

Using consistency of the scheme (26) with (16), we are able to prove convergence of its solution to the function  $v: \bar{\Omega} \rightarrow \mathbb{R}$  defined by (5).

**Theorem 3.6.** *Let Assumption 3.2 and Assumption 3.4 hold. Then for small  $h > 0$ , the unique solution  $u^h: \bar{\mathcal{G}}_h \rightarrow \mathbb{R}_-$  to (17) for  $\varepsilon = \varepsilon_h$  takes negative values, and  $v^h := -\varepsilon_h \log(-u^h)$  converges uniformly on compact subsets of  $\Omega$  as  $\varepsilon$  approaches zero to the viscosity solution  $v$  to (7) defined by (5).*

*Proof.* We use the following properties of the numerical scheme (26):

*Monotonicity.* For any  $h > 0$ ,  $x \in \bar{\mathcal{G}}_h$ ,  $t \in \mathbb{R}$ , and functions  $\bar{v}, \underline{v}: \bar{\mathcal{G}}_h \rightarrow \mathbb{R}$  such that  $\bar{v} \leq \underline{v}$  on  $\bar{\mathcal{G}}_h$ , it holds that

$$S_h(x, t, \bar{v}) \geq S_h(x, t, \underline{v}).$$

*Consistency.* Let  $\varphi \in C^\infty(\bar{\Omega})$ . For any  $h > 0$ ,  $x \in \bar{\Omega}$ , and  $\xi \in \mathbb{R}$ ,

$$S_h(x, \varphi(x) + \xi, \varphi + \xi) = S_h(x, \varphi(x), \varphi).$$

Thus, by Theorem 3.5 and since the application

$$x \mapsto |D\varphi(x)|_{A_b(x)}^2 - 2\langle b(x), D\varphi(x) \rangle - 1$$

is uniformly continuous over  $x \in \bar{\Omega}$ , it holds that for any  $x \in \bar{\Omega}$ ,

$$\begin{aligned}
&\limsup_{\substack{h \rightarrow 0, \xi \rightarrow 0 \\ y \in \bar{\mathcal{G}}_h, y \rightarrow x}} S_h(y, \varphi(y) + \xi, \varphi + \xi) \\
&\leq \begin{cases} |D\varphi(x)|_{A_b(x)}^2 - 2\langle b(x), D\varphi(x) \rangle - 1 & \text{if } x \in \Omega, \\ (\varphi(x) - g(x)) \vee (|D\varphi(x)|_{A_b(x)}^2 - 2\langle b(x), D\varphi(x) \rangle - 1) & \text{if } x \in \partial\Omega, \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
&\liminf_{\substack{h \rightarrow 0, \xi \rightarrow 0 \\ y \in \bar{\mathcal{G}}_h, y \rightarrow x}} S_h(y, \varphi(y) + \xi, \varphi + \xi) \\
&\geq \begin{cases} |D\varphi(x)|_{A_b(x)}^2 - 2\langle b(x), D\varphi(x) \rangle - 1 & \text{if } x \in \Omega, \\ (\varphi(x) - g(x)) \wedge (|D\varphi(x)|_{A_b(x)}^2 - 2\langle b(x), D\varphi(x) \rangle - 1) & \text{if } x \in \partial\Omega. \end{cases}
\end{aligned}$$

*Stability.* Let us prove that  $u^h$  takes negative values — so that  $v^h$  is well-defined — and that  $v^h$  is uniformly bounded over  $h$  and its argument  $x$ . By (13), we may choose  $p \in \mathbb{R}^d$  with norm large enough so that

$$|p|_{A_b(x)}^2 - 2\langle b(x), p \rangle - 1 > 0 \quad (27)$$

uniformly over  $x \in \bar{\Omega}$ . Let us define  $\bar{w}, \underline{w}: \bar{\mathcal{G}}_h \rightarrow \mathbb{R}$  by

$$\bar{w}(x) := g_{\min}, \quad \underline{w}(x) := \langle p, x \rangle + \sup_{y \in \partial\Omega} (g(y) - \langle p, y \rangle).$$

Then  $S^h(x, \bar{w}(x), \bar{w}) \leq 0$  on  $\bar{\mathcal{G}}_h$  and since  $h$  is assumed to be small, by Theorem 3.5 and (27),  $S^h(x, \underline{w}(x), \underline{w}) \geq 0$  on  $\bar{\mathcal{G}}_h$ . Using notations of (18) and letting  $\bar{u}^h := -\exp(-\bar{w}/\varepsilon_h)$ ,  $\underline{u}^h := -\exp(-\underline{w}/\varepsilon_h)$ ,  $\bar{U}^h := (\bar{u}^h(x))_{x \in \mathcal{G}_h}$ , and  $\underline{U}^h := (\underline{u}^h(x))_{x \in \mathcal{G}_h}$ , it follows that  $M^h \bar{U}^h \leq G_h$  and  $M^h \underline{U}^h \geq G^h$ . Since  $M^h$  is a nonsingular  $M$ -matrix, the unique solution  $U^h \in \mathbb{R}^{\mathcal{G}_h}$  to  $M^h U^h = G^h$ , defined by  $U_x^h := u^h(x)$ , satisfies  $\bar{U}^h \leq U^h \leq \underline{U}^h$ . Thus  $\bar{u}^h \leq u^h \leq \underline{u}^h < 0$  and  $\bar{w} \leq v^h \leq \underline{w}$ .

Using the above properties of the scheme and following the proof of [4, Theorem 2.1], we deduce that functions  $\bar{v}, \underline{v}: \bar{\Omega} \rightarrow \mathbb{R}$  defined by

$$\bar{v}(x) := \limsup_{\substack{h \rightarrow 0 \\ y \in \bar{\mathcal{G}}_h, y \rightarrow x}} v^h(y), \quad \underline{v}(x) := \liminf_{\substack{h \rightarrow 0 \\ y \in \bar{\mathcal{G}}_h, y \rightarrow x}} v^h(y),$$

are respectively a subsolution and supersolution to (16), or equivalently to (7). By Theorem 2.6,  $\bar{v} \leq v_* \leq v^* \leq \underline{v}$  on  $\Omega$ , and by definition  $\bar{v} \geq \underline{v}$  on  $\bar{\Omega}$ . Thus  $\bar{v} = v = \underline{v}$  on  $\Omega$  and the locally uniform convergence follows from the definitions of  $\bar{v}$  and  $\underline{v}$ .  $\square$

### 3.3 Approximation of the Randers distance to a finite set

Let  $Y \subset \Omega$  be a finite set, and assume that there is  $h > 0$  such that  $Y \subset \mathcal{G}_h$ . We are interested in approximating the Randers distance to  $Y$ , defined by

$$\text{dist}_{A,b}(x, Y) := \min_{y \in Y} \text{dist}_{A,b}(x, y) \quad \text{on } \bar{\Omega}.$$

To this end, we use that

$$\text{dist}_{A,b}(x, Y) = v(x) \quad \text{on } \bar{\Omega}$$

if  $v: \bar{\Omega} \rightarrow \mathbb{R}$  is defined by (5) with  $\Omega$  replaced by  $\Omega \setminus Y$ ,  $g = 0$  on  $Y$ , and

$$g(x) \geq \sup_{y \in Y} \text{dist}_{A,b}(x, y) \quad \text{on } \partial\Omega.$$

Note that contrary to the assumption that  $\partial\Omega$  is of class  $W^{3,\infty}$  in (5), we allow  $\partial(\Omega \setminus Y)$  to contain isolated points.

Let  $u^h: \bar{\mathcal{G}}_h \rightarrow \mathbb{R}$  and  $U^h \in \mathbb{R}^{\mathcal{G}_h}$  be defined by  $U_x^h := u^h(x)$ . We define  $M^h \in \mathbb{R}^{\mathcal{G}_h \times \mathcal{G}_h}$  and  $G^h \in \mathbb{R}^{\mathcal{G}_h}$  as in (18), and  $M^{h,Y} \in \mathbb{R}^{\mathcal{G}_h \times \mathcal{G}_h}$  and  $G^{h,Y} \in \mathbb{R}^{\mathcal{G}_h}$  by

$$M^{h,Y} := M^h + \sum_{y \in Y} e_y e_y^\top (I_{|\mathcal{G}_h|} - M^h), \quad G_x^{h,Y} := \begin{cases} -1 & \text{if } x \in Y, \\ G_x^h & \text{else,} \end{cases}$$

where  $(e_x)_{x \in \mathcal{G}_h}$  is the canonical basis of  $\mathbb{R}^{\mathcal{G}_h}$ . Then  $u^h$  is solution to (17) with  $\Omega$  replaced by  $\Omega \setminus Y$  if and only if  $U^h$  is solution to

$$M^{h,Y} U^h = G^{h,Y}.$$

Let  $E_Y \in \mathbb{R}^{\mathcal{G}_h \times Y}$  be the matrix whose columns are  $(e_y)_{y \in Y}$ . Then

$$M^{h,Y} = M^h + E_Y E_Y^\top (I_{|\mathcal{G}_h|} - M^h).$$

In order to efficiently compute several Randers distances on  $\Omega$  to target sets  $Y$  of small cardinal, we may use the Woodbury matrix identity, according to which

$$(M^{h,Y})^{-1} = (M^h)^{-1} - (M^h)^{-1} E_Y (E_Y^\top (M^h)^{-1} E_Y)^{-1} E_Y^\top ((M^h)^{-1} - I_{|\mathcal{G}_h|}), \quad (28)$$

along with a prefactorization of the matrix  $M^h$ .

Although we assumed earlier that the function  $g$  is finite, in practice we may also solve the scheme associated to  $g = +\infty$  on  $\partial\Omega$ , in which case

$$G^h = 0, \quad G^{h,Y} = - \sum_{y \in Y} e_y.$$

### 3.4 Approximation of the gradient of the Randers distance

We have two motivations for approximating numerically the gradient of the function  $v: \bar{\Omega} \rightarrow \mathbb{R}$  defined by (5).

First, it is well-known that for any  $x \in \bar{\Omega}$  and  $0 \leq t < \tau^{x,\alpha}$ , if  $\alpha: \mathbb{R}_+ \rightarrow B_d(0,1)$  is optimal in (6), then

$$\begin{aligned} \alpha(t) &= \operatorname{argmax}_{\alpha' \in B_d(0,1)} -\langle A^{1/2}(\gamma^{x,\alpha}(t))\alpha' + b(\gamma^{x,\alpha}(t)), Dv(\gamma^{x,\alpha}(t)) \rangle - 1 \\ &= -\frac{A^{1/2}(\gamma^{x,\alpha}(t))Dv(\gamma^{x,\alpha}(t))}{|Dv(\gamma^{x,\alpha}(t))|_{A(\gamma^{x,\alpha}(t))}} \end{aligned}$$

and thus

$$\dot{\gamma}^{x,\alpha}(t) = A^{1/2}(\gamma^{x,\alpha}(t))\alpha(t) + b(\gamma^{x,\alpha}(t)) = b(\gamma^{x,\alpha}(t)) - \frac{A(\gamma^{x,\alpha}(t))Dv(\gamma^{x,\alpha}(t))}{|Dv(\gamma^{x,\alpha}(t))|_{A(\gamma^{x,\alpha}(t))}}.$$

Hence we may use the approximation of  $Dv$  to reconstruct the minimal path  $\gamma^{x,\alpha}$  numerically.

Second, it was noticed in [7] that on Riemannian manifolds, the numerical error in approximations of  $Dv$  often seems lesser than the one in the approximation of  $v$  by  $v^h := -\varepsilon \log(-u^h)$ , where  $u^h$  solves a discretisation of (10) — in our case, this would be (17). Thus the authors proposed instead to approximate  $v$  by the solution to a discretisation of Poisson's equation on the manifold, with a right-hand side equal to a discretized divergence of an approximation of  $Dv$ .

Let Assumption 3.2 and Assumption 3.4 hold. For small  $h > 0$  and  $\varepsilon = \varepsilon_h$ , let  $u_\varepsilon: \bar{\Omega} \rightarrow \mathbb{R}$  be defined by (9) and  $v_\varepsilon := -\varepsilon \log(-u_\varepsilon)$ . Since  $v_\varepsilon$  converges to  $v$  on  $\Omega$ , our strategy is to approximate  $Dv$  by  $Dv_\varepsilon$ , which by definition of  $v_\varepsilon$  is a positive multiple of  $Du_\varepsilon$ . We in turn approximate  $Du_\varepsilon$  by  $D^h u^h$  on  $\mathcal{G}_h$ , where  $u^h: \bar{\mathcal{G}}_h \rightarrow \mathbb{R}$  solves (17) and  $D^h$  is one of the operators  $D_{I_d}^h$  and  $D_{A_b}^h$  defined, for  $u: \bar{\mathcal{G}}_h \rightarrow \mathbb{R}$  and  $x \in \mathcal{G}_h$ , and  $(e_1, \dots, e_d)$  the canonical basis of  $\mathbb{R}^d$ , by

$$D_{I_d}^h u(x) := \sum_{i=1}^d \frac{\delta_{x,h}^{e_i} u - \delta_{x,h}^{-e_i} u}{\tilde{h}_{x,h}^{e_i} + \tilde{h}_{x,h}^{-e_i}} e_i, \quad D_{A_b}^h u(x) := \sum_{e \in \mathbb{Z}^d} a_x^e \frac{\delta_{x,h}^e u - \delta_{x,h}^{-e} u}{\tilde{h}_{x,h}^e + \tilde{h}_{x,h}^{-e}} A_b^{-1}(x)e.$$

To derive the second operator, we used that for  $u: \bar{\Omega} \rightarrow \mathbb{R}$ ,

$$Du(x) = A_b^{-1}(x)A_b(x)Du(x) = \sum_{e \in \mathbb{Z}^d} a_x^e \langle e, Du(x) \rangle A_b^{-1}(x)e,$$

which has the benefit of yielding a discretisation using the same finite differences than the centered version of scheme (17). If  $x \in \bar{\Omega}$ ,  $0 \leq t < \tau_{x,\alpha}$ ,  $\alpha: \mathbb{R}_+ \rightarrow B_d(0,1)$  is optimal in (6), and  $y \in \mathcal{G}_h$  is such that  $\gamma_{x,\alpha}(t) \approx y$ , we approximate  $\dot{\gamma}^{x,\alpha}(t)$  by

$$b(y) - \frac{A(y)D^h u^h(y)}{|D^h u^h(y)|_{A(y)}}. \quad (29)$$

If  $x \in \mathcal{G}_h$ , using the equation in  $\Omega$  in (7) to normalise  $D^h u^h(x)$ , we approximate  $Dv(x)$  itself by

$$X^h(x) := \frac{D^h u^h(x)}{|D^h u^h(x)|_{A(x)} - \langle b(x), D^h u^h(x) \rangle}.$$

In the setting of section 3.3, let  $v: \bar{\Omega} \rightarrow \mathbb{R}$  be defined by (5) with  $\Omega$  replaced by  $\Omega \setminus Y$ ,  $g = 0$  on  $Y$ , and  $g = +\infty$  on  $\partial\Omega$ . Following the idea of [7], we let the approximation  $X^h: \mathcal{G}_h \setminus Y \rightarrow \mathbb{R}^d$  of  $Dv$  be defined as above but with  $\Omega$  replaced by  $\Omega \setminus Y$ , and we approximate  $v$  on  $\mathcal{G}_h$  by one of the solutions  $w_{I_d}^h: \mathcal{G}_h \rightarrow \mathbb{R}$  and  $w_{A_b}^h: \mathcal{G}_h \rightarrow \mathbb{R}$  to the minimization problems

$$\inf_{\substack{w_{I_d}^h: \mathcal{G}_h \rightarrow \mathbb{R} \\ w_{I_d}^h = 0 \text{ on } Y}} \sum_{\substack{x \in \mathcal{G}_h \setminus Y \\ 1 \leq i \leq d \\ \sigma \in \{-1,1\} \\ x + \sigma \tilde{h}_{x,h}^{\sigma e_i} e_i \in \mathcal{G}_h}} \left| \frac{\delta_{x,h}^{\sigma e_i} w_{I_d}^h}{\tilde{h}} - \sigma \langle e_i, X^h(x) \rangle \right|^2,$$

$$\inf_{\substack{w_{A_b}^h : \mathcal{G}_h \rightarrow \mathbb{R} \\ w_{A_b}^h = 0 \text{ on } Y}} \sum_{\substack{x \in \mathcal{G}_h \setminus Y \\ e \in \mathbb{Z}^d \\ x + \tilde{h}_{x,h}^e e \in \mathcal{G}_h}} a_x^e \left| \frac{\delta_{x,h}^e w_{A_b}^h}{h} - \langle e, X^h(x) \rangle \right|^2.$$

Writing the optimality condition for these minimization problems, we see that  $w_{I_d}^h$  and  $w_{A_b}^h$  solve the numerical schemes

$$\begin{cases} \sum_{\substack{1 \leq i \leq d \\ \sigma \in \{-1,1\} \\ x + \sigma \tilde{h}_{x,h}^{\sigma e_i} e_i \in \mathcal{G}_h}} \left( \frac{\sigma}{h} \left\langle e_i, \frac{X^h(x) + X^h(x + \sigma h e_i)}{2} \right\rangle - \frac{\delta_{x,h}^{\sigma e_i} w_{I_d}^h}{h^2} \right) = 0 & \text{in } \mathcal{G}_h \setminus Y, \\ w_{I_d}^h(x) = 0 & \text{on } Y, \end{cases} \quad (30)$$

$$\begin{cases} \sum_{\substack{e \in \mathbb{Z}^d \\ x + \sigma \tilde{h}_{x,h}^e e \in \mathcal{G}_h}} a_x^e \left( \frac{1}{h} \left\langle e, \frac{X^h(x) + X^h(x + h e)}{2} \right\rangle - \frac{\delta_{x,h}^e w_{A_b}^h}{h^2} \right) = 0 & \text{in } \mathcal{G}_h \setminus Y, \\ w_{A_b}^h(x) = 0 & \text{on } Y. \end{cases} \quad (31)$$

## 4 Application to regularized optimal transport

Let  $\mu, \nu$  belong to the set  $\mathcal{P}_1(\Omega)$  of probability measures on  $\Omega$  with finite first order moments. The 1-Wasserstein distance  $W_1(\mu, \nu)$  between  $\mu$  and  $\nu$  associated to  $A$  and  $b$  is defined by

$$W_1(\mu, \nu) := \inf_{\gamma \in \Pi(\mu, \nu)} \int_{\Omega \times \Omega} \text{dist}_{A,b}(x, y) d\gamma(x, y), \quad (32)$$

where  $\Pi(\mu, \nu)$  is the set of probability measures on  $\Omega \times \Omega$  whose first and second marginals coincide respectively with  $\mu$  and  $\nu$ , called *transport plans* between  $\mu$  and  $\nu$ . Note that since  $\text{dist}_{A,b}$  is not symmetric, neither is  $W_1$ , which is only a quasidistance on  $\mathcal{P}(\Omega)$ .

For  $h > 0$ , we define discretizations  $\alpha, \beta \in \mathbb{R}_+^{\mathcal{G}_h}$  of measures  $\mu$  and  $\nu$  by

$$\alpha_x := \mu(\Omega \cap x + [-h/2, h/2]^d), \quad \beta_y := \nu(\Omega \cap y + [-h/2, h/2]^d),$$

and study the discretized optimal transport problem

$$\inf_{P \in U(\alpha, \beta)} \langle P, C \rangle,$$

where  $U(\alpha, \beta)$  and  $C \in \mathbb{R}_+^{\mathcal{G}_h \times \mathcal{G}_h}$  are defined by

$$U(\alpha, \beta) := \{P \in \mathbb{R}_+^{\mathcal{G}_h \times \mathcal{G}_h} \mid P\mathbf{1} = \alpha, P^\top \mathbf{1} = \beta\}, \quad C_{x,y} := \text{dist}_{A,b}(x, y).$$

To solve this problem numerically, we use the entropic regularization method proposed in [8]: for any  $P \in \mathbb{R}_+^{\mathcal{G}_h \times \mathcal{G}_h}$ , we define

$$\text{Ent}(P) := - \sum_{x, y \in \mathcal{G}_h} P_{x,y} \log(P_{x,y}),$$

and for  $\varepsilon > 0$  — we will choose  $\varepsilon = \varepsilon_h$  as above — we consider the regularized optimal transport problem

$$\inf_{P \in U(\alpha, \beta)} \langle P, C \rangle - \varepsilon \text{Ent}(P). \quad (33)$$

Introducing dual variables  $f, g \in \mathbb{R}^{\mathcal{G}_h}$ , we write the Lagrangian associated to the regularized problem:

$$\mathcal{L}(P, f, g) := \langle P, C \rangle - \varepsilon \text{Ent}(P) - \langle f, P\mathbf{1} - \alpha \rangle - \langle g, P^\top \mathbf{1} - \beta \rangle. \quad (34)$$

We deduce the first-order optimality condition: if  $P$  is optimal in (33), then there are  $f, g \in \mathbb{R}^{\mathcal{G}_h}$  such that for any  $x, y \in \mathcal{G}_h$ ,

$$C_{x,y} + \varepsilon \log(P_{x,y}) + \varepsilon - f_x - g_y = 0.$$



By the change of variable  $t \mapsto \exp(t/\varepsilon)$ , it follows that  $P$  satisfies

$$P = \text{diag}(\hat{u})K \text{diag}(\hat{v}), \quad (35)$$

where  $K \in \mathbb{R}_+^{\mathcal{G}_h \times \mathcal{G}_h}$  and  $\hat{u}, \hat{v} \in \mathbb{R}_+^{\mathcal{G}_h}$  are defined by

$$K_{x,y} := \exp\left(-\frac{C_{x,y}}{\varepsilon}\right) = \exp\left(-\frac{\text{dist}_{A,b}(x,y)}{\varepsilon}\right), \quad (36)$$

and, up to multiplication of  $\hat{u}$  and division of  $\hat{v}$  by the same positive factor,

$$\hat{u}_x := \exp\left(\frac{f_x}{\varepsilon} - \frac{1}{2}\right), \quad \hat{v}_y := \exp\left(\frac{g_y}{\varepsilon} - \frac{1}{2}\right).$$

The matrix  $K$  depends only on parameters of the problem, so the unknowns are  $\hat{u}$  and  $\hat{v}$ . Injecting (35) in the definition of  $U(\alpha, \beta)$ , we see  $\hat{u}$  and  $\hat{v}$  must satisfy

$$\hat{u} \odot (K\hat{v}) = \alpha, \quad \hat{v} \odot (K^\top \hat{u}) = \beta,$$

where  $\odot$  represents the elementwise product of vectors.

The standard way to estimate  $\hat{u}$  and  $\hat{v}$  numerically is *Sinkhorn's algorithm*: let  $\hat{v}^{(0)} \in (\mathbb{R}_+^*)^{\mathcal{G}_h}$  and, for any  $n \in \mathbb{N}$ , let

$$\hat{u}^{(n)} := \frac{\alpha}{K\hat{v}^{(n)}}, \quad \hat{v}^{(n+1)} := \frac{\beta}{K^\top \hat{u}^{(n)}},$$

where fractions between vectors represent elementwise division. Then sequences  $(\hat{u}^{(n)})_n$  and  $(\hat{v}^{(n)})_n$  converge respectively to  $\hat{u}$  and  $\hat{v}$ , up to multiplication of  $\hat{u}$  and division of  $\hat{v}$  by the same positive factor.

The more computationally intensive part of this algorithm is to compute the matrix-vector products  $K\hat{v}$  and  $K^\top \hat{u}$ , for  $\hat{v}, \hat{u} \in \mathbb{R}_+^{\mathcal{G}_h}$ , since the matrix  $K$  is dense. An efficient way to approximate those products using Varadhan's formula was proposed in [11], in the case of Riemannian manifolds. Adapting it to our setting yields the approximation  $K \approx (M^h)^{-1}$ , where  $M^h \in \mathbb{R}^{\mathcal{G}_h \times \mathcal{G}_h}$  is defined as in (18), with coefficients  $\eta_{x,h}^e$  satisfying Assumption 3.2. We justify this approximation below, but first let us note that using Proposition 2.2, we also have  $K^\top \approx (\hat{M}^h)^{-1}$ , where  $\hat{M}^h$  is defined as  $M^h$  with the exception that  $b$  is replaced by  $-b$ . We replace products  $K\hat{v}$  and  $K^\top \hat{u}$  in Sinkhorn's algorithm by their approximations  $(M^h)^{-1}\hat{u}$  and  $(\hat{M}^h)^{-1}\hat{v}$  and use prefactorizations of  $M^h$  and  $\hat{M}^h$  to solve those linear systems efficiently at each iteration.

To justify the approximation of  $K$  by  $(M^h)^{-1}$ , let  $y \in \mathcal{G}_h$  and, in the setting of section 3.3 with  $Y = \{y\}$ , let  $v: \bar{\Omega} \rightarrow \mathbb{R}$  and  $u^h: \bar{\mathcal{G}}_h \rightarrow \mathbb{R}_-$  be respectively defined by (5) and solution to (17) with  $\Omega$  replaced by  $\Omega \setminus \{y\}$ ,  $g(y) = 0$ ,  $g = +\infty$  on  $\partial\Omega$ . We define  $U^h \in \mathbb{R}^{\mathcal{G}_h}$  by  $U_x^h := u^h(x)$ . Then  $U^h = -(M^{h,\{y\}})^{-1}e_y$ , which together with (28) yields that

$$U^h = -\frac{(M^h)^{-1}e_y}{((M^h)^{-1})_{y,y}}.$$

Therefore, at least for  $x$  and  $y$  far from  $\partial\Omega$ ,

$$\begin{aligned} ((M^h)^{-1})_{x,y} &= -U_x^h((M^h)^{-1})_{y,y} = -u^h(x)((M^h)^{-1})_{y,y} \approx \exp(-v(x)/\varepsilon)((M^h)^{-1})_{y,y} \\ &= \exp\left(-\frac{\text{dist}_{A,b}(x,y) - \varepsilon \log(((M^h)^{-1})_{y,y})}{\varepsilon}\right). \end{aligned}$$

Using the previous equation and (36), we see that  $(M^h)^{-1}$  approximates a matrix  $K$  which would be defined with a perturbation  $\varepsilon \log(((M^h)^{-1})_{y,y})$  in the cost of the optimal transport problem. Let us show that this perturbation is small when choosing  $\varepsilon$  according to Assumption 3.4.

**Proposition 4.1.** *Let Assumption 3.2 and Assumption 3.4 hold. For any  $h > 0$ , let  $M^h \in \mathbb{R}^{\mathcal{G}_h \times \mathcal{G}_h}$  be defined as in (18) with  $\varepsilon = \varepsilon_h$ . Let  $y \in \Omega$  be such that  $y \in \mathcal{G}_{h_0}$  for some  $h_0 > 0$ . Then*

$$\lim_{h \rightarrow 0, y \in \mathcal{G}_h} \varepsilon_h \log(((M^h)^{-1})_{y,y}) = 0.$$

*Proof.* Since  $M^h \mathbf{1} \geq \mathbf{1} \geq e_y$  and  $M^h$  is a nonsingular  $M$ -matrix, it holds that  $(M^h)^{-1} e_y \leq \mathbf{1}$ , thus  $((M^h)^{-1})_{y,y} \leq 1$  and

$$\varepsilon_h \log(((M^h)^{-1})_{y,y}) \leq 0.$$

The vector  $\bar{U}^h \in \mathbb{R}^{\mathcal{G}_h}$  defined by

$$\bar{U}_x^h := \begin{cases} \left(1 + \sum_{e \in \mathbb{Z}^d} \eta_{y,h}^e\right)^{-1} & \text{if } x = y, \\ 0 & \text{else,} \end{cases}$$

is such that  $M^h \bar{U}^h \leq e_y$ . Thus  $(M^h)^{-1} e_y \geq \bar{U}^h$  and

$$\varepsilon_h \log(((M^h)^{-1})_{y,y}) \geq -\varepsilon_h \log \left(1 + \sum_{e \in \mathbb{Z}^d} \eta_{y,h}^e\right).$$

We may assume that  $h$  is small enough so that  $\tilde{h}_{y,h}^e = h$  whenever  $a_x^e > 0$  or  $b_x^e \neq 0$ . Then, using (19) and that  $\varepsilon_h = Ch^r$ ,

$$\begin{aligned} & \lim_{h \rightarrow 0, y \in \mathcal{G}_h} \varepsilon_h \log \left(1 + \sum_{e \in \mathbb{Z}^d} \eta_{y,h}^e\right) \\ &= \lim_{h \rightarrow 0, y \in \mathcal{G}_h} h^r \log \left(1 + \frac{2C}{h^{1-r}} \sum_{e \in \mathbb{Z}^d} b_y^e + \frac{2C^2}{h^{2-2r}} \sum_{e \in \mathbb{Z}^d} a_y^e\right) = 0, \end{aligned}$$

which concludes the proof.  $\square$

Note that the limit is not uniform near  $\partial\Omega$  due to the assumption that  $\tilde{h}_{y,h}^e = h$  whenever  $a_x^e > 0$  or  $b_x^e \neq 0$ . This suggests applying the proposed numerical method to optimal transport problems where measures  $\mu$  and  $\nu$  are supported on compact subsets of  $\Omega$ .

Injecting (35) into (34), we deduce the dual optimization problem to (33):

$$\sup_{f \in \mathbb{R}^{\mathcal{G}_h}, g \in \mathbb{R}^{\mathcal{G}_h}} \langle f, \alpha \rangle + \langle g, \beta \rangle - \varepsilon \left\langle \exp \left( \frac{f}{\varepsilon} - \frac{1}{2} \right), K \exp \left( \frac{g}{\varepsilon} - \frac{1}{2} \right) \right\rangle,$$

where operations on vectors are elementwise. Thus if  $\hat{u}^h, \hat{v}^h \in \mathbb{R}_+^{\mathcal{G}_h}$  are approximations of  $\hat{u}$  and  $\hat{v}$  in (35), we approximate the Wasserstein distance  $W_1(\mu, \nu)$  by

$$W_1^h(\mu, \nu) := \varepsilon (\langle \log \hat{u}^h, \alpha \rangle + \langle \log \hat{v}^h, \beta \rangle) + 1 - \langle \hat{u}^h, (M^h)^{-1} \hat{v}^h \rangle. \quad (37)$$

## 5 Numerical results

We apply the proposed numerical scheme to two two-dimensional problems expressed in the setting of section 3.3. For both problems, we choose the domain  $\Omega = B_2(0, 1)$ , and  $g(x) = +\infty$  on  $\partial\Omega$ .

In the first problem, we consider a flat manifold described by constant matrix and vector fields  $A = A_{(1)}$  and  $b = b_{(1)}$  defined by

$$A_{(1)} := \begin{pmatrix} 0.5 & 0.6 \\ 0.6 & 1.0 \end{pmatrix}, \quad b_{(1)} := \begin{pmatrix} 0.3 \\ 0.4 \end{pmatrix},$$

and the target set  $Y = Y_{(1)}$  defined by

$$Y_{(1)} := \left\{ \begin{pmatrix} -0.6 \\ 0.6 \end{pmatrix}, \begin{pmatrix} -0.6 \\ -0.6 \end{pmatrix}, \begin{pmatrix} 0.6 \\ -0.6 \end{pmatrix}, \begin{pmatrix} 0.6 \\ 0.6 \end{pmatrix} \right\}.$$

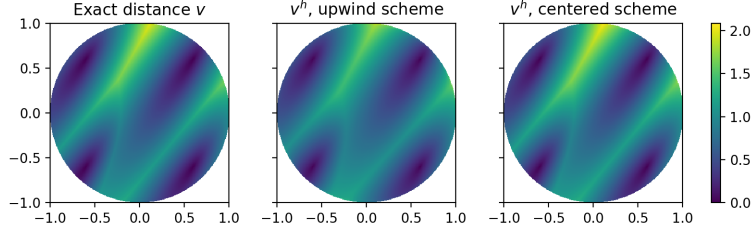


Figure 1: Randers distance to the target set  $Y_{(1)}$  on a manifold with parameters  $A_{(1)}$  and  $b_{(1)}$ , computed exactly (left) and approximated using the centered (middle,  $h = 0.00625$ ,  $\varepsilon = 0.5h^{2/3}$ ) and upwind (right,  $h = 0.00625$ ,  $\varepsilon = 0.5h^{1/2}$ ) schemes.

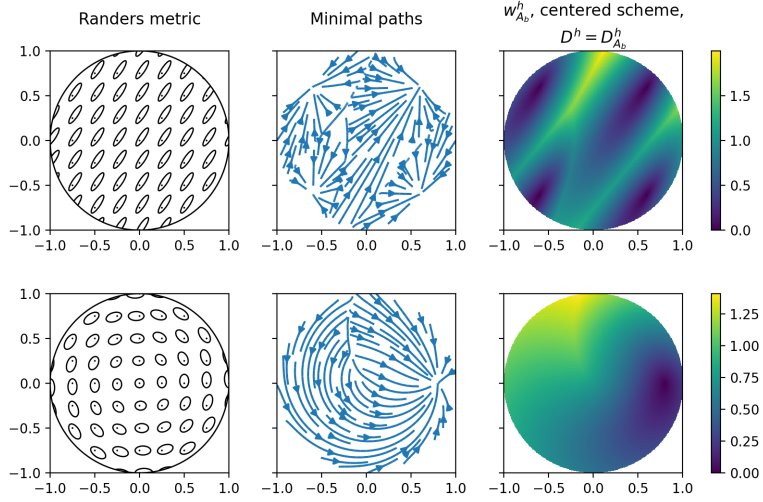


Figure 2: Representation of the Randers metric and approximations of minimal paths and of the Randers distance for parameters  $A_{(1)}$ ,  $b_{(1)}$ , and  $Y_{(1)}$  (top), and  $A_{(2)}$ ,  $b_{(2)}$ , and  $Y_{(2)}$  (right), with  $h = 0.00625$  and  $\varepsilon = 0.5h^{2/3}$ .

In this case, and more generally when  $A$  and  $b$  are constant functions, the Randers distance function may be computed exactly using the formula

$$\text{dist}_{A,b}(x, y) = \frac{\|x - y\|_{A^{-1}}^2}{\langle b, A^{-1}(x - y) \rangle + \sqrt{\langle b, A^{-1}(x - y) \rangle^2 + (1 - |b|_{A^{-1}}^2)|x - y|_{A^{-1}}^2}}$$

when  $x \neq y$ . This formula is obtained by choosing a constant function  $\alpha \in \mathcal{A}$  in (4). This allows us to compare our numerical results with the exact solution.

In the second problem, we consider parameters  $A = A_{(2)}$  and  $b = b_{(2)}$  defined by

$$A_{(2)}(x) := \begin{pmatrix} 1 + \frac{2x_2^2}{|x|} + x_2^2 & -\frac{2x_1x_2}{|x|} - x_1x_2 \\ -\frac{2x_1x_2}{|x|} - x_1x_2 & 1 + \frac{2x_1^2}{|x|} + x_1^2 \end{pmatrix}, \quad b_{(2)}(x) := x^\perp = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix},$$

where  $A_{(2)}(0)$  is extended by continuity, and the target set  $Y = Y_{(2)} := \{(0.8, 0)\}$ .

On Figure 1, we display the exact function  $v: x \mapsto \text{dist}_{A_{(1)}, b_{(1)}}(x, Y_{(1)})$  and the solutions  $v^h: \mathcal{G}_h \rightarrow \mathbb{R}$  to (26) for the upwind and centered schemes.

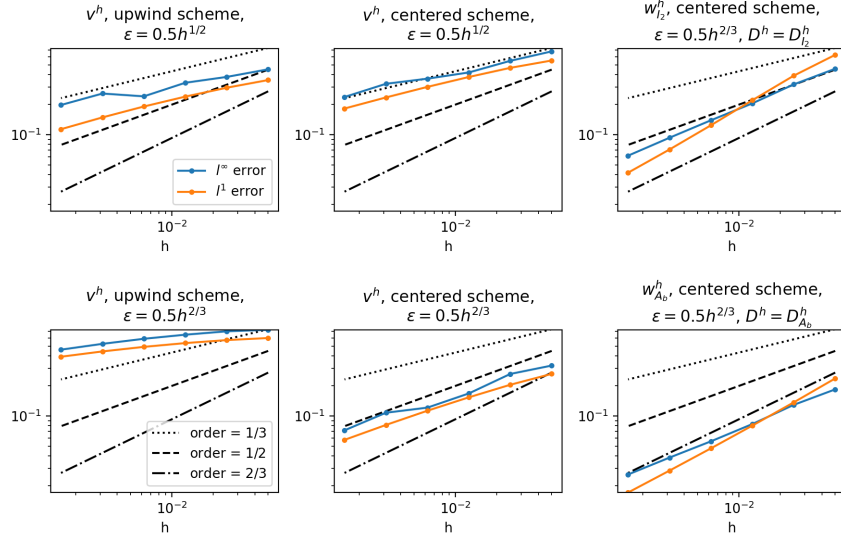


Figure 3: Error between the exact distance  $v$  and its approximations on the manifold with parameters  $A_{(1)}$ ,  $b_{(1)}$ , and  $Y_{(1)}$ , depending on  $h$ .

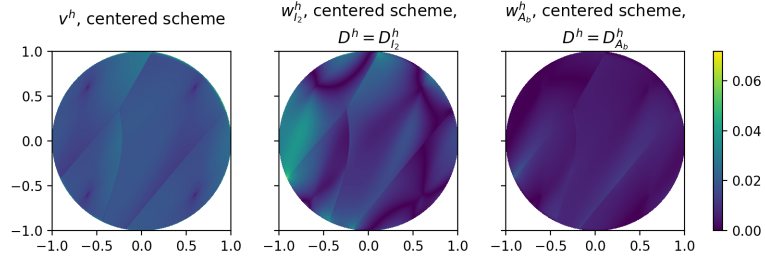


Figure 4: Error between the exact distance  $v$  and its approximations on the manifold with parameters  $A_{(1)}$ ,  $b_{(1)}$ , and  $Y_{(1)}$ , for  $h = 0.0015625$  and  $\varepsilon_h = 0.5h^{2/3}$ .

On Figure 2, we depict on the left the Randers metrics associated to our two sets of parameters: for some points  $x \in B_2(0, 1)$ , marked by dots, we display the ellipses

$$\begin{aligned} &\{y \in \mathbb{R}^2 \mid |y - x|_{A_{(1)}^{-1}(x)} - \langle b_{(1)}(x), y - x \rangle = 0.1\}, \\ &\{y \in \mathbb{R}^2 \mid |y - x|_{A_{(2)}^{-1}(x)} - \langle b_{(2)}(x), y - x \rangle = 0.05\}. \end{aligned}$$

On the middle, we show approximations of minimal paths to the target sets  $Y_{(1)}$  and  $Y_{(2)}$ , obtained using (29) with  $D^h = D_{A_b}^h$ . On the right, we display the solution  $w_{A_b}^h : \mathcal{G}_h \rightarrow \mathbb{R}$  to (31), where  $X_h$  is computed using  $D^h = D_{A_b}^h$ .

On Figure 3, we study the convergence of some approximations of the exact distance function  $v$ . We define respectively the  $l^\infty$  and  $l^1$  errors between  $v$  and one of its approximations  $v^h$  as

$$\max_{x \in \mathcal{G}_h} |v^h(x) - v(x)|, \quad h^2 \sum_{x \in \mathcal{G}_h} |v^h(x) - v(x)|.$$

The results seem consistent with the optimal choices for the exponent  $r$  in Theorem 3.5 for the upwind and centered schemes, although the numerically observed rate of convergence does not attain the order of consistency of the scheme far from  $\partial\Omega$ . The curves on the right suggest preferring the scheme (31) and  $D^h = D_{A_b}^h$  to (30) and  $D^h = D_{I_2}^h$ .

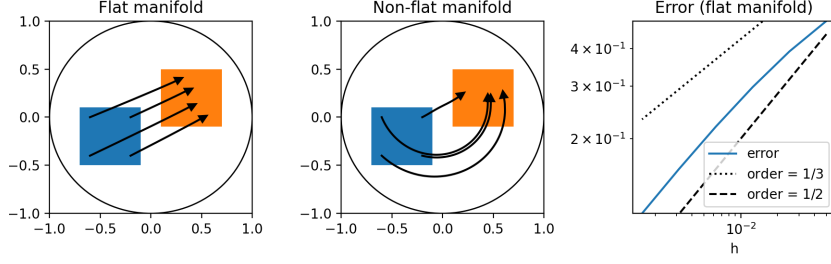


Figure 5: Numerical solution of the optimal transport problem (32) on manifolds with parameters  $A_{(1)}$  and  $b_{(1)}$  (left:  $h = 0.00625$ ,  $\varepsilon = h^{2/3}$ ; right: convergence,  $\varepsilon = h^{2/3}$ ) and  $A_{(2)}$  and  $b_{(2)}$  (middle,  $h = 0.00625$ ,  $\varepsilon = h^{2/3}$ ).

On Figure 4, we display the absolute value of the error between the exact distance function  $v$  and the solutions  $v^h$ ,  $w_{I_2}^h$ , and  $w_{A_b}^h$  to (26), (30), and (31). We observe that approximating  $v$  by  $w_{A_b}^h$  instead of  $v^h$  helps avoiding large numerical errors far from the target set  $Y$ , and also near the boundary  $\partial\Omega$ .

On Figure 5, we solve numerically the optimal transport problem (32), where  $\mu$  and  $\nu$  are uniform probability measures on  $[-0.7, -0.1] \times [-0.5, 0.1]$  and  $[0.1, 0.7] \times [-0.1, 0.5]$  respectively. We compute approximations  $\hat{u}^h, \hat{v}^h \in \mathbb{R}_+^{\mathcal{G}_h}$  of  $\hat{u}$  and  $\hat{v}$  in (35). The arrows on the figure represent the application

$$\mathcal{G}_h \cap \text{supp}(\mu) \rightarrow \mathbb{R}^d, \quad x \mapsto \frac{\hat{u}_x^h \sum_{y \in \mathcal{G}_h} (M^{-1})_{x,y} \hat{v}_y^h y}{\alpha_x},$$

which is an approximation of  $x \mapsto \sum_{y \in \mathcal{G}_h} y P_{x,y} / \alpha_x$ , where  $P$  is optimal in (33). Note that this application is not a translation even in the case of constant parameters  $A_{(1)}$  and  $b_{(1)}$ . This is a property of the 1-Wasserstein distance, even in the continuous setting, and is not related to our choice of approximation of the matrix  $K$  in (35). On the right, we display the error between  $W_1(\mu, \nu)$  and its approximation  $W^h(\mu, \nu)$  defined by (37), depending on the parameter  $h$ .

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