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Quantum potential and free classical particles

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Abstract

A quantum potential $V(x, t)$ of δ -function type is introduced, to describe the motion of a particle. Quantum-mechanically, it is in a bound state (potential trap of Hojman and Asenjo [6]), though classically one seems to be free. The motion of the object takes place according to the Huygens-Fresnel principle. The new position of the object (wave front) plays the role of the secondary sources that maintain the propagation. We found that the de Broglie - Bohm quantum potential is the difference between the constant bound energy $E = -mc^2/2$ from the stationary case and our potential $V(x, t)$.

1 Introduction

It is well-known that Quantum Field Theory (QFT) put together classical fields, Special Relativity (SR) and Quantum Mechanics (QM). It treats particles (called field quanta) as excited states of an underlying field. The way it works mathematically is by placing a quantum harmonic oscillator at each point in space. As Griffiths has noticed [1], observers not only disturb what is to be measured, but also produce it. We enforce the particle to have a definite position. The measurement process "creates" a particular value of the quantity to be measured (see also [2]). Araya and Bars [3] use analyticity properties of quantum fields to uncover a new multiverse geometry when the classical field has horizon or singularities. They applied their ideas to extended Rindler space which is classically equivalent to flat Minkowski space. Quantum-mechanically, the sheet structure is naturally interpreted as an infinite stack of identically Minkowski geometries, or "universes". They also showed that information does not flow from one Rindler sheet to another.

In [4] the author chose a constant value of the Simula [5] "quantum pressure" (which is in fact a quantum potential, having units of energy) for to give a physical interpretation of his scalar field $\Phi(x)$. Recently Hojman and Asenjo

[6] studied the existence of nonvanishing Bohm (quantum) potentials, in the Madelung version of the Schrödinger equation, allowing for the construction of particular solutions for states of quantum particles interacting with nontrivial external potentials whose propagation is equivalent to the one for classical free particles. To achieve that aim, they enforced the condition that Bohm potential cancels out any contribution of the external potential.

We study in this paper the motion of a particle in a δ -Dirac function well. Our purpose is to show that an inertial particle is not free at all but one moves in a quantum δ -function type potential. We are motivated by the well-known idea of the QFT that particles are excited states of the quantum vacuum fields. Another motivation comes from the wave-particle duality, a property which is not restricted to microscopic particles. We therefore stressed that the wave-like properties (for example, the propagation according to the Huygens-Fresnel principle) are to be valid for macroscopic objects as well. To reach our goal, one follows Griffiths' reasoning [1]. We consider, however, the velocity v from the expression of the potential well to be the particle velocity¹ and not the velocity of the well. We compute the r.m.s. of the position and momentum operators and find that they obey the Uncertainty Principle. In our model, the "strength" of the potential well depends on \hbar but the mean value of the potential V is classical, with $\langle V(x,t) \rangle = -mc^2$, m being the particle mass and c - the velocity of light. It is worth to mention that we shall make use of the fundamental constants c and \hbar , from SR and QM, and not the Newton constant G . Nevertheless, the mean value of $V(x,t)/m$ will be interpreted as a constant gravitational potential. Thanks to the Huygens-Fresnel principle, we show that an inertial particle moves like a wave: each new position of the particle on the wave front represents, in our view, another identical particle that acts as a new source which propagates unidirectionally, with velocity v .

2 δ -function potential

We begin with the time-dependent Schrödinger equation

$$i\hbar \frac{\partial \Psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi(x,t)}{\partial x^2} + V(x,t)\Psi(x,t), \quad (2.1)$$

describing a particle moving with velocity v along the x -axis. We assume that the potential energy V is given by

$$V(x,t) = -a\delta(x-vt), \quad (2.2)$$

where a is a positive constant and δ is the Dirac function (distribution). Griffiths found that the solution of (2.1) with V given by (2.2) has the exact (normalizable) expression

$$\Psi(x,t) = \frac{\sqrt{ma}}{\hbar} e^{-\frac{ma|x-vt|}{\hbar^2}} e^{-\frac{i}{\hbar}[(E+\frac{mv^2}{2})t-mvx]}, \quad (2.3)$$

¹Velocity is defined as $v = \langle p \rangle / m$, where $\langle p \rangle$ is the expectation value of the momentum of the particle in the state described by the wave function $\Psi(x,t)$.

where $E = -ma^2/2\hbar^2$ (which is unique) is the bound-state energy of the stationary δ -function, $V(x) = -a\delta(x)$. Griffiths gives no estimation of the value of the "strength" a . From (2.2) it is clear that a must have units of energy \times distance. One assumes the model we treat here has a general meaning², being valid for any constants m and v , so a should depend on fundamental constants. Having only two fundamental constant at our disposal, \hbar and the velocity of light c , the single option is $a = \hbar c$. Therefore, (2.2) becomes

$$V(x, t) = -\hbar c \delta(x - vt). \quad (2.4)$$

Here is the first equation where we make use of SR, through the constant³ c . In addition, classically, $V = 0$. In other words, even though the particle is (macroscopically) inertial, it is not free but embedded in the exterior potential (2.4). Quantum-mechanically, it is in a bound state. The mean value of the potential gives us

$$\begin{aligned} \langle V(x, t) \rangle &= -\hbar c \int_{-\infty}^{\infty} \Psi^* \delta(x - vt) \Psi dx \\ &= -mc^2 \int_{-\infty}^{\infty} e^{-b|x-vt|} \delta(x - vt) dx = -mc^2, \end{aligned} \quad (2.5)$$

where the fundamental property of the δ -function has been used in the last step and $b = 2mc/\hbar$. One sees that the expectation value of V is proportional to the mass m of the object and plays the role of a potential energy. The bound state of the stationary case becomes now $E = -mc^2/2$. It is worth noting that the rest (potential) energy mc^2 is the only element we have used from the SR, the rest being non-relativistic QM. We also observe that the rest energy is not a correction to some formula from classical mechanics at high velocities where v/c is not negligible; it is there even at small velocities. Even in Newtonian Mechanics we have the freedom to add a constant potential mc^2 to $V(r) = -Gm/r$ [7]. Therefore, we assume that inertial motion is, in fact, a motion in a constant gravitational potential, given by (2.5). That interpretation justifies the appearance of the inertial forces in accelerated frames: the variation of the gravitational potential generates those forces. Therefore, quantum-mechanically, an inertial object is not free but bound, though macroscopically it appears to be free. We will justify this viewpoint later, making a comparison with the Hojman and Asenjo recipe [6].

The probability density of localization of the particle $P = |\Psi(x, t)|^2$ is not differentiable at $x = vt$ because of the modulus at the exponent. The width of the curve at half of P_{max} (at constant t) is $(mc/\hbar) \ln 2$. A plot of $P(x)$ versus x has a similar form with $\Psi(x)$ from the static situation (see Ref.1, p.54).

²If we take, for instance, $a = \hbar v$, then $\Psi(x, t)$ will vanish for $v = 0$, which is not admitted. Therefore, a may not depend on v .

³One could, of course, take c from Maxwell's equations, with no use of SR, but in that case the choice is arbitrary and c will lose its universal character.

3 Mean Hamiltonian

Our next task is to compute the expectation values of the main physical quantities and compare them with the corresponding classical expressions. Let us calculate, firstly, the expectation value of the Hamiltonian operator in the state Ψ from (2.3). We have

$$\langle \hat{H} \rangle = \int_{-\infty}^{\infty} \Psi^* \hat{H} \Psi dx = i\hbar \int_{-\infty}^{\infty} \Psi^* \frac{\partial \Psi}{\partial t} dx. \quad (3.1)$$

To find $\partial \Psi / \partial t$ we remind that $(\partial / \partial t)|x - vt| = -v \operatorname{sgn}(x - vt)$, where $\operatorname{sgn}(x)$ is -1 for $x < 0$ and equals 1 for $x > 0$. But $\operatorname{sgn}(x)$ is an odd function, so that

$$\int_{-\infty}^{\infty} e^{-b|x-vt|} \operatorname{sgn}(x - vt) dx = 0. \quad (3.2)$$

The other integral gives us

$$\int_{-\infty}^{\infty} e^{-b|x-vt|} dx = 2 \int_0^{\infty} e^{-b|x-vt|} dx = \frac{2}{b}. \quad (3.3)$$

One obtains finally

$$\langle \hat{H} \rangle = -\frac{mc^2}{2} \left(1 - \frac{v^2}{c^2}\right) = E \left(1 - \frac{v^2}{c^2}\right). \quad (3.4)$$

One verifies that, when $v = 0$, $\langle \hat{H} \rangle = E$, the stationary case. The same result emerges when $v \ll c$. Moreover, $v \rightarrow c$ leads to $\langle \hat{H} \rangle \rightarrow 0$, as if the particle were not bound. This is equivalent to the massless case (for example, a photon)⁴. Anyway, an increasing velocity gives an increasing energy and the particle becomes less bound (the "effective" bound energy $E(1 - v^2/c^2)$ tends to zero).

Another aim concerns the r.m.s. of the Hamiltonian \hat{H} . Firstly we consider

$$\langle \hat{H}^2 \rangle = -\hbar^2 \int_{-\infty}^{\infty} \Psi^* \frac{\partial^2 \Psi}{\partial t^2} dx. \quad (3.5)$$

To find $\partial^2 \Psi / \partial t^2$, we remind that the 1st time derivative gave $\operatorname{sgn}(y)$. The 2nd derivative leads therefore to $\partial(\operatorname{sgn}(y)) / \partial y = 2\delta(y)$, with $y = x - vt$. Hence

$$\frac{\partial^2 \Psi}{\partial t^2} = \left[-bv^2 \delta(y) + \frac{b^2 v^2}{4} - \frac{(E + \frac{mv^2}{2})^2}{\hbar^2} - \frac{ibv}{\hbar} \left(E + \frac{mv^2}{2}\right) \operatorname{sgn}(y) \right] \Psi. \quad (3.6)$$

Using similar calculations as before, we have finally

$$\langle \hat{H}^2 \rangle = \left(\frac{mc^2}{2}\right)^2 \left(1 + \frac{v^2}{c^2}\right), \quad (3.7)$$

whence

$$\Delta H = \sqrt{\langle \hat{H}^2 \rangle - \langle \hat{H} \rangle^2} = mcv. \quad (3.8)$$

⁴The massless situation is not, of course, valid to our problem, because Schrodinger's equation applies to massive particles only.

4 Uncertainty relation between position and momentum

Our next goal is to find the mean values of the particle position and momentum. We have

$$\langle \hat{x} \rangle = \int_{-\infty}^{\infty} \Psi^* x \Psi dx = \frac{mc}{\hbar} \int_{-\infty}^{\infty} x e^{-b|x-vt|} dx. \quad (4.1)$$

We change the variable of integration from x to $y = x - vt$ and obtain

$$\langle \hat{x} \rangle = \frac{mc}{\hbar} \left(\int_{-\infty}^{\infty} y e^{-b|y|} dy + vt \int_{-\infty}^{\infty} e^{-b|y|} dy \right). \quad (4.2)$$

The 1st integral is null. Hence, (4.2) yields

$$\langle \hat{x} \rangle = vt. \quad (4.3)$$

For the momentum $\langle \hat{p} \rangle$ we get

$$\begin{aligned} \langle \hat{p} \rangle &= \int_{-\infty}^{\infty} \Psi^* \left(-i\hbar \frac{\partial \Psi}{\partial x} \right) dx \\ &= \frac{-ima}{\hbar} \int_{-\infty}^{\infty} \left[\frac{-ma}{\hbar^2} \text{sgn}(x - vt) + \frac{i}{\hbar} mv \right] e^{-b|x-vt|} dx. \end{aligned} \quad (4.4)$$

The 1st integral gives zero and from the 2nd one obtains

$$\langle \hat{p} \rangle = \frac{bmv}{2} \int_{-\infty}^{\infty} e^{-b|x-vt|} dx = mv. \quad (4.5)$$

From (4.3) and (4.5) it is clear that v represents the classical velocity of the particle. Moreover, $d\langle \hat{p} \rangle / dt = 0$, though the potential V is time dependent.

Having the above results we can compute the r.m.s. deviations Δx and Δp . We have

$$\langle \hat{x}^2 \rangle = \int_{-\infty}^{\infty} \Psi^* x^2 \Psi dx = \frac{b}{2} \int_{-\infty}^{\infty} x^2 e^{-b|x-vt|} dx. \quad (4.6)$$

Passing to the variable y and using integration by parts, the result appears as

$$\langle \hat{x}^2 \rangle = \frac{2}{b^2} + v^2 t^2 = \frac{1}{2} \lambda_C^2 + v^2 t^2, \quad (4.7)$$

where $\lambda_C = \hbar/mc$ is the Compton wavelength of the particle. One now obtains

$$\Delta x = \sqrt{\langle \hat{x}^2 \rangle - \langle \hat{x} \rangle^2} = \frac{\lambda_C}{\sqrt{2}} \quad (4.8)$$

To calculate $\langle \hat{p}^2 \rangle$, we make use of the expression (3.4)

$$\langle \hat{p}^2 \rangle = 2m(\langle \hat{H} \rangle - \langle V \rangle) = m^2 v^2 + m^2 c^2, \quad (4.9)$$

whence

$$\Delta p = \sqrt{\langle \hat{p}^2 \rangle - \langle \hat{p} \rangle^2} = mc. \quad (4.10)$$

We notice that Δx and Δp are not time dependent (no spreading), as for a Gaussian wave packet ⁵. This has its origin in the fact that $\Psi(x, t)$ represents a monochromatic wave (the energy E is unique - no energetic spectrum, continuous or discret). It is therefore similar with the de Broglie (dB) plane wave. However, $\Psi(x, t)$ is normalized to one, in contrast with the dB wave, which is not. From (4.8) and (4.10) we find that

$$\Delta x \Delta p = \frac{\hbar}{\sqrt{2}} > \frac{\hbar}{2}, \quad (4.11)$$

and the Heisenberg relation is satisfied. It is worth to observe that Δp is very large, as a consequence of a very tiny Δx (of the order of the Compton wavelength). A large spread in momentum produces a large uncertainty in wavelength (from the dB relation) and so the particle is very localized. That corresponds to the Fig.1.7 from Griffiths book [1], where a sudden jerk has been given to the rope and a very narrow lump (our particle) travels along the x - direction. That is in the spirit of the δ -function source.

According to the viewpoint expressed until now, it turns out that our quantum object does appear as representing a wave, the velocity of which is determined by the initial conditions (the value of vector v). Once the wave propagates (along the x-direction) the Huygens-Fresnel principle applies: every new position of the wave-front represents a new source of propagation, as if the object were regenerating continuously. That is in the spirit of Everett's QM [8], where any object evolves into a macroscopic superposition of different versions of themselves. By performing a measurement we make only visible one branch of the superposition of identical objects.

If the above recipe is valid, then we may explain the motion of a particle: once put in motion with constant velocity, it actually propagates as a wave in accordance with Huygens-Fresnel principle and, as any wave, preserves its constant velocity of propagation, if not perturbed by external agents. Of course, we have to apply a force to stop that wave which, fortunately, propagates along one direction. This is a different property compared to standard waves, which propagate in all directions. In addition, their velocity is not given by the properties of the medium (quantum vacuum in our situation) but by the initial conditions (initial velocity).

5 deB-Bohm quantum potential

Let us see now an eventual connection between our results and the deB-Bohm mechanics. For that purpose, we write $\Psi(x, t)$ as

$$\Psi(x, t) = \rho(x, t) e^{\frac{i}{\hbar} S(x, t)}, \quad (5.1)$$

⁵We mention that a Gaussian-type packet is a choice: we suppose that a particle is described by it - a conjecture that is in accordance with experimental facts.

with

$$\rho(x, t) = \frac{\sqrt{ma}}{\hbar} e^{-\frac{ma|x-vt|}{\hbar^2}}, \quad S(x, t) = -(E + \frac{mv^2}{2})t + mvx. \quad (5.2)$$

The Schrodinger equation (2.1) gives us [9]

$$\frac{\partial S}{\partial t} + \frac{1}{2m} \left(\frac{\partial S}{\partial x} \right)^2 + V(x, t) + Q(x, t) = 0, \quad (5.3)$$

and

$$\frac{\partial \rho^2}{\partial t} + \frac{1}{m} \frac{\partial}{\partial x} \left(\rho^2 \frac{\partial S}{\partial x} \right) = 0, \quad (5.4)$$

where Q is the quantum potential

$$Q = -\frac{\hbar^2}{2m} \frac{1}{\rho} \frac{\partial^2 \rho}{\partial x^2}. \quad (5.5)$$

We have also

$$\frac{\partial S}{\partial t} = - \left(E + \frac{mv^2}{2} \right), \quad \frac{\partial S}{\partial x} = mv. \quad (5.6)$$

The expression for Q will be given by

$$Q = \frac{a}{2} \left[2\delta(x - vt) - \frac{ma}{\hbar^2} \right]. \quad (5.7)$$

With $a = \hbar c$, one obtains

$$Q = \hbar c \delta(x - vt) - \frac{mc^2}{2} = E - V. \quad (5.8)$$

Hence, the dB-Bohm quantum potential is the difference between the bound energy E [1] and our δ -function potential $V(x, t) = -\hbar c \delta(x - vt)$. In addition, we have $\langle Q \rangle = E - \langle V \rangle = mc^2/2$. One may easily check that, with the action S from (5.2) and Q from (5.7), Eqs. (5.3) and (5.4) are fulfilled.

As we already noticed, Hojman and Asenjo [6] enforced the condition that Bohm quantum potential cancels the external potential in order to find whether a quantum interacting particle can behave as a free classical particle (the particle becomes classical because Planck's constant appears in Bohm potential only). In our situation, the two potentials do not cancel but, nevertheless, their sum is a constant ($Q + V = E = const.$). With $a = \hbar c$ and for $|x - vt| \ll \hbar/mc$, the expression for the amplitude $\rho(x, t)$ acquires the approximate form

$$\rho(x, t) \approx \frac{1}{\sqrt{\lambda_C}} \left(1 - \frac{|x - vt|}{\lambda_C} \right) > 0, \quad (5.9)$$

where $\lambda_C = \hbar/mc$. It is worth observing that $\rho(x, t)$ from (5.9) equals the amplitude $A(z)$ from Eq.14 of [6] (excepting a minus sign), provided one consider $\gamma = \hbar c$ and $\beta = 1/\sqrt{\lambda_C}$ (with $z = x - vt$, $zsgn(z) = |x - vt|$). We notice that $-A(z)$ is also a solution of Eq.12 from [6], such that one could change the sign of the solution (14) for A , so becoming a positive quantity with our choices for β and γ , as our ρ with the approximation used in (5.9).

6 Conclusions

QFT shows us that particles represent excited states of vacuum quantum fields. That seems to suggest that quantum vacuum is filled with particles in ground (virtual) states. We tried to extend that viewpoint to macroscopic objects, whose motion is viewed as following the Huygens-Fresnel principle. It propagates as an unidirectional wave, to which the velocity is decided by the initial conditions (initial velocity in our situation). The object is moving as a wave to which propagation is determined by the secondary waves (sources) from the new wave front. We proposed an universal "strength" for the Griffiths potential, inspired by the necessity of having the correct physical units. Our model is developed on the grounds of QM and the rest energy from SR, because of the presence of the velocity of light. We found that our (normalizable) wave function is not dispersing (the wave is monochromatic, like the deB plane wave), even though $\langle x \rangle$ is time-dependent: Δx and Δp are constant and obey the position-momentum Heisenberg relation.

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