Functional calculus and dilation for C_0-groups of polynomial growth

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FUNCTIONAL CALCULUS AND DILATION FOR C_0-GROUPS OF POLYNOMIAL GROWTH

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ABSTRACT. Let $U(t) = e^{itB}$ be a C_0-group on a Banach space X. Let further $\phi \in C_c^\infty(\mathbb{R})$ satisfy $\sum_{n\in\mathbb{Z}} \phi(\cdot - n) \equiv 1$. For $\alpha \geq 0$, we put

$$E_\infty^\alpha = \{f \in C_b(\mathbb{R}) : \ \|f\|_{E_\infty^\alpha} = \sum_{n\in\mathbb{Z}} (1+|n|)^\alpha \|f * [\phi(\cdot - n)]\check{}\|_{L^\infty(\mathbb{R})} < \infty\},$$

which is a Banach algebra. It is shown that $\|U(t)\| \leq C(1+|t|)^\alpha$ for all $t \in \mathbb{R}$ if and only if the generator B has a bounded E_∞^α functional calculus, under some minimal hypothesis, which exclude simple counterexamples. A third equivalent condition is that $U(t)$ admits a dilation to a shift group on some space of functions $\mathbb{R} \to X$. In the case $U(t) = A^{it}$ with some sectorial operator A, we use this calculus to show optimal bounds for fractions of the semigroup generated by A, resolvent functions and variants of it. Finally, the E_∞^α calculus is compared with Besov functional calculi as considered in [4, 16].

1. INTRODUCTION

Throughout the article, we let X be a Banach space and $U(t)$ a strongly continuous group on X. Our major issue is to investigate groups satisfying

$$\|U(t)\| \leq C(1+|t|)^\alpha \quad (t \in \mathbb{R}) \tag{1.1}$$

for some constants $C, \alpha \geq 0$. In Theorem 4.9, a characterization of the polynomial growth (1.1) will be given in terms of the functional calculus of its generator.

Let iB be the generator of $U(t)$, i.e. $U(t) = e^{itB}$. Condition (1.1) implies that $\|U(t)\| \lesssim e^{\omega|t|}$, so that the spectrum of B is contained in the closure of $\mathrm{Str}_\omega = \{z \in \mathbb{C} : \ |\operatorname{Im} z| < \omega\}$. This holds for any $\omega > 0$, so that it is contained in fact in $\mathbb{R}$. If $f \in H^\infty(\mathrm{Str}_\omega)$, i.e. f is a bounded holomorphic function on Str_ω for some $\omega > 0$, then, with the Cauchy Integral Formula in mind, it is reasonable to define

$$f(B) = \frac{1}{2\pi i} \int_{\partial\,\mathrm{Str}_\omega} f(z)(z-B)^{-1} dz. \tag{1.2}$$

As the resolvents of B are uniformly bounded on the contour $\partial\,\mathrm{Str}_\omega$, the right hand side is well defined at least if $|f(z)| \leq Ce^{-\epsilon|\operatorname{Re} z|}$ for some $C, \epsilon > 0$. If an estimate $\|f(B)\| \leq C\|f\|_{\infty,\omega}$ holds for some $C > 0$, where $\|f\|_{\infty,\omega} = \sup_{z\in\mathrm{Str}_\omega} |f(z)|$, then, by an approximation procedure (see Proposition 2.2), the operator $f(B)$ can be defined for any $f \in H^\infty(\mathrm{Str}_\omega)$, and $f \mapsto f(B)$ is an algebra homomorphism $H^\infty(\mathrm{Str}_\omega) \to B(X)$, called the H^∞ calculus. The H^∞ calculus

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has by now a well established history [4, 11, 14, 19] and important applications e.g. in the spectral theory of elliptic differential equations and maximal regularity of parabolic evolution equations. Here the boundedness of the functional calculus is crucial, and the algebra of functions it is defined on reflects its quality.

Connections between growth conditions on the group $U(t)$ and functional calculus have been investigated in several articles [2, 10, 13]: First, if X is a Hilbert space, then B has a bounded $C_0(\mathbb{R})$ calculus (and consequently is of scalar type) if and only if $U(t)$ is a uniformly bounded group. This has been extended by Boyadzhiev and deLaubenfels [2] to arbitrary groups. Namely,

$$\{e^{-\omega|t|}U(t) : t \in \mathbb{R}\} \text{ is a bounded subset of } B(X) \text{ for any } \omega > \theta \tag{1.3}$$

if and only if

$$B \text{ has a bounded } H^\infty(\mathrm{Str}_\omega) \text{ calculus for any } \omega > \theta. \tag{1.4}$$

If X is a general Banach space, this is false. A group generator need not to have a bounded $H^\infty(\mathrm{Str}_\omega)$ calculus at all, even for large ω. A counterexample is the shift group $U(t)f(s) = f(s+t)$ on $X = L^p(\mathbb{R})$. In this case, $\|U(t)\|_{B(X)} = 1$, so that one could even choose $\theta = 0$ in (1.3), but the generator has a bounded H^∞ calculus if and only if $p = 2$ [4]. However it is true in any Banach space X that if the sets in (1.3) are γ-bounded (see Subsection 2.1 for the definition), then (1.4) holds. Moreover, if the space X has Pisier's geometric property (α) (see Subsection 2.3 for the definition), then also the converse holds [13, Theorem 6.8]. A different approach avoiding the notion of γ-boundedness is given in [9, Theorem 3.2].

The following class of functions on $\mathbb{R}$ will be at the center of our interest. Let $\alpha \geq 0$ and $p \in [1, \infty]$ be given parameters. Let $(\phi_n)_{n\in\mathbb{Z}}$ be a smooth partition of unity on the real line such that $\operatorname{supp} \phi_n \subset [n-1, n+1]$, i.e. $\sum_{n\in\mathbb{Z}} \phi_n(t) = 1$ for all $t \in \mathbb{R}$. Then we set

$$E_p^\alpha = \left\{ f \in L^p(\mathbb{R}) : \|f\|_{E_p^\alpha} = \sum_{n\in\mathbb{Z}} (1+|n|)^\alpha \|f * \check{\phi}_n\|_{L^p(\mathbb{R})} < \infty \right\}.$$

This definition resembles formally that of classical Besov spaces $\mathcal{B}_{p,1}^\alpha$, with the only difference that $(\phi_n)_{n\in\mathbb{Z}}$ have their support in equidistant intervals $[n-1, n+1]$. As for Besov spaces, the holomorphic and bounded functions on Str_ω form a dense subspace of E_p^α, and E_∞^α is a Banach algebra. However, the Fourier analysis and differentiability properties of the two concepts E_p^α and $\mathcal{B}_{p,1}^\alpha$ are quite different and this requires particular attention (Section 3).

The main result which is in Section 4 states that $U(t) = e^{itB}$ satisfies the polynomial growth (1.1) if and only if B has a E_∞^α calculus, under some further minimal calculus hypothesis: e.g. $\{U(t) : t \in [0,1]\}$ is γ-bounded, or alternatively, if X has Pisier's property (α), then it suffices that B has a $H^\infty(\mathrm{Str}_\omega)$ calculus. However some additional condition is necessary, as the above shift group example shows.

We will prove the non-trivial direction "only if" by showing that $U(t)$ admits a dilation to a shift group $W(t)$ acting on a further space Z which contains $L^2(\mathbb{R}, (1+|t|)^{-2\alpha}dt) \otimes X$. This means that there exist an injection $I : X \to Z$ and a surjection $P : Z \to X$ (both

continuous), such that

$$\begin{array}{ccc} X & \xrightarrow{U(t)} & X \\ {\scriptstyle I}\downarrow & & \uparrow{\scriptstyle P} \\ Z & \xrightarrow[W(t)]{} & Z \end{array}$$

commutes for any $t \in \mathbb{R}$. The shift group $W(t)$ satisfies again the polynomial bound (1.1). Moreover, the above diagram also holds with $f(B_U)$ and $f(B_W)$ in place of $U(t)$ and $W(t)$, and thus, any functional calculus of $W(t)$ transfers to a functional calculus of $U(t)$. In this sense, the shift group on Z is an extremal example of a group satisfying (1.1). The space Z will be a variant of the so-called Gaussian function space. These have been introduced in a series of articles, e.g. [14, 12], notably in connetion with H^∞ functional calculus.

The use of such dilations is not new in spectral theory. For example it is shown in [8], that if $-A$ is the generator of a bounded analytic semigroup $U(t)$ on a UMD space X, then the H^∞ calculus of A is bounded if and only if $U(t)$ admits a dilation to a bounded group on $Z = L^2([0,1], X)$. A survey on dilations of (semi)groups can be found in Davies's book [5].

We close this introduction with a short overview of Sections 2 – 6.

In Section 2, we introduce the necessary background on Banach space geometry, i.e. the notions of Gaussian function spaces, γ-boundedness, as well as type, cotype and Pisier's property (α). We will explain the H^∞ functional calculus which will serve as a foundation for all functional calculi we consider.

Section 3 is entirely devoted to the function spaces E_p^α. In particular, for $p \in [2, \infty]$, we show that a function $f \in E_p^\alpha$ admits an atomic decomposition

$$f(t) = \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} a_{nm} e^{int} \check{\phi}_0(t - \pi m), \tag{1.5}$$

where $\check{\phi}_0$ is a smooth rapidly decreasing function, and the coefficients satisfy

$$\sum_{n\in\mathbb{Z}} \langle n\rangle^\alpha \|(a_{nm})_m\|_{\ell^p} \cong \|f\|_{E_p^\alpha} < \infty.$$

This resembles somewhat a wavelet decomposition of Besov spaces as e.g. in [26, Theorem 3.5], but containing a Fourier sum component in place of a (space) dilation component as for wavelets.

Section 4 contains the main Theorem 4.9 of the article, i.e. the dilation of the polynomially bounded group and the equivalence to its E_∞^α functional calculus.

In Section 5, the E_∞^α calculus is extended to functions with values in the commutant set $\{T \in B(X) : TU(t) = U(t)T \ \forall\, t \in \mathbb{R}\}$, which is called operator valued calculus in the literature. As a by-product, we obtain γ-bounded operator families associated with $U(t)$. Here we focus on spaces X having property (α).

In Section 6, E_∞^α norms of some typical functions are explicitly calculated. Here we have in mind the case $U(t) = A^{it}$, where A is a sectorial operator and functions corresponding to the analytic semigroup generated by $-A$, as well as variants of its resolvent. This will give bounds and γ-bounds of operators such as $z^\beta A^\beta \exp(-zA)$, $\operatorname{Re} z > 0$, and $\lambda^\beta A^\gamma(\lambda + A^\delta)^{-1}$,

and these bounds are optimal within the class of sectorial operators such that A^{it} satisfies (1.1). Then we compare the E^α_∞ calculus with the $\mathcal{B}^\alpha_{\infty,1}$ calculus. In [4, Theorem 4.10], the following interesting equivalence was established:

Theorem 1.1. Let $U(t) = e^{itB}$ be a C_0-group. Let $\alpha > 0$. Then the following are equivalent:

- $(1)_\alpha$ B has a bounded $\mathcal{B}^\alpha_{\infty,1}$ functional calculus.
- $(2)_\alpha$ For any $\omega > 0$, B has a bounded $H^\infty(\mathrm{Str}_\omega)$ calculus, say $\Psi_\omega : H^\infty(\mathrm{Str}_\omega) \to B(X)$, and $\|\Psi_\omega\| \leq C\omega^{-\alpha}$ as $\omega \to 0$.

Let us contrast this with our result.

Theorem 1.2. Let $U(t) = e^{itB}$ be a C_0-group on a space X with property (α). Let $\beta \geq 0$. Then the following are equivalent

- $(1)'_\beta$ B has a bounded E^β_∞ functional calculus.
- $(2)'_\beta$ For any $\omega > 0$, B has a bounded $H^\infty(\mathrm{Str}_\omega)$ calculus, and $\|e^{itB}\| = \|\Psi_\omega(f_t)\| \leq C\omega^{-\beta}\|f_t\|_{\omega,\infty}$ for $f_t = e^{it(\cdot)}$ and any $t \in \mathbb{R}$.

This follows immediately from Theorem 4.9, noting that $\inf_{\omega\in(0,1)} \omega^{-\beta}e^{\omega|t|} \cong (1+|t|)^\beta$ and $\|f_t\|_{\infty,\omega} = e^{\omega|t|}$. For $\alpha = \beta$, condition $(2)'_\beta$ is clearly a weaker condition than $(2)_\alpha$. Thus, E^α_∞ functional calculus is weaker than $\mathcal{B}^\alpha_{\infty,1}$ calculus, i.e.

$$(1)_\alpha \Longrightarrow (1)'_\alpha. \tag{1.6}$$

In [4, Theorem 5.4], it is shown that the functional calculus of the f_t can be used to measure the H^∞ calculus angle:

Theorem 1.3. Let $U(t) = e^{itB}$ be a C_0-group and assume that B has a bounded $H^\infty(\mathrm{Str}_\omega)$ calculus for some $\omega \in (0,\pi)$. Let $\mu \in (0,\omega)$. The following are equivalent.

1. B has a bounded $H^\infty(\mathrm{Str}_\theta)$ calculus for any $\theta > \mu$.
2. For any $\theta > \mu$, there exists C_θ such that $\|e^{itB}\| \leq C_\theta \exp(\theta|t|)$.

Theorem 1.2 shows that replacing the exponential growth of the norm of e^{itB} in Theorem 1.3 (2) by a polynomial growth, the E^α_∞ calculus takes a (refined) role of the H^∞ calculus.

At the end of Section 6, we will focus on the reverse of (1.6) and study the gap between α and β for the implication $(1')_\alpha \Longrightarrow (1)_\beta$ where β depends on X (and α). First we will consider the most general case and allow any Banach space X, in which case $\beta = \alpha + 1$. Second we treat spaces X having non-trivial type and cotype. In this case, the geometry of X allows to narrow the gap between group growth α and Besov calculus order β, see Theorem 6.8.

2. Preliminaries and notations

2.1. **Gaussian function spaces.** We recall the construction of Gaussian function spaces from [13], see also [27].

Let H be a Hilbert space. We consider the tensor product $H \otimes X$ as a subspace of $B(H,X)$ in the usual way, i.e. by identifying $\sum_{k=1}^n h_k \otimes x_k \in H \otimes X$ with the mapping $u : h \mapsto \sum_{k=1}^n \langle h, h_k\rangle x_k$ for any finite families $h_1,\dots,h_n \in H$ and $x_1,\dots,x_n \in X$. Choose such families with corresponding u, where the h_k shall be orthonormal. Let $\gamma_1,\dots,\gamma_n$ be

independent standard Gaussian random variables over some probability space. We equip $H \otimes X$ with the norm

$$\big\|u\big\|_{\gamma(H,X)} = \big\|\sum_k \gamma_k \otimes x_k\big\|_{\mathrm{Gauss}(X)} = \left(\mathbb{E}\big\|\sum_k \gamma_k \otimes x_k\big\|_X^2\right)^{\frac{1}{2}}.$$

By [6, Corollary 12.17], this is independent of the choice of the h_k representing u. Then we let $\gamma_+(H,X)$ be the space of all $u \in B(H,X)$ such that

$$\|u\|_{\gamma(H,X)} = \sup\big\{\|uP\|_{\gamma(H,X)} :\ P : H \to H \text{ finite rank orthogonal projection}\big\} < \infty.$$

Further we denote $\gamma(H,X)$ the closure of $H \otimes X$ in $\gamma_+(H,X)$. We refer to $\gamma(H,X)$ as a Gaussian function space. If X does not contain c_0 isomorphically, then $\gamma(H,X) = \gamma_+(H,X)$ [13, Remark 4.2]. In the sequel, we will only make use of $\gamma(H,X)$ and not of $\gamma_+(H,X)$. If H is separable, then $\|u\|_{\gamma(H,X)} = \big\|\sum_k \gamma_k \otimes u(e_k)\big\|_{\mathrm{Gauss}(X)}$, where the e_k form an orthonormal basis of H [13, Remark 4.2].

Assume that (Ω,μ) is a σ-finite measure space and $H = L^2(\Omega)$. Denote $P_2(\Omega,X)$ the space of Bochner-measurable functions $f : \Omega \to X$ such that $x' \circ f \in L^2(\Omega)$ for all $x' \in X'$. We identify $P_2(\Omega,X)$ with a subspace of $B(L^2(\Omega),X'')$ by assigning to f the operator u_f defined by

$$\langle u_f h, x'\rangle = \int_\Omega \langle f(t), x'\rangle h(t) d\mu(t). \tag{2.1}$$

An application of the uniform boundedness principle shows that, in fact, u_f belongs to $B(L^2(\Omega),X)$ [13, Section 4], [7, Section 5.5]. Then we let

$$\gamma(\Omega,X) = \big\{f \in P_2(\Omega,X) :\ u_f \in \gamma(L^2(\Omega),X)\big\}$$

and set

$$\|f\|_{\gamma(\Omega,X)} = \|u_f\|_{\gamma(L^2(\Omega),X)}.$$

The space $\{u_f :\ f \in \gamma(\Omega,X)\}$ is a proper subspace of $\gamma(L^2(\Omega),X)$ in general. It is dense in $\gamma(L^2(\Omega),X)$ as it contains $L^2(\Omega) \otimes X$.

Resuming the above, we have the following embeddings of spaces, cf. also [21, Section 3].

$$L^2(\Omega) \otimes X \to \gamma(\Omega,X) \to \gamma(L^2(\Omega),X) \to B(L^2(\Omega),X).$$

In some cases, $\gamma(L^2(\Omega),X)$ and $\gamma(\Omega,X)$ can be identified with more classical spaces. If X is a Banach function space with finite cotype, e.g. an L^p space for some $p \in [1,\infty)$, then for any step function $f = \sum_{k=1}^n x_k \chi_{A_k} : \Omega \to X$, where $x_k \in X$ and the $A_k's$ are measurable and disjoint with $\mu(A_k) \in (0,\infty)$, we have (cf. [13, Remark 3.6, Example 4.6])

$$\|f\|_{\gamma(\Omega,X)} = \Big\|\sum_k \gamma_k \otimes \mu(A_k)^{\frac{1}{2}} x_k\Big\|_{\mathrm{Gauss}(X)} \cong \left\|\left(\sum_k \mu(A_k)|x_k|^2\right)^{\frac{1}{2}}\right\|_X = \left\|\left(\int_\Omega |f(t)(\cdot)|^2 d\mu(t)\right)^{\frac{1}{2}}\right\|_X.$$

The second equivalence follows from [6, Theorem 16.18]. The last expression above is a (classical) square function (see e.g. [4, Section 6]), whence for an arbitrary space X, $\|u\|_{\gamma(H,X)}$ is called generalized square function [13, Section 4]. In particular, if X is a Hilbert space,

then $\gamma(\Omega, X) = L^2(\Omega, X)$ with equal norms.

A further notion which is important for our purposes is the γ-boundedness.

Let $\tau \subset B(X)$. Then τ is called γ-bounded if there is a constant $C < \infty$ such that for any finite families $T_1, \dots, T_n \in \tau$ and $x_1, \dots, x_n \in X$,

$$\big\|\sum_k \gamma_k \otimes T_k x_k\big\|_{\mathrm{Gauss}(X)} \leq C \big\|\sum_k \gamma_k \otimes x_k\big\|_{\mathrm{Gauss}(X)}.$$

We denote by $\gamma(\tau)$ the smallest admissible constant C above, and set $\gamma(\tau) = \infty$ if τ is not γ-bounded. Note that one always has $\gamma(\tau) \geq \sup\{\|T\| : T \in \tau\}$. We record some properties which will be useful in subsequent sections.

Lemma 2.1. Let (Ω, μ) be a σ-finite measure space.

(1) Suppose that $(f_n)_n$ is a sequence in $L^\infty(\Omega)$ with $\sup_n \|f_n\|_\infty < \infty$ and $f_n(t) \to 0$ for almost all $t \in \Omega$. Then $\|f_n g\|_{\gamma(\Omega,X)} \to 0$ for all $g \in \gamma(\Omega, X)$.

(2) For $f \in \gamma(\Omega, X)$ and $g \in \gamma(\Omega, X')$, we have

$$\int_\Omega |\langle f(t), g(t)\rangle| d\mu(t) \leq \|f\|_{\gamma(\Omega,X)} \|g\|_{\gamma(\Omega,X')}.$$

(3) Suppose that X does not contain c_0 isomorphically. If $t \mapsto N(t)$ is strongly continuous $\Omega \to B(X)$ and $\{N(t) : t \in \Omega\}$ is γ-bounded with constant C, then for any $g \in \gamma(\Omega, X)$, $N(t)g(t)$ belongs to $\gamma(\Omega, X)$, and

$$\|N(t)g(t)\|_{\gamma(\Omega,X)} \leq C \|g\|_{\gamma(\Omega,X)}.$$

(4) Let $T \in B(X)$ and $K \in B(H_2, H_1)$, where H_1, H_2 are Hilbert spaces. Then for $u \in \gamma(H_1, X)$, we have $TuK \in \gamma(H_2, X)$ and

$$\|TuK\|_{\gamma(H_2,X)} \leq \|T\| \, \|u\|_{\gamma(H_1,X)} \|K\|.$$

Thus, any $K \in B(H_2, H_1)$ can be lifted to an operator

$$K^\otimes : \gamma(H_1, X) \to \gamma(H_2, X),\ u \mapsto u \circ K',$$

with $\|K^\otimes\| = \|K\|$. In particular,

(a) if $m \in L^\infty(\Omega)$ and $u \in \gamma(L^2(\Omega), X)$, then $m \cdot u$, defined by $[m \cdot u](h) = u(m \cdot h)$, satisfies $\|m \cdot u\|_\gamma \leq \|m\|_\infty \|u\|_\gamma$.

(b) $\|u\|_{\gamma(L^2(g(t)^2 dt),X)} = \|g \cdot u\|_{\gamma(L^2(dt),X)}$, for $g : \Omega \to (0, \infty)$ and $u \in \gamma(L^2(g(t)^2 dt), X)$.

(5) *Extended triangle inequality.* Let (Ω_1, μ_1) and (Ω_2, μ_2) be σ-finite measure spaces. Let $g : \Omega_1 \times \Omega_2 \to X$ be weakly measurable and assume that for any $x' \in X'$, we have

$$\int_{\Omega_1} \left(\int_{\Omega_2} |\langle g(t,s), x'\rangle| ds \right)^2 dt < \infty.$$

Then

$$\int_{\Omega_2} g(\cdot, s) ds \in \gamma(\Omega_1, X) \text{ and } \| \int_{\Omega_2} g(\cdot, s) ds\|_{\gamma(\Omega_1,X)} \leq \int_{\Omega_2} \|g(\cdot, s)\|_{\gamma(\Omega_1,X)} ds$$

hold as soon as the right most expression is finite.

Proof. For a proof of (1)-(4), we refer to [13, Lemma 4.10, Corollary 5.5, Proposition 4.11, Example 5.7, Proposition 4.3]. The equalities in (4) (a) (resp. (b)) follow from the boundedness (resp. isometry) of the mappings $L^2(\Omega) \to L^2(\Omega)$, $f \mapsto m \cdot f$ and $L^2(g(t)^2 dt) \to L^2(dt)$, $f \mapsto g \cdot f$.

(5) We have $\int g(\cdot, s)ds \in P_2(\Omega_1, X)$, since

$$\int \left| \int \langle g(t,s), x' \rangle ds \right|^2 dt \leq \int \left(\int |\langle g(t,s), x' \rangle| ds \right)^2 dt,$$

where the right expression is finite due to the assumptions.

Further, for any $x' \in X'$ and $h \in L^2(\Omega_1)$, we have

$$\int \int \langle g(t,s), x' \rangle h(t) ds dt = \int \int \langle g(t,s), x' \rangle h(t) dt ds.$$

Indeed, we can apply Fubini's theorem, since

$$\int \int |\langle g(t,s), x' \rangle h(t)| \, ds dt = \int \left(\int |\langle g(t,s), x' \rangle| ds \right) |h(t)| dt$$

is finite due to the same assumption as above.

Then the claim follows from the following calculation:

$$\begin{aligned}
\| \sum_k \gamma_k \otimes \int g(t,s) ds h_k(t) dt \|_{\mathrm{Gauss}(X)} &= \| \sum_k \gamma_k \otimes \int \int g(t,s) h_k(t) dt ds \|_{\mathrm{Gauss}(X)} \\
&= \left(\int \| \sum_k \gamma_k(\lambda) \int \int g(t,s) h_k(t) dt ds \|_X^2 d\lambda \right)^{\frac{1}{2}} \\
&\leq \int \left(\int \| \sum_k \gamma_k(\lambda) \int g(t,s) h_k(t) dt \|_X^2 d\lambda \right)^{\frac{1}{2}} ds \\
&\leq \int \| g(\cdot, s) \|_{\gamma(\Omega_1, X)} ds,
\end{aligned}$$

where γ_k are independent standard Gaussians and h_k form any orthonormal system in $L^2(\Omega_1)$.

□

2.2. **Preliminaries on the H^∞ calculus.** Let $\theta > 0$. We define: $\mathrm{Str}_\theta := \{x + iy : \ x \in \mathbb{R}, |y| < \theta\}$ and $H^\infty(\mathrm{Str}_\theta) := \{f : \mathrm{Str}_\theta \to \mathbb{C} : \ f \text{ is analytic and bounded}\}$. This space is equipped with the norm $\|f\|_{\infty,\theta} = \sup_{\lambda \in \mathrm{Str}_\theta} |f(\lambda)|$ and is complete. Put $H_0^\infty(\mathrm{Str}_\theta) := \{f \in H^\infty(\mathrm{Str}_\theta) : \exists \epsilon, C > 0 \text{ s.th. } |f(z)| \leq Ce^{-\epsilon |\operatorname{Re} z|}\}$. Let B be a densely defined operator on some Banach space X. We say that B is a *strip-type* operator, if $\sigma(B) \subset \mathrm{Str}_\omega$ for some $\omega > 0$, and if for any $\theta > \omega$, there exists C such that

$$\|(\lambda - B)^{-1}\| \leq C$$

for any $\lambda \in \mathbb{C} \backslash \mathrm{Str}_\theta$. One denotes by $\omega(B)$ the strip height, which is the infimum of all $\omega > 0$ satisfying the above.

An important class of strip-type operators is given by C_0-groups: If $U(t)$ is a strongly continuous group with infinitesimal generator C, such that $\|U(t)\| \lesssim e^{\omega|t|}$, then $B = -iC$ is a strip-type operator with $\omega(B) \leq \omega$ [9, p. 91].

A strip-type operator has a holomorphic functional calculus [9]. Namely, for $\omega > \omega(B)$ and $f \in H_0^\infty(\mathrm{Str}_\omega)$, one defines the operator

$$f(B) = \frac{1}{2\pi i} \int_\Gamma f(\lambda) R(\lambda, B) d\lambda,$$

where Γ is the boundary $\partial \mathrm{Str}_\gamma$ with $\omega(B) < \gamma < \omega$ oriented counterclockwise.

The definitions do not depend on γ and the resulting mapping $f \mapsto f(B)$ is an algebra homomorphism $H_0^\infty(\mathrm{Str}_\omega) \to B(X)$.

One says that B has a (bounded) $H^\infty(\mathrm{Str}_\omega)$ calculus if $H_0^\infty(\mathrm{Str}_\omega) \to B(X)$, $f \mapsto f(B)$ is bounded. In this case, one can extend that mapping to a bounded homomorphism $H^\infty(\mathrm{Str}_\omega) \to B(X)$. It satisfies the following proposition, usually called convergence lemma in the literature, (see e.g. [4, Lemma 2.1] and [9, Proposition 5.1.7]). In particular, one recovers the C_0-group from the functional calculus by $U(t) = e^{it(\cdot)}(B)$.

Proposition 2.2. Let B be a strip-type operator with bounded $H^\infty(\mathrm{Str}_\omega)$ calculus for some $\omega > \omega(B)$. Let further $(f_k)_k$ be a sequence in $H^\infty(\mathrm{Str}_\omega)$ such that $f_k(z) \to f(z)$ for any $z \in \mathrm{Str}_\omega$ as $k \to \infty$, and such that $\sup_k \|f_k\|_{\infty,\omega} < \infty$.

Then f belongs to $H^\infty(\mathrm{Str}_\omega)$ and $f(B)x = \lim_k f_k(B)x$ for arbitrary $x \in X$.

As a particular example, note that for any $\omega > 0$, the sequence

$$\psi_k(z) = \left(\frac{e^z}{(1+e^z)^2} \right)^{\frac{1}{k}} \quad (z \in \mathbb{C},\, k \in \mathbb{N}) \tag{2.2}$$

approximates the function $z \mapsto 1$, and by the above proposition, for any $f \in H^\infty(\mathrm{Str}_\omega)$, the operator $f(B)$ can be recovered as the strong limit $\lim_k (\psi_k f)(B)$.

We remark that there is a sectorial version of all the above strip-type definitions, which plays an important role in the literature (see e.g. [4], [19]). More precisely, let Σ_ω be the image of Str_ω under the conformal mapping $\exp : z \mapsto e^z$. Let also $H_{(0)}^\infty(\Sigma_\omega) = H_{(0)}^\infty(\mathrm{Str}_\omega) \circ \{\exp\}$. An injective and densely defined operator A is called sectorial if $\sigma(A) \subset \overline{\Sigma_\omega}$ for some $\omega \in (0, \pi)$, and if for any $\theta > \omega$, there exists $C > 0$ such that $\|\lambda R(\lambda, A)\| \leq C$ for any $\lambda \in \mathbb{C} \backslash \overline{\Sigma_\theta}$.

Then for $f \in H_0^\infty(\Sigma_\theta)$, one sets

$$f(A) = \frac{1}{2\pi i} \int_\Gamma f(\lambda) R(\lambda, A) d\lambda,$$

where $\Gamma = \partial \Sigma_\gamma$ and $\omega < \gamma < \theta$. If $\|f(A)\| \leq C \|f\|_{\infty,\theta}$, then A is said to have an $H^\infty(\Sigma_\theta)$ calculus. In this case, similar to the strip case, one defines $f(A)$ for $f \in H^\infty(\Sigma_\theta)$.

If A is a sectorial operator (having an $H^\infty(\Sigma_\theta)$ calculus), then there is a strip-type operator B such that $f(A) = (f \circ \exp)(B)$ for $f \in H_0^\infty(\Sigma_\omega)$ (having an $H^\infty(\mathrm{Str}_\theta)$ calculus [9, Corollary 4.2.5].

For more information on the H^∞ calculus, see e.g. [4], especially Section 2 and [19], especially Chapter 9.

2.3. **Property** (α)**.** Following [23], we say that X has property (α) if there is a constant $C \geq 1$ such that for any finite family (x_{ij}) in X and any finite family (t_{ij}) of complex numbers,

$$\Big\|\sum_{i,j} \gamma_i \otimes \gamma_j \otimes t_{ij}x_{ij}\Big\|_{\mathrm{Gauss}(\mathrm{Gauss}(X))} \leq C \sup_{i,j}|t_{ij}| \, \Big\|\sum_{i,j} \gamma_i \otimes \gamma_j \otimes x_{ij}\Big\|_{\mathrm{Gauss}(\mathrm{Gauss}(X))}. \tag{2.3}$$

Equivalently, X has property (α) if and only if we have a uniform equivalence

$$\Big\|\sum_{i,j} \gamma_i \otimes \gamma_j \otimes x_{ij}\Big\|_{\mathrm{Gauss}(\mathrm{Gauss}(X))} \cong \Big\|\sum_{i,j} \gamma_{ij} \otimes x_{ij}\Big\|_{\mathrm{Gauss}(X)},$$

where $(\gamma_{ij})_{i,j\geq 1}$ be a doubly indexed family of independent standard Gaussian variables.

Property (α) is inherited by closed subspaces and isomorphic spaces as one can see directly from Definition (2.3). Let (Ω, μ) be a measure space and $1 \leq p < \infty$. Using Kahane's inequality [20, Corollary 3.4.1], one can show that the spaces $L^p(\Omega)$ have property (α), and moreover, if X has property (α), then also $L^p(\Omega, X)$ has property (α).

2.4. **Type and cotype.** A Banach space X is said to have type $p \in [1, 2]$ if there exists $C > 0$ such that for any finite family $x_1, \ldots, x_n \in X$ we have

$$\Big\|\sum_k \gamma_k \otimes x_k\Big\|_{\mathrm{Gauss}(X)} \leq C \left(\sum_k \|x_k\|_X^p\right)^{\frac{1}{p}}. \tag{2.4}$$

Further, X is said to have cotype $q \in [2, \infty]$ if there exists $C > 0$ such that for any finite family $x_1, \ldots, x_n \in X$, we have

$$\left(\sum_k \|x_k\|_X^q\right)^{\frac{1}{q}} \leq C\Big\|\sum_k \gamma_k \otimes x_k\Big\|_{\mathrm{Gauss}(X)} \tag{2.5}$$

(left hand side replaced by the sup norm for $q = \infty$).

Any Banach space X always has type 1 and cotype ∞, whence these are also called trivial type and cotype. The notions become more restrictive for larger p and smaller q. A space X has type 2 and cotype 2 if and only if it is isomorphic to a Hilbert space [6, Corollary 12.20]. It is clear that type and cotype are inherited by subspaces and isomorphic spaces.

Let (Ω, μ) be a measure space and $1 \leq p < \infty$. Then $L^p(\Omega)$ has type $\min(p, 2)$ and cotype $\max(p, 2)$ [6, Corollary 11.7]. More generally, for a Banach space X, $L^p(\Omega, X)$ has type $\min(p, \operatorname{type} X)$ and cotype $\max(p, \operatorname{cotype} X)$ [6, Theorem 11.12]. This is false in general for $p = \infty$, and the space $L^\infty(\Omega)$ has trivial type 1 (and no better type), if it is infinite dimensional.

Property (α), type and cotype are usually defined in terms of Rademacher instead of Gaussian variables. One obtains equivalent notions [6, Theorem 12.26, 12.28].

Let us resume the geometric properties of X that we considered in this section by putting them into a logical order.

$$\begin{array}{ccccc} X \text{ has non-trivial type} & \Longrightarrow & X \text{ has non-trivial cotype} & \Longrightarrow & X \not\supseteq c_0 \\ & & \Uparrow (*) & & \\ & & X \text{ has property } (\alpha) & & \end{array}$$

As references for the above implications, we refer to [6, Chapters 11, 13 and 14]. In $(*)$, the converse holds if X is a Banach function space [23].

3. The spaces E_p^α

In this section, we develop the Fourier analysis of the spaces E_p^α, and compare them to the Besov spaces $\mathcal{B}_{p,1}^\alpha = \mathcal{B}_{p,1}^\alpha(\mathbb{R})$. The spaces E_p^α are defined by a summability condition for a decomposition in the Fourier image, in a similar way to the Besov spaces.

Let $(\psi_n)_{n\in\mathbb{Z}}$ be a dyadic partition of unity, i.e. the following conditions hold (cf. [1, Lemma 6.1.7]): $\psi_n \in C_c^\infty(\mathbb{R})$, $\operatorname{supp}\psi_0 \subset [-1,1]$, $\operatorname{supp}\psi_n \subset [2^{n-2}, 2^n]$ for $n \geq 1$ and $\operatorname{supp}\psi_n \subset [-2^{|n|}, -2^{|n|-2}]$ for $n \leq -1$. Further, $\sum_{n\in\mathbb{Z}} \psi_n(t) = 1$ for all $t \in \mathbb{R}$, $\psi_n = \psi_1(2^{-n}\cdot)$ for $n \geq 1$, and $\psi_{-n} = \psi_n(-\cdot)$ for $n \in \mathbb{Z}$. Then one defines the Besov space [25, p. 45]

$$\begin{aligned} \mathcal{B}_{p,1}^\alpha = \{ & f : \mathbb{R} \to \mathbb{C},\ f \text{ uniformly continuous and bounded,} \\ & \|f\|_{\mathcal{B}_{p,1}^\alpha} = \sum_{n\in\mathbb{Z}} 2^{|n|\alpha} \|f * \check{\psi}_n\|_{L^p(\mathbb{R})} < \infty \}, \end{aligned}$$

The new spaces E_p^α are defined with an equidistant partition of unity $(\phi_n)_{n\in\mathbb{Z}}$ in place of $(\psi_n)_{n\in\mathbb{Z}}$, which gives a new scale with different properties.

Let us give an overview of the section. After definition and some useful elementary properties of E_p^α, we show that the norm of E_p^α admits an atomic decomposition in the case $2 \leq p \leq \infty$. Namely, f can be written as an (infinite) linear combination of one rapidly decreasing function shifted in space and Fourier phase, see (3.2). The decomposition reduces for $p = \infty$ to the Fourier series in the case that f is periodic. Subsequently we show optimal embedding results between the classical spaces $\mathcal{B}_{\infty,1}^\alpha$ and the new ones E_∞^α.

We write in short $\langle t \rangle = 1 + |t|$. We have the following elementary inequalities:

$$\langle k + l \rangle \lesssim \langle k \rangle + \langle l \rangle \lesssim \langle k \rangle \cdot \langle l \rangle.$$

Definition 3.1. Let $\phi_0 \in C_c^\infty(\mathbb{R})$, and assume that $\operatorname{supp}\phi_0 \subset [-1,1]$ and $\sum_{n=-\infty}^{\infty} \phi_0(t-n) = 1$ for all $t \in \mathbb{R}$. For $n \in \mathbb{Z}$ we put $\phi_n = \phi_0(\cdot - n)$ and $\widetilde{\phi}_n = \sum_{k=-1}^{1} \phi_{n+k}$. We call the collection $(\phi_n)_{n\in\mathbb{Z}}$ an equidistant partition of unity. Note that $\widetilde{\phi}_m \phi_n = \phi_n$ for $m = n$ and $\widetilde{\phi}_m \phi_n = 0$ for $|n - m| \geq 2$.

Let $\alpha \geq 0$ be a parameter, $1 \leq p \leq \infty$ and $(\phi_n)_n$ an equidistant partition of unity. We define

$$E_p^\alpha = \{f \in L^p(\mathbb{R}) : \|f\|_{E_p^\alpha} = \sum_{n\in\mathbb{Z}} \langle n\rangle^\alpha \|f * \check{\phi}_n\|_p < \infty\} \tag{3.1}$$

and equip it with the norm $\|f\|_{E_p^\alpha}$. Further, we set

$$E_0^\alpha = \{f \in E_\infty^\alpha : \lim_{|t|\to\infty} f(t) = 0\}$$

with the norm $\|f\|_{E_0^\alpha} = \|f\|_{E_\infty^\alpha}$.

Let us record some elementary properties of the spaces E_p^α and E_0^α.

Proposition 3.2. Let $\alpha \geq 0$ and $1 \leq p \leq \infty$.

(1) E_p^α and E_0^α are Banach spaces.
(2) *Dense subset.* For any $\omega > 0$, $H^\infty(\mathrm{Str}_\omega) \cap E_p^\alpha$ is dense in E_p^α. More precisely, for $f \in E_p^\alpha$, $\sum_{n=-N}^M f * \check{\phi}_n$ converges to f in E_p^α $(N, M \to \infty)$.
(3) *Multiplicativity.* For $f \in E_\infty^\alpha$ and $g \in E_p^\alpha$, we have $fg \in E_p^\alpha$ and $\|fg\|_{E_p^\alpha} \lesssim \|f\|_{E_\infty^\alpha}\|g\|_{E_p^\alpha}$. Similarly, for $f \in E_\infty^\alpha$ and $g \in E_0^\alpha$, we have $fg \in E_0^\alpha$.
(4) E_p^α is independent of the choice of the partition of unity, and different choices give equivalent norms.
(5) Translations $f \mapsto f(\cdot - t)$ and dilations $f \mapsto f(a\cdot)$ $(t \in \mathbb{R}, a > 0)$ are isomorphisms $E_p^\alpha \to E_p^\alpha$.
(6) Let $J_\alpha(f) = (\langle\cdot\rangle^{-\alpha}\hat{f})^\vee$. Then J_α is an isomorphism $E_p^0 \to E_p^\alpha$ and $\mathcal{B}_{\infty,1}^0 \to \mathcal{B}_{\infty,1}^\alpha$.

Proof. (1): For E_p^α, this can be shown as for Besov spaces. Then E_0^α is a Banach space because $\|\cdot\|_\infty \leq \|\cdot\|_{E_\infty^\alpha}$, and thus, convergence to 0 at infinity is preserved by limits in E_0^α.

(2): Let $\psi = \sum_{k=-N}^M \phi_k \in C_c^\infty(\mathbb{R})$. As $f * \check{\psi} = (\hat{f}\psi)^\vee$ is the Fourier transform of a distribution with compact support, the Paley-Wiener theorem yields that $f * \check{\psi}$ is an entire function given by $f * \check{\psi}(t + is) = \int_{\mathbb{R}} f(r - t)[e^{-s(\cdot)}\psi]^\vee(r)dr$. In particular, for $|s| < \omega$,

$$\|f * \check{\psi}(\cdot + is)\|_{L^\infty(\mathbb{R})} \leq \|f\|_{L^p(\mathbb{R})}\|[e^{-s(\cdot)}\psi]^\vee\|_{L^{p'}(\mathbb{R})} \leq C(f, N, M, \omega) < \infty.$$

Therefore, $f * \check{\psi} \in H^\infty(\mathrm{Str}_\omega)$ for any $\omega > 0$. As $\psi(t) = 0$ for $t \in [-N - 1, M + 1]^c$ (resp. $\psi(t) = 1$ for $t \in [-N, M]$) and $\operatorname{supp}\phi_n \subset [n - 1, n + 1]$, we have $\check{\psi} * \check{\phi}_n = 0$ for $n \in [-N - 2, M + 2]^c$ (resp. $\check{\psi} * \check{\phi}_n = \check{\phi}_n$ for $n \in [-N + 1, M - 1]$). Thus

$$\|f - f * \check{\psi}\|_{E_p^\alpha} = \sum_{n\in\mathbb{Z}} \langle n\rangle^\alpha \|f * \check{\phi}_n - f * \check{\psi} * \check{\phi}_n\|_p \leq \sum_{n\in F} \langle n\rangle^\alpha \|f * \check{\phi}_n\|_p + \sum_{n\in G} \langle n\rangle^\alpha \|f * \check{\phi}_n\|_p (1 + 3\|\check{\phi}_0\|_1),$$

where $F = \mathbb{Z}\backslash\{-N-2, -N-1, \ldots, M+1, M+2\}$ and $G = \{-N-2, -N-1, -N, M, M+1, M+2\}$. We used that $\|\check{\phi}_n\|_1 = \|\check{\phi}_0\|_1$ is a constant. For $N, M \to \infty$, the first sum converges to 0 since $f \in E_p^\alpha$, and the second sum converges to 0 by the same reason.

(3): Let $f \in E_\infty^\alpha$ and $g \in E_p^\alpha$. For $k, l \in \mathbb{Z}$, we put $f_k = f * \check{\phi}_k$ and $g_l = g * \check{\phi}_l$. By (2), it suffices to consider the case that there exist $K, L \in \mathbb{N}$ such that $f = \sum_{-K}^K f_k$, $g = \sum_{-L}^L g_l$.

For $n \in \mathbb{Z}$, we have

$$\|(f_k g_l) * \check{\phi}_n\|_p \leq \|\check{\phi}_n\|_1 \|f_k g_l\|_p \leq \|\check{\phi}_1\|_1 \|f_k\|_\infty \|g_l\|_p.$$

Note that

$$\operatorname{supp}(f_k g_l)\hat{} = \operatorname{supp}(\hat{f}_k * \hat{g}_l) \subset \operatorname{supp}(\phi_k) + \operatorname{supp}(\phi_l) \subset [k+l-2, k+l+2].$$

Thus if $|n-(k+l)| > 2$, we have $(f_k g_l) * \check{\phi}_n = 0$. If $|n-(k+l)| \leq 2$, then $\langle n\rangle^\alpha \lesssim \langle k\rangle^\alpha + \langle l\rangle^\alpha \lesssim \langle k\rangle^\alpha \langle l\rangle^\alpha$. Therefore,

$$\begin{aligned}\|fg\|_{E_p^\alpha} &= \sum_{n\in\mathbb{Z}} \langle n\rangle^\alpha \|(fg) * \check{\phi}_n\|_p \leq \sum_{|k|\leq K} \sum_{|l|\leq L} \sum_{|n-(k+l)|\leq 2} \langle n\rangle^\alpha \|f_k g_l * \check{\phi}_n\|_p \\ &\lesssim \sum_{|k|\leq K} \sum_{|l|\leq L} \langle k\rangle^\alpha \langle l\rangle^\alpha \|f_k\|_\infty \|g_l\|_p = \|f\|_{E_\infty^\alpha} \|g\|_{E_p^\alpha}.\end{aligned}$$

If $g \in E_0^\alpha$, then $fg \in E_\infty^\alpha$ by the above, and clearly $\lim_{|t|\to\infty} f(t)g(t) = 0$. Thus, $fg \in E_0^\alpha$.

(4): Let $(\phi_n)_n$ and $(\psi_n)_n$ be two equidistant partitions of unity. Then $f * \check{\phi}_n = f * \check{\phi}_n * \tilde{\psi}_n\check{}$ and consequently,

$$\begin{aligned}\sum_{n\in\mathbb{Z}} \langle n\rangle^\alpha \|f * \check{\phi}_n\|_p &\leq \sum_{n\in\mathbb{Z}} \langle n\rangle^\alpha \|\check{\phi}_n\|_1 \|f * \tilde{\psi}_n\check{}\,\|_p \leq \sum_{k=-1}^{1} \sum_{n+k\in\mathbb{Z}} \langle n+k\rangle^\alpha \|f * \check{\psi}_{n+k}\|_p \\ &\lesssim \sum_{n\in\mathbb{Z}} \langle n\rangle^\alpha \|f * \check{\psi}_n\|_p.\end{aligned}$$

Exchanging the roles of $(\phi_n)_n$ and $(\psi_n)_n$ gives the result.

(5): For translations, the isomorphism follows directly from $f(\cdot - t) * \check{\phi}_n = (f * \check{\phi}_n)(\cdot - t)$. For dilations, note that $f(a\cdot) * \check{\phi}_n = (f * \check{\psi}_n)(a\cdot)$ with $\psi_n = \phi_n(a\cdot)$. There exists $N = N(a)$ such that $\psi_n = \psi_n \cdot \left(\sum_{k=-N}^N \phi_{\lfloor n/a\rfloor + k}\right)$. Since $\|\check{\psi}_n\|_1 = \|\check{\psi}_0\|_1$ we have

$$\|f(a\cdot) * \check{\phi}_n\|_p \lesssim \|f * \check{\psi}_n\|_p = \left\| f * \check{\psi}_n * \left(\sum_{k=-N}^N \check{\phi}_{\lfloor \frac{n}{a}\rfloor + k} \right) \right\|_p \leq \sum_{k=-N}^N \|f * \check{\phi}_{\lfloor \frac{n}{a}\rfloor + k}\|_p \|\check{\psi}_0\|_1.$$

As moreover $\langle n\rangle \cong \langle \lfloor n/a\rfloor + k\rangle$ for $|k| \leq N$, we get

$$\|f(a\cdot)\|_{E_p^\alpha} = \sum_{n\in\mathbb{Z}} \langle n\rangle^\alpha \|f(a\cdot) * \check{\phi}_n\|_p \lesssim \sum_{n\in\mathbb{Z}} \sum_{|k|\leq N} \langle n\rangle^\alpha \|f * \check{\phi}_{\lfloor n/a\rfloor + k}\|_p \lesssim \sum_{n\in\mathbb{Z}} \langle n\rangle^\alpha \|f * \check{\phi}_n\|_p = \|f\|_{E_p^\alpha}.$$

Exchanging a and $1/a$ gives the estimate the other way around.

(6): If $\langle n\rangle^\alpha \|J_\alpha f * \check{\phi}_n\|_p \cong \|f * \check{\phi}_n\|_p$ for any $f \in E_p^0$, then clearly, $J_\alpha : E_p^0 \to E_p^\alpha$ is an isomorphism. We have

$$\begin{aligned}\|J_\alpha f * \check{\phi}_n\|_p &= \|f * \check{\phi}_n * (\langle\cdot\rangle^{-\alpha} \widetilde{\phi}_n)\check{}\,\|_p \\ &\leq \|f * \check{\phi}_n\|_p \|(\langle\cdot\rangle^{-\alpha} \widetilde{\phi}_n)\check{}\,\|_1\end{aligned}$$

and

$$\|(\langle\cdot\rangle^{-\alpha}\widetilde{\phi_n})^{\check{}}\|_1 \leq \|\langle\cdot\rangle^{-\alpha}\widetilde{\phi_n}\|_{W_p^1} \lesssim \langle n\rangle^{-\alpha}.$$

Since this is also valid for negative α, we get

$$\|J_\alpha f * \check{\phi}_n\|_p \lesssim \langle n\rangle^{-\alpha}\|f * \check{\phi}_n\|_p = \langle n\rangle^{-\alpha}\|J_{-\alpha}J_\alpha f * \check{\phi}_n\|_p \lesssim \|J_\alpha f * \check{\phi}_n\|_p.$$

The case $J_\alpha : \mathcal{B}^0_{\infty,1} \to \mathcal{B}^\alpha_{\infty,1}$ is proved in the same manner, see [1, Lemma 6.2.7 and Theorem 6.2.7]. □

For $2 \leq p \leq \infty$, the space E_p^α can be described in terms of an atomic decomposition, both in space variable and, once spatially localized, in phase variable in the spirit of the Fourier series.

We give first a heuristic view of these two steps. Let $f \in E_p^\alpha$.

(1) Space decomposition. Consider the rapidly decreasing function $\check{\phi}_0$ on $\mathbb{R}$. Discrete translations of it sum up to a constant. Indeed, we have,

$$\frac{1}{2\pi} = \frac{1}{2\pi}\phi_0(0) = [\phi_0\delta_0]^{\check{}} = [\sum_{k\in\mathbb{Z}} \phi_0(\cdot)\delta_0(\cdot - 2k)]^{\check{}} = \sum_{m\in\mathbb{Z}} \check{\phi}_0(\cdot - \pi m),$$

where we used the Poisson summation formula in the fourth equation.

(2) Fourier series expansion. Then split up $f = \sum_{m\in\mathbb{Z}} f_m$, where $f_m = f\check{\phi}_0(\cdot - m\pi)$ is concentrated around $m\pi$. Consider the Fourier series expansion $f_m = \sum_{n\in\mathbb{Z}} a_{nm}e^{in(\cdot)}$ valid on $((m - \frac{1}{2})\pi, (m + \frac{1}{2})\pi)$.

The next proposition will show that the two steps give a representation $f = \sum_{m,n\in\mathbb{Z}} a_{nm}e^{in(\cdot)}$ $\check{\phi}_0(\cdot - \pi m)$ and that the norm of f can be expressed in terms of the a_{nm}'s.

We let

$$\ell^1_\alpha(\ell^p) = \ell^1(\mathbb{Z}, \langle n\rangle^\alpha, \ell^p(\mathbb{Z})) = \left\{(a_{nm}) \in \mathbb{C}^{\mathbb{Z}\times\mathbb{Z}} : \ \|a_{nm}\|_{\ell^1_\alpha(\ell^p)} = \sum_{n\in\mathbb{Z}} \langle n\rangle^\alpha \|(a_{nm})_m\|_{\ell^p(\mathbb{Z})} < \infty\right\},$$

equipped with the norm $\|a_{nm}\|_{\ell^1_\alpha(\ell^p)}$ above.

The decomposition in the next proposition is not independent of the choice of the $(\phi_n)_n$, so we fix one. We further assume that $\operatorname{supp}\phi_0 \subset (-1, 1)$ and choose some $\eta_0 \in C_c^\infty(\mathbb{R})$ such that $\operatorname{supp}\eta_0 \subset [-1, 1]$ and $\eta_0(t) = 1$ for all $t \in \operatorname{supp}\phi_0$. Put $\eta_n = \eta_0(\cdot - n)$.

Proposition 3.3. Let $\alpha \geq 0$ and $2 \leq p \leq \infty$.

(1) The linear mappings

$$T : E_p^\alpha \to \ell^1_\alpha(\ell^p),\ f \mapsto (Tf)_{nm} = \frac{1}{2}\int_{\mathbb{R}} f(t)e^{-int}\hat{\eta}_0(t - \pi m)dt$$

and

$$S : \ell^1_\alpha(\ell^p) \to E_p^\alpha,\ (a_{nm}) \mapsto \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} a_{nm}e^{in(\cdot)}\check{\phi}_0(\cdot - \pi m) \tag{3.2}$$

are well-defined, continuous and $ST = \mathrm{Id}_{E_p^\alpha}$. Here, in the definition of S, the sum over m converges locally uniformly, and the sum over n converges in E_p^α. In particular, every $f \in E_p^\alpha$ has a decomposition

$$f(t) = \sum_{nm} a_{nm} e^{int} \check{\phi}_0(t - \pi m)$$

such that

$$\|f\|_{E_p^\alpha} \cong \sum_{n\in\mathbb{Z}} \langle n\rangle^\alpha \|(a_{nm})_m\|_p.$$

(2) In the particular case that $p = \infty$ and $f \in E_\infty^\alpha$ is π-periodic, $(Tf)_{nm} = (-1)^{nm} a_n$ with $a_n = \frac{1}{2}\int_{\mathbb{R}} f(t) e^{-int} \hat{\eta}_0(t) dt$, and the above decomposition of f reduces to the Fourier series:

$$f(t) = STf(t) = \sum_{n\in\mathbb{Z}} \left(\sum_{m\in\mathbb{Z}} (-1)^{nm} \check{\phi}_0(t - \pi m) \right) a_n e^{int} = \sum_{n\in 2\mathbb{Z}} a_n e^{int}.$$

Conversely, any π-periodic function $f(t) = \sum_{n\in 2\mathbb{Z}} a_n e^{int}$ belongs to E_∞^α if and only if we have $\sum_{n\in 2\mathbb{Z}} \langle n\rangle^\alpha |a_n| < \infty$, and in this case, $\|f\|_{E_\infty^\alpha} \cong \sum_{n\in 2\mathbb{Z}} \langle n\rangle^\alpha |a_n|$.

Proof. (1): Let $f \in E_p^\alpha$ and $a_{nm} = (Tf)_{nm}$. For any $n \in \mathbb{Z}$, $\|(a_{nm})_m\|_p \lesssim \|f * \widetilde{\phi_n}\check{}\,\|_p$. Indeed, if $p = \infty$, then

$$\begin{aligned}
|a_{nm}| &= \frac{1}{2} \Big| \int_{\mathbb{R}} f(t) [\eta_n e^{i\pi m(\cdot)}]\hat{}\,(t) dt \Big| \\
&\le |\langle \hat{f}, \eta_n e^{i\pi m(\cdot)} \rangle| \\
&= |\langle \hat{f}\widetilde{\phi_n}, \eta_n e^{i\pi m(\cdot)} \rangle| \\
&= |\langle f * \widetilde{\phi_n}\check{}\,, [\eta_n e^{i\pi m(\cdot)}]\hat{}\,\rangle| \\
&\le \|f * \widetilde{\phi_n}\check{}\,\|_\infty \|[\eta_n e^{i\pi m(\cdot)}]\hat{}\,\|_1,
\end{aligned}$$

and the last factor is a constant. Thus, $\|a_{nm}\|_\infty \lesssim \|f * \widetilde{\phi_n}\check{}\,\|_\infty$. If $p = 2$, then by the Plancherel identity for both Fourier series and Fourier transforms,

$$\|(a_{nm})_m\|_2 = \|\hat{f}\eta_n\|_{L^2(\mathbb{R})} = \|f * \check{\eta}_n\|_{L^2(\mathbb{R})} = \|f * \check{\eta}_n * \widetilde{\phi_n}\check{}\,\|_{L^2(\mathbb{R})} \le \|\check{\eta}_0\|_{L^1(\mathbb{R})} \|f * \widetilde{\phi_n}\check{}\,\|_{L^2(\mathbb{R})}.$$

Finally, the case $2 < p < \infty$ follows by complex interpolation. Thus,

$$\|Tf\|_{\ell^1_\alpha(\ell^p)} = \sum_{n\in\mathbb{Z}} \langle n\rangle^\alpha \|(a_{nm})_m\|_p \lesssim \sum_{n\in\mathbb{Z}} \langle n\rangle^\alpha \|f * \widetilde{\phi_n}\check{}\,\|_p \le \sum_{k=-1}^{1} \sum_{n\in\mathbb{Z}} \langle n+k\rangle^\alpha \|f * \check{\phi}_n\|_p \lesssim \|f\|_{E_p^\alpha},$$

so T is a bounded operator.

We turn to the operator S and consider a fixed $(a_{nm}) \in \ell^1_\alpha(\ell^p)$.

- Step 1: For any $n \in \mathbb{Z}$, $\sum_{m=-M}^{M} a_{nm} e^{in(\cdot)} \check{\phi}_0(\cdot - \pi m)$ converges locally uniformly as $M \to \infty$ and $\| \sum_{m\in\mathbb{Z}} a_{nm} e^{in(\cdot)} \check{\phi}_0(\cdot - \pi m)\|_p \lesssim \|(a_{nm})_m\|_p$.
- Step 2: $\sum_{n=-N}^{N} \sum_{m\in\mathbb{Z}} a_{nm} e^{in(\cdot)} \check{\phi}_0(\cdot - \pi m)$ converges in E_p^α as $N \to \infty$.
- Step 3: If $a_{nm} = (Tf)_{nm}$ for some $f \in E_p^\alpha$, then $\sum_{m\in\mathbb{Z}} a_{nm} e^{in(\cdot)} \check{\phi}_0(\cdot - \pi m) = f * \check{\phi}_n$ for any $n \in \mathbb{Z}$.

- Step 4: Conclusion.

Step 1: Since $\phi_0 \in C_0^\infty(\mathbb{R})$, $\check{\phi}_0$ is rapidly decreasing, and thus in particular, for any $R > 0$,

$$\sum_{m\in\mathbb{Z}} \sup_{|t|\le R} |\check{\phi}_0(t-\pi m)| \lesssim \sum_{m\in\mathbb{Z}} \sup_{|t|\le R} \langle t-\pi m\rangle^{-2} < \infty.$$

Hence for any $F \subset \mathbb{Z}$ finite,

$$\sup_{|t|\le R} |\sum_{m\in F} a_{nm} e^{int}\check{\phi}_0(t-\pi m)| \le \|(a_{nm})_m\|_\infty \sum_{m\in F} \sup_{|t|\le R} |\check{\phi}_0(t-\pi m)|.$$

This shows the locally uniform convergence (note that $\|(a_{nm})_m\|_\infty \le \|(a_{nm})_m\|_p$). For the claimed estimate, we proceed again by complex interpolation. If $p = \infty$, for any $t \in \mathbb{R}$ fixed,

$$\begin{aligned} |\sum_{m\in\mathbb{Z}} a_{nm} e^{int}\check{\phi}_0(t-\pi m)| &\le \|(a_{nm})_m\|_\infty \sum_{m\in\mathbb{Z}} |\check{\phi}_0(t-\pi m)| \\ &\le \|(a_{nm})_m\|_\infty \sum_{m\in\mathbb{Z}} \sup_{|t|\le\pi} |\check{\phi}_0(t-\pi m)| \\ &\lesssim \|(a_{nm})_m\|_\infty. \end{aligned}$$

For $p = 1$,

$$\begin{aligned} \int_{\mathbb{R}} |\sum_{m\in\mathbb{Z}} a_{nm} e^{int}\check{\phi}_0(t-\pi m)|dt &\le \sum_{m\in\mathbb{Z}} |a_{nm}| \int_{\mathbb{R}} |\check{\phi}_0(t-\pi m)|dt \\ &\lesssim \|(a_{nm})_m\|_1, \end{aligned}$$

since $\int_{\mathbb{R}} |\check{\phi}_0(t-\pi m)|dt$ does not depend on m and is finite.

Step 2: Let $n, n_0 \in \mathbb{Z}$. Since the series in step 1 converges locally uniformly and is bounded in $L^\infty(\mathbb{R})$, by dominated convergence

$$\left[\sum_{m\in\mathbb{Z}} a_{nm}\check{\phi}_0(\cdot-\pi m)e^{in(\cdot)}\right] * \check{\phi}_{n_0} = \sum_{m\in\mathbb{Z}} a_{nm}\left[\check{\phi}_0(\cdot-\pi m)e^{in(\cdot)} * \check{\phi}_{n_0}\right].$$

We keep in mind that

$$[\check{\phi}_0(\cdot-\pi m)e^{in(\cdot)}]\hat{} = e^{i\pi nm}\phi_n e^{-i\pi m(\cdot)}.$$

The support condition on $(\phi_n)_n$ yields that the sum vanishes for $|n-n_0| \ge 2$. For $|n-n_0| \le 1$, by step 1,

$$\begin{aligned} \|[\sum_{m\in\mathbb{Z}} a_{nm}\check{\phi}_0(\cdot-\pi m)e^{in(\cdot)}] * \check{\phi}_{n_0}\|_p &\le \|\sum_{m\in\mathbb{Z}} a_{nm}\check{\phi}_0(\cdot-\pi m)e^{in(\cdot)}\|_p \|\check{\phi}_{n_0}\|_1 \\ &\lesssim \|(a_{nm})_m\|_\infty. \end{aligned}$$

Thus for any $F \subset \mathbb{Z}$ finite and $F^* = \{n+k : n \in F, k = -1, 0, 1\}$,

$$\begin{aligned} \|\sum_{n\in F}\sum_{m\in\mathbb{Z}} a_{nm}\check{\phi}_0(\cdot-\pi m)e^{in(\cdot)}\|_{E_p^\alpha} &= \sum_{n_0\in\mathbb{Z}} \langle n_0\rangle^\alpha \|\sum_{n\in F}\sum_{m\in\mathbb{Z}} a_{nm}\check{\phi}_0(\cdot-\pi m)e^{in(\cdot)} * \check{\phi}_{n_0}\|_p \\ &\lesssim \sum_{k=-1}^{1}\sum_{n\in F} \langle n+k\rangle^\alpha \|(a_{nm})_m\|_p \end{aligned}$$

$$\lesssim \sum_{n\in F^*} \langle n\rangle^\alpha \|(a_{nm})_m\|_p.$$

This shows the stated convergence of the double series defining $S(a_{nm})$ and that S is a bounded operator.

Step 3: Let $\psi \in C_c^\infty(\mathbb{R})$.

$$\begin{aligned}
\langle f * \check{\phi}_n, \psi\rangle &= \langle \hat{f}\phi_n, \check{\psi}\rangle \\
&= \langle \hat{f}\eta_n, \phi_n\check{\psi}\rangle \\
&= \langle \hat{f}\eta_n, \sum_{m\in\mathbb{Z}} \frac{1}{2}\langle e^{-i\pi m(\cdot)}, \phi_n\check{\psi}\rangle e^{i\pi m(\cdot)}\rangle \\
&= \sum_m \frac{1}{2}\langle \hat{f}\eta_n, e^{i\pi m(\cdot)}\rangle\langle \phi_n e^{-i\pi m(\cdot)}, \check{\psi}\rangle \\
&= \sum_m \frac{1}{2}\langle \hat{f}\eta_n, e^{i\pi m(\cdot)}\rangle e^{-i\pi nm}\langle \check{\phi}_0(\cdot - \pi m)e^{in(\cdot)}, \psi\rangle \\
&= \sum_m a_{nm}\langle e^{in(\cdot)}\check{\phi}_0(\cdot - \pi m), \psi\rangle \\
&= \langle \sum_m a_{nm} e^{in(\cdot)}\check{\phi}_0(\cdot - \pi m), \psi\rangle.
\end{aligned}$$

In the third equality, we develop $\phi_n\check{\psi}$ in a Fourier series. Note that its coefficients are rapidly decreasing, since $\phi_n\check{\psi} \in C_c^\infty(\mathbb{R})$. Hence, the sum can be taken out in the subsequent equality. In the last equality, the sum can be taken inside by dominated convergence, because $\|(a_{nm})_m\|_\infty \le \|(a_{nm})_m\|_p < \infty$.

Step 4: According to step 3, for any $f \in E_p^\alpha$ and $n \in \mathbb{Z}$,

$$f * \check{\phi}_n = \sum_m a_{nm} e^{in(\cdot)}\check{\phi}_0(\cdot - \pi m).$$

Take the sum over $n \in \mathbb{Z}$ on both sides of this equation. The left hand side converges to f in E_p^α after Proposition 3.2, and the right hand side converges to STf. This shows $f = STf$.

(2): Let $f \in E_\infty^\alpha$ be π-periodic.

$$\begin{aligned}
(Tf)_{nm} &= \int_{\mathbb{R}} f(t)e^{-int}\hat{\eta}_0(t - \pi m)dt = \int_{\mathbb{R}} f(t - \pi m)e^{-int}\hat{\eta}_0(t - \pi m)dt \\
&= e^{-inm\pi}\int_{\mathbb{R}} f(t)e^{-int}\hat{\eta}_0(t)dt = (-1)^{nm}a_n.
\end{aligned}$$

We show that

$$\sum_m (-1)^{nm}\check{\phi}_0(t - \pi m) = \begin{cases} \pi & n \text{ even} \\ 0 & n \text{ odd} \end{cases}.$$

For the case n even, note that $\sum_m \check{\phi}_0(t-\pi m)$ is a π-periodic function and thus admits a Fourier series expansion $\sum_{k\in\mathbb{Z}} \alpha_k e^{2ikt}$. Hence it suffices to check that $\alpha_k = \delta_{k=0}$, for which we refer to [16, Proof of 5.3]. For the case n odd, note that $\sum_m (-1)^m \check{\phi}_0(t-\pi m)$ is 2π-periodic. Similarly, one can compute the Fourier coefficients [16, Proof of 5.3].

This shows that STf is the Fourier series of f, and the "only if" part. For the "if" part, define $a_{nm} = (-1)^{nm} a_n$ for even n and $a_{nm} = 0$ for odd n. Then $f = S(a_{nm})$ and $\|f\|_{E^\alpha_\infty} \lesssim \|a_{nm}\|_{\ell^1_\alpha(\ell^\infty)} = \sum_{n\in 2\mathbb{Z}} \langle n\rangle^\alpha |a_n| \lesssim \sum_{n\in\mathbb{Z}} \langle n\rangle^\alpha |a_{2n}| \lesssim \|f\|_{E^\alpha_\infty}$. □

Remark 3.4. The decomposition $f(t) = \sum_{n,m} a_{nm} e^{int} \check{\phi}_0(t-\pi m)$ is not unique. Indeed, let $\eta_0^{(1)} \geq \eta_0^{(2)}$ be two different choices of η_0 with corresponding $T^{(1)}, T^{(2)}$ as in the above proposition. Let $f \in E^\alpha_p$ such that $\hat{f}(t) > 0$ for $t \in [-1,1]$. Then

$$(T^{(1)}f)_{00} = \frac{1}{2}\langle \hat{f}, \eta_0^{(1)}\rangle > \frac{1}{2}\langle \hat{f}, \eta_0^{(2)}\rangle = (T^{(2)}f)_{00}.$$

From Proposition 3.3 we obtain the following optimal embedding result between E^α_∞ and the classical Besov spaces $\mathcal{B}^\alpha_{\infty,1}$.

Proposition 3.5. Let $\alpha \geq 0$.

(1) If $\beta \geq \alpha + 1$, then $\mathcal{B}^\beta_{\infty,1} \hookrightarrow E^\alpha_\infty \hookrightarrow \mathcal{B}^\alpha_{\infty,1}$.
(2) If $\beta < \alpha + 1$, then $\mathcal{B}^\beta_{\infty,1} \not\hookrightarrow E^\alpha_\infty$.
(3) If $\beta > \alpha + \frac{1}{2}$, then for any $f \in \mathcal{B}^\beta_{\infty,1}$ periodic or with compact support, $f \in E^\alpha_\infty$ and there exists $C > 0$ depending on the period or the length of the support such that $\|f\|_{E^\alpha_\infty} \leq C\|f\|_{\mathcal{B}^\beta_{\infty,1}}$.
(4) If $\beta < \alpha + \frac{1}{2}$, then there exists a periodic $f \in \mathcal{B}^\beta_{\infty,1}$ such that $f \notin E^\alpha_\infty$.

Proof. (1): Let $(\phi_k)_{k\in\mathbb{Z}}$ be an equidistant partition of unity and $(\psi_n)_{n\in\mathbb{Z}}$ a dyadic one, as at the beginning of Section 3. Further, let $\widetilde{\psi}_n = \sum_{k=n-1}^{n+1} \psi_k$. For $n \in \mathbb{N}$, we let $A_n = \{k \in \mathbb{N} : 2^{n-1} \leq k \leq 2^n - 1\}$, $A_{-n} = -A_n$ and $A_0 = \{0\}$. Further we let $\widetilde{A_n} = A_{n-1} \cup A_n \cup A_{n+1}$. Then $\bigcup_{n\in\mathbb{Z}} A_n = \mathbb{Z}$ as a disjoint union, $\sum_{k\in\widetilde{A_n}} \phi_k \psi_n = \psi_n$ and for $k \in \widetilde{A_n}$, $\langle k\rangle \cong 2^{|n|}$. Therefore,

$$\begin{aligned}\|f\|_{\mathcal{B}^\alpha_{\infty,1}} &= \sum_{n\in\mathbb{Z}} 2^{|n|\alpha} \|f * \check{\psi}_n\|_\infty \leq \sum_{n\in\mathbb{Z}} 2^{|n|\alpha} \sum_{k\in\widetilde{A_n}} \|f * \check{\psi}_n * \check{\phi}_k\|_\infty \\ &\lesssim \sum_{k\in\mathbb{Z}} \langle k\rangle^\alpha \|f * \check{\phi}_k\|_\infty \|\check{\psi}_n\|_1 \lesssim \|f\|_{E^\alpha_\infty}.\end{aligned}$$

In the other direction, as $\mathrm{card} A_n \lesssim 2^{|n|}$ and for $k \in A_n$, $\widetilde{\psi}_n \phi_k = \phi_k$,

$$\|f\|_{E^\alpha_\infty} = \sum_{n\in\mathbb{Z}} \sum_{k\in A_n} \langle k\rangle^\alpha \|f * \check{\phi}_k * \widetilde{\psi}_n\check{}\,\|_\infty \lesssim \sum_{n\in\mathbb{Z}} 2^{|n|\alpha} 2^{|n|} \|f * \widetilde{\psi}_n\check{}\,\|_\infty \lesssim \|f\|_{\mathcal{B}^{\alpha+1}_{\infty,1}}.$$

(2): Since J_α in Proposition 3.2 (6) is an isomorphism $\mathcal{B}^0_{\infty,1} \to \mathcal{B}^\alpha_{\infty,1}$ and $E^0_\infty \to E^\alpha_\infty$ simultaneously, it suffices to consider the case $\alpha = 0$, $\beta < 1$. Let

$$f_N(t) = \sum \langle n\rangle^{-1} e^{int} \check{\phi}_0(t - \pi n),$$

where the sum ranges over all even $n \in \mathbb{N}$ such that $1 \leq n \leq N$. For such an n fixed, we have in turn

$$\|f_N * \check{\phi}_n\|_\infty = \| \sum_{k \leq N \text{ even}} \langle k \rangle^{-1} (\phi_k \phi_n e^{i\pi n(\cdot)})\check{} \|_\infty = \langle n \rangle^{-1} \|(\phi_n^2)\check{}(t - \pi n)\|_\infty,$$

and the last factor is a constant. Thus, $\|f_N\|_{E^0_\infty} \geq \sum_{n=1}^N \|f_N * \check{\phi}_n\|_\infty \gtrsim \sum_{n \leq N \text{ even}} \langle n \rangle^{-1} \to \infty$ for $N \to \infty$. On the other hand, f_N is bounded in $\mathcal{B}^\beta_{\infty,1}$. Indeed, $\|f_N\|_{\mathcal{B}^\beta_{\infty,1}} \lesssim \|f_N\|_\infty + \|f_N'\|_\infty$ by [25]. But

$$\begin{aligned}
\|f_N\|_\infty &\leq \sup_{t \in \mathbb{R}} |\sum_{n=1}^N \langle n \rangle^{-1} e^{int} \check{\phi}_0(t - \pi n)| \leq \sup_{t \in \mathbb{R}} \sum_{n=1}^\infty |\check{\phi}_0(t - \pi n)| < \infty, \text{ and} \\
\|f_N'\|_\infty &\leq \sup_{t \in \mathbb{R}} |\sum_{n=1}^N \langle n \rangle^{-1} n e^{int} \check{\phi}_0(t - \pi n)| + \sup_{t \in \mathbb{R}} |\sum_{n=1}^N \langle n \rangle^{-1} e^{int} (\check{\phi}_0)'(t - \pi n)| \\
&\leq \sup_{t \in \mathbb{R}} \sum_{n=1}^\infty |\check{\phi}_0(t - \pi n)| + |(\check{\phi}_0)'(t - \pi n)| < \infty.
\end{aligned}$$

(3): Since J_α maps periodic functions to periodic functions, we can again assume $\alpha = 0$. By [25], any $f \in \mathcal{B}^\beta_{\infty,1}$ is β'-Hölder continuous with $\beta' \in (\frac{1}{2}, \beta)$. Then by [15, p. 34], the Fourier coefficients a_n of f are absolutely summable. By Proposition 3.2 (5), we can assume that f is π-periodic, and by Proposition 3.3, $\|f\|_{E^0} \cong \sum_{n \in \mathbb{Z}} |a_n| < \infty$.

If f has compact support, say $[N\frac{\pi}{2}, (N+1)\frac{\pi}{2}]$ for some $N \in \mathbb{Z}$, then $g = \sum_{m \in \mathbb{Z}} f(\cdot - m\pi)$ is periodic, and by [25, p. 110], $\|g\|_{\mathcal{B}^\beta_{\infty,1}} \cong \|f\|_{\mathcal{B}^\beta_{\infty,1}}$. Thus, the first part yields $g \in E^\alpha_\infty$, and consequently $\|f\|_{E^\alpha_\infty} = \|g\varphi\|_{E^\alpha_\infty} \lesssim \|g\|_{E^\alpha_\infty} \|\varphi\|_{E^\alpha_\infty} < \infty$, where $\varphi \in C^\infty_c(\mathbb{R})$ is chosen appropriately.

(4): Again we can assume $\alpha = 0$. Choose a periodic $\frac{1}{2}$-Hölder continuous function f whose Fourier coefficients are not absolutely summable [15, p. 36]. By [25], $f \in \mathcal{B}^\beta_{\infty,1}$, and by Proposition 3.3 (2), $f \notin E^0_\infty$. □

4. The E^α_∞ calculus

Throughout the section, we let iB be a generator of a C_0-group $U(t) = e^{itB}$ on some space X not containing c_0 isomorphically. Then B is a strip-type operator and we can consider its extended holomorphic calculus from Section 2.2. Boyadzhiev and deLaubenfels [2], see also [4, Theorem 2.4], have shown that if X is a Hilbert space, then an exponential growth $\|U(t)\| \lesssim e^{\omega|t|}$ can be characterized by the strip height ω of the bounded H^∞ functional calculus of B (up to $\omega \pm \epsilon$). This result has been extended by Kalton and Weis to spaces X with property (α), where the boundedness of $e^{-\omega|t|}U(t)$ is replaced by γ-boundedness [13, Theorem 6.8].

We will show in this section that there is an analogous characterization of polynomial group growth

$$\|U(t)\| \leq C(1 + |t|)^\alpha \tag{4.1}$$

for some $C, \alpha > 0$, in terms of a E^α_∞ functional calculus. In fact, we shall construct this calculus for one distinguished group $W(t)$, which will be a shift group satisfying (4.1), on a space $\ell^\infty_\gamma(L^2, X)$ defined in terms of a Gaussian function space. The E^α_∞ functional calculus for a general group $U(t)$ is then obtained by a dilation to that case, in the sense that there exists an injection I from X to $\ell^\infty_\gamma(L^2, X)$ and a surjection P from $\ell^\infty_\gamma(L^2, X)$ back to X such that the groups $U(t)$ and $W(t)$ are linked by the equation $U(t) = PW(t)I$. Dilations are a classical tool in the study of (semi)groups, see e.g. the book of Davies [5, Chapter 6] for a survey.

Heuristically, we want to define $f(B)$ by making sense of the "windowed Fourier inversion formula"

$$f(B)x = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(t)U(t)x\,dt = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}}\int_{\mathbb{R}} \hat{f}(t)\phi_n(t)U(t)x\,dt, \tag{4.2}$$

where $\hat{f}$ is partitioned by some equidistant Fourier partition of unity $(\phi_n)_{n\in\mathbb{Z}}$. We shall see that $f \in E^\alpha_\infty$ is a natural condition in order that the above sum converge.

In Definition 4.1 through Proposition 4.7, we assume that $U(t)$ satisfies the polynomial bound (4.1) for some $\alpha \geq 0$, and that $\{U(t) : t \in [0,1]\}$ is γ-bounded. Recall the Gaussian function spaces and γ-boundedness from Section 2.

Definition 4.1. Let $(\phi_n)_n$ be an equidistant partition of unity. We set $\ell^1(L^2) = \{f : \mathbb{R} \to \mathbb{C} : \|f\|_{\ell^1(L^2)} = \sum_{n\in\mathbb{Z}} \|\phi_n f\|_{L^2(\langle t\rangle^{-2\alpha}dt)} < \infty\}$ which is a Banach space with respect to $\|f\|_{\ell^1(L^2)}$. For an element $u \in B(\ell^1(L^2), X)$, we have $\phi_n u \in B(L^2(\langle t\rangle^{-2\alpha}dt), X)$ where we put $[\phi_n u](f) = u(\phi_n f)$. Then we let

$$\ell^\infty_\gamma(L^2, X) = \{u \in B(\ell^1(L^2), X) : \|u\|_{\ell^\infty_\gamma(L^2,X)} = \sup_{n\in\mathbb{Z}} \|\phi_n u\|_{\gamma(L^2(\langle t\rangle^{-2\alpha}dt),X)} < \infty\}.$$

We let $\widetilde{\ell^\infty_\gamma}$ be the subspace of $\ell^\infty_\gamma(L^2, X)$ consisting of elements of the form $f : \mathbb{R} \to X$ such that $\phi_n f \in P_2(\mathbb{R}, X)$, so a $g \in \ell^1(L^2)$ is mapped to $\int_{\mathbb{R}} g(t)f(t)\langle t\rangle^{-2\alpha}dt$.

Lemma 4.2.

(1) The space $\ell^\infty_\gamma(L^2, X)$ is complete, and $\widetilde{\ell^\infty_\gamma}$ is a dense subspace of $\ell^\infty_\gamma(L^2, X)$.
(2) The norm of $\ell^\infty_\gamma(L^2, X)$ is independent of the choice of (ϕ_n) and moreover, one has the norm equivalence

$$\|u\|_{\ell^\infty_\gamma(L^2,X)} \cong \sup_{n\in\mathbb{Z}} \langle n\rangle^{-\alpha}\|\phi_n u\|_{\gamma(L^2(dt),X)}.$$

Proof. We show the completeness of $\ell^\infty_\gamma(L^2, X)$. Let $(u_k)_{k\in\mathbb{N}}$ be a sequence in $\ell^\infty_\gamma(L^2, X)$ such that $\sum_k \|u_k\|_{\ell^\infty_\gamma(L^2,X)}$ is finite. It is easy to check that $\|u\|_{B(\ell^1(L^2),X)} \lesssim \|u\|_{\ell^\infty_\gamma(L^2,X)}$ (see also [27, p. 12]). Thus, there exists $u \in B(\ell^1(L^2), X)$ such that $\sum_k u_k$ converges to u in the latter space. We show that u belongs to $\ell^\infty_\gamma(L^2, X)$ and that the convergence holds in $\ell^\infty_\gamma(L^2, X)$. It is easy to check that we have $\phi_n \sum_{k=1}^N u_k \to \phi_n u$ in $\gamma(L^2(\langle t\rangle^{-2\alpha}dt), X)$. Thus,

$$\begin{aligned}\sup_{n\in\mathbb{Z}} \|\phi_n u\|_{\gamma(L^2(\langle t\rangle^{-2\alpha}dt),X)} &= \sup_n \|\sum_k \phi_n u_k\|_\gamma \\ &\leq \sup_n \sum_k \|\phi_n u_k\|_\gamma\end{aligned}$$

$$\leq \sum_k \sup_n \|\phi_n u_k\|_\gamma$$
$$= \sum_k \|u_k\|_{\ell^\infty_\gamma(L^2,X)} < \infty,$$

and therefore, $u \in \ell^\infty_\gamma(L^2,X)$. Now replacing in the above argument u by $u - \sum_{n=1}^N u_k$, which equals $\sum_{N+1}^\infty u_k$ in $B(\ell^1(L^2),X)$, shows the convergence $u = \sum_k u_k$ in $\ell^\infty_\gamma(L^2,X)$. We have shown that $\ell^\infty_\gamma(L^2,X)$ is complete and turn to the density statement.

Let $u \in \ell^\infty_\gamma(L^2,X)$ arbitrarily. Decomposing $u = \sum_{k=0}^2 \sum_{n\equiv k\,(3)} \phi_n u =: u_0 + u_1 + u_2$, one sees that it is enough to show that u_0, u_1, u_2 can be approximated by elements in $\widetilde{\ell^\infty_\gamma}$. We consider only u_0 in the sequel, as u_1 and u_2 can be treated similarly. Fix $\epsilon > 0$. For $n \equiv 0\,(3)$, let $v_n \in L^2(\langle t\rangle^{-2\alpha}dt) \otimes X$ with $\|v_n - \tilde{\phi}_n u_0\|_{\gamma(L^2(\langle t\rangle^{-2\alpha}dt),X)} < \epsilon$. Such a v_n exists, as $L^2(\langle t\rangle^{-2\alpha}dt) \otimes X$ is dense in $\gamma(L^2(\langle t\rangle^{-2\alpha}dt),X)$. Let $v = \sum_{n\equiv 0\,(3)} v_n$. We claim that $v \in \ell^\infty_\gamma(L^2,X)$. For $f \in \ell^1(L^2)$ with compact support, one has

$$\|v(f)\|_X = \|\sum_{n\equiv 0\,(3)} v_n(f)\| = \|\sum_{n\equiv 0\,(3)} v_n(\sum_{k=n-1}^{n+1} \phi_k f)\|,$$

where the sum is finite if we assume that $\phi \cdot v_n = 0$ for $\operatorname{supp}\phi \cap [n-2, n+2] = \emptyset$, which is obviously possible. Thus

$$\|v(f)\|_X \leq \sup_n \|v_n\|_{L^2(\langle t\rangle^{-2\alpha}dt)\to X} \sum_n \|\phi_n f\|_{L^2(\langle t\rangle^{-2\alpha}dt)}$$
$$\leq \left(\sup_{n\in\mathbb{Z}} \|\tilde{\phi}_n u_0\|_{\gamma(L^2(\langle t\rangle^{-2\alpha}dt),X)} + \epsilon\right) \|f\|_{\ell^1(L^2)} \leq \left(2\|u_0\|_{\ell^\infty_\gamma(L^2,X)} + \epsilon\right) \|f\|_{\ell^1(L^2)}.$$

As such f form a dense subset of $\ell^1(L^2)$, v belongs to $B(\ell^1(L^2),X)$. Moreover, putting $v_n = 0$ for $n \not\equiv 0\,(3)$, we have

$$\|v\|_{\ell^\infty_\gamma(L^2,X)} = \sup_n \|\phi_n v\|_\gamma = \sup_n \|\phi_n(v_{n-1} + v_n + v_{n+1})\|_\gamma \leq \sup_{n\equiv 0\,(3)} \|v_n\|_\gamma \leq 3\|u_0\|_{\ell^\infty_\gamma(L^2,X)} + \epsilon.$$

Thus v belongs to $\ell^\infty_\gamma(L^2,X)$. Also

$$\|v - u_0\|_{\widetilde{\ell^\infty_\gamma}} = \sup_n \|\phi_n v - \phi_n u_0\|_\gamma = \sup_{n\in\mathbb{Z}} \|\phi_n(v_{n-1} + v_n + v_{n+1}) - \phi_n\tilde{\phi}_n u_0\|_\gamma \leq \left(\sup_{n\in\mathbb{Z}} \|\phi_n\|_\infty\right) \epsilon \lesssim \epsilon.$$

Part (2) of the lemma is left to the reader. □

We define the operators

$$I = I(U) : X \longrightarrow \ell^\infty_\gamma(L^2,X),\ x \mapsto U(-t)x$$

and

$$P = P(U) : \ell^\infty_\gamma(L^2,X) \longrightarrow X,\ f \mapsto \int_{\mathbb{R}} \chi_{[0,1]}(t)U(t)[f(t)]dt,$$

where $f \in \widetilde{\ell^\infty_\gamma}$. Then P extends boundedly to $\ell^\infty_\gamma(L^2,X)$, by density $\widetilde{\ell^\infty_\gamma} \subset \ell^\infty_\gamma(L^2,X)$ and the next lemma.

Lemma 4.3. The operators I and P are bounded.

Proof. Since $U(t)$ is a group, $\{U(t) : t \in [n, n+1]\} = U(n) \circ \{U(t) : t \in [0,1]\}$, and therefore,

$$\gamma(\{U(t) : t \in [n, n+1]\}) \leq \|U(n)\| \gamma(\{U(t) : t \in [0,1]\}) \lesssim \langle n \rangle^{\alpha}.$$

Thus, for $n \in \mathbb{Z}$, by Lemma 2.1 (3), $\phi_n U(-\cdot)x$ belongs to $\gamma(L^2(dt), X)$ and satisfies

$$\|\phi_n(t)U(-t)x\|_{\gamma(L^2(dt),X)} \leq \|\phi_n\|_{L^2(dt)} \gamma(\{U(-t) : t \in [n-1, n+1]\})\|x\| \lesssim \langle n \rangle^{\alpha} \|x\|.$$

Thus, I is bounded. Now let $f \in \widetilde{\ell^\infty_\gamma}$ and $x' \in X'$. We have by Lemma 2.1

$$\begin{aligned}
|\langle Py, x' \rangle| &= |\int_{\mathbb{R}} \chi_{[0,1]}(t) \langle U(t) f(t), x' \rangle| \\
&\leq \int_{\mathbb{R}} |\langle \chi_{[0,1]}(t) U(t) f(t), \chi_{[0,1]}(t) x' \rangle| dt \\
&\leq \|\chi_{[0,1]} U(\cdot) f\|_{\gamma(L^2(dt),X)} \|\chi_{[0,1]} x'\|_{\gamma(L^2(dt),X')} \\
&\leq \gamma(\{U(t) : t \in [0,1]\}) \|\chi_{[0,1]} \left(\phi_{-1} + \phi_0 + \phi_1\right) f\|_{\gamma(L^2(dt),X)} \|\chi_{[0,1]}\|_{L^2(dt)} \|x'\|_{X'} \\
&\lesssim \sum_{n=-1}^{1} \|\phi_n f\|_{\gamma(L^2(dt),X)} \|x'\|_{X'} \\
&\lesssim \|f\|_{\ell^\infty_\gamma(L^2,X)} \|x'\|_{X'}.
\end{aligned}$$

Taking the supremum over $\|x'\| \leq 1$ shows the boundedness of P. □

For $t \in \mathbb{R}$, let

$$W(t) : \ell^\infty_\gamma(L^2, X) \to \ell^\infty_\gamma(L^2, X),\ f \mapsto f(\cdot - t)$$

be the shift operator, where the definition extends boundedly to $\ell^\infty_\gamma(L^2, X)$, by the next proposition.

Proposition 4.4. The group $t \mapsto W(t)$ just defined satisfies $\|W(t)\| \leq C(1 + |t|)^{\alpha}$.

Proof. Let $n \in \mathbb{Z}$ and $t \in \mathbb{R}$, and put $m_0 = \lfloor n - t \rfloor$ the rounded down. Then $\sum_{m=m_0-1}^{m_0+2} \phi_m$ equals 1 on $\operatorname{supp} \phi_n(\cdot + t) \subset [(n-t)-1, (n-t)+1]$. Let $y \in \widetilde{\ell^\infty_\gamma}$. Note that the shift $f \mapsto f(\cdot - t)$ is an isometry on $L^2(dt)$. Thus, by Lemma 2.1, $y \mapsto y(\cdot - t)$ also defines an isometry on $\gamma(L^2(dt), X)$. Therefore,

$$\begin{aligned}
\|\phi_n W(t) y\|_{\gamma(L^2(dt),X)} &= \|\phi_n y(\cdot - t)\|_\gamma \\
&= \|\phi_n(\cdot + t) y\|_\gamma \\
&= \|\sum_{m=m_0-1}^{m_0+2} \phi_m \phi_n(\cdot + t) y\|_\gamma \\
&\leq \|\phi_n(\cdot + t)\|_\infty \sum_{m=m_0-1}^{m_0+2} \|\phi_m y\|_\gamma.
\end{aligned}$$

Note that $\sup\left\{\frac{\langle m\rangle}{\langle n\rangle}:\ n\in\mathbb{Z},\, m-\lfloor n-t\rfloor\in\{-1,0,1,2\}\right\}\cong\langle t\rangle$. Then

$$\begin{aligned}\|W(t)y\|_{\ell^\infty_\gamma(L^2,X)} &\lesssim \sup_{n\in\mathbb{Z}}\langle n\rangle^{-\alpha}\sum_{m=m_0-1}^{m_0+2}\|\phi_m y\|_\gamma\\ &= \sup_{n\in\mathbb{Z}}\langle n\rangle^{-\alpha}\sum_{m=m_0-1}^{m_0+2}\langle m\rangle^{\alpha}\left(\langle m\rangle^{-\alpha}\|\phi_m y\|_\gamma\right)\\ &\le \sup_{n\in\mathbb{Z}}\langle n\rangle^{-\alpha}\sum_{m=m_0-1}^{m_0+2}\langle m\rangle^{\alpha}\|y\|_{\ell^\infty_\gamma(L^2,X)}\\ &\lesssim \langle t\rangle^{\alpha}\|y\|_{\ell^\infty_\gamma(L^2,X)}.\end{aligned}$$

□

As $W(t)$ might not be strongly continuous on $\ell^\infty_\gamma(L^2,X)$, we consider the subspace

$$Z=\{u\in\ell^\infty_\gamma(L^2,X):\ W(t)u\to u \text{ in } \ell^\infty_\gamma(L^2,X) \text{ as } t\to 0\}.$$

It is easy to check that Z is closed and $W(t)$-invariant for any $t\in\mathbb{R}$. By construction, $W(t)$ is strongly continuous on Z. Furthermore, $\mathrm{Im}(I)\subset Z$, since

$$\begin{aligned}\|W(t)U(-\cdot)x-U(-\cdot)x\|_{\ell^\infty_\gamma(L^2,X)} &= \sup_{n\in\mathbb{Z}}\langle n\rangle^{-\alpha}\|\phi_n\left(U(t-\cdot)x-U(-\cdot)x\right)\|_{\gamma(L^2(dt),X)}\\ &\le \sup_{n\in\mathbb{Z}}\gamma\left(\{U(s):\ s\in[n-1,n+1]\}\right)\langle n\rangle^{-\alpha}\|U(t)x-x\|_X\\ &\to 0\quad(t\to 0).\end{aligned}$$

Also $Z\cap\widetilde{\ell^\infty_\gamma}$ is dense in Z, the verification is left to the reader.

In the sequel, the shift group $W(t)$ will serve as a universal model in the class of polynomially bounded, γ-bounded groups. The reason is that it dilates the original group $U(t)$ in the sense of the following proposition.

Proposition 4.5. Let I,P and $W(t)$ be defined as above. Then for any $t\in\mathbb{R}$, the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{U(t)} & X\\ {\scriptstyle I}\downarrow & & \uparrow{\scriptstyle P}\\ \ell^\infty_\gamma(L^2,X) & \xrightarrow[W(t)]{} & \ell^\infty_\gamma(L^2,X)\end{array}$$

Proof. For $x\in X$ and $t\in\mathbb{R}$, we have

$$\begin{aligned}PW(t)I(x)=PW(t)U(-\cdot)x=PU(t-\cdot)x&=\int_{\mathbb{R}}\chi_{[0,1]}(s)U(s)U(t-s)x\,ds\\ &=\int_{\mathbb{R}}\chi_{[0,1]}(s)U(t)x\,ds=U(t)x.\end{aligned}$$

□

Let us now investigate the functional calculus of the strongly continuous part of $W(t)$ on $Z \subset \ell^\infty_\gamma(L^2, X)$ (or its generator $\widetilde{B}$, where $W(t) = \exp(it\widetilde{B})$).

Proposition 4.6. Let $\omega > 0$.

(1) For any $f \in H^\infty_0(\mathrm{Str}_\omega)$, the following representation formula holds

$$f(\widetilde{B})y = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(t)W(t)y\,dt \quad (y \in \widetilde{\ell^\infty_\gamma} \cap Z). \tag{4.3}$$

(2) There exists some $C > 0$ such that for any $f \in H^\infty(\mathrm{Str}_\omega)$, we have $\|f(\widetilde{B})\| \le C\|f\|_{E^\alpha_\infty}$. In particular, $\widetilde{B}$ has a bounded H^∞ calculus.

(3) Moreover, this calculus extends continuously to a mapping $\Psi : E^\alpha_\infty \to B(Z)$, and

$$\Psi(f) = \sum_{n=-\infty}^{\infty} \left(f * \check{\phi}_n\right)(\widetilde{B}) \quad (f \in E^\alpha_\infty).$$

Proof. (1): We let $f \in H^\infty_0(\mathrm{Str}_\omega)$. By the Cauchy integral formula, also $f^{(k)} \in H^\infty_0(\mathrm{Str}_\omega)$ for any $k \in \mathbb{N}$, so $|f(z)|, |f'(z)|, \dots, |f^{(k)}(z)| \lesssim e^{-\epsilon|\operatorname{Re} z|}$. This implies $\langle t\rangle^k \hat{f}(t) \in C_0(\mathbb{R})$, so choosing $k > \alpha + 1$, this shows that $\hat{f}$ belongs to $L^1(\langle t\rangle^\alpha dt)$. It is easy to check that for such functions, the holomorphic functional calculus coincides with the Philipps calculus, which are the left hand resp. right hand side of (4.3).

(2): Let $n \in \mathbb{Z}$ and $y \in \widetilde{\ell^\infty_\gamma} \cap Z$. First assume $f \in H^\infty_0(\mathrm{Str}_\omega)$. In the following calculation, we use that $\phi_n \cdot (\psi * \rho) = 0$ as soon as $\operatorname{supp}\psi \subset [m-1, m+1]$, $\operatorname{supp}\rho \subset [k-1, k+1]$ and $|k - (n-m)| > 3$.

$$\begin{aligned}
\|\phi_n \int_{\mathbb{R}} \hat{f}(t)W(t)y\,dt\|_{\gamma(L^2(dt),X)} &= \|\phi_n \int_{\mathbb{R}} \hat{f}(t)y(\cdot - t)\,dt\|_\gamma \\
&= \|\phi_n \left(\sum_{m\in\mathbb{Z}} \phi_m \hat{f}\right) * \left(\sum_{|k-(n-m)|\le 3} \phi_k y\right)\|_\gamma \\
&\le \sum_{m,k} \|\phi_n \left(\phi_m \hat{f}\right) * (\phi_k y)\|_\gamma \\
&\le \sum_{m,k} \|\phi_n\|_\infty \|\phi_m \hat{f} * (\cdot)\|_{\gamma(L^2(dt),X)\to\gamma(L^2(dt),X)} \|\phi_k y\|_\gamma.
\end{aligned}$$

The above norm of a convolution transfers from the scalar valued L^2 space to the Gaussian function space in the following sense: Let $T : L^2(\mathbb{R}, dt) \to L^2(\mathbb{R}, dt)$ be the convolution operator $g \mapsto (\phi_m \hat{f}) * g$. We have $2\pi\|T\| = \|(\phi_m \hat{f})^\vee\|_\infty = \|\check{\phi_m} * f\|_\infty$, since $2\pi Tg = F(\phi_m \hat{f})^\vee F^{-1} g$, where F is the Fourier transform. It is easy to check that for $g \in \gamma(\mathbb{R}, dt, X)$, we have $T^\otimes g = (\phi_m \hat{f}) * g$, where $T^\otimes$ is defined in Lemma 2.1. Thus, we have $\|\phi_m \hat{f} * (\cdot)\|_{\gamma(L^2(dt),X)\to\gamma(L^2(dt),X)} = 2\pi\|\check{\phi_m} * f\|_\infty$. Consequently,

$$\|f(\widetilde{B})y\|_{\ell^\infty_\gamma(L^2,X)} \le \sup_{n\in\mathbb{Z}} \langle n\rangle^{-\alpha} \sum_{m,k} \|\check{\phi_m} * f\|_\infty \|\phi_k y\|_{\gamma(L^2(dt),X)}$$

$$\begin{aligned}
&= \sup_{n\in\mathbb{Z}} \langle n\rangle^{-\alpha} \sum_{m,k} \|\check{\phi_m} * f\|_\infty \langle m\rangle^\alpha \langle m\rangle^{-\alpha} \|\phi_k y\|_{\gamma(L^2(dt),X)} \\
&\lesssim \|f\|_{E^\alpha_\infty} \sup_{k\in\mathbb{Z}} \langle k\rangle^{-\alpha} \|\phi_k y\|_{\gamma(L^2(dt),X)} \\
&= \|f\|_{E^\alpha_\infty} \|y\|_{\ell^\infty_\gamma(L^2,X)},
\end{aligned}$$

where we have used that $\langle k\rangle \lesssim \langle n\rangle \cdot \langle m\rangle$ for the above summing range $m \in \mathbb{Z}$, $|k-(n-m)| \leq 3$.

Now let $f \in H^\infty(\mathrm{Str}_\omega)$. Recall the functions ψ_k from (2.2) and that $y = \lim_k \psi_k(\widetilde{B})y$ by Proposition 2.2. We have $f\psi_k \in H^\infty_0(\mathrm{Str}_\omega)$, and thus by the above, for any $y \in Z$,

$$\begin{aligned}
\|f(\widetilde{B})y\| &= \lim_k \|f(\widetilde{B})\psi_k(\widetilde{B})y\| \lesssim \liminf_k \|f\psi_k\|_{E^\alpha_\infty} \|y\| \lesssim \|f\|_{E^\alpha_\infty} \liminf_k \|\psi_k\|_{E^\alpha_\infty} \|y\| \\
&\lesssim \|f\|_{E^\alpha_\infty} \liminf_k \|\psi_k\|_{H^\infty(\mathrm{Str}_\omega)} \|y\| \lesssim \|f\|_{E^\alpha_\infty} \|y\|.
\end{aligned}$$

(3): This follows from Proposition 3.2 (2). □

Then the functional calculus Ψ of $W(t)$ has the following convergence property.

Proposition 4.7. If $(f_k)_k \subset E^\alpha_\infty$ with

(1) $\sum_{n\in\mathbb{Z}} \langle n\rangle^\alpha \sup_k \|f_k * \check{\phi}_n\|_\infty < \infty$,
(2) $f_k(t) \to f(t)$ for all $t \in \mathbb{R}$ for some function f,

then $f \in E^\alpha_\infty$ and

$$\Psi(f_k)y \to \Psi(f)y \quad (y \in Z). \tag{4.4}$$

Note that $H^\infty(\mathrm{Str}_\omega) \subset E^\alpha_\infty$, and that the above extends the H^∞ calculus convergence lemma: If $(f_k)_k \subset H^\infty(\mathrm{Str}_\omega)$ is a bounded sequence for some $\omega > 0$, and $f_k(z) \to f(z)$ pointwise on Str_ω, then (1) and (2) above are satisfied for any $\alpha \geq 0$.

Proof. We have $f * \check{\phi}_n(t) = \lim_k f_k * \check{\phi}_n(t)$ for any $t \in \mathbb{R}$ by dominated convergence. In particular, $|f * \check{\phi}_n(t)| \leq \sup_k |f_k * \check{\phi}_n(t)| \leq \sup_k \|f_k * \check{\phi}_n\|_\infty$. Thus, $f \in E^\alpha_\infty$. As the sequence $(f_k)_k$ is bounded in E^α_∞ by condition (1), it suffices to check (4.4) on the dense subset $\widetilde{\ell^\infty_\gamma} \cap Z$. If $y \in \widetilde{\ell^\infty_\gamma} \cap Z$ and $N \in \mathbb{N}$, then

$$\|\Psi(f)y - \Psi(f_k)y\| \leq \| \sum_{|n|\leq N} (f - f_k) * \check{\phi}_n(\widetilde{B})y\| + \sum_{|n|>N} \|(f - f_k) * \check{\phi}_n(\widetilde{B})y\|.$$

For the first sum, note that $(f-f_k)*\check{\phi}_n \overset{k}{\to} 0$ pointwise on $\mathbb{R}$ and that $\sup_k \|(f-f_k)*\check{\phi}_n\|_\infty < \infty$. Thus, for any $g \in L^2(dt)$, $\left[(f - f_k) * \check{\phi}_n\right] g \to 0$ in $L^2(dt)$. This carries over to

$$2\pi \left[(f - f_k) * \check{\phi}_n\right](\widetilde{B})(\phi_j y) = \left((f - f_k) * \check{\phi}_n\right)\hat{} * (\phi_j y) \to 0 \quad \text{in } \gamma(L^2(dt), X),$$

for $j, n \in \mathbb{Z}$, where we use once again Lemma 2.1 and the fact that pointwise multiplication and convolution are conjugated by the Fourier transform. One now easily deduces that $(f - f_k) * \check{\phi}_n(\widetilde{B})y \overset{k}{\to} 0$ in Z.

The second sum, we simply estimate by $2C \sum_{n\geq|N|} \langle n\rangle^\alpha \sup_k \|f_k * \check{\phi}_n\|_\infty \|y\|$, which converges to 0 for $N \to \infty$. We have proved the convergence property (4.4).

It remains to show that this entails the H^∞ calculus convergence property. Let $f \in H^\infty(\mathrm{Str}_\omega)$ and $|\theta| < \omega$. By the Cauchy integral formula, $f * \check{\phi}_n(t) = f(\cdot - i\theta) * \check{\phi}_n(\cdot - i\theta)(t)$,

so that $\|f * \check{\phi}_n\|_{L^\infty(\mathbb{R})} \leq \|f\|_{\infty,\theta} \|\check{\phi}_n(\cdot - i\theta)\|_{L^1(\mathbb{R})}$. But $|\check{\phi}_n(t+i\theta)| = |\check{\phi}_0(t+i\theta)|e^{n\theta}$. Choosing $n\theta < 0$, we get $\|f * \check{\phi}_n\|_{L^\infty(\mathbb{R})} \lesssim e^{-|n|\theta} \|f\|_{\infty,\theta}$. Now consider the sequence $(f_k)_k \subset H^\infty(\mathrm{Str}_\omega)$. Applying the above to $f = f_k$ gives

$$\sum_{n\in\mathbb{Z}} \langle n\rangle^\alpha \sup_k \|f_k * \check{\phi}_n\|_\infty \leq \left(\sum_{n\in\mathbb{Z}} e^{-n|\theta|}\langle n\rangle^\alpha\right) \sup_k \|f_k\|_{\infty,\omega}, \tag{4.5}$$

and the last sum is finite for any $\alpha \geq 0$. □

Definition 4.8. Let $U(t)$ be a C_0-group and B its generator. Let $\omega > 0$. Then B is said to have a bounded E^α_∞ calculus if there exists $C > 0$ such that

$$\|f(B)\| \leq C\|f\|_{E^\alpha_\infty} \quad (f \in H^\infty(\mathrm{Str}_\omega)).$$

By density of $H^\infty(\mathrm{Str}_\omega)$ in E^α_∞ for any ω, this definition is independent of ω.

The main theorem of this section reads as follows.

Theorem 4.9. Let B be a 0-strip-type operator on some space X not containing c_0 isomorphically such that iB generates a strongly continuous group $U(t) = e^{itB}$, and let further $\alpha \geq 0$. Assume that $\{U(t) : t \in [0,1]\}$ is γ-bounded. Let $\ell^\infty_\gamma(L^2, X)$ be as in Definition 4.1 and $W(t)$ as before. Then the following are equivalent.

(1) $\|U(t)\| \lesssim \langle t\rangle^\alpha$ and B has a bounded $H^\infty(\mathrm{Str}_\omega)$ calculus for some $\omega > 0$.
(2) B has a bounded E^α_∞ calculus.
(3) $U(t)$ has a dilation to the shift group $W(t)$ on $\ell^\infty_\gamma(L^2, X)$, i.e. there exist some bounded operators $I : X \to \ell^\infty_\gamma(L^2, X)$ and $P : \ell^\infty_\gamma(L^2, X) \to X$ such that

$$\begin{array}{ccc} X & \xrightarrow{\quad U(t) \quad} & X \\ {\scriptstyle I}\downarrow & & \uparrow{\scriptstyle P} \\ \ell^\infty_\gamma(L^2, X) & \xrightarrow[W(t)]{} & \ell^\infty_\gamma(L^2, X) \end{array}$$

Proof. (1) $\Longrightarrow$ (3): This is the content of Proposition 4.5.
(3) $\Longrightarrow$ (2): By Proposition 4.6, the generator $\widetilde{B}$ of $W(t)$ on $Z \subseteq \widetilde{\ell^\infty_\gamma}$ has a bounded E^α_∞ calculus. Let $f \in H^\infty_0(\mathrm{Str}_\omega)$ for some $\omega > 0$. As in the proof of Proposition 4.6 (1), one can show that for arbitrary $x \in X$, $f(B)x = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(t)U(t)x\,dt$. Thus, $f(B)x = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(t)PW(t)Ix\,dt = Pf(\widetilde{B})Ix$. From this, we immediately deduce that $\|f(B)\| \leq \|P\|\,\|f(\widetilde{B})\|\,\|I\| \lesssim \|f\|_{E^\alpha_\infty}$. As $H^\infty(\mathrm{Str}_\omega) \hookrightarrow E^\alpha_\infty$, this entails that B has a bounded H^∞ calculus. It is easy to check now that Definition 4.8 is satisfied.
(2) $\Longrightarrow$ (1): If B has a bounded E^α_∞ calculus, then it has also a bounded $H^\infty(\mathrm{Str}_\omega)$ calculus for any $\omega > 0$, since $H^\infty(\mathrm{Str}_\omega) \subset E^\alpha_\infty$. Further, we have $U(t) = f_t(B)$ with $f_t(s) = e^{its}$, so it only remains to estimate $\|f_t\|_{E^\alpha_\infty}$. But $f_t * \check{\phi}_n(s) = e^{its}\phi_n(t)$, whence $\|f_t\|_{E^\alpha_\infty} = \sum_{n:\,|n-t|\leq 1} \langle n\rangle^\alpha |\phi_n(t)| \cong \langle t\rangle^\alpha$. □

Remark 4.10.

(1) Note that if the underlying Banach space X has in addition property (α), then the above theorem can be stated without the assumption that $\{U(t) : t \in [0,1]\}$ is γ-bounded. Indeed, condition (2) and thus also (3) still implies that B has a bounded $H^\infty(\mathrm{Str}_\omega)$ calculus, which yields by property (α) that $\{U(t) : t \in [0,1]\}$ is γ-bounded [13, Corollary 6.6].

(2) The polynomial growth in condition (1) of the theorem does not guarantee the boundedness of the H^∞ calculus. A counterexample is the shift group $U(t)f(x) = f(x-t)$ on $L^p(\mathbb{R})$, which is even uniformly bounded. In [4, Lemma 5.3], it is shown that its generator does not have a bounded H^∞ calculus unless $p = 2$.

(3) Compare Theorem 4.9 to [4, Theorem 4.10]. The latter tells that B has a bounded $\mathcal{B}^\alpha_{\infty,1}$ calculus for some $\alpha > 0$, if and only if B has a bounded H^∞ calculus $\Psi_\omega : H^\infty(\mathrm{Str}_\omega) \to B(X)$ for any $\omega > 0$ and

$$\|\Psi_\omega\| \lesssim \omega^{-\alpha}.$$

Let $\mathrm{BIP} = \{e^{it(\cdot)} : t \in \mathbb{R}\} \subset H^\infty(\mathrm{Str}_\omega)$. Theorem 4.9 shows that replacing $\mathcal{B}^\alpha_{\infty,1}$ by the smaller class E^α_∞ is equivalent to the restricted condition

$$\|\Psi_\omega|_{\mathrm{BIP}}\| \lesssim \omega^{-\alpha}.$$

(4) Assume that the equivalent conditions of the theorem hold. Then it is easy to see that the dilation can be transferred to the functional calculus by $f(B) = Pf(\widetilde{B})I$ for any $f \in E^\alpha_\infty$. We immediately deduce the following properties of that calculus.
   (a) The windowed Fourier inversion formula (4.2) holds, where the right hand side converges in X for any $x \in X$ and any $f \in E^\alpha_\infty$.
   (b) The convergence property, Proposition 4.7, holds literally with $U(t)$ in place of $W(t)$ and X in place of Z.

5. Operator valued and γ-bounded functional calculus

In this section, we let again X be a space not containing c_0 isomorphically and fix some $\alpha \geq 0$. In [19, Theorem 12.7], it is shown that if a sectorial operator A has a bounded H^∞ calculus, then this calculus extends boundedly to a certain class of operator valued holomorphic functions.

We will show a similar result for 0-strip-type operators B and the E^α_∞ calculus. As for the H^∞ calculus, if the space X has property (α), this procedure can be used to obtain γ-bounded families of the type $\{f(B) : f \in \tau\}$ with convenient $\tau \subset E^\alpha_\infty$.

Let $[B]' = \{T \in B(X) : TR(\lambda, B) = R(\lambda, B)T \ \forall\, \lambda \in \mathbb{C}\backslash\mathbb{R}\}$ denote the commutant set of B. Let further $\mathcal{H}^\infty(\mathrm{Str}_\omega) = H^\infty(\mathrm{Str}_\omega, [B]')$

$$= \{F : \mathrm{Str}_\omega \to [B]' : F \text{ analytic and } \gamma(\{F(z) : z \in \mathrm{Str}_\omega\}) < \infty\}$$

and $\mathcal{H}^\infty_0(\mathrm{Str}_\omega) = H^\infty_0(\mathrm{Str}_\omega, [B]') = \big\{F \in \mathcal{H}^\infty(\mathrm{Str}_\omega) : \|F(z)\| \lesssim e^{-\epsilon|\operatorname{Re} z|} \text{ for some } \epsilon > 0\big\}$. We define an operator valued calculus by

$$\mathcal{H}^\infty_0(\mathrm{Str}_\omega) \to B(X),\ F \mapsto F(B) = \frac{1}{2\pi i}\int_\Gamma F(\lambda)(\lambda - B)^{-1} d\lambda \tag{5.1}$$

for the usual contour Γ. Here the restriction of functions with values in $[B]'$ rather than $B(X)$ ensures the multiplicativity of the calculus. As in [19, Theorem 12.7], one can show that if B has a bounded H^∞ calculus to some strip height $\theta < \omega$, then the calculus in (5.1) extends boundedly to $\mathcal{H}^\infty(\mathrm{Str}_\omega)$. We will not make use of this fact, but rather show that under polynomial growth of the group $U(t) = e^{itB}$, there is an extension to an operator valued variant $\mathcal{E}^\alpha$ of the space E^α_∞. We define

$$\mathcal{E}^\alpha = \{F : \mathbb{R} \to [B]' \text{ strongly cont. and bdd, } \|F\|_{\mathcal{E}^\alpha} = \sum_{n\in\mathbb{Z}} \langle n\rangle^\alpha \gamma(\{F * \check{\phi}_n(t) : t \in \mathbb{R}\}) < \infty\}$$

and equip it with the norm $\|F\|_{\mathcal{E}^\alpha}$. At first, we record the following properties of $\mathcal{E}^\alpha$.

Proposition 5.1. (1) $\mathcal{E}^\alpha$ is a Banach algebra.

(2) $\mathcal{E}^\alpha$ contains $\mathcal{H}^\infty_0(\mathrm{Str}_\omega)$.

(3) For any $F \in \mathcal{E}^\alpha$, $n \in \mathbb{Z}$ and $\omega > 0$, $F * \check{\phi}_n$ belongs to $\mathcal{H}^\infty(\mathrm{Str}_\omega)$, and $\sum_{n=-N}^M F * \check{\phi}_n$ converges to F in $\mathcal{E}^\alpha$ $(N, M \to \infty)$.

(4) Different choices of the partition $(\phi_n)_n$ give the same space $\mathcal{E}^\alpha$ with equivalent norms.

The proof is mostly a copy of that of Proposition 3.2, replacing $L^\infty(\mathbb{R})$ norms by γ-bounds, and is outlined in [16, Proposition 5.12]. For $F : \mathbb{R} \to B(X)$ such that $\{F(t) : t \in \mathbb{R}\}$ is γ-bounded, we put

$$M_F : \gamma(\mathbb{R}, X) \to \gamma(\mathbb{R}, X),\ g(t) \mapsto F(t)g(t).$$

By Lemma 2.1 (3), $\|M_F g\| \le \gamma(\{F(t) : t \in \mathbb{R}\})\|g\|$. Since $\gamma(\mathbb{R}, X)$ is dense in $\gamma(L^2(\mathbb{R}), X)$, M_F extends uniquely to a bounded operator

$$M_F : \gamma(L^2(\mathbb{R}), X) \to \gamma(L^2(\mathbb{R}), X).$$

Let $\mathcal{F}$ be the Fourier transform on $L^2(\mathbb{R})$ and denote $\mathcal{F}^\otimes$ the extension to $\gamma(L^2(\mathbb{R}), X)$ given as in Lemma 2.1 (4). Further let

$$S_F = (\mathcal{F}^\otimes)^{-1} M_F \mathcal{F}^\otimes :\ \gamma(L^2(\mathbb{R}), X) \to \gamma(L^2(\mathbb{R}), X)$$

and

$$T_F g(t) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{F}(t-s) g(s) ds \quad (g \in \gamma(\mathbb{R}, X) \cap L^1(\mathbb{R}, X),\ g \text{ compact support}).$$

The remaining technicalities for the construction of the calculus are collected in the next lemma.

Lemma 5.2.

(1) Assume that $F \in L^1(\mathbb{R}, B(X))$ and $\hat{F}(t) = \int_{\mathbb{R}} e^{-its} F(s) ds$ has compact support. Then for any $g \in \gamma(\mathbb{R}, X) \cap L^1(\mathbb{R}, X)$ with compact support, we have

$$T_F g \text{ belongs to } \gamma(\mathbb{R}, X) \text{ and } u_{T_F g} = S_F u_g. \tag{5.2}$$

(2) Let $(F_k)_k$ be a sequence of functions $\mathbb{R} \to B(X)$ such that $\sup_k \gamma(\{F_k(t) : t \in \mathbb{R}\}) < \infty$ and $F_k(t) \to 0$ strongly for almost all $t \in \mathbb{R}$. Then for any $\psi \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and any $R > 0$, $\gamma(\{F_k * \psi(t) : |t| \le R\}) \to 0$. Consequently, for any $g \in \gamma(\mathbb{R}, X)$, $\|M_{F_k * \psi} g\|_{\gamma(\mathbb{R},X)} \to 0$ as $k \to \infty$.

Proof. (1): First note that $\{\hat{F}(t) : t \in \mathbb{R}\}$ is γ-bounded by the convex hull lemma of γ-bounded sets, see [3, Lemma 3.2] or [21, Lemma 2.2]. Then by Lemma 2.1 (5) and the estimate

$$\int_{\mathbb{R}} \|\hat{F}(s)g(t-s)\|_{\gamma(\mathbb{R},X)}ds \leq C\gamma(\{\hat{F}(s) : s \in \operatorname{supp}\hat{F}\})\|g\|_{\gamma(\mathbb{R},X)} < \infty,$$

we deduce immediately that $T_F g$ belongs to $\gamma(\mathbb{R}, X)$. If we let $\mathcal{F}_X : L^1(\mathbb{R}, X) \to C_0(\mathbb{R}, X)$ be the vector valued Fourier transform, i.e. $\mathcal{F}_X g(t) = \int_{\mathbb{R}} e^{-its}g(s)ds$, then we can express

$$\mathcal{F}_X T_F g(t) = \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-itr}\hat{F}(r-s)g(s)drds = \int_{\mathbb{R}} e^{-its}F(t)g(s)ds = F(t)\mathcal{F}_X g(t).$$

It is easy to check (see also [13, Example 4.9 b)]) that

$$u_{\mathcal{F}_X f} = \mathcal{F}^{\otimes} u_f$$

as soon as $f \in \gamma(\mathbb{R}, X) \cap L^1(\mathbb{R}, X)$. Applying this to both $f = T_F g$ and $f = g$, we deduce

$$\mathcal{F}^{\otimes} u_{T_F g} = u_{\mathcal{F}_X T_F g} = u_{F(\cdot)\mathcal{F}_X g(\cdot)} = M_F u_{\mathcal{F}_X g} = M_F \mathcal{F}^{\otimes} u_g.$$

Now the claim follows by applying $(\mathcal{F}^{\otimes})^{-1}$ to both sides.

(2): Write $F_k * \psi(t) = \int_{|s|\leq C} F_k(s)\psi(t-s)ds + \int_{|s|\geq C} F_k(s)\psi(t-s)ds$. For the first integral, note that for any $x \in X$, $\chi_{[-C,C]}(\cdot)F_k(\cdot)x \to 0$ in $L^1(\mathbb{R}, X)$ by assumption and dominated convergence. Since $\psi \in L^\infty(\mathbb{R})$, by [19, Corollary 2.17],

$$\gamma(\{(\chi_{[-C,C]}F_k) * \psi(t) : |t| \leq R\}) \to 0 \text{ for } k \to \infty.$$

For the second integral, we appeal again to [3, Lemma 3.2], noting that

$$\sup_{|t|\leq R} \|\psi(t-\cdot)(1-\chi_{[-C,C]})\|_1 \to 0 \text{ for } C \to \infty,$$

and thus $\sup_{k,\,|t|\leq R} \gamma(\{(1-\chi_{[C,C]})F_k * \psi(t) : |t| \leq R\}) \to 0$. Then the rest follows from Lemma 2.1. □

Recall $\ell^\infty_\gamma(L^2, X)$ from Definition 4.1 and its dense subspace $\widetilde{\ell^\infty_\gamma}$ of elements that allow a representation as a function $f : \mathbb{R} \to X$. We also let $W(t) = e^{it\widetilde{B}}$ be the shift group on $Z \subset \ell^\infty_\gamma(L^2, X)$, where it is strongly continuous.

Proposition 5.3. Let $\omega > 0$.

(1) For $F \in \mathcal{H}^\infty_0(\mathrm{Str}_\omega)$ and $y \in Z \cap \widetilde{\ell^\infty_\gamma}$, we have

$$F(\widetilde{B})y = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{F}(t)W(t)ydt,$$

where $F(\widetilde{B})$ is defined in (5.1).

(2) There exists a $C > 0$ such that for any $F \in \mathcal{H}^\infty_0(\mathrm{Str}_\omega)$, we have $\|F(\widetilde{B})\| \leq C\|F\|_{\mathcal{E}^\alpha}$.

(3) Recall the functions $\psi_k(z) = (e^z/(1+e^z)^2)^{1/k} \in H^\infty(\mathrm{Str}_\omega)$ from (2.2). For any $F \in \mathcal{H}^\infty(\mathrm{Str}_\omega)$, the strong limit $\Psi(F) = \lim_k (\psi_k F)(\widetilde{B})$ exists, and extends continuously to a mapping $\Psi : \mathcal{E}^\alpha \to B(Z)$ such that

$$\Psi(F) = F(\widetilde{B}) \quad (F \in \mathcal{H}^\infty_0(\mathrm{Str}_\omega)). \tag{5.3}$$

(4) *Convergence Property.* If $(F_k)_k \subset \mathcal{E}^\alpha$ with $\sum_{n\in\mathbb{Z}}\langle n\rangle^\alpha \sup_k \gamma(\{F_k * \check{\phi}_n(t) : t\in\mathbb{R}\}) < \infty$ and $F_k(t)y \to F(t)y$ for any $y \in Z$ and any $t \in \mathbb{R}$, then $F \in \mathcal{E}^\alpha$ and $\Psi(F_k)y \to \Psi(F)y$ for any $y \in Z$.

Proof. (1): This can be shown exactly as in the scalar case, cf. Proposition 4.6.

(2): Let $n \in \mathbb{Z}$ and $y \in Z \cap \widetilde{\ell^\infty_\gamma}$. As in the scalar case in the proof of Proposition 4.6, we get

$$\|\phi_n \int_{\mathbb{R}} \hat{F}(t)W(t)y dt\|_{\gamma(L^2(dt),X)} \le \sum \|\phi_n\|_\infty \|\phi_m \hat{F} * (\cdot)\|_{\gamma(L^2(dt),X)\to\gamma(L^2(dt),X)} \|\phi_k y\|_{\gamma(L^2(dt),X)},$$

where the sum is over $m \in \mathbb{Z}$ and $k \in \mathbb{Z}$ such that $|k-(n-m)| \le 3$. Lemma 5.2 (1) shows that $\|\phi_m \hat{F} * (\cdot)\|_{\gamma(L^2(dt),X)\to\gamma(L^2(dt),X)} = 2\pi\gamma(\{(\phi_m \hat{F})\check{}(t) : t\in\mathbb{R}\})$. As in the scalar case, we conclude that $\|\Psi(F)y\| \lesssim \|F\|_{\mathcal{E}^\alpha}\|y\|$.

(3): Recall that $\lim_k \psi_k(\widetilde{B}) = \mathrm{Id}_Z$ strongly. Thus, for $F \in \mathcal{H}^\infty_0(\mathrm{Str}_\omega)$, we have by the multiplicativity of the $\mathcal{H}^\infty_0$ calculus $\Psi(F) = \lim_k (\psi_k F)(\widetilde{B}) = \lim_k F(\widetilde{B})\psi_k(\widetilde{B}) = F(\widetilde{B})$. Let now $F \in \mathcal{H}^\infty(\mathrm{Str}_\omega)$. Then by Proposition 5.1, we have $\|(\psi_k F)(\widetilde{B})\| \lesssim \|\psi_k F\|_{\mathcal{E}^\alpha} \lesssim \|\psi_k\|_{E^\alpha_\infty}\|F\|_{\mathcal{E}^\alpha} \lesssim \|F\|_{\mathcal{E}^\alpha}$, so that $(\psi_k F)(\widetilde{B})$ is a uniformly bounded sequence. For $y = \psi_l(\widetilde{B})z$ for some $z \in Z$ and $l \in \mathbb{N}$,

$$(\psi_k F)(\widetilde{B})y = (\psi_k F)(\widetilde{B})\psi_l(\widetilde{B})z = (\psi_k\psi_l F)(\widetilde{B})z = (\psi_l F)(\widetilde{B})\psi_k(\widetilde{B})z \to (\psi_l F)(\widetilde{B})z \quad (k\to\infty).$$

As moreover, such y form a dense subset of Z, we can easily conclude with Proposition 5.1.

(4): We have $F*\check{\phi}_n(t)y = \lim_k F_k * \check{\phi}_n(t)y$ for any $y \in Z$ and $t \in \mathbb{R}$ by dominated convergence. It is easy from the definition of γ-boundedness to check that this implies $\gamma(\{F * \check{\phi}_n(t) : t \in \mathbb{R}\}) \le \sup_k \gamma(\{F_k * \check{\phi}_n(t) : t\in\mathbb{R}\})$. Now argue as in the proof of Proposition 4.7, using Lemma 5.2 (2) instead of Lemma 2.1. □

Remark 5.4. The bounded homomorphism $\Psi : \mathcal{E}^\alpha \to B(Z)$ is uniquely determined by (5.3) and the Convergence Property (4) in the above proposition, and we shall call such a Ψ the $\mathcal{E}^\alpha$ calculus of the group generator $\widetilde{B}$. We write henceforth $F(\widetilde{B})$ instead of $\Psi(F)$.

Indeed, if $F \in \mathcal{H}^\infty(\mathrm{Str}_\omega)$, then $F_k(z) = \psi_k(z)F(z)$ defines a sequence in $\mathcal{H}^\infty_0(\mathrm{Str}_\omega)$ which approximates F in the sense of (4) in the above proposition: Argue as in the proof of Proposition 4.7 to show that

$$\sum_{n\in\mathbb{Z}}\langle n\rangle^\alpha \sup_k \gamma(\{F_k * \check{\phi}_n(t) : t\in\mathbb{R}\}) \le \left(\sum_{n\in\mathbb{Z}} e^{-n|\omega-\epsilon|}\langle n\rangle^\alpha\right) \sup_k \gamma(\{F_k(z) : z \in \mathrm{Str}_\omega\}).$$

This determines $\Psi|_{\mathcal{H}^\infty(\mathrm{Str}_\omega)}$. By Proposition 5.1 (3), a bounded extension to $\mathcal{E}^\alpha$ is unique.

In the main theorem of this section below, we now are able to produce γ-bounded operators by means of the $\mathcal{E}^\alpha$ calculus.

Theorem 5.5. Let $U(t) = e^{itB}$ be a C_0-group on some space X not containing c_0 isomorphically such that $\{U(t) : t \in [0,1]\}$ is γ-bounded. Then the following conditions are equivalent:

(1) $\|U(t)\| \lesssim \langle t\rangle^\alpha$.
(2) B has a bounded E^α_∞ calculus.
(3) B has a bounded $\mathcal{E}^\alpha$ calculus.

If in addition, X has property (α), then the following conditions are also equivalent to (1)-(3).

(4) B has a bounded E^α_∞ calculus and moreover, for any $\mathcal{G} \subset E^\alpha_\infty$ with

$$\sum_{n\in\mathbb{Z}} \langle n\rangle^\alpha \sup_{f\in\mathcal{G}} \|f * \check{\phi}_n\|_\infty = C_{\mathcal{G}} < \infty,$$

$\{f(B) :\ f \subset \mathcal{G}\}$ is γ-bounded with constant $\lesssim C_{\mathcal{G}}$.

(5) B has a bounded $\mathcal{E}^\alpha$ calculus and moreover, for any $\mathcal{G} \subset \mathcal{E}^\alpha$ with

$$\sum_{n\in\mathbb{Z}} \langle n\rangle^\alpha \gamma(\{F * \check{\phi}_n(t) :\ t\in\mathbb{R},\ F\in\mathcal{G}\}) = C_{\mathcal{G}} < \infty,$$

$\{F(B) :\ F \subset \mathcal{G}\}$ is γ-bounded with constant $\lesssim C_{\mathcal{G}}$.

Proof. (1) $\Longrightarrow$ (3): Denote as before space and operators of the dilation of $U(t)$ by Z, P, I and $W(t) = e^{it\widetilde{B}}$. By Proposition 5.3, the generator $\widetilde{B}$ of $W(t)$ has a $\widetilde{\mathcal{E}^\alpha}$ calculus, where $\widetilde{\mathcal{E}^\alpha}$ is the operator valued space associated with $\widetilde{B}$. Let $J : B(X) \to B(Z)$ be defined first for $u \in Z \cap \widetilde{\ell^\infty_\gamma}$ by $J(T)[u] = T \circ u$. It is easy to check with Lemma 2.1 (4) that $\|J(T)[u]\|_{\ell^\infty_\gamma(L^2,X)} \le \|T\|_{B(X)}\|u\|_{\ell^\infty_\gamma(L^2,X)}$. Thus $J(T)$ extends by density of $Z \cap \widetilde{\ell^\infty_\gamma}$ to Z, and J is contractive. Further $J(B(X)) \subset [\widetilde{B}]'$. For $F \in \mathcal{E}^\alpha$, put $F_J(t) = J(F(t))$ which belongs to $\widetilde{\mathcal{E}^\alpha}$. It is a simple matter to check that $\Psi(F) = PF_J(B)I$ defines the $\mathcal{E}^\alpha$ calculus for B.

Then by Theorem 4.9, (1) - (3) are equivalent. Assume now in addition that X has property (α). We clearly have the implications (5) $\Longrightarrow$ (4) $\Longrightarrow$ (2), so that it remains to show

(1) $\Longrightarrow$ (5) : Write $\mathrm{Gauss}(X) = \gamma(\ell^2, X)$. Clearly, the span of elements of the form $e_n \otimes x_n$, $n \in \mathbb{N}$, $x_n \in X$ form a dense subset. For $t \in \mathbb{R}$, let $V(t) \in B(\mathrm{Gauss}(X))$ be defined by $V(t)(e_n \otimes x_n) = e_n \otimes U(t)x_n$. By Lemma 2.1 (4), $V(t)$ is a C_0-group and $\|V(t)\| = \|U(t)\| \lesssim \langle t\rangle^\alpha$. It is immediately checked that $\gamma(\{V(t) :\ t \in [0,1]\}) = \gamma(\{U(t) :\ t \in [0,1]\}) < \infty$. By the first part of the proof, the generator C of $V(t) = e^{itC}$ has a bounded $\mathcal{E}^\alpha_C$ calculus, where $\mathcal{E}^\alpha_C$ is the operator valued space associated with C. We consider $F_1, \dots, F_N \in \mathcal{G}$, where $\mathcal{G} \subset \mathcal{E}^\alpha$ is given in (5). For $t \in \mathbb{R}$ and $n = 1, \dots, N$, put

$$F(t)(e_n \otimes x_n) = e_n \otimes F_n(t)x_n$$

and $F(t)(e_n \otimes x_n) = 0$ $(n > N)$. Clearly, $F(t)$ extens linearly to $\ell^2 \otimes X$ and then boundedly to $\mathrm{Gauss}(X)$, and moreover, belongs to $[C]'$. Also $F \in \mathcal{E}^\alpha_C$, since

$$\begin{aligned}
\gamma\left(\{(F * \check{\phi}_m)(t) :\ t\in\mathbb{R}\}\right) &= \sup \|\sum_k \gamma_k \otimes (F * \check{\phi}_m)(t_k)y_k\|_{\mathrm{Gauss}(\mathrm{Gauss}(X))} \\
&= \sup \|\sum_{k,n} \gamma_k \otimes \gamma_n \otimes (F_n * \check{\phi}_m)(t_k)x_{n,k}\|_{\mathrm{Gauss}(\mathrm{Gauss}(X))} \\
&\lesssim \gamma\left(\{(\widetilde{F} * \check{\phi}_m)(t) :\ t\in\mathbb{R},\ \widetilde{F}\in\mathcal{G}\}\right),
\end{aligned}$$

and the last quantity is in $\ell^1(\mathbb{Z}, \langle m\rangle^\alpha)$ by assumption. Here, the supremum runs over all finite sums in k, $t_k \in \mathbb{R}$, $\sum_k e_k \otimes y_k = \sum_k e_k \otimes e_n \otimes x_{n,k}$ in Gauss(Gauss(X)) of norm less than 1. The last estimate above follows from property (α). Using Remark 5.4, it is easy to check that $F(C)(\sum_n \gamma_n \otimes x_n) = \sum_{n=1}^N \gamma_n \otimes F_n(B)x_n$. Now

$$\begin{aligned}\gamma(\{F_1(B), \dots, F_N(B)\}) = \|F(C)\|_{B(\mathrm{Gauss}(X))} &\lesssim \|F\|_{\mathcal{E}_C^\alpha} \\ &\le \sum_n \langle n\rangle^\alpha \gamma\left(\{(\widetilde{F} * \check{\phi}_n)(t) :\ t \in \mathbb{R},\ \widetilde{F} \in \mathcal{G}\}\right).\end{aligned}$$

Since $\gamma(\{F(B) :\ F \in \mathcal{G}\})$ equals the supremum of $\gamma(\{F_1(B), \dots, F_N(B)\})$ for all choices of N and F_n, (5) follows. □

The following γ-boundedness result is shown for the H^∞ calculus in [14] (see also [19, Theorem 12.8]).

Theorem 5.6. Let A be a sectorial operator on a Banach space with property (α). Assume that for some angle $\omega \in (0, \pi)$, A has an $H^\infty(\Sigma_\omega)$ calculus. Then for any $\theta \in (\omega, \pi)$, we have

$$\gamma(\{f(A) :\ \|f\|_{\infty,\theta} \le 1\}) < \infty.$$

In view of Theorem 5.5, we have the following partial extension to 0-sectorial operators with polynomially bounded imaginary powers.

Corollary 5.7. Let A be a sectorial operator on a Banach space with property (α). Assume that for some angle $\omega \in (0, \pi)$, A has an $H^\infty(\Sigma_\omega)$ calculus and that $\|A^{it}\| \lesssim \langle t\rangle^\alpha$. Then there is a constant $C > 0$ such that for any $f : (0, \infty) \to \mathbb{C}$ with $f \circ \exp \in E_\infty^\alpha$, we have

$$\gamma(\{f(tA) :\ t > 0\}) \le C\|f \circ \exp\|_{E_\infty^\alpha}.$$

Proof. Clearly, $B = \log(A)$ and $U(t) = A^{it}$ satisfy the assumptions of Theorem 5.5. For $t > 0$, let $f_t(s) = f(te^s)$. By the implication (1) ⇒ (4), it only remains to show that

$$\sum_{n\in\mathbb{Z}} \langle n\rangle^\alpha \sup_{t>0} \|f_t * \check{\phi}_n\|_\infty \lesssim \|f_1\|_{E_\infty^\alpha}. \tag{5.4}$$

But $f_t * \check{\phi}_n(s) = f_1 * \check{\phi}_n(s + \log(t))$, so that clearly, for any $t > 0$, $\|f_t * \check{\phi}_n\|_\infty = \|f_1 * \check{\phi}_n\|_\infty$, and thus, the left hand side of (5.4) in fact equals $\|f_1\|_{E_\infty^\alpha}$. □

6. E_∞^α norms of particular functions and Besov calculus

In this section, we show how the functional calculus obtained in Sections 4 and 5 can be applied to classical operators and functional calculi. Firstly, we deduce optimal bounds for semigroup and resolvents if the operator A is of the type considered so far. Secondly, we compare the E_∞^α calculus with Besov functional calculus in several situations.

We start with calculating the E_∞^α norms for some special functions. They correspond to semigroup operators generated by $A = e^B$, resolvents of A and variants of these. By Corollary 5.7, we will deduce γ-boundedness results for semigroup and resolvent operators under the condition that A has (norm) polynomially bounded imaginary powers and a bounded H^∞ calculus. The E_∞^α norm estimates are obtained by elementary calculus, yet they are sharp.

We will repeatedly make use of the following lemma whose proof is elementary [16, Lemma 5.20].

Lemma 6.1. Let $g : \mathbb{R} \to \mathbb{C}$ be integrable. Assume that there exists an interval I of length strictly less than π such that for any $t \in \mathbb{R}$ with $g(t) \neq 0$ we have $\arg(g(t)) \in I$. Then $\|\check{g}\|_\infty \cong \|g\|_1$, where the equivalence constants only depend on the length $|I| < \pi$.

Consider the operator

$$(rA)^\beta \exp(-aA),$$

where A is 0-sectorial, $a = re^{i\theta}$ such that $r > 0$ and $|\theta| < \frac{\pi}{2}$, and $\beta \geq 0$. We have $(rA)^\beta \exp(-aA) = f_a(B)$, where $B = \log(A)$, and

$$f_a(t) = (ae^t)^\beta \exp(-ae^t).$$

Proposition 6.2. $\|f_a\|_{E^\alpha_\infty} \cong (\frac{\pi}{2} - |\theta|)^{-(\alpha+\beta+\frac{1}{2})}$.

Proof. We have $\|f\|_{E^\alpha_\infty} = \sum_{n\in\mathbb{Z}} \langle n\rangle^\alpha \left\| \left[\hat{f}_a \cdot \phi_n\right]^\vee \right\|_\infty$. The proof consists now in determining $\hat{f}_a$ and using Lemma 6.1. This is outlined in [16, Proposition 5.19]. □

Combining the above proposition with Corollary 5.7, we get

Corollary 6.3. Let A be a 0-sectorial operator on a space X with property (α). Assume that A has a bounded $H^\infty(\Sigma_\omega)$ calculus for some $\omega \in (0, \pi)$ and $\|A^{it}\| \lesssim \langle t\rangle^\alpha$ for some $\alpha \geq 0$. Then for any $\beta \geq 0$, there exists $C > 0$ such that

$$\gamma\left(\{(tA)^\beta \exp(-te^{i\theta}A) : \ t > 0\}\right) \leq C\left(\frac{\pi}{2} - |\theta|\right)^{-(\alpha+\beta+\frac{1}{2})}.$$

Next consider the operators

$$\lambda^{1-\gamma}A^\gamma(\lambda + A)^{-1} \text{ and } \lambda(\lambda + A^\delta)^{-1}$$

where $\lambda = re^{i\theta}$ with $r > 0$ and $|\theta| < \pi$, $\gamma \in [0, 1]$ and $\delta \geq 1$. We have $\lambda^{1-\gamma}A^\gamma(\lambda + A)^{-1} = f_\gamma(B)$ and $\lambda(\lambda + A^\delta)^{-1} = g_\delta(B)$ with $f_\gamma(t) = \lambda^{1-\gamma}e^{\gamma t}(\lambda + e^t)^{-1}$ and $g_\delta(t) = \lambda(\lambda + e^{\delta t})^{-1}$.

Proposition 6.4.

(1) For any $\gamma \in [0, 1]$ and $\theta \in (-\pi, \pi)$, we have $\|f_\gamma\|_{E^\alpha_\infty} \cong (\pi - |\theta|)^{-(\alpha+1)}$ with equivalence constants independent of γ.
(2) For any $\theta \in (-\pi, \pi)$ and $\delta \geq 1$, we have

$$\|g_\delta\|_{E^\alpha_\infty} \cong \begin{cases} \delta^\alpha(\pi - |\theta|)^{-(\alpha+1)} & \text{for } \alpha > 0 \\ \log(\delta) + (\pi - |\theta|)^{-1}\left(|\log(\pi - |\theta|)| + 1\right) & \text{for } \alpha = 0 \end{cases}.$$

The proof follows the same idea as the one of Proposition 6.2 and is outlined in [16, Proposition 5.22]. The immediate consequence of Corollary 5.7 and Proposition 6.4 for 0-sectorial operators is the following.

Corollary 6.5. Let A be a 0-sectorial operator on a space X with property (α). Assume that A has a bounded H^∞ calculus and $\|A^{it}\| \lesssim \langle t\rangle^\alpha$ for some $\alpha \geq 0$.

(1) There exists $C > 0$ such that for any $\gamma \in [0,1]$ and $\theta \in [0,\pi)$, we have

$$\gamma\left(\{\lambda^{1-\gamma}A^{\gamma}(\lambda+A)^{-1} : |\arg\lambda| = \theta\}\right) \leq C\,(\pi-\theta)^{-(\alpha+1)}.$$

(2) There exists $C > 0$ such that for any $\delta \geq 1$ and $\theta \in [0,\pi)$, we have

$$\gamma\left(\{\lambda(\lambda+A^{\delta})^{-1} : |\arg\lambda| = \theta\}\right) \leq C \begin{cases} \delta^{\alpha}(\pi-\theta)^{-(\alpha+1)} & \text{for } \alpha > 0 \\ \log(\delta) + (\pi-\theta)^{-1}(|\log(\pi-\theta)|+1) & \text{for } \alpha = 0. \end{cases}$$

Let us now have a look at Besov functional calculus. Let B be a 0-strip-type operator generating the group $U(t) = e^{itB}$. Up to now we know that the polynomial growth of $U(t)$ is equivalent to the E_∞^α calculus of B (Theorem 4.9). We recall the classical definition of a Besov space $\mathcal{B}_{p,1}^{\beta}$ which is given at the beginning of Section 3, and focus on the case $\beta > 0$, $p = \infty$. While Proposition 3.3 shows that the E_∞^α norm is related to summability of the Fourier coefficients representing the function f, it is well known that $\|f\|_{\mathcal{B}_{\infty,1}^{\beta}}$ satisfies the almost norm equivalence

$$\|f\|_{C^{\alpha_0}} \lesssim \|f\|_{\mathcal{B}_{\infty,1}^{\beta}} \lesssim \|f\|_{C^{\alpha_1}}$$

for $\alpha_0 < \beta < \alpha_1$, where C^{α_0} and C^{α_1} are spaces of Hölder continuous functions, see [25]. Therefore, the $\mathcal{B}_{\infty,1}^{\beta}$ norm is closely related to differentiability of functions, which in examples is often easier to recognize than the summability of Fourier series. Thus, we shall investigate the $\mathcal{B}_{\infty,1}^{\beta}$ functional calculus and compare it with the E_∞^α calculus. Let us start with some observations on the multiplication operator B_p, where $B_p g(t) = tg(t)$ on $X = E_p^\alpha$ ($X = E_0^\alpha$ for $p = \infty$).

Proposition 6.6. Let $1 \leq p \leq \infty$ and $\alpha \geq 0$. If $p < \infty$, we let $X = E_p^\alpha$, and if $p = \infty$, we let $X = E_0^\alpha = E_\infty^\alpha \cap C_0(\mathbb{R})$. Consider the group $(U_p(t))_{t\in\mathbb{R}}$ defined by

$$U_p(t) : X \to X,\ g \mapsto e^{it(\cdot)}g.$$

Then $(U_p(t))_{t\in\mathbb{R}}$ is a C_0-group with $\|U_p(t)\| \cong \langle t\rangle^\alpha$. The associated 0-strip-type operator B_p has a E_∞^α calculus which is given by

$$f(B_p)g = fg \quad (f \in E_\infty^\alpha,\ g \in X).$$

Proof. Since $e^{it(\cdot)} \in E_\infty^\alpha$ with $\|e^{it(\cdot)}\|_{E_\infty^\alpha} \cong \langle t\rangle^\alpha$ according to the proof of Theorem 4.9, Proposition 3.2 yields $\|U_p(t)\|_{B(X)} \lesssim \langle t\rangle^\alpha$. This estimate is also optimal. Indeed, we have $(e^{it(\cdot)}g) * \check{\phi}_n(s) = e^{-its}[g * \phi_n(\cdot - t)\check{}\,](s)$. Thus if $\hat{g}$ has its support in $[-1,1]$, then $\|U_p(t)g\|_X \cong \langle t\rangle^\alpha \|g\|_X$. It is clear that $t \mapsto U_p(t)$ is a group. It is further strongly continuous. Indeed, for any $g \in X$ and $n \in \mathbb{Z}$, $\|(e^{it(\cdot)}g - g) * \check{\phi}_n\|_p \leq \|\check{\phi}_n\|_1 \|(e^{it(\cdot)} - 1)g\|_p \to 0$ as $t \to 0$ by dominated convergence for $p < \infty$, and by the fact that $\lim_{|t|\to\infty} g(t) = 0$ for the case $X = E_0^\alpha$. Now the strong continuity follows from

$$\sum_{n\in\mathbb{Z}} \langle n\rangle^\alpha \sup_{|t|\leq 1} \|(U_p(t)g) * \check{\phi}_n\|_p = \sum_{n\in\mathbb{Z}} \langle n\rangle^\alpha \sup_{|t|\leq 1} \|g * \phi_n(\cdot + t)\check{}\,\|_p < \infty.$$

Denote iB_p the generator of $U_p(t)$. For $f \in H_0^\infty(\mathrm{Str}_\omega)$ for some $\omega > 0$ and $g \in X$, we have

$$f(B)g = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(t)U_p(t)g\,dt$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(t) e^{it(\cdot)} g dt$$
$$= fg.$$

In particular, $\|f(B)g\| \lesssim \|f\|_{E^\alpha_\infty} \|g\|_X \lesssim \|f\|_{\infty,\omega} \|g\|_X$, so that B has a E^α_∞ calculus (in particular, an H^∞ calculus). Then by an approximation argument, using Proposition 4.7, $f(B)g = fg$ for any $f \in E^\alpha_\infty$ and $g \in X$. □

At first, we focus on the multiplication operator B_∞ on E^α_0. This is an extremal example in the following sense.

Proposition 6.7. Let $\alpha \geq 0$. Let B_∞ be the multiplication operator on $X = E^\alpha_0$ as in Proposition 6.6. Then

$$\|f(B_\infty)\| \cong \|f\|_{E^\alpha_\infty} \quad (f \in E^\alpha_\infty).$$

In particular, if C is a further 0-strip-type operator on some Banach space Y also having a bounded E^α_∞ calculus, then

$$\|f(C)\|_{B(Y)} \lesssim \|f(B_\infty)\|_{B(X)} \quad (f \in E^\alpha_\infty).$$

Further, B_∞ has a $\mathcal{B}^\beta_{\infty,1}$ calculus if and only if

$$\beta \geq \alpha + 1.$$

Proof. In view of Proposition 3.5, all we have to show is

$$\|f\|_{E^\alpha_\infty} \cong \sup\{\|fg\|_{E^\alpha_\infty} : g \in E^\alpha_0, \|g\|_{E^\alpha_0} \leq 1\}.$$

The inequality "$\gtrsim$" follows from the fact that E^α_∞ is a Banach algebra. For the other estimate, consider a sequence χ_k in E^α_0 with the properties $\chi_k(t) \to 1$ for any $t \in \mathbb{R}$ and $\sup_{k\in\mathbb{N}} \|\chi_k\|_{E^\alpha_\infty} < \infty$ (e.g. $\chi_k(t) = \chi(\frac{t}{k})$ for some $\chi \in C^\infty_0(\mathbb{R})$ such that $\chi(0) = 1$). Then for any $n \in \mathbb{Z}$ and $t \in \mathbb{R}$, by dominated convergence $(f\chi_k) * \check{\phi}_n(t) \to f * \check{\phi}_n(t)$, so that $\limsup_k \|(f\chi_k) * \check{\phi}_n\|_\infty \geq \|f * \check{\phi}_n\|_\infty$. Thus,

$$\sup_k \|f\chi_k\|_{E^\alpha_\infty} = \sup_{k,N} \sum_{|n|\leq N} \langle n\rangle^\alpha \|(f\chi_k) * \check{\phi}_n\|_\infty \geq \sup_N \sum_{|n|\leq N} \langle n\rangle^\alpha \|f * \check{\phi}_n\|_\infty = \|f\|_{E^\alpha_\infty}.$$

□

Proposition 6.7 shows that in full generality, there is a gap of $\beta - \alpha = 1$ between the Besov functional calculus and polynomial group growth. This result can be refined in terms of type and cotype of the underlying space X.

Theorem 6.8. Let X be a Banach space with property (α). Let further B be a 0-strip-type operator on X having an H^∞ calculus such that $\|e^{itB}\| \lesssim \langle t\rangle^\alpha$ for some $\alpha \geq 0$. Then B has a $\mathcal{B}^\beta_{\infty,1}$ calculus for

$$\beta > \alpha + \max\left(\frac{1}{\operatorname{type} X} - \frac{1}{\operatorname{cotype} X}, \frac{1}{2}\right).$$

On the other hand, for $2 \leq p < \infty$, the multiplication operator B_p on $X = E_p^\alpha$ from Proposition 6.6 does not have a $\mathcal{B}_{\infty,1}^\beta$ calculus for any

$$\beta < \alpha + \frac{1}{\operatorname{type} X} - \frac{1}{\operatorname{cotype} X}.$$

Theorem 6.8 will be shown in a forthcoming article.

Remark 6.9. (1) The second part of the theorem could also be stated as follows: The pointwise multiplication $E_\infty^\alpha \cdot E_p^\alpha$ maps to E_p^α, whereas

$$\mathcal{B}_{\infty,1}^\beta \cdot E_p^\alpha \to E_p^\alpha \text{ only if } \beta \geq \alpha + 1 - \frac{1}{p}.$$

(2) If the difference $\frac{1}{\operatorname{type} X} - \frac{1}{\operatorname{cotype} X}$ is less than $\frac{1}{2}$, then it is not clear what the optimal order β_0 for the Besov calculus in Theorem 6.8 is. Theorem 6.8 only yields the range $\beta_0 - \alpha \in [0, \frac{1}{2}]$.

(3) If iB is the generator of a uniformly bounded C_0-group on a Hilbert space, then by the transference principle of Coifman and Weiss, B has a bounded $H^\infty(\mathrm{Str}_\omega)$ calculus for any $\omega > 0$ and the norm of this calculus is independent of ω [19, Theorem 10.5]. Thus, $\beta_0 - \alpha = 0$ in this case (see [17, Lemma 3.2] for the sectorial counterpart of that last conclusion).

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References

[1] J. Bergh and J. Löfström. *Interpolation spaces. An introduction.* Grundlehren der mathematischen Wissenschaften, 223. Berlin etc.: Springer, 1976.
[2] K. Boyadzhiev and R. deLaubenfels. Spectral theorem for unbounded strongly continuous groups on a Hilbert space. *Proc. Am. Math. Soc.* 120(1):127–136, 1994.
[3] P. Clément, B. de Pagter, F. Sukochev and H. Witvliet. Schauder decomposition and multiplier theorems. *Studia Math.* 138:135–163, 2000.
[4] M. Cowling, I. Doust, A. McIntosh and A. Yagi. Banach space operators with a bounded H^∞ functional calculus. *J. Aust. Math. Soc., Ser. A* 60(1):51–89, 1996.
[5] E. Davies. *One-parameter semigroups.* London Mathematical Society, Monographs, No.15. London etc.: Academic Press, A Subsidiary of Harcourt Brace Jovanovich, Publishers. VIII, 1980.
[6] J. Diestel, H. Jarchow and A. Tonge. *Absolutely summing operators.* Cambridge Studies in Advanced Mathematics, 43. Cambridge: Cambridge Univ. Press, 1995.
[7] A. Fröhlich. *H^∞-Kalkül und Dilatationen.* PhD thesis, Universität Karlsruhe, 2003.
[8] A. Fröhlich and L. Weis. H^∞ calculus and Dilations. *Bull. Soc. math. France* 134 (4), 487–508, 2006.
[9] M. Haase. *The functional calculus for sectorial operators.* Operator Theory: Advances and Applications, 169. Basel: Birkhäuser, 2006.
[10] M. Haase. A transference principle for general groups and functional calculus on UMD spaces. *Math. Ann.* 345(2):245–265, 2009.
[11] N. Kalton, P. Kunstmann and L. Weis. Perturbation and interpolation theorems for the H^∞-calculus with applications to differential operators. *Math. Ann.* 336(4):747–801, 2006.

[12] N. Kalton, J. van Neerven, M. Veraar and L. Weis. Embedding vector-valued Besov spaces into spaces of γ-radonifying operators. *Math. Nachr.* 281(2):238–252, 2008.
[13] N. Kalton and L. Weis. The H^∞-calculus and square function estimates, preprint.
[14] N. Kalton and L. Weis. The H^∞-calculus and sums of closed operators. *Math. Ann.* 321(2):319–345, 2001.
[15] Y. Katznelson. *An introduction to harmonic analysis. 3rd ed.* Cambridge Mathematical Library, Cambridge: Cambridge University Press, 2004.
[16] C. Kriegler. Spectral multipliers, R-bounded homomorphisms, and analytic diffusion semigroups. PhD-thesis, online at http://digbib.ubka.uni-karlsruhe.de/volltexte/1000015866
[17] C. Kriegler and C. Le Merdy. Tensor extension properties of $C(K)$-representations and applications to unconditionality. *J. Aust. Math. Soc.* 88:205–230, 2010.
[18] C. Kriegler and L. Weis. The Hörmander functional calculus. In preparation.
[19] P. C. Kunstmann and L. Weis. Maximal L_p-regularity for parabolic equations, Fourier multiplier theorems and H^∞-functional calculus. *Functional analytic methods for evolution equations. Based on lectures given at the autumn school on evolution equations and semigroups, Levico Terme, Trento, Italy, October 28–November 2, 2001.* Berlin: Springer, Lect. Notes Math. 1855, 65–311, 2004.
[20] S. Kwapień and W. A. Woyczyński. *Random series and stochastic integrals: single and multiple.* Probability and Its Applications. Boston: Birkhäuser, 1992.
[21] C. Le Merdy. γ-bounded representations of amenable groups. Preprint on arxiv.org
[22] N. Lebedev. *Special functions and their applications. Rev. engl. ed. Translated and edited by Richard A. Silverman.* New York: Dover Publications, 1972.
[23] G. Pisier. Some results on Banach spaces without local unconditional structure. *Compos. Math.* 37:3–19, 1978.
[24] B. Sz.-Nagy and C. Foias. *Harmonic analysis of operators on Hilbert space.* Akademiai Kiadó, Budapest, 1970.
[25] H. Triebel. *Theory of function spaces.* Monographs in Mathematics, 78. Basel etc.: Birkhäuser, 1983.
[26] H. Triebel. *Theory of function spaces. III.* Monographs in Mathematics, 100. Basel etc.: Birkhäuser, 2006.
[27] J. van Neerven. γ-radonifying operators: a survey. *Proc. Centre Math. Appl. Austral. Nat. Univ.* 44:1–61, 2010.

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