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► **To cite this version:**

Hugo Gangloff, Jean-Baptiste Courbot, Emmanuel Monfrini, Christophe Collet. Spatial Triplet Markov Trees for auxiliary variational inference in Spatial Bayes Networks. SMTDA 2020: 6th international conference on Stochastic Modeling Techniques and Data Analysis, Jun 2020, Barcelone (online), Spain. pp.237-249. hal-03122810

HAL Id: hal-03122810

<https://hal.science/hal-03122810>

Submitted on 30 Jan 2023

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Spatial Triplet Markov Trees for Auxiliary Variational Inference in Spatial Bayes Networks

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Stochastic Modeling Techniques and Data Analysis,
Virtual Conference, June, 2-5 2020

Outline

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 - Spatial Bayes Network
 - STMT network
- 3 Variational Inference**
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 - Several VIs on a small example
 - Experimental results
- 4 Conclusion & Perspectives**

Introduction

Bayesian Networks: main definitions

- Also known as acyclic Directed Graphical Models (DGM)
- Let $\mathcal{G} = (\mathcal{S}, \mathcal{E})$ be a directed graph:
 - Directed edges link *parents* vertices $\mathcal{P}(s)$ and son vertice $s \in \mathcal{S}$
 - Vertices without any parent are *root* vertices (denoted r)
- Random variables associated to each vertice
- Define *local conditional probabilities* $p(x_r)$ or $p(x_s | \mathbf{x}_{\mathcal{P}(s)})$
→ They define joint distribution on \mathcal{G} :

$$p(\mathbf{x}) = \prod_{r \in \text{roots}} p(x_r) \prod_{s \in \mathcal{S} \setminus \text{roots}} p(x_s | \mathbf{x}_{\mathcal{P}(s)}) \quad (1)$$

Inference basics

Inference tasks:

- compute marginals $p(x_s)$
- compute conditionals $p(x_s | \mathbf{x}_{p(s)})$
- compute a mode of the distribution
(e.g. Maximum A Posteriori (MAP): $\hat{\mathbf{x}} = \operatorname{argmax}_{\mathbf{x}} p(\mathbf{x})$,
Maximum Posterior Modes (MPM): $\hat{x}_s = \operatorname{argmax}_{x_s} p(x_s)$)



Direct computations in Bayes Networks:

- Forward Backward (Stratonovich 1965) / Belief Propagation (Pearl 1982)

Approximate computations in general DGM:

- Loopy Belief Propagation (Weiss 2000)
- MCMC: Gibbs sampling (Geman et al. 1984)
- Variational Inference (Jordan et al. 1999)

Bayesian Networks: popularity

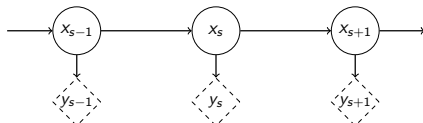
Hidden Markov Models (Baum et al. 1966) (Dymarski 2011)

Bayesian Networks with latent \mathbf{X} and observed variables \mathbf{Y} . Widely known for:

- Stock index forecasting (Gorynin et al. 2017)
- Speech processing (Toda 2011)
- Gene prediction (Stanke et al. 2006)

Hidden Markov Chain with Independent Noise:

$$p(\mathbf{x}, \mathbf{y}) = p(x_r) \prod_{s \in \mathcal{S}} p(x_s | x_{s-1}) \prod_{s \in \mathcal{S}} p(y_s | x_s) \quad (2)$$



Hidden Markov Chain (*direct computations*)

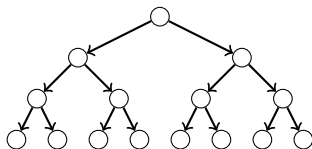
The probabilistic models

Markov Tree (MT) Bayesian Networks

Vertices (except roots) have one parent.

MT joint distribution (Monfrini et al. 2003):

$$p(\mathbf{x}) = p(x_r) \prod_{s \in \bar{S}} p(x_s | x_{s-}). \quad (3)$$



Dyadic MT (3 layers)
(direct computations)

- *Dyadic tree*: 1 root and 2 sons/vertice (except last layer).
- *Quadtree*: 1 root and 4 sons/vertice (except last layer).

Exact inference dyadic/quad trees:

- Upward-Downward (Durand et al. 2004) (principled approach to get the marginals in dyadic MT).

$$p(x_s) = \sum_{x_{s-}} p(x_{s-}) p(x_s | x_{s-}). \quad (4)$$

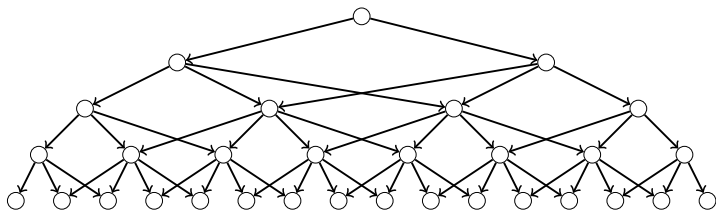
Spatial Bayesian Networks (SBN)

A SBN is a Bayesian network with joint distribution ([this paper](#)):

$$p(\mathbf{x}) = p(x_r) \prod_{s \in \bar{\mathcal{S}}} p(x_s | x_{s^-}, x_{v(s)}), \quad (5)$$

v associates a father's neighbour: not a Markov Tree!

The *spatial context* is modeled but the numerous loops makes inference complex and approximate.



SBN (based on a dyadic tree) with 4 layers
(approximate computations)

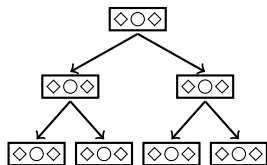
Spatial Triplet Markov Tree Bayesian Networks

In STMT, $\mathcal{T} = (\mathbf{X}, \mathbf{V})$ and \mathbf{V} auxiliary process ([this paper](#)) and ([Courbot et al. 2018](#)).

\mathcal{T} is a *Markov Tree* with joint distribution:

$$p(\mathbf{t}) = p(\mathbf{t}_r) \prod_{s \in \bar{\mathcal{S}}} p(\mathbf{t}_s | \mathbf{t}_{s-}), \quad (6)$$

with \mathbf{t}_s a triplet $(x_s, v_s^{\leftarrow}, v_s^{\rightarrow})$



STMT (Markov property)
(*direct computations*)

Spatial Triplet Markov Tree Bayesian Networks

In STMT, $\mathcal{T} = (\mathbf{X}, \mathbf{V})$ and \mathbf{V} auxiliary process ([this paper](#)) and ([Courbot et al. 2018](#)).

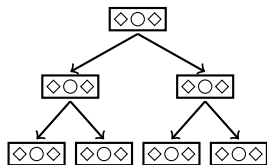
\mathcal{T} is a *Markov Tree* with joint distribution:

$$p(\mathbf{t}) = p(\mathbf{t}_r) \prod_{s \in \bar{\mathcal{S}}} p(\mathbf{t}_s | \mathbf{t}_{s^-}), \quad (6)$$

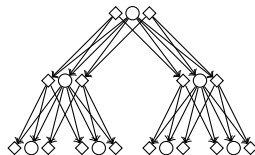
with \mathbf{t}_s a triplet $(x_s, v_s^{\leftarrow}, v_s^{\rightarrow})$ such that:

$$p(\mathbf{t}_s | \mathbf{t}_{s^-}) = p(x_s | x_{s^-}, \mathbf{v}_{s^-}) p(v_s^{\leftarrow} | x_{s^-}, \mathbf{v}_{s^-}) p(v_s^{\rightarrow} | x_{s^-}, \mathbf{v}_{s^-}). \quad (7)$$

The transitions are specially designed to model the *spatial context*.



STMT (Markov property)
(direct computations)



STMT (all edges)
(direct computations)

Sampling from the models

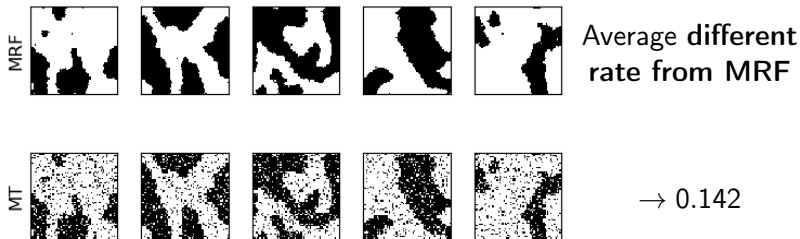
Test the ability to capture spatial context by *clamped* sampling from Markov Random Field (MRF) realizations



Average different
rate from MRF

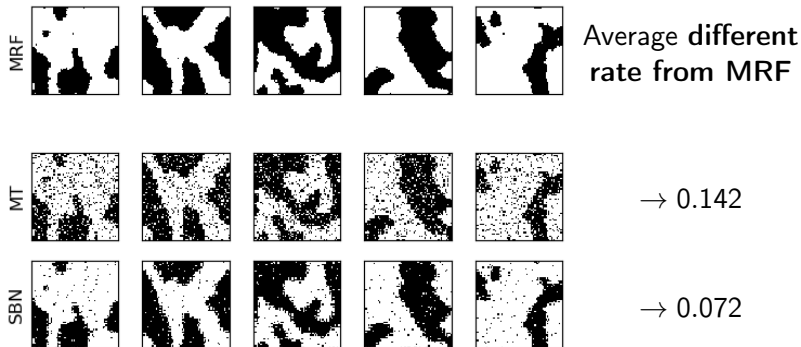
Sampling from the models

Test the ability to capture spatial context by *clamped* sampling from Markov Random Field (MRF) realizations



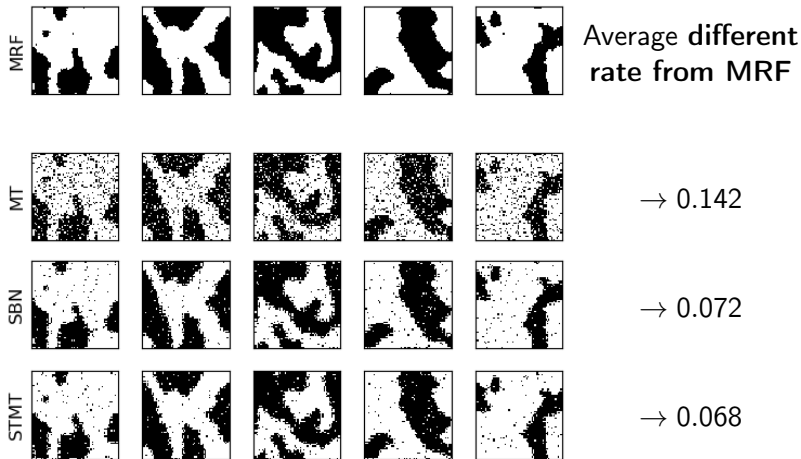
Sampling from the models

Test the ability to capture spatial context by *clamped* sampling from Markov Random Field (MRF) realizations



Sampling from the models

Test the ability to capture spatial context by *clamped* sampling from Markov Random Field (MRF) realizations



- SBN and STMT seem to capture the *spatial context*

Variational Inference

The method

VI recasts inference into a maximization problem (Jordan et al. 1999):

$$-\text{KL}(q(\mathbf{x})||p(\mathbf{x})) = \mathbb{E}_q[\log p(\mathbf{x})] - \mathbb{E}_q[\log q(\mathbf{x})], \quad (8)$$

where:

- p is the *target* distribution (SBN here)
- q is the *variational* distribution (**structured** or not)

The method

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where:

- p is the *target* distribution (SBN here)
- q is the *variational* distribution (**structured** or not)

Extended to integrate auxiliary variables (Agakov et al. 2004)

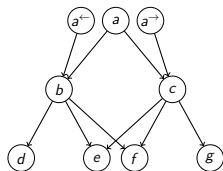
$$\begin{aligned} -\text{KL}(q(\mathbf{x}, \mathbf{v})||\tilde{p}(\mathbf{x}, \mathbf{v})) &= \mathbb{E}_q[\log \tilde{p}(\mathbf{x}, \mathbf{v})] - \mathbb{E}_q[\log q(\mathbf{x}, \mathbf{v})], \\ &= \mathbb{E}_q[\log \tilde{p}(\mathbf{x})] + \mathbb{E}_q[\log \tilde{p}(\mathbf{v}|\mathbf{x})] - \mathbb{E}_q[\log q(\mathbf{x}, \mathbf{v})]. \end{aligned} \quad (9)$$

\tilde{p} is extended with auxiliary variables (with condition: $\tilde{p}(\mathbf{x}) = p(\mathbf{x})$)

A small SBN example I

- Target distribution:

Let $\mathbf{x} = (a, a^{\leftarrow}, a^{\rightarrow}, b, c, d, e, f, g)$,
 $p(\mathbf{x})$ has a SBN factorization

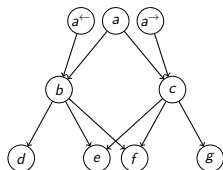


*(approximate
computations)*

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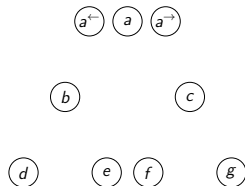
*(approximate
computations)*

- Variational approximation:

Mean-Field (MF) approximation:

$$q^{MF}(\mathbf{x}) = \prod_i q(x_i) \quad (10)$$

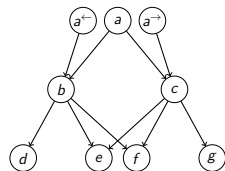
→ Independent random variables



*MF approximation
(direct computations)*

A small SBN example I

- Target distribution:



*(approximate
computations)*

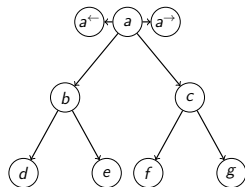
Let $\mathbf{x} = (a, a^{\leftarrow}, a^{\rightarrow}, b, c, d, e, f, g)$,
 $p(\mathbf{x})$ has a SBN factorization

- Variational approximation:

Markov Tree (MT) approximation:

$$q^{MT}(\mathbf{x}) = q(x_r) \prod_{s \in \bar{\mathcal{S}}} q(x_s | x_{s^-}). \quad (10)$$

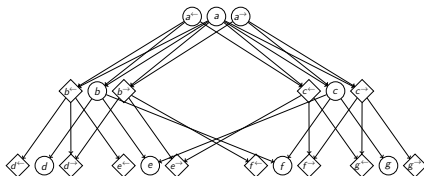
$\rightarrow q^{MT}$ is MT structured



*MT approximation
(direct computations)*

A small SBN example II

- Target distribution:



(approximate computations)

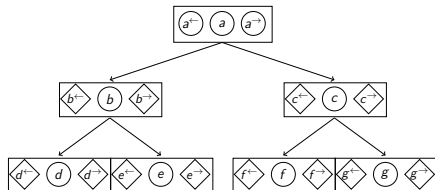
$\tilde{p}(\mathbf{x}, \mathbf{v})$ is a SBN with auxiliary variables (and $\tilde{p}(\mathbf{x}) = p(\mathbf{x})$)

- Variational approximation:

STMT approximation:

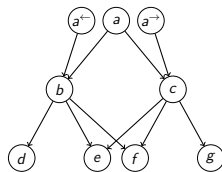
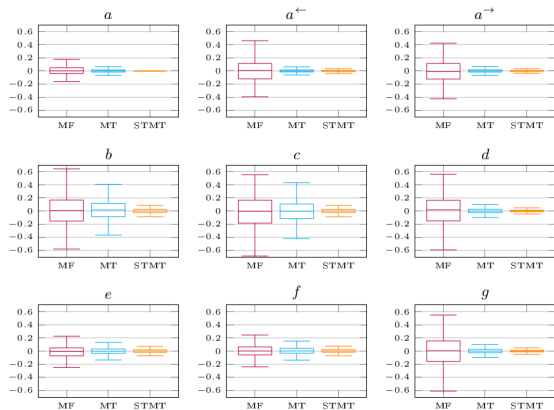
$$q^{STMT}(\mathbf{t}) = q(\mathbf{t}_r) \prod_{s \in \bar{S}} q(\mathbf{t}_s | \mathbf{t}_{s-}) \quad (11)$$

$\rightarrow q^{STMT}$ is STMT structured



STMT approximation
(direct computations)

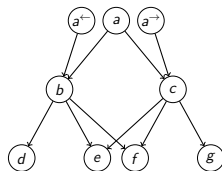
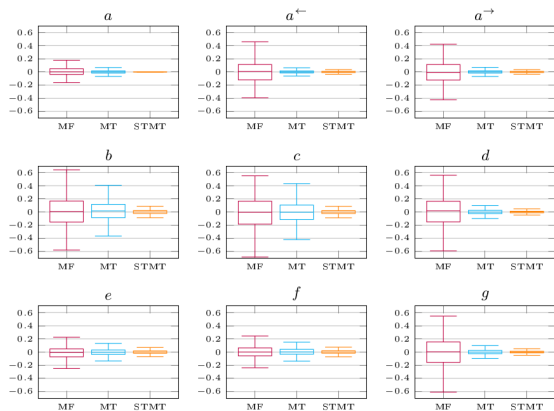
Small network example: model proximity



SBN

→ Dispersions of errors for true marginals estimation (1000 trials)

Small network example: model proximity



SBN

→ Dispersions of errors for true marginals estimation (1000 trials)

- Error dispersion : MF VI > MT VI > STMT VI
- STMT seems to best capture the enhanced correlations of SBN

Conclusion & Perspectives

Conclusion & Perspectives

Conclusions:

- 2 new tree-structured models with rich correlations
- Importance of the triplet structure for exact approaches in more complex networks.
- Auxiliary variables in probabilistic models can be an efficient asset if the model is well-thought!

Perspectives:

- Analytical comparison STMT/SBN
- Extend the work to quadtrees for image processing

References I

- [1] Felix V Agakov and David Barber. "An auxiliary variational method". In: *International Conference on Neural Information Processing*. Springer. 2004, pp. 561–566.
- [2] Leonard E Baum and Ted Petrie. "Statistical inference for probabilistic functions of finite state Markov chains". In: *The annals of mathematical statistics* 37.6 (1966), pp. 1554–1563.
- [3] Jean-Baptiste Courbot et al. "Triplet markov trees for image segmentation". In: *SSP 2018: IEEE Workshop on Statistical Signal Processing*. 2018, pp. 233–237.
- [4] J-B Durand, Paulo Goncalves, and Yann Guédon. "Computational methods for hidden Markov tree models-An application to wavelet trees". In: *IEEE Transactions on Signal Processing* 52.9 (2004), pp. 2551–2560.
- [5] Przemyslaw Dymarski. *Hidden Markov Models: Theory and Applications*. IntechOpen, 2011.
- [6] Stuart Geman and Donald Geman. "Stochastic relaxation, Gibbs distributions, and the Bayesian restoration of images". In: *IEEE Transactions on pattern analysis and machine intelligence* 6 (1984), pp. 721–741.
- [7] Ivan Gorynin, Emmanuel Monfrini, and Wojciech Pieczynski. "Pairwise Markov models for stock index forecasting". In: *25th European Signal Processing Conference (EUSIPCO)*. IEEE. 2017, pp. 2041–2045.
- [8] Michael I Jordan et al. "An introduction to variational methods for graphical models". In: *Machine learning* 37.2 (1999), pp. 183–233.
- [9] E Monfrini et al. "Image and signal restoration using pairwise Markov trees". In: *IEEE Workshop on Statistical Signal Processing, 2003*. IEEE. 2003, pp. 174–177.
- [10] Judea Pearl. "Reverend bayes on inference engines: a distributed hierarchical approach". In: *Proceedings of the Second AAAI Conference on Artificial Intelligence*. 1982, pp. 133–136.

References II

- [11] Mario Stanke et al. "Gene prediction in eukaryotes with a generalized hidden Markov model that uses hints from external sources". In: *BMC bioinformatics* 7.1 (2006), p. 62.
- [12] Ruslan Leont'evich Stratonovich. "Conditional markov processes". In: *Non-linear transformations of stochastic processes*. Elsevier, 1965, pp. 427–453.
- [13] Tomoki Toda. "Modeling of speech parameter sequence considering global variance for HMM-based speech synthesis". In: *Hidden Markov Models, Theory and Applications*. InTech, 2011, pp. 131–150.
- [14] Yair Weiss. "Correctness of local probability propagation in graphical models with loops". In: *Neural computation* 12.1 (2000), pp. 1–41.