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# Spatial Triplet Markov Trees for Auxiliary Variational Inference in Spatial Bayes Networks

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# Outline

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- 2 The probabilistic models**
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  - Spatial Bayes Network
  - STMT network
- 3 Variational Inference**
  - The method
  - Several VIs on a small example
  - Experimental results
- 4 Conclusion & Perspectives**

# Introduction

# Bayesian Networks: main definitions

- Also known as acyclic Directed Graphical Models (DGM)
- Let  $\mathcal{G} = (\mathcal{S}, \mathcal{E})$  be a directed graph:
  - Directed edges link *parents* vertices  $\mathcal{P}(s)$  and son vertice  $s \in \mathcal{S}$
  - Vertices without any parent are *root* vertices (denoted  $r$ )
- Random variables associated to each vertice
- Define *local conditional probabilities*  $p(x_r)$  or  $p(x_s | \mathbf{x}_{\mathcal{P}(s)})$   
→ They define joint distribution on  $\mathcal{G}$ :

$$p(\mathbf{x}) = \prod_{r \in \text{roots}} p(x_r) \prod_{s \in \mathcal{S} \setminus \text{roots}} p(x_s | \mathbf{x}_{\mathcal{P}(s)}) \quad (1)$$

# Inference basics

Inference tasks:

- compute marginals  $p(x_s)$
- compute conditionals  $p(x_s | \mathbf{x}_{p(s)})$
- compute a mode of the distribution  
(e.g. Maximum A Posteriori (MAP):  $\hat{\mathbf{x}} = \operatorname{argmax}_{\mathbf{x}} p(\mathbf{x})$ ,  
Maximum Posterior Modes (MPM):  $\hat{x}_s = \operatorname{argmax}_{x_s} p(x_s)$ )



Direct computations in Bayes Networks:

- Forward Backward (Stratonovich 1965) / Belief Propagation (Pearl 1982)

Approximate computations in general DGM:

- Loopy Belief Propagation (Weiss 2000)
- MCMC: Gibbs sampling (Geman et al. 1984)
- Variational Inference (Jordan et al. 1999)

# Bayesian Networks: popularity

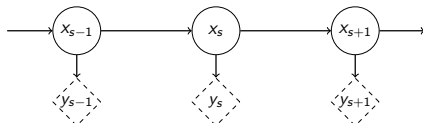
**Hidden Markov Models** (Baum et al. 1966) (Dymarski 2011)

Bayesian Networks with latent  $\mathbf{X}$  and observed variables  $\mathbf{Y}$ . Widely known for:

- Stock index forecasting (Gorynin et al. 2017)
- Speech processing (Toda 2011)
- Gene prediction (Stanke et al. 2006)

Hidden Markov Chain with Independent Noise:

$$p(\mathbf{x}, \mathbf{y}) = p(x_r) \prod_{s \in \bar{\mathcal{S}}} p(x_s | x_{s-1}) \prod_{s \in \mathcal{S}} p(y_s | x_s) \quad (2)$$



Hidden Markov Chain (*direct computations*)

## The probabilistic models

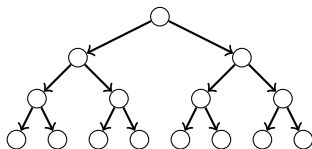


# Markov Tree (MT) Bayesian Networks

Vertices (except roots) have one parent.

MT joint distribution (Monfrini et al. 2003):

$$p(\mathbf{x}) = p(x_r) \prod_{s \in \bar{S}} p(x_s | x_{s-}). \quad (3)$$



*Dyadic MT (3 layers)*  
*(direct computations)*

- *Dyadic tree*: 1 root and 2 sons/vertice (except last layer).
- *Quadtree*: 1 root and 4 sons/vertice (except last layer).

Exact inference dyadic/quad trees:

- Upward-Downward (Durand et al. 2004) (principled approach to get the marginals in dyadic MT).

$$p(x_s) = \sum_{x_{s-}} p(x_{s-}) p(x_s | x_{s-}). \quad (4)$$

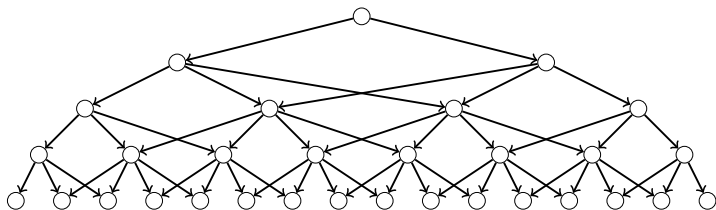
# Spatial Bayesian Networks (SBN)

A SBN is a Bayesian network with joint distribution ([this paper](#)):

$$p(\mathbf{x}) = p(x_r) \prod_{s \in \bar{\mathcal{S}}} p(x_s | x_{s^-}, x_{v(s)}), \quad (5)$$

$v$  associates a father's neighbour: not a Markov Tree!

The *spatial context* is modeled but the numerous loops makes inference complex and approximate.



SBN (based on a dyadic tree) with 4 layers  
(*approximate computations*)

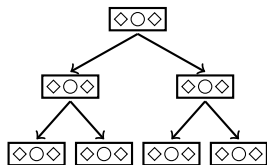
# Spatial Triplet Markov Tree Bayesian Networks

In STMT,  $\mathcal{T} = (\mathbf{X}, \mathbf{V})$  and  $\mathbf{V}$  auxiliary process ([this paper](#)) and ([Courbot et al. 2018](#)).

$\mathcal{T}$  is a *Markov Tree* with joint distribution:

$$p(\mathbf{t}) = p(\mathbf{t}_r) \prod_{s \in \bar{\mathcal{S}}} p(\mathbf{t}_s | \mathbf{t}_{s-}), \quad (6)$$

with  $\mathbf{t}_s$  a triplet  $(x_s, v_s^{\leftarrow}, v_s^{\rightarrow})$



STMT (Markov property)  
(*direct computations*)

# Spatial Triplet Markov Tree Bayesian Networks

In STMT,  $\mathcal{T} = (\mathbf{X}, \mathbf{V})$  and  $\mathbf{V}$  auxiliary process ([this paper](#)) and ([Courbot et al. 2018](#)).

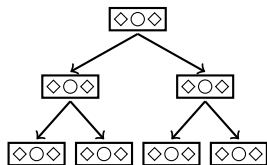
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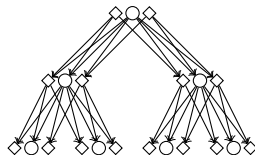
with  $\mathbf{t}_s$  a triplet  $(x_s, v_s^{\leftarrow}, v_s^{\rightarrow})$  such that:

$$p(\mathbf{t}_s | \mathbf{t}_{s^-}) = p(x_s | x_{s^-}, \mathbf{v}_{s^-}) p(v_s^{\leftarrow} | x_{s^-}, \mathbf{v}_{s^-}) p(v_s^{\rightarrow} | x_{s^-}, \mathbf{v}_{s^-}). \quad (7)$$

The transitions are specially designed to model the *spatial context*.



STMT (Markov property)  
(direct computations)



STMT (all edges)  
(direct computations)

## Sampling from the models

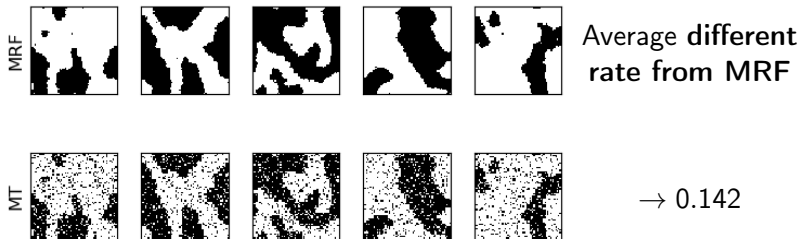
Test the ability to capture spatial context by *clamped* sampling from Markov Random Field (MRF) realizations



Average different  
rate from MRF

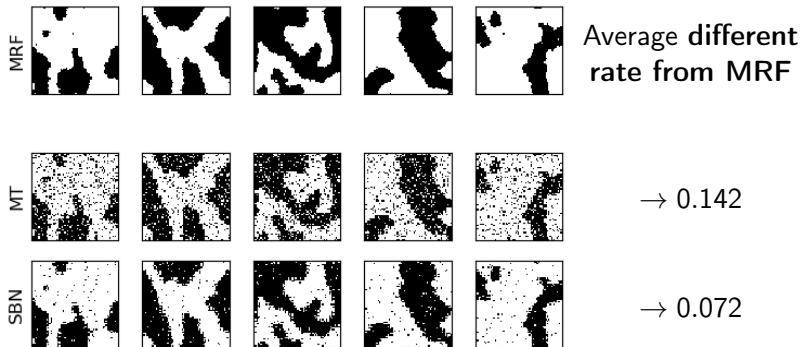
## Sampling from the models

Test the ability to capture spatial context by *clamped* sampling from Markov Random Field (MRF) realizations



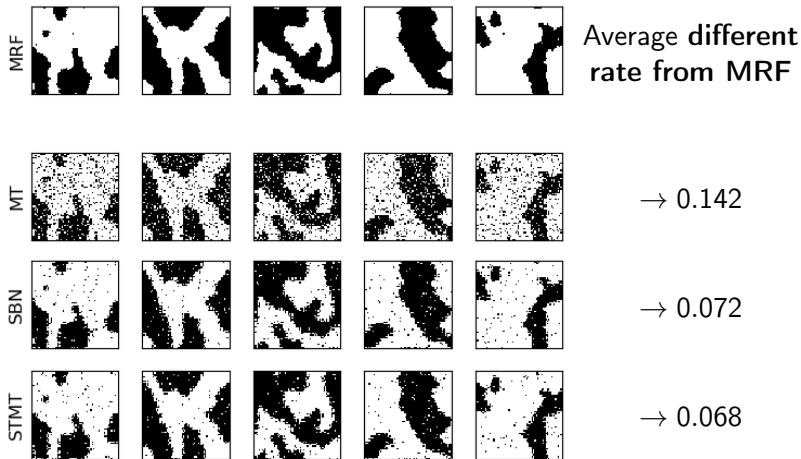
## Sampling from the models

Test the ability to capture spatial context by *clamped* sampling from Markov Random Field (MRF) realizations



## Sampling from the models

Test the ability to capture spatial context by *clamped* sampling from Markov Random Field (MRF) realizations



- SBN and STMT seem to capture the *spatial context*



# Variational Inference

# The method

VI recasts inference into a maximization problem (Jordan et al. 1999):

$$-\text{KL}(q(\mathbf{x})||p(\mathbf{x})) = \mathbb{E}_q[\log p(\mathbf{x})] - \mathbb{E}_q[\log q(\mathbf{x})], \quad (8)$$

where:

- $p$  is the *target* distribution (SBN here)
- $q$  is the *variational* distribution (**structured** or not)

# The method

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where:

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- $q$  is the *variational* distribution (**structured** or not)

Extended to integrate auxiliary variables (Agakov et al. 2004)

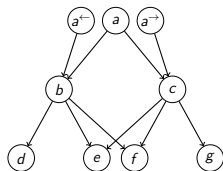
$$\begin{aligned} -\text{KL}(q(\mathbf{x}, \mathbf{v})||\tilde{p}(\mathbf{x}, \mathbf{v})) &= \mathbb{E}_q[\log \tilde{p}(\mathbf{x}, \mathbf{v})] - \mathbb{E}_q[\log q(\mathbf{x}, \mathbf{v})], \\ &= \mathbb{E}_q[\log \tilde{p}(\mathbf{x})] + \mathbb{E}_q[\log \tilde{p}(\mathbf{v}|\mathbf{x})] - \mathbb{E}_q[\log q(\mathbf{x}, \mathbf{v})]. \end{aligned} \quad (9)$$

$\tilde{p}$  is extended with auxiliary variables (with condition:  $\tilde{p}(\mathbf{x}) = p(\mathbf{x})$ )

## A small SBN example I

- Target distribution:

Let  $\mathbf{x} = (a, a^{\leftarrow}, a^{\rightarrow}, b, c, d, e, f, g)$ ,  
 $p(\mathbf{x})$  has a SBN factorization

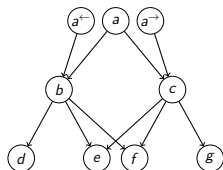


*(approximate  
computations)*

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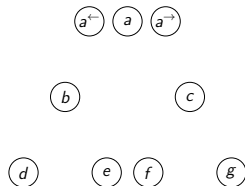
*(approximate  
computations)*

- Variational approximation:

Mean-Field (MF) approximation:

$$q^{MF}(\mathbf{x}) = \prod_i q(x_i) \quad (10)$$

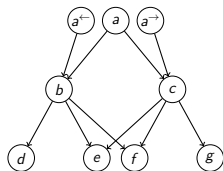
→ Independent random variables



*MF approximation  
(direct computations)*

# A small SBN example I

- Target distribution:



*(approximate  
computations)*

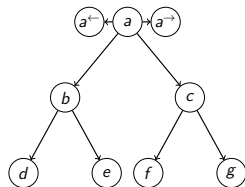
Let  $\mathbf{x} = (a, a^{\leftarrow}, a^{\rightarrow}, b, c, d, e, f, g)$ ,  
 $p(\mathbf{x})$  has a SBN factorization

- Variational approximation:

Markov Tree (MT) approximation:

$$q^{MT}(\mathbf{x}) = q(x_r) \prod_{s \in \bar{\mathcal{S}}} q(x_s | x_{s^-}). \quad (10)$$

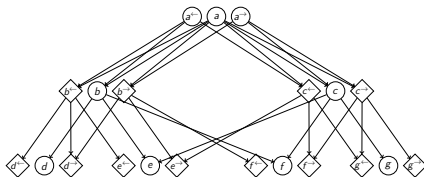
$\rightarrow q^{MT}$  is MT structured



*MT approximation  
(direct computations)*

## A small SBN example II

- Target distribution:



*(approximate computations)*

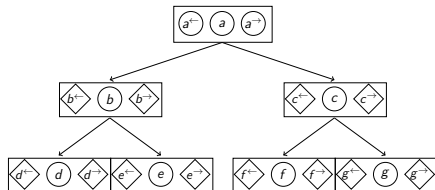
$\tilde{p}(\mathbf{x}, \mathbf{v})$  is a SBN with auxiliary variables (and  $\tilde{p}(\mathbf{x}) = p(\mathbf{x})$ )

- Variational approximation:

STMT approximation:

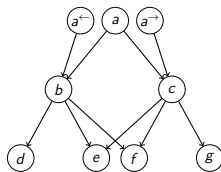
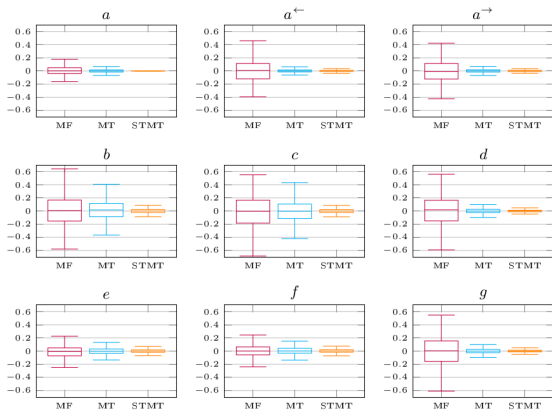
$$q^{STMT}(\mathbf{t}) = q(\mathbf{t}_r) \prod_{s \in \bar{S}} q(\mathbf{t}_s | \mathbf{t}_{s-}) \quad (11)$$

$\rightarrow q^{STMT}$  is STMT structured



STMT approximation  
*(direct computations)*

# Small network example: model proximity

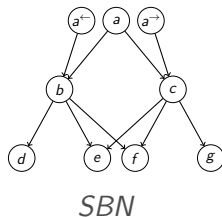
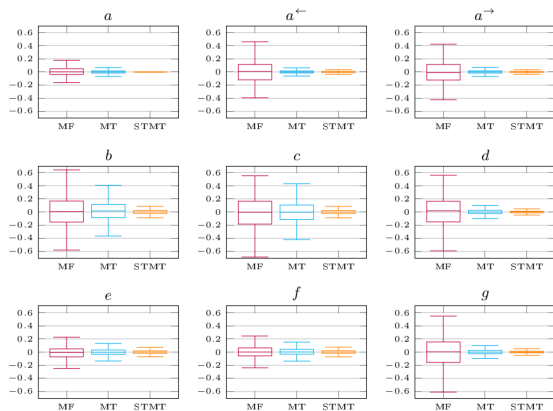


SBN

→ Dispersions of errors for true marginals estimation (1000 trials)



# Small network example: model proximity



→ Dispersions of errors for true marginals estimation (1000 trials)

- Error dispersion : MF VI > MT VI > STMT VI
- STMT seems to best capture the enhanced correlations of SBN

## Conclusion & Perspectives

# Conclusion & Perspectives

## Conclusions:

- 2 new tree-structured models with rich correlations
- Importance of the triplet structure for exact approaches in more complex networks.
- Auxiliary variables in probabilistic models can be an efficient asset if the model is well-thought!

## Perspectives:

- Analytical comparison STMT/SBN
- Extend the work to quadrees for image processing

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