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# Existence results for the $\mathbf{A} - \varphi - \mathbf{B}$ magnetodynamic formulation of the Maxwell system with skin and proximity effects

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## Abstract

The  $\mathbf{A} - \varphi - \mathbf{B}$  magnetodynamic Maxwell system given in its potential and space-time formulation is a popular model considered in the engineering community. It allows to model some phenomena such as eddy current losses in multiple turn winding. Indeed, in some cases, they can significantly alter the performance of the devices, and consequently can no more be neglected. It turns out that this model is not yet analytically studied, therefore we here consider its well-posedness. First, the existence of strong solutions with the help of the theory of Showalter on degenerated parabolic problems is established. Second, using energy estimates, existence and uniqueness of the weak solution of the  $\mathbf{A} - \varphi - \mathbf{B}$  is deduced.

**Key Words :** Maxwell equations, potential formulation, degenerate parabolic problems, Eddy current losses.

**AMS (MOS) subject classification** 35H65; 35Q60; 78A25.

## 1 Introduction

Let  $T > 0$  and  $\Omega \subset \mathbb{R}^3$  be an open simply connected bounded domain with a Lipschitz boundary  $\Gamma$  that is also connected. The usual Maxwell system is given in  $\Omega \times [0, T]$  by Faraday's law :

$$\operatorname{curl} \mathbf{E} = -\partial_t \mathbf{B}, \quad (1)$$

and Ampère-Maxwell's law :

$$\operatorname{curl} \mathbf{H} = \partial_t \mathbf{D} + \mathbf{J}, \quad (2)$$

with initial and boundary conditions to be specified. Here,  $\mathbf{E}$  stands for the electric field,  $\mathbf{H}$  for the magnetic field,  $\mathbf{B}$  for the magnetic flux density,  $\mathbf{J}$  for the current flux density (or eddy current) and  $\mathbf{D}$  for the displacement flux density. The Maxwell system also includes Gauss' laws, i.e., the divergence equations  $\operatorname{div} \mathbf{B} = 0$  and  $\operatorname{div} \mathbf{D} = q$ , with  $q$  the charge density, here supposed to be zero.

In the low frequency regime, the magneto-quasistatic approximation can be applied, which consists in neglecting the temporal variation of the displacement flux density with respect to the current density [5, 2], see also [1, p. 743], so that the propagation phenomena are not taken into account. Consequently, Ampère-Maxwell's equation (2) reduces to Ampère's equation

$$\operatorname{curl} \mathbf{H} = \mathbf{J}. \quad (3)$$

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The current density  $\mathbf{J}$  can be decomposed in two terms such that  $\mathbf{J} = \mathbf{J}_s + \mathbf{J}_e$ .  $\mathbf{J}_s$  is a known distribution current density generally generated by a coil, which is supposed to be divergence free in  $\Omega$  and such that  $\text{supp}(\mathbf{J}_s) = \Omega_s \subset \Omega$ .  $\mathbf{J}_e$  represents the unknown eddy current generated in the conductive part  $\Omega_c \subset \Omega$ , in which the electrical conductivity  $\sigma_c$  is not equal to zero. Both equations (1) and (3) are linked by the material constitutive laws (here we restrict ourselves to the isotropic and linear case):

$$\mathbf{B} = \nu^{-1} \mathbf{H}, \quad (4)$$

$$\mathbf{J}_e = \sigma_c \mathbf{E}, \quad (5)$$

where  $\nu$  stands for the reluctivity of the material. Figure 1 displays the domain configuration we are interested in, in the case  $\Omega_s \cap \Omega_c = \emptyset$ . Boundary conditions associated with the previous system are given by  $\mathbf{B} \cdot \mathbf{n} = 0$  on  $\Gamma$ , where  $\mathbf{n}$  denotes the unit outward normal along the boundary of  $\Omega$ .

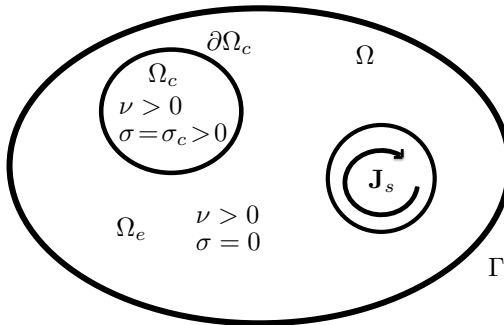


Figure 1: Domains configuration.

In order to solve the problem with the magneto-quasistatic approximation, a formulation which is able to take into account the eddy currents in  $\Omega_c$  and which verifies Maxwell's equations in  $\Omega_e = \Omega \setminus \Omega_c$  must be developed. This can be obtained by choosing the potential formulation often used for electromagnetic problems [13]. Indeed from the divergence free property of  $\mathbf{B}$ , namely  $\text{div } \mathbf{B} = 0$  in  $\Omega$ , the boundary condition  $\mathbf{B} \cdot \mathbf{n} = 0$  on  $\Gamma$ , and the fact that  $\Omega$  is a simply connected domain, by Theorem 3.17 of [3], a magnetic vector potential  $\mathbf{A}$  can be introduced such that

$$\mathbf{B} = \text{curl } \mathbf{A} \text{ in } \Omega, \quad (6)$$

with the boundary condition  $\mathbf{A} \times \mathbf{n} = 0$  on  $\Gamma$ . Like  $\mathbf{B}$ , the magnetic vector potential  $\mathbf{A}$  exists in the whole domain  $\Omega$ . To ensure its uniqueness, it is necessary to impose a gauge condition. The most popular one is  $\text{div } \mathbf{A} = 0$  (the so-called Coulomb gauge). Moreover, from equations (1) and (6), an electric scalar potential  $\varphi$  can be introduced in  $\Omega_c$  so that the electric field takes the form:

$$\mathbf{E} = -\partial_t \mathbf{A} - \nabla \varphi \text{ in } \Omega_c. \quad (7)$$

Similarly to the magnetic vector potential, the electric scalar potential must be gauged as well. To obtain uniqueness, the averaged value of the potential  $\varphi$  on  $\Omega_c$  is taken equal to zero. From (4),(5), (6) and (7), equation (3) leads to the so-called  $\mathbf{A} - \varphi$  formulation:

$$\text{curl}(\nu \text{curl } \mathbf{A}) + \sigma \left( \partial_t \mathbf{A} + \nabla \varphi \right) = \mathbf{J}_s. \quad (8)$$

The great interest of this formulation relies in its effectivity in both domain  $\Omega_c$  and  $\Omega_e$ . Indeed, in  $\Omega_c$  we have  $\sigma = \sigma_c$ , and in  $\Omega_e$  we have  $\sigma = 0$  so that the second term vanishes and the  $\mathbf{A} - \varphi$  formulation becomes the classical  $\mathbf{A}$  formulation used in the magnetostatic case. In a previous work [15], existence results for problem (8) with appropriated boundary conditions have been derived, using a weak formulation that is used for the numerical resolution by the Finite Element Method in the context of electromagnetic problems [8].

Actually,  $\Omega_s$  is usually composed of multiple-turn windings, in which some eddy currents are also generated if  $\Omega_s \cap \Omega_c \neq \emptyset$ . Consequently, when the working frequency (from imposed source) increases, the eddy currents/induced currents in the windings must be considered to ensure a good description of the involved physical phenomena. In particular, eddy current losses, composed of skin and proximity effect losses, have to be investigated. A first possibility is to discretize each separate turn of each winding and to use the above  $\mathbf{A} - \varphi$  formulation. Unfortunately, such a way to proceed is seldom if ever an option, given the extremely high computational cost in terms of memory requirements and computation time for fully-fledged 3D finite element models of industrial configurations. Consequently, a homogenization is used in order to replace the (litz-)wire bundle by a homogeneous domain. It consists in transforming a heterogeneous material like a winding region (consisting of conductors, insulation and e.g. air) into a homogeneous material. In the frequency domain, it amounts to the use of complex frequency-dependent proximity-effect reluctivity and skin-effect impedance values. In the time domain, one way to model such a skin-effect is to employ an RL ladder network of which the order determines the modeling accuracy [11, 21, 18]. By analogy, these authors introduce auxiliary inductions to approach the proximity-effect reluctivity. This allows to consider conductors of arbitrary cross-section and packing [10], and is used in 2D [11] as well as in 3D [21] finite element time dependent models. Numerical simulations allow to validate the relevance of this approach by comparison with experimental data, to evaluate for instance the leakage inductances in magnetic components [14], the impedance of multi-turn coils [12], the effective resistance of the windings [17, 19], or skin and proximity losses [20].

In the time domain, the skin effect is accounted for by directly adding the RL ladder circuit to the supply circuit. In what follows, for the sake of simplicity, the skin effect is no further considered as it does not influence the formulation. In many practical electrotechnical applications, the skin effect is negligible with regard to the proximity effect [11, 21, 18].

From the mathematical point of view, the time dependent  $\mathbf{A} - \varphi$  formulation has consequently to be reformulated and generalized, by adding additional unknowns coming from the homogenization method. The right time dependent  $\mathbf{A} - \varphi - \mathbf{B}$  formulation is then obtained by considering a new constitutive law, simply replacing (4) by

$$\begin{pmatrix} \mathbf{H} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{pmatrix} = \nu \begin{pmatrix} \text{curl } \mathbf{A} \\ \mathbf{B}_2 \\ \vdots \\ \mathbf{B}_n \end{pmatrix} + \sigma_s \mathcal{P} \partial_t \begin{pmatrix} \text{curl } \mathbf{A} \\ \mathbf{B}_2 \\ \vdots \\ \mathbf{B}_n \end{pmatrix}, \quad (9)$$

where the additional unknown  $\mathbf{B} = (\mathbf{B}_i)_{i=2}^n$  are auxiliary induction components in the winding region  $\Omega_s \subset \Omega$ , while  $\mathcal{P}$  is a given symmetric positive definite matrix of size  $3n \times 3n$  and  $\sigma_s$  is a given positive function in  $\Omega_s$  that depends on some material properties (see the precise assumptions below) and on the number  $n$  that characterizes the accuracy of the approximation (the number of auxiliary induction components being clearly equal to  $n - 1$ ), see the identities (10) and (17)-(18) in [11] or the identity (10) in [18]. Note that the first line of (9) is an extension of (4), a modification of the material law accounting for the proximity-effect losses. Therefore using this first line, Ampère's law (3), the splitting  $\mathbf{J} = \mathbf{J}_s + \mathbf{J}_e$ , the identity (5), and (7) lead to the differential equation

$$\text{curl}(\nu \text{curl } \mathbf{A}) + \sigma_c (\partial_t \mathbf{A} + \nabla \varphi) + \sigma_s \left( P_{11} \partial_t (\text{curl } \text{curl } \mathbf{A}) + \sum_{i=2}^n P_{1i} \partial_t (\text{curl } \mathbf{B}_i) \right) = \mathbf{J}_s, \quad (10)$$

instead of (8). On the other hand one directly sees that the  $n - 1$  last lines of (9) yield a system of differential equations in  $\mathbf{B}_i, i = 2, \dots, n$

$$\sigma_s \hat{\mathcal{P}} \partial_t \mathbf{B} + \sigma_s \mathcal{P}_1 \partial_t \text{curl } \mathbf{A} + \nu \mathbf{B} = \mathbf{0}, \quad (11)$$

where  $\hat{\mathcal{P}} = (\mathcal{P}_{ij})_{2 \leq i, j \leq n}$  is the "submatrix" of  $\mathcal{P}$ , where the first line and the first column of  $\mathcal{P}$  are skipped. Altogether using magnetic Gauss' law, and the boundary and initial conditions, we arrive at the set of equations (12)-(18) below. The advantage of the time dependent formulation is that it allows to consider non-sinusoidal sources, e.g. pulse width modulation, as well as non-linearities in material laws. As said before, the well-posedness of this problem is an open question. Therefore the goal of this paper is to establish the existence and uniqueness of strong and weak solutions to this system.

As previously, the domain  $\Omega$  contains the conductor part  $\Omega_c$  assumed to be simply connected, and the (not necessarily connected) source domain  $\Omega_s$  both with a Lipschitz boundary such that  $\overline{\Omega_c} \subset \Omega$  and  $\overline{\Omega_s} \subset \Omega$ . The electrical conductivity is not equal to zero in  $\Omega_c$  (as previously), but this time is also considered different from zero in  $\Omega_s$ , so that eddy currents may appear in  $\Omega_c \cup \Omega_s$ . The domain  $\Omega_e = \Omega \setminus (\overline{\Omega_c} \cap \overline{\Omega_s})$  is now defined as the part of  $\Omega$  where the electrical conductivity is identically equal to zero. Two geometrical configurations between  $\Omega_c$  and  $\Omega_s$  are supposed: either  $\overline{\Omega_c} \cap \overline{\Omega_s} = \emptyset$ , i.e.,  $\Omega_c$  and  $\Omega_s$  are disjoint or  $\Omega_s \subset \Omega_c$ , i.e.,  $\Omega_s$  is included into  $\Omega_c$ .

Let us finish this introduction by some notation used in the whole paper. On a given domain  $D$ , the  $L^2(D)$  norm is denoted by  $\|\cdot\|_D$ , and the corresponding  $L^2(D)$  inner product by  $(\cdot, \cdot)_D$ . The usual norm and semi-norm on  $H^1(D)$  are respectively denoted by  $\|\cdot\|_{1,D}$  and  $|\cdot|_{1,D}$ . In the case  $D = \Omega$ , we drop the index  $\Omega$ . Recall that  $H_0^1(D)$  is the subspace of  $H^1(D)$  with vanishing trace on  $\partial D$ . Finally, the notation  $a \lesssim b$  and  $a \sim b$  means the existence of positive constants  $C_1$  and  $C_2$ , which are independent of the quantities  $a$  and  $b$  under consideration such that  $a \leq C_2 b$  and  $C_1 b \leq a \leq C_2 b$ , respectively.

The paper is organized as follows. In section 2, the strong and weak formulations of the problem are presented and the existence result is stated (see Theorem 2.1). Then, section 3 is devoted to the proof of some preliminary results in order to apply a result of Showalter on degenerated parabolic problems. Finally, in section 4 we prove energy estimates and the main result of our paper. Let us note that the case corresponding to  $\Omega_c$  not simply connected as well as numerical aspects will be treated in a forthcoming work.

## 2 Formulation of the problem and the main result

Before stating the problem, let us specify the assumptions satisfied by the involved parameters introduced before. We suppose that  $\nu \in L^\infty(\Omega)$  and that there exists  $\nu_0 \in \mathbb{R}_+^*$  such that  $\nu > \nu_0$  in  $\Omega$ . We also assume that  $\sigma_c \in L^\infty(\Omega_c)$  and  $\sigma_s \in L^\infty(\Omega_s)$  and that there exists  $\sigma_0 \in \mathbb{R}_+^*$  such that  $\sigma_c > \sigma_0$  in  $\Omega_c$  and  $\sigma_s > \sigma_0$  in  $\Omega_s$ .  $\mathcal{P}$  is a symmetric positive definite matrix of size  $3n \times 3n$  given by

$$\mathcal{P} = (\mathcal{P}_{ij})_{1 \leq i, j \leq n},$$

where each  $\mathcal{P}_{ij}$  is a  $3 \times 3$  symmetric matrix such that  $\mathcal{P}_{ij} = \mathcal{P}_{ji}$ . Below  $\hat{\mathcal{P}} = (\mathcal{P}_{ij})_{2 \leq i, j \leq n}$  is the "submatrix" of  $\mathcal{P}$ , where the first line and the first column of  $\mathcal{P}$  are skipped, that is still a symmetric positive definite matrix, while  $\mathcal{P}_1$  is the first column of  $\mathcal{P}$  under  $\mathcal{P}_{11}$ , namely

$$\mathcal{P}_1 = (\mathcal{P}_{i1})_{2 \leq i \leq n} = \begin{pmatrix} \mathcal{P}_{21} \\ \mathcal{P}_{31} \\ \vdots \\ \mathcal{P}_{n1} \end{pmatrix}.$$

According to the introduction, the  $\mathbf{A} - \varphi - \mathbf{B}$  formulation of the magnetodynamic problem with skin and proximity effects can be formulated as follows: Given the source  $\mathbf{J}_s$  that is divergence free in  $\Omega$  (supported in  $\Omega_s$ ) and an initial datum  $\mathbf{A}_0$  (both in an appropriate spaces described below), we look for  $\mathbf{A}, \varphi$ , and

$\mathbf{B} = (\mathbf{B}_i)_{i=2}^n$  (also in appropriate spaces described below) solutions of

$$\operatorname{curl}(\nu \operatorname{curl} \mathbf{A}) + \sigma_c(\partial_t \mathbf{A} + \nabla \varphi) + \sigma_s \left( P_{11} \partial_t(\operatorname{curl} \operatorname{curl} \mathbf{A}) + \sum_{i=2}^n P_{1i} \partial_t(\operatorname{curl} \mathbf{B}_i) \right) = \mathbf{J}_s \text{ in } \Omega \times (0, T), \quad (12)$$

$$\operatorname{div}(\sigma_c(\partial_t \mathbf{A} + \nabla \varphi)) = 0 \text{ in } \Omega_c \times (0, T), \quad (13)$$

$$\sigma_s \hat{\mathcal{P}} \partial_t \mathbf{B} + \sigma_s \mathcal{P}_1 \partial_t \operatorname{curl} \mathbf{A} + \nu \mathbf{B} = 0 \text{ in } \Omega_s \times (0, T), \quad (14)$$

$$\mathbf{A} \times \mathbf{n} = 0 \text{ on } \Gamma \times (0, T), \quad (15)$$

$$\sigma_c(\partial_t \mathbf{A} + \nabla \varphi) \cdot \mathbf{n} = 0 \text{ on } \partial \Omega_c \times (0, T), \quad (16)$$

$$\mathbf{A}(t=0, \cdot) = \mathbf{A}_0 \text{ in } \Omega, \quad (17)$$

$$\mathbf{B}(t=0) = \mathbf{0} \text{ in } \Omega_s. \quad (18)$$

At last, we recall the gauge conditions. Like mentioned in section 1, we choose the Coulomb one  $\operatorname{div} \mathbf{A} = 0$  in  $\Omega$ , and we ask for the averaged value of  $\varphi$  in  $\Omega_c$  to be equal to zero.

We now define  $\mathbf{L}^2(\Omega) = L^2(\Omega)^3$ ,

$$\begin{aligned} X(\Omega) &= H_0(\operatorname{curl}, \Omega) = \left\{ \mathbf{A} \in \mathbf{L}^2(\Omega); \operatorname{curl} \mathbf{A} \in \mathbf{L}^2(\Omega) \text{ and } \mathbf{A} \times \mathbf{n} = \mathbf{0} \text{ on } \Gamma \right\}, \\ X_N(\Omega) &= \{ \mathbf{A} \in X(\Omega) : \operatorname{div} \mathbf{A} \in L^2(\Omega) \}, \\ X^0(\Omega) &= \{ \mathbf{A} \in X(\Omega) : \operatorname{div} \mathbf{A} = 0 \text{ in } \Omega \}, \\ \widetilde{H}^1(\Omega_c) &= \left\{ \varphi \in H^1(\Omega_c); \int_{\Omega_c} \varphi \, d\mathbf{x} = 0 \right\}, \end{aligned}$$

equipped with their usual norm

$$\begin{aligned} \|\mathbf{A}\|_{X(\Omega)}^2 &= \|\mathbf{A}\|^2 + \|\operatorname{curl} \mathbf{A}\|^2, \forall \mathbf{A} \in X(\Omega), \\ \|\mathbf{A}\|_{X^0(\Omega)}^2 &= \|\mathbf{A}\|^2 + \|\operatorname{curl} \mathbf{A}\|^2, \forall \mathbf{A} \in X^0(\Omega), \\ \|\mathbf{A}\|_{X_N(\Omega)}^2 &= \|\mathbf{A}\|_{X(\Omega)}^2 + \|\operatorname{div} \mathbf{A}\|^2, \forall \mathbf{A} \in X_N(\Omega), \\ \|\varphi\|_{\widetilde{H}^1(\Omega_c)} &= |\varphi|_{1, \Omega_c}, \forall \varphi \in \widetilde{H}^1(\Omega_c). \end{aligned}$$

Let us note that  $X_N(\Omega)$  is compactly embedded into  $(L^2(\Omega))^3$  because the boundary  $\partial \Omega$  is supposed to be Lipschitz regular, see [23] or [3, Theorem 2.8]. Similarly, we set

$$H(\operatorname{div} = 0, \Omega) = \{ \mathbf{A} \in \mathbf{L}^2(\Omega) : \operatorname{div} \mathbf{A} = 0 \text{ in } \Omega \},$$

that is a closed subspace of  $\mathbf{L}^2(\Omega)$ . Note that, here and below,  $\operatorname{div} \mathbf{A} = 0$  in  $\Omega$  means equivalently that

$$(\mathbf{A}, \nabla \xi) = 0 \quad \forall \xi \in H_0^1(\Omega).$$

We also need to introduce the following closed subspace of  $H(\operatorname{div} = 0, \Omega)$ :

$$H_0(\operatorname{div} = 0, \Omega) = \{ \mathbf{A} \in H(\operatorname{div} = 0, \Omega) : \mathbf{A} \cdot \mathbf{n} = 0 \text{ on } \partial \Omega \}.$$

The variational (or weak) formulation associated with (12)-(18) is obtained in a usual way, multiplying (12) by a test function  $\mathbf{A}' \in X_N(\Omega)$  (resp. (13) by a test function  $\varphi' \in \widetilde{H}^1(\Omega_c)$  and (14) by a test function  $\mathbf{B}' \in L^2(\Omega_s)^{3(n-1)}$ ), integrating the results in  $\Omega$ , formal integrations by parts and taking the sum we find

$$\begin{aligned} &(\sigma_c(\partial_t \mathbf{A} + \nabla \varphi), \bar{\mathbf{A}}' + \nabla \varphi')_{\Omega_c} + (\sigma_s \mathcal{P}(\operatorname{curl} \partial_t \mathbf{A}, \partial_t \mathbf{B})^\top, (\operatorname{curl} \mathbf{A}', \mathbf{B}')^\top)_{\Omega_s} + (\nu \operatorname{curl} \mathbf{A}, \operatorname{curl} \mathbf{A}')_{\Omega} \\ &+ (\nu \mathbf{B}, \mathbf{B}')_{\Omega_s} = (\mathbf{J}_s, \mathbf{A}')_{\Omega_s}, \quad \forall \mathbf{A}' \in X_N(\Omega), \varphi' \in \widetilde{H}^1(\Omega_c), \mathbf{B}' \in L^2(\Omega_s)^{3(n-1)}. \end{aligned} \quad (19)$$

An existence result for this problem can be stated as follows

**Theorem 2.1.** *Let us assume that  $\mathbf{J}_s \in H^1((0, T); H_0(\operatorname{div} = 0, \Omega_s))$  and set  $\mathbf{J}_{s,0} = \mathbf{J}_s(t = 0)$ . Let  $\mathbf{A}_0 \in X^0(\Omega)$  be the unique solution of*

$$(\nu \operatorname{curl} \mathbf{A}_0, \operatorname{curl} \mathbf{A}') = (\mathbf{J}_{s,0}, \mathbf{A}')_{\Omega_s}, \forall \mathbf{A}' \in X_N(\Omega).$$

*Then problem (19) has a unique solution  $(\mathbf{A}, \varphi, \mathbf{B})$  in  $H^1(0, T; X^0(\Omega)) \times L^2(0, T; \widetilde{H}^1(\Omega_c)) \times H^1(0, T; L^2(\Omega_s)^{3(n-1)})$  with  $\mathbf{A}(t = 0) = \mathbf{A}_0$  and  $\mathbf{B}(t = 0) = \mathbf{0}$ .*

*Proof.* The proof is postponed to section 4. ■

By the uniqueness of the solution, this local existence result directly allows to obtain a global one.

Note that by Theorem 1.17 of [4] (see also Corollary A.3 of [6]), any  $\mathbf{u} \in H^1(0, T; E)$ , where  $E$  is a Hilbert space, is absolutely continuous on  $[0, T]$  with values in  $E$  giving a meaning to the initial conditions in the above Theorem.

### 3 Preparations for the application of a theorem by Showalter

Our results on existence and uniqueness rely on the following theorem:

**Theorem 3.1** ([22], Theorem V4.B). *Let  $V_m$  be a seminorm space obtained from a symmetric and non-negative sesquilinear form  $m(\cdot, \cdot)$ , and let  $\mathcal{M} \in \mathcal{L}(V_m, V'_m)$  be the corresponding operator given by  $\mathcal{M}x(y) = m(x, y)$ , for all  $x, y \in V_m$ . Let  $V$  be a Hilbert space which is dense and continuously embedded into  $V_m$ . Let  $\mathcal{A}$  be a continuous, sesquilinear and elliptic form on  $V$  and denote by  $\mathcal{A}$  the corresponding isomorphism from  $V$  onto  $V'$ . Let  $D = \{u \in V : \mathcal{A}u \in V'_m\}$ . Then for any  $f \in C^1([0, \infty), V'_m)$  and  $y_0 \in V_m$ , there exists a unique solution  $y$  to*

$$\begin{cases} (\mathcal{M}y)_t(t) + \mathcal{A}y(t) = f(t) & \text{in } V'_m, \quad \forall t > 0, \\ \mathcal{M}y(0) = \mathcal{M}y_0 & \text{in } V'_m, \end{cases} \quad (20)$$

with the regularity

$$\mathcal{M}y \in C([0, \infty), V'_m) \cap C^1((0, \infty), V'_m)$$

and such that

$$y(t) \in D, \forall t > 0.$$

Before going on, let us mention some hidden regularity of the solution  $y$  of the previous problem (20).

**Lemma 3.2.** *Under the assumption of Theorem 3.1,  $y \in C((0, \infty), V)$ ,  $m(y, y) \in C^1(0, \infty)$  with*

$$\frac{d}{dt} m(y(t), y(t)) = 2\Re \langle (\mathcal{M}y)_t(t), y(t) \rangle, \forall t > 0, \quad (21)$$

where  $\langle \cdot, \cdot \rangle$  means the duality pairing between  $V'_m$  and  $V_m$  and finally

$$m(y(t), y(t)) \rightarrow m(y_0, y_0) \text{ as } t \rightarrow 0. \quad (22)$$

*Proof.* The first identity of (20) can be equivalently written as

$$y(t) = \mathcal{A}^{-1}(f(t) - (\mathcal{M}y)_t(t)), \forall t > 0.$$

Since  $f - (\mathcal{M}y)_t$  belongs to  $C((0, \infty), V'_m)$  and  $\mathcal{A}^{-1}$  is continuous from  $V'_m$  into  $V$ , the first assertion is proved.

For the second assertion, let us fix  $t > 0$  and an arbitrary real number  $h > -t$ . Then we may write

$$\begin{aligned} \frac{m(y(t+h), y(t+h)) - m(y(t), y(t))}{h} &= \left\langle \frac{\mathcal{M}y(t+h) - \mathcal{M}y(t)}{h}, y(t) \right\rangle + \overline{\left\langle \frac{\mathcal{M}y(t+h) - \mathcal{M}y(t)}{h}, y(t+h) \right\rangle} \\ &= \left\langle \frac{\mathcal{M}y(t+h) - \mathcal{M}y(t)}{h}, y(t) \right\rangle + \overline{\left\langle \frac{\mathcal{M}y(t+h) - \mathcal{M}y(t)}{h}, y(t) \right\rangle} \\ &= \left\langle \frac{\mathcal{M}y(t+h) - \mathcal{M}y(t)}{h}, y(t) - y(t+h) \right\rangle. \end{aligned}$$

The first two terms of this right-hand side clearly converge as  $h$  goes to zero, hence it remains to show that the last term tends to zero. Indeed by the definition of the norm in  $V'_m$ , one has

$$\left| \left\langle \frac{\mathcal{M}y(t+h) - \mathcal{M}y(t)}{h}, y(t) - y(t+h) \right\rangle \right| \leq \left\| \frac{\mathcal{M}y(t+h) - \mathcal{M}y(t)}{h} \right\|_{V'_m} |y(t) - y(t+h)|_{V_m}$$

where  $|z|_{V_m} = m(z, z)^{\frac{1}{2}}$  is the semi-norm associated with  $m$ . Since  $V$  is continuously embedded into  $V_m$ , one deduces that

$$\left| \left\langle \frac{\mathcal{M}y(t+h) - \mathcal{M}y(t)}{h}, y(t) - y(t+h) \right\rangle \right| \lesssim \left\| \frac{\mathcal{M}y(t+h) - \mathcal{M}y(t)}{h} \right\|_{V'_m} \|y(t) - y(t+h)\|_V.$$

By the regularity of  $\mathcal{M}y \in C^1((0, \infty), V'_m)$ , the first factor of this right-hand side remains bounded as  $h$  goes to zero, while the second factor tends to zero as one just shows that  $y \in C((0, \infty), V)$ .

Altogether we deduce that

$$\frac{m(y(t+h), y(t+h)) - m(y(t), y(t))}{h} \rightarrow \langle (\mathcal{M}y)_t(t), y(t) \rangle + \overline{\langle (\mathcal{M}y)_t(t), y(t) \rangle},$$

which shows that  $m(y, y)$  is  $C^1$  and that (21) holds.

Let us go on with the third assumption. First notice that for any  $z \in V_m$ , we have

$$|z|_{V_m} = \|\mathcal{M}z\|_{V'_m}. \quad (23)$$

Indeed we directly have

$$|z|_{V_m}^2 = m(z, z) = \langle \mathcal{M}z, z \rangle \leq \|\mathcal{M}z\|_{V'_m} |z|_{V_m},$$

which implies that

$$|z|_{V_m} \leq \|\mathcal{M}z\|_{V'_m}.$$

On the other hand, one has

$$\|\mathcal{M}z\|_{V'_m} = \sup_{w \in V_m: |w|_{V_m} \leq 1} |\langle \mathcal{M}z, w \rangle| = \sup_{w \in V_m: |w|_{V_m} \leq 1} |m(z, w)| \leq |z|_{V_m}.$$

Applying the identity (23) to  $y(t) - y_0$  for any  $t > 0$ , we get

$$|y(t) - y_0|_{V_m} = \|\mathcal{M}y(t) - \mathcal{M}y_0\|_{V'_m}.$$

Therefore  $|y(t) - y_0|_{V_m}$  tends to zero as  $t$  goes to zero, because this right-hand side does due to the regularity  $\mathcal{M}y \in C([0, \infty), V'_m)$  and the initial condition  $\mathcal{M}y(0) = \mathcal{M}y_0$ . Finally as the triangle inequality is valid for a semi-norm, we have

$$||y(t)|_{V_m} - |y_0|_{V_m}| \leq |y(t) - y_0|_{V_m},$$

which guarantees that (22) holds. ■

In order to apply this theorem, we show that problem (12)-(18) fits in the associated framework. This requires some preliminary results. First as in [15], for  $\mathbf{A} \in L^2(\Omega)^3$ , we consider the unique solution  $\varphi_{\mathbf{A}} \in \widetilde{H}^1(\Omega_c)$  of

$$\int_{\Omega_c} \sigma_c \nabla \varphi_{\mathbf{A}} \cdot \nabla \bar{\chi} \, dx = - \int_{\Omega_c} \sigma_c \mathbf{A} \cdot \nabla \bar{\chi} \, dx, \forall \bar{\chi} \in \widetilde{H}^1(\Omega_c). \quad (24)$$

Such a solution exists by Lax-Milgram lemma and furthermore, by Cauchy-Schwarz's inequality, we have

$$\|\sigma_c^{1/2} \nabla \varphi_{\mathbf{A}}\|_{\Omega_c} \leq \|\sigma_c^{1/2} \mathbf{A}\|_{\Omega_c}. \quad (25)$$

By (24), we deduce that the field  $\sigma_c(\mathbf{A} + \nabla \varphi_{\mathbf{A}})$  is divergence free in  $\Omega_c$ , i.e.,

$$\operatorname{div}(\sigma_c(\mathbf{A} + \nabla \varphi_{\mathbf{A}})) = 0 \text{ in } \Omega_c, \quad (26)$$



and satisfies the boundary condition

$$\sigma_c(\mathbf{A} + \nabla\varphi_{\mathbf{A}}) \cdot \mathbf{n} = 0 \text{ on } \partial\Omega_c. \quad (27)$$

Now we introduce the space

$$V_m := \{(\mathbf{A}, \mathbf{B})^\top \in \mathbf{L}^2(\Omega) \times \mathbf{L}^2(\Omega_s)^{n-1} : \text{curl } \mathbf{A} \in \mathbf{L}^2(\Omega_s)\},$$

and introduce the sesquilinear form

$$m((\mathbf{A}, \mathbf{B})^\top, (\mathbf{A}', \mathbf{B}')^\top) = \int_{\Omega_c} \sigma_c(\mathbf{A} + \nabla\varphi_{\mathbf{A}}) \cdot \bar{\mathbf{A}}' dx + \int_{\Omega_s} \sigma_s \mathcal{P} \mathbf{B}_{\mathbf{A}} \cdot \bar{\mathbf{B}}'_{\mathbf{A}'} dx, \forall (\mathbf{A}, \mathbf{B})^\top, (\mathbf{A}', \mathbf{B}')^\top \in V_m,$$

where  $\mathbf{B}_{\mathbf{A}}$  means the  $3n$  column

$$\mathbf{B}_{\mathbf{A}} = \begin{pmatrix} \text{curl } \mathbf{A} \\ \mathbf{B} \end{pmatrix}.$$

Note that by (24) with  $\chi = \varphi_{\mathbf{A}'}$ , we have

$$\int_{\Omega_c} \sigma_c(\mathbf{A} + \nabla\varphi_{\mathbf{A}}) \cdot \bar{\mathbf{A}}' dx = \int_{\Omega_c} \sigma_c(\mathbf{A} + \nabla\varphi_{\mathbf{A}}) \cdot (\bar{\mathbf{A}}' + \nabla\bar{\varphi}_{\mathbf{A}'}) dx,$$

and consequently the form  $m$  is symmetric and non-negative (recall that  $\mathcal{P}$  is symmetric positive definite) with

$$m((\mathbf{A}, \mathbf{B})^\top, (\mathbf{A}, \mathbf{B})^\top) \sim \|\mathbf{A} + \nabla\varphi_{\mathbf{A}}\|_{\Omega_c}^2 + \|\text{curl } \mathbf{A}\|_{\Omega_s}^2 + \|\mathbf{B}\|_{\Omega_s}^2, \forall (\mathbf{A}, \mathbf{B})^\top \in V_m.$$

From its definition and from the linearity of the mapping  $\mathbf{A} \rightarrow \nabla\varphi_{\mathbf{A}}$ , the expression  $|\cdot|_m := m(\cdot, \cdot)^{\frac{1}{2}}$  is a seminorm on  $V_m$ . It is indeed a seminorm but not a norm because by taking  $\mathbf{B} = \mathbf{0}$  and  $\mathbf{A} = \nabla\varphi$ , with  $\varphi \in H^1(\Omega)$  different from zero, we find that  $|(\nabla\varphi, \mathbf{0})^\top|_m = 0$ , while the pair  $(\nabla\varphi, \mathbf{0})^\top$  is different from zero.

Now recall that (see for instance [22]) the dual space  $V'_m$  of  $V_m$  is a Hilbert space that will be characterized in the next lemma. We start with the case  $\bar{\Omega}_c \cap \bar{\Omega}_s = \emptyset$ .

**Lemma 3.3.** *If  $\bar{\Omega}_c \cap \bar{\Omega}_s = \emptyset$ , then it holds*

$$V'_m = \{(\mathbf{A}', (\mathbf{B}'_i)_{i=1}^n) \in H_0(\text{div} = 0, \Omega_c) \times \mathbf{L}^2(\Omega_s)^n : \mathbf{B}'_1 \in H_c(\text{div} = 0, \Omega_s)\}, \quad (28)$$

*In other words,  $l \in V'_m$  if and only if there exist  $\mathbf{A}' \in H_0(\text{div} = 0, \Omega_c)$ ,  $\mathbf{B}'_1 \in H_c(\text{div} = 0, \Omega_s)$  and  $\mathbf{B}'_i \in \mathbf{L}^2(\Omega_s)$ ,  $i = 2, \dots, n$  such that  $l = l_{(\mathbf{A}', (\mathbf{B}'_i)_{i=1}^n)}$ , where*

$$l_{(\mathbf{A}', (\mathbf{B}'_i)_{i=1}^n)}(\mathbf{A}, \mathbf{B}) := \int_{\Omega_c} \mathbf{A}' \cdot \bar{\mathbf{A}} dx + \int_{\Omega_s} (\mathbf{B}'_i)_{i=1}^n \cdot \bar{\mathbf{B}}_A dx, \forall (\mathbf{A}, \mathbf{B}) \in V_m, \quad (29)$$

and

$$\|l\|_{V'_m} \sim \|\mathbf{A}'\|_{\Omega_c} + \sum_{i=1}^n \|\mathbf{B}'_i\|_{\Omega_s}.$$

*Proof.* Denote the right-hand side of (28) by

$$W_m = \{(\mathbf{A}', (\mathbf{B}'_i)_{i=1}^n) \in H_0(\text{div} = 0, \Omega_c) \times \mathbf{L}^2(\Omega_s)^n : \mathbf{B}'_1 \in H_c(\text{div} = 0, \Omega_s)\},$$

that is a Hilbert space equipped with the inner product

$$((\mathbf{A}', (\mathbf{B}'_i)_{i=1}^n), (\mathbf{A}'', (\mathbf{B}''_i)_{i=1}^n))_{W_m} = \int_{\Omega_c} \mathbf{A}' \cdot \bar{\mathbf{A}}'' dx + \int_{\Omega_s} (\mathbf{B}'_i)_{i=1}^n \cdot \bar{\mathbf{B}}'' dx.$$

The inclusion  $W_m \subset V'_m$  is direct since for  $(\mathbf{A}', (\mathbf{B}'_i)_{i=1}^n) \in W_m$ , the linear form  $l_{(\mathbf{A}', (\mathbf{B}'_i)_{i=1}^n)}$  defined above is continuous on  $V_m$ . Indeed since  $\mathbf{A}'$  belongs to  $H_0(\text{div} = 0, \Omega_c)$ , for any  $(\mathbf{A}, \mathbf{B}) \in V_m$ , we have

$$\int_{\Omega_c} \mathbf{A}' \cdot \bar{\mathbf{A}} dx = \int_{\Omega_c} \mathbf{A}' \cdot (\bar{\mathbf{A}} + \nabla\bar{\varphi}_{\mathbf{A}}) dx.$$

Inserting this identity in (29) and applying Cauchy-Schwarz's inequality we find that

$$|l(\mathbf{A}', (\mathbf{B}'_i)_{i=1}^n)(\mathbf{A}, \mathbf{B})| \lesssim (\|\mathbf{A}'\|_{\Omega_c} + \|(\mathbf{B}'_i)_{i=1}^n\|_{\Omega_s}) m((\mathbf{A}, \mathbf{B}), (\mathbf{A}, \mathbf{B}))^{1/2}.$$

For the converse inclusion, let us fix  $l \in V'_m$ , which, with the convention of [22, p. 5, 9], means that  $l$  is a conjugate linear functional from  $V_m$  into  $\mathbb{C}$  such that

$$|l(\mathbf{A}, \mathbf{B})| \lesssim m((\mathbf{A}, \mathbf{B}), (\mathbf{A}, \mathbf{B}))^{1/2}, \forall (\mathbf{A}, \mathbf{B}) \in V_m. \quad (30)$$

From this estimate, we first notice that

$$l(\mathbf{A}, \mathbf{B}) = l(\mathbf{A}'', \mathbf{B}), \forall (\mathbf{A}, \mathbf{B}), (\mathbf{A}'', \mathbf{B}) \in V_m \text{ s.t. } \mathbf{A} + \nabla\varphi_{\mathbf{A}} = \mathbf{A}'' + \nabla\varphi_{\mathbf{A}''} \text{ in } \Omega_c, \text{curl } \mathbf{A}' = \text{curl } \mathbf{A}'' \text{ in } \Omega_s. \quad (31)$$

Now let us fix an extension operator  $E$  from  $H^1(\Omega_s)$  to  $H^1(\mathbb{R}^3)$  (see [9, p. 25]) that is a linear continuous operator  $H^1(\Omega_s)$  to  $H^1(\mathbb{R}^3)$  satisfying  $(E\varphi)|_{\Omega_s} = \varphi$ , for all  $\varphi \in H^1(\Omega_s)$ . We further fix two (smooth) functions  $\eta_s, \eta_c \in \mathcal{D}(\Omega)$  such that  $\eta_s = 1$  in  $\Omega_s$  (resp.  $\eta_c = 1$  in  $\Omega_c$ ) and  $\eta_s = 0$  in  $\Omega_c$  (resp.  $\eta_c = 0$  in  $\Omega_s$ ). Then we set

$$E_s\varphi = \eta_s E\varphi, \forall \varphi \in H^1(\Omega_s).$$

This define a linear operator from  $H^1(\Omega_s)$  to  $H_0^1(\Omega)$ . We further notice that for any  $\varphi \in H^1(\Omega_s)$ ,  $E_s\varphi$  coincides with  $\varphi$  on  $\Omega_s$  and is equal to zero on  $\Omega_c$ .

Now we introduce a linear mapping  $F$  from  $W_m$  into  $V_m$  defined as follows: for any  $(\mathbf{A}'', (\mathbf{B}''_i)_{i=1}^n) \in W_m$ , we set

$$F(\mathbf{A}'', (\mathbf{B}''_i)_{i=1}^n) = (\mathbf{A}, \mathbf{B}),$$

where  $\mathbf{B} = (\mathbf{B}''_i)_{i=2}^n$ , and

$$\mathbf{A} = \eta_c \widetilde{\mathbf{A}''} + E_s R \mathbf{B}''_1,$$

where  $\widetilde{\mathbf{A}''}$  means the extension of  $\mathbf{A}''$  by zero outside  $\Omega_c$  and the operator  $R$  is defined in Lemma 3.4 below. Note that

$$\mathbf{A} = \mathbf{A}'' \text{ in } \Omega_c, \text{curl } \mathbf{A} = \mathbf{B}''_1 \text{ in } \Omega_s. \quad (32)$$

Now introduce the mapping

$$l_0 : W_m \rightarrow \mathbb{C} : (\mathbf{A}'', (\mathbf{B}''_i)_{i=1}^n) \rightarrow l(F(\mathbf{A}'', (\mathbf{B}''_i)_{i=1}^n)),$$

that is clearly conjugate linear and continuous from  $W_m$  to  $\mathbb{C}$ . Hence, by Riesz's representation theorem, there exists  $(\mathbf{A}', (\mathbf{B}'_i)_{i=1}^n) \in W_m$  such that

$$l_0(\mathbf{A}'', (\mathbf{B}''_i)_{i=1}^n) = \int_{\Omega_c} \mathbf{A}' \cdot \bar{\mathbf{A}}'' dx + \int_{\Omega_s} (\mathbf{B}'_i)_{i=1}^n \cdot (\bar{\mathbf{B}}''_i)_{i=1}^n dx, \forall (\mathbf{A}'', (\mathbf{B}''_i)_{i=1}^n) \in W_m. \quad (33)$$

Now for  $(\mathbf{A}, \mathbf{B}) \in V_m$ , we set

$$\mathbf{A}'' = \mathbf{A}|_{\Omega_c} + \nabla\varphi_{\mathbf{A}}, \mathbf{B}''_1 = \text{curl}(\mathbf{A}|_{\Omega_s}), \mathbf{B}''_i = \mathbf{B}_i, \forall i = 2, \dots, n. \quad (34)$$

This yields a pair  $(\mathbf{A}'', (\mathbf{B}''_i)_{i=1}^n)$  in  $W_m$  such that

$$l(\mathbf{A}, \mathbf{B}) = l(F(\mathbf{A}'', (\mathbf{B}''_i)_{i=1}^n)),$$

due to (31), (32) and (34). This means that

$$l(\mathbf{A}, \mathbf{B}) = l_0((\mathbf{A}'', (\mathbf{B}''_i)_{i=1}^n)),$$

and, by (33) and (34), that

$$\begin{aligned} l(\mathbf{A}, \mathbf{B}) &= \int_{\Omega_c} \mathbf{A}' \cdot \bar{\mathbf{A}}'' dx + \int_{\Omega_s} (\mathbf{B}'_i)_{i=1}^n \cdot (\bar{\mathbf{B}}''_i)_{i=1}^n dx \\ &= \int_{\Omega_c} \mathbf{A}' \cdot \bar{\mathbf{A}} dx + \int_{\Omega_s} (\mathbf{B}'_i)_{i=1}^n \cdot \bar{\mathbf{B}}_{\mathbf{A}} dx. \end{aligned}$$

This ends the proof.  $\blacksquare$

To complete the proof of the previous Lemma we need the next result.

**Lemma 3.4.** *There exists a linear and continuous operator  $R$  from  $H_c(\operatorname{div} = 0, \Omega_s)$  to  $H^1(\Omega_s)^3$  such that*

$$\operatorname{curl} R\mathbf{A} = \mathbf{A} \text{ in } \Omega_s. \quad (35)$$

*Proof.* By Lemma 3.5 of [3], the linear mapping

$$C : H^1(\Omega_s)^3 \rightarrow H_c(\operatorname{div} = 0, \Omega_s) : \Psi \rightarrow \operatorname{curl} \Psi,$$

is continuous and onto. Hence it admits a right inverse  $R$ , see [7, Thm 2.12], that is linear and continuous from  $H_c(\operatorname{div} = 0, \Omega_s)$  to  $H^1(\Omega_s)^3$ . ■

Note that Theorem 3.12 of [3] implies that  $C$  is also continuous from  $H(\operatorname{curl}, \Omega_s)$  into  $H_c(\operatorname{div} = 0, \Omega_s)$ . We go on with the case  $\Omega_s \subset \Omega_c$ .

**Lemma 3.5.** *If  $\Omega_s \subset \Omega_c$ , then it holds*

$$V'_m = \{(\mathbf{A}', (\mathbf{B}'_i)_{i=2}^n) \in H_0(\operatorname{div} = 0, \Omega_c) \times \mathbf{L}^2(\Omega_s)^{n-1} : \operatorname{curl} \mathbf{A}' \in \mathbf{L}^2(\Omega_s)\}, \quad (36)$$

*in other words,  $l \in V'_m$  if and only if there exist  $\mathbf{A}' \in H_0(\operatorname{div} = 0, \Omega_c)$  such that  $\operatorname{curl} \mathbf{A}' \in \mathbf{L}^2(\Omega_s)$  and  $\mathbf{B}' = (\mathbf{B}'_i)_{i=2}^n$  such that  $l = l_{(\mathbf{A}', (\mathbf{B}'_i)_{i=2}^n)}$ , where*

$$l_{(\mathbf{A}', \mathbf{B}')}(\mathbf{A}, \mathbf{B}) := \int_{\Omega_c} \mathbf{A}' \cdot \bar{\mathbf{A}} \, dx + \int_{\Omega_s} \mathbf{B}'_{\mathbf{A}'} \cdot \bar{\mathbf{B}}_{\mathbf{A}} \, dx, \forall (\mathbf{A}, \mathbf{B}) \in V_m, \quad (37)$$

and

$$\|l\|_{V'_m} \sim \|\mathbf{A}'\|_{\Omega_c} + \|\mathbf{B}'_{\mathbf{A}'}\|_{\Omega_s}.$$

*Proof.* Denote the right-hand side of (36) by

$$W_m = \{(\mathbf{A}', (\mathbf{B}'_i)_{i=2}^n) \in H_0(\operatorname{div} = 0, \Omega_c) \times \mathbf{L}^2(\Omega_s)^{n-1} : \operatorname{curl} \mathbf{A}' \in \mathbf{L}^2(\Omega_s)\},$$

that is a Hilbert space equipped with the inner product

$$((\mathbf{A}', \mathbf{B}'), (\mathbf{A}'', \mathbf{B}''))_{W_m} = \int_{\Omega_c} \mathbf{A}' \cdot \bar{\mathbf{A}}'' \, dx + \int_{\Omega_s} \mathbf{B}'_{\mathbf{A}'} \cdot \bar{\mathbf{B}}''_{\mathbf{A}''} \, dx$$

The inclusion  $W_m \subset V'_m$  is proved as before. So we concentrate on the converse inclusion. Let us fix  $l \in V'_m$  and introduce the mapping

$$l_0 : W_m \rightarrow \mathbb{C} : (\mathbf{A}'', \mathbf{B}'') \rightarrow l(\widetilde{\mathbf{A}}'', \mathbf{B}''),$$

that is clearly conjugate linear and continuous from  $W_m$  to  $\mathbb{C}$ . Hence, by Riesz's representation theorem, there exists  $(\mathbf{A}', \mathbf{B}') \in W_m$  such that

$$l_0(\mathbf{A}'', (\mathbf{B}''_i)_{i=1}^n) = \int_{\Omega_c} \mathbf{A}' \cdot \bar{\mathbf{A}}'' \, dx + \int_{\Omega_s} \mathbf{B}'_{\mathbf{A}'} \cdot \bar{\mathbf{B}}''_{\mathbf{A}''} \, dx, \forall (\mathbf{A}'', \mathbf{B}'') \in W_m. \quad (38)$$

Now for  $(\mathbf{A}, \mathbf{B}) \in V_m$ , we set

$$\mathbf{A}'' = \mathbf{A}|_{\Omega_c} + \nabla \varphi_{\mathbf{A}}, \text{ and } \mathbf{B}'' = \mathbf{B}. \quad (39)$$

This yields a pair  $(\mathbf{A}'', \mathbf{B}'')$  in  $W_m$  such that

$$l(\mathbf{A}, \mathbf{B}) = l(\widetilde{\mathbf{A}}'', \mathbf{B}''),$$

due to (31) and (39). This means that

$$l(\mathbf{A}, \mathbf{B}) = l_0(\mathbf{A}'', \mathbf{B}''),$$

and we conclude as before. ■

For further purposes, we also need the following density result.

**Lemma 3.6.** *The space  $H_0^1(\Omega)^3 \times \mathcal{D}(\Omega_s)^{3(n-1)}$  is dense in  $V_m$ .*

*Proof.* As  $\mathcal{D}(\Omega_s)$  is dense in  $L^2(\Omega_s)$ , it suffices to show that  $H_0^1(\Omega)^3$  is dense in

$$Y := \{\mathbf{A} \in \mathbf{L}^2(\Omega) : \operatorname{curl} \mathbf{A} \in \mathbf{L}^2(\Omega_s)\},$$

equipped with the norm

$$\|\mathbf{A}\|_Y = \|\mathbf{A}\|_\Omega + \|\operatorname{curl} \mathbf{A}\|_{\Omega_s}.$$

Note that

$$m((\mathbf{A}, \mathbf{B})^\top, (\mathbf{A}', \mathbf{B}')^\top)^{\frac{1}{2}} \lesssim \|\mathbf{A}\|_Y + \|\mathbf{B}\|_{\Omega_s}.$$

Hence convergence in the norm of the right-hand side norm implies convergence in the left-hand side semi-norm.

So let us fix  $\mathbf{A} \in Y$  and take an arbitrary positive real number  $\varepsilon$ . First by Proposition 2.3 of [3], there exists  $\mathbf{A}' \in \mathcal{D}(\bar{\Omega}_s)^3$  such that

$$\|\mathbf{A} - \mathbf{A}'\|_{H(\operatorname{curl}, \Omega_s)} \leq \varepsilon. \quad (40)$$

Now as before by using an extension operator  $E_{\Omega_s}$  from  $H^1(\Omega_s)$  to  $H^1(\mathbb{R}^3)$  (see [9, p. 25]) and a (smooth) functions  $\eta_{\Omega_s} \in \mathcal{D}(\Omega)$  such that  $\eta_{\Omega_s} = 1$  in  $\Omega_s$ , we can consider  $\mathbf{A} - \eta_{\Omega_s} E_{\Omega_s} \mathbf{A}'$  that belongs to  $\mathbf{L}^2(\Omega \setminus \Omega_s)$ . Therefore there exists  $\mathbf{C} \in \mathcal{D}(\Omega \setminus \Omega_s)^3$  such that

$$\|\mathbf{A} - \eta_{\Omega_s} E_{\Omega_s} \mathbf{A}' - \mathbf{C}\|_{\Omega \setminus \Omega_s} \leq \varepsilon. \quad (41)$$

Then we set

$$\mathbf{D} = \begin{cases} \eta_{\Omega_s} E_{\Omega_s} \mathbf{A}' + \mathbf{C} & \text{in } \Omega \setminus \Omega_s, \\ \mathbf{A}' & \text{in } \Omega_s. \end{cases}$$

By construction  $\mathbf{D}$  belongs to  $H_0^1(\Omega)^3$ . Furthermore by (40) and (41), we see that

$$\|\mathbf{A} - \mathbf{D}\|_\Omega^2 = \|\mathbf{A} - \mathbf{A}'\|_{\Omega_s}^2 + \|\mathbf{A} - \eta_{\Omega_s} E_{\Omega_s} \mathbf{A}' - \mathbf{C}\|_{\Omega \setminus \Omega_s}^2 \leq 2\varepsilon^2,$$

while by (40)

$$\|\operatorname{curl} \mathbf{A} - \operatorname{curl} \mathbf{D}\|_{\Omega_s} \leq \varepsilon.$$

These two estimates yield

$$\|\mathbf{A} - \mathbf{D}\|_Y \leq (1 + \sqrt{2})\varepsilon.$$

The proof is then complete.  $\blacksquare$

At this stage in order to apply Theorem 3.1, we define the Hilbert space

$$V = \{(\mathbf{A}, \mathbf{B}) \in V_m : \mathbf{A} \in X_N(\Omega)\}$$

equipped with the inner product

$$\begin{aligned} ((\mathbf{A}, \mathbf{B}), (\mathbf{A}', \mathbf{B}'))_V &= \int_\Omega \left( \operatorname{curl} \mathbf{A} \cdot \operatorname{curl} \bar{\mathbf{A}}' + \operatorname{div} \mathbf{A} \operatorname{div} \bar{\mathbf{A}}' + \mathbf{A} \cdot \bar{\mathbf{A}}' \right) dx \\ &+ \int_{\Omega_s} \mathbf{B} \cdot \bar{\mathbf{B}}' dx, \forall (\mathbf{A}, \mathbf{B}), (\mathbf{A}', \mathbf{B}') \in V. \end{aligned}$$

We also introduce the sesquilinear form

$$\begin{aligned} a((\mathbf{A}, \mathbf{B}), (\mathbf{A}', \mathbf{B}')) &= \int_\Omega \left( \nu \operatorname{curl} \mathbf{A} \cdot \operatorname{curl} \bar{\mathbf{A}}' + \operatorname{div} \mathbf{A} \operatorname{div} \bar{\mathbf{A}}' \right) dx \\ &+ \int_{\Omega_s} \nu \mathbf{B} \cdot \bar{\mathbf{B}}' dx, \forall (\mathbf{A}, \mathbf{B}), (\mathbf{A}', \mathbf{B}') \in V, \end{aligned}$$

that is clearly continuous and coercive on  $V$  due the compact embedding of  $X_N(\Omega)$  into  $L^2(\Omega)^3$ .

Furthermore  $V$  is clearly continuously embedded into  $V_m$  and is dense in  $V_m$  due to Lemma 3.6. Therefore we have checked all assumptions of Theorem 3.1. Before stating a consequence of this theorem, we introduce

$$D = \{(\mathbf{A}, \mathbf{B}) \in V : \exists l \in V'_m : a((\mathbf{A}, \mathbf{B}), (\mathbf{A}', \mathbf{B}')) = l(\mathbf{A}', \mathbf{B}'), \forall (\mathbf{A}', \mathbf{B}') \in V\},$$

and show that elements of such a set are divergence free.

**Lemma 3.7.** *For any  $(\mathbf{A}, \mathbf{B}) \in D$ ,  $\mathbf{A}$  is divergence free in  $\Omega$ .*

*Proof.* Fix  $(\mathbf{A}, \mathbf{B}) \in D$ , then there exists  $l \in V'_m$  such that

$$a((\mathbf{A}, \mathbf{B}), (\mathbf{A}'', \mathbf{B}'')) = l(\mathbf{A}'', \mathbf{B}''), \forall (\mathbf{A}'', \mathbf{B}'') \in V. \quad (42)$$

For any  $f \in L^2(\Omega)$ , we take the unique solution  $u_f \in H_0^1(\Omega)$  of

$$\Delta u_f = f \text{ in } \Omega,$$

or equivalently the unique solution of

$$\int_{\Omega} \nabla u_f \cdot \nabla \bar{v} \, d\mathbf{x} = - \int_{\Omega} f \bar{v} \, d\mathbf{x}, \forall v \in H_0^1(\Omega).$$

By taking  $\mathbf{A}'' = \nabla u_f$  in (42) (allowed since  $\nabla u_f \in X_N(\Omega)$ ) and  $\mathbf{B}'' = \mathbf{0}$ , we find

$$\int_{\Omega} \operatorname{div} \mathbf{A} \bar{f} \, d\mathbf{x} = l(\nabla u_f, \mathbf{0}).$$

But in the case  $\bar{\Omega}_c \cap \bar{\Omega}_s = \emptyset$  by Lemma 3.3 (resp. in the case  $\Omega_s \subset \Omega_c$  by Lemma 3.5),  $l$  is in the form (29) with  $(\mathbf{A}', (\mathbf{B}'_i)_{i=1}^n) \in W_m$  (resp. in the form (37) with  $(\mathbf{A}', (\mathbf{B}'_i)_{i=2}^n) \in W_m$ ). But in both case, we will have

$$l(\nabla u_f, \mathbf{0}) = \int_{\Omega_c} \mathbf{A}' \cdot \nabla \bar{u}_f \, dx,$$

for  $\mathbf{A}' \in H_0(\operatorname{div} = 0, \Omega_c)$ . This right-hand side being zero by Green's formula, we deduce that

$$\int_{\Omega} \operatorname{div} \mathbf{A} \bar{f} \, d\mathbf{x} = 0.$$

Since  $f$  is arbitrary in  $L^2(\Omega)$ , we conclude that  $\operatorname{div} \mathbf{A} = 0$  in  $\Omega$ . ■

## 4 Some existence results

Let us now give the first consequence of Theorem 3.1.

**Theorem 4.1.** *Let  $T > 0$  be fixed and  $\mathbf{F} \in C^1([0, T], V'_m)$  and  $\mathbf{X}_0 \in V_m$ . Then there exists a unique solution  $\mathbf{X} = (\mathbf{A}, \mathbf{B})$  to*

$$\begin{cases} \partial_t(\mathcal{M}\mathbf{X})(t) + \mathcal{A}\mathbf{X}(t) = \mathbf{F}(t) & \text{in } V'_m, \forall t \in (0, T), \\ (\mathcal{M}\mathbf{X})(t=0) = \mathcal{M}\mathbf{X}_0, \end{cases} \quad (43)$$

with the regularity  $\mathcal{M}\mathbf{X} \in C([0, T], V'_m) \cap C^1((0, T], V'_m)$ , and

$$\mathbf{X}(t) \in D, \forall t \in (0, T].$$

Furthermore one has (recalling that  $\langle \cdot, \cdot \rangle$  means the duality pairing between  $V'_m$  and  $V_m$ )

$$\langle \partial_t(\mathcal{M}\mathbf{X})(\cdot, t), \mathbf{Z} \rangle + \int_{\Omega} \nu \operatorname{curl} \mathbf{A} \cdot \operatorname{curl} \bar{\mathbf{A}}'' \, dx + \int_{\Omega_s} \nu \mathbf{B} \cdot \bar{\mathbf{B}}'' \, dx = (\mathbf{F}(t), \mathbf{Z}), \forall \mathbf{Z} = (\mathbf{A}'', \mathbf{B}'') \in V, \forall t \in (0, T), \quad (44)$$

with the estimate

$$\|\mathbf{A}\|_{L^2(0, T; X^0(\Omega))} + \|\mathbf{B}\|_{L^2(0, T; L^2(\Omega_s)^{3(n-1)})} + \|\mathcal{M}\mathbf{X}\|_{C([0, T]; V'_m)} \lesssim \|\mathbf{F}\|_{L^2(0, T; V'_m)} + |\mathbf{X}_0|_{V_m}. \quad (45)$$

*Proof.* The existence and uniqueness result directly follows from Theorem 3.1, by taking an appropriate extension of  $\mathbf{F}$  in the whole  $[0, \infty)$ . Hence it remains to prove (44) and the estimate (45).

For that purpose, we closely follow the proof of Corollary 3.8 of [16] or of Theorem 4.1 of [15]. As  $V \subset V_m$ , the first identity of (43) implies that

$$\langle \partial_t(\mathcal{M}\mathbf{X})(\cdot, t), \mathbf{Z} \rangle + a(\mathbf{X}, \mathbf{Z}) = (\mathbf{F}(t), \mathbf{Z}), \forall \mathbf{Z} \in V, \forall t \in (0, T).$$

By Lemma 3.7 we directly arrive at (44).

To prove (45), we first take  $\mathbf{Z} = \mathbf{X}$  in (44) and take the real part of the obtained identity, to get, due to Lemma 3.2

$$\frac{1}{2} \partial_t m(\mathbf{X}, \mathbf{X}) + \|\nu^{1/2} \operatorname{curl} \mathbf{A}\|_{\Omega}^2 + \|\nu_0^{1/2} \mathbf{B}\|_{\Omega_s}^2 = \Re \langle \mathbf{F}(t), \mathbf{X} \rangle.$$

Hence by the definition of the norm in  $V'_m$ , we get

$$\frac{1}{2} \partial_t m(\mathbf{X}, \mathbf{X}) + \|\nu^{1/2} \operatorname{curl} \mathbf{A}\|_{\Omega}^2 + \|\nu_0^{1/2} \mathbf{B}\|_{\Omega_s}^2 \leq \|\mathbf{F}(t)\|_{V'_m} |\mathbf{X}|_{V_m},$$

and with the estimate  $|\mathbf{X}|_{V_m} \leq C_1 \|\nu^{1/2} \operatorname{curl} \mathbf{A}\|_{\Omega} + \|\mathbf{B}\|_{\Omega_s}$ , for some  $C_1 > 0$  and Young's inequality we obtain

$$\partial_t m(\mathbf{X}, \mathbf{X}) + \|\nu^{1/2} \operatorname{curl} \mathbf{A}\|_{\Omega}^2 + \|\nu_0^{1/2} \mathbf{B}\|_{\Omega_s}^2 \leq C^2 \|\mathbf{F}(t)\|_{V'_m}^2,$$

for some  $C > 0$ . Integrating this estimate in  $t \in (\eta, u)$ , with  $\eta, u \in (0, T]$  arbitrary but such that  $\eta < u$ , we get

$$\begin{aligned} m(\mathbf{X}(\cdot, u), \mathbf{X}(\cdot, u)) + \int_{\eta}^u \|\nu^{1/2} \operatorname{curl} \mathbf{A}(\cdot, t)\|_{\Omega}^2 dt + \int_{\eta}^u \|\nu_0^{1/2} \mathbf{B}(\cdot, t)\|_{\Omega_s}^2 dt \\ \leq C^2 \int_{\eta}^u \|\mathbf{F}(t)\|_{V'_m}^2 dt + m(\mathbf{X}(\cdot, \eta), \mathbf{X}(\cdot, \eta)). \end{aligned}$$

Letting  $\eta$  tends to zero and recalling the regularity  $\mathcal{M}\mathbf{X} \in C([0, T], V'_m)$  and using again Lemma 3.2, we find that

$$\begin{aligned} m(\mathbf{X}(\cdot, u), \mathbf{X}(\cdot, u)) + \int_0^u \|\nu^{1/2} \operatorname{curl} \mathbf{A}(\cdot, t)\|_{\Omega}^2 dt + \int_0^u \|\nu_0^{1/2} \mathbf{B}(\cdot, t)\|_{\Omega_s}^2 dt \\ \lesssim \int_0^T \|\mathbf{F}(t)\|_{V'_m}^2 dt + |\mathbf{X}_0|_{V_m}^2. \end{aligned} \quad (46)$$

In a first step by taking  $u = T$ , this shows that

$$\|\mathbf{A}\|_{L^2(0, T; X^0(\Omega))} + \|\mathbf{B}\|_{L^2(0, T; L^2(\Omega_s)^{3(n-1)})} \lesssim \|\mathbf{F}\|_{L^2(0, T; V'_m)} + |\mathbf{X}_0|_{V_m}. \quad (47)$$

In a second step, for any  $u \in (0, T]$ , we also have

$$|m(\mathbf{X}(\cdot, u), \mathbf{X}(\cdot, u))| \lesssim \int_0^T \|\mathbf{F}(t)\|_{V'_m}^2 dt + |\mathbf{X}_0|_{V_m}^2 + \int_0^u \|\nu^{1/2} \operatorname{curl} \mathbf{A}(\cdot, t)\|_{\Omega}^2 dt + \int_0^u \|\nu_0^{1/2} \mathbf{B}(\cdot, t)\|_{\Omega_s}^2 dt,$$

and therefore by (47)

$$|m(\mathbf{X}(\cdot, u), \mathbf{X}(\cdot, u))| \lesssim \|\mathbf{F}\|_{L^2(0, T; V'_m)} + |\mathbf{X}_0|_{V_m}, \forall u \in [0, T]. \quad (48)$$

The proof of the estimate (45) is then complete.  $\blacksquare$

Now we can prove an existence result to problem (12)-(18).

**Theorem 4.2.** *Let us suppose that  $\mathbf{J}_s \in C^2([0, T], H_0(\operatorname{div} = 0, \Omega_s))$ . Then, there exists one and only one solution  $(\mathbf{A}, \varphi, \mathbf{B}) \in H^1(0, T; X^0(\Omega)) \times L^2(0, T; \widetilde{H}^1(\Omega_c)) \times H^1(0, T; L^2(\Omega_s))$  to problem (12)-(18) with the (additional) initial condition*

$$\mathbf{A}(t=0) = \mathbf{A}_0, \mathbf{B}(t=0) = \mathbf{0}. \quad (49)$$

and that satisfies the estimate

$$\|\mathbf{A}\|_{H^1(0, T; X^0(\Omega))} + \|\varphi\|_{L^2(0, T; \widetilde{H}^1(\Omega_s))} + \|\mathbf{B}\|_{H^1(0, T; L^2(\Omega_s)^{3(n-1)})} \lesssim \|\mathbf{J}_s\|_{H^1(0, T; L^2(\Omega_s)^3)}. \quad (50)$$

*Proof.* Assume that a solution  $(\mathbf{A}, \varphi, \mathbf{B})$  of problem (12)-(18) exists and is sufficiently regular, then comparing (13) and (16) with (26) and (27), we can say that  $\varphi = \partial_t \varphi_{\mathbf{A}}$ ,  $\varphi_{\mathbf{A}}$  being the unique solution of (24). With this property, problem (12)-(18) is then (formally) equivalent to

$$\begin{cases} \partial_t(\mathcal{M}(\mathbf{A}, \mathbf{B})) + \mathcal{A}(\mathbf{A}, \mathbf{B}) = (\tilde{\mathbf{J}}_s, \mathbf{0}) & \text{in } \Omega \times (0, T), \\ \mathbf{A}(t=0) = \mathbf{A}_0, \mathbf{B}(t=0) = \mathbf{0} & \text{in } \Omega. \end{cases} \quad (51)$$

Now in order to apply Theorem 3.1 (or equivalently Theorem 4.1), we need that the above right-hand side  $\tilde{\mathbf{J}}_s$  corresponds to an element of  $V'_m$ . Hence we perform a sort of elliptic lifting. More precisely in the case  $\bar{\Omega}_c \cap \bar{\Omega}_s = \emptyset$ , for any  $t \in [0, T]$ , we consider the unique solution  $\mathbf{L}_{\mathbf{J}_s(t)} \in X^0(\Omega_s)$  of

$$\begin{cases} \operatorname{curl}(\operatorname{curl} \mathbf{L}_{\mathbf{J}_s(t)}) = \mathbf{J}_s(t) & \text{in } \Omega_s, \\ \mathbf{L}_{\mathbf{J}_s(t)} \times \mathbf{n} = 0 & \text{on } \partial\Omega_s, \end{cases} \quad (52)$$

or in a weak form

$$\int_{\Omega_s} \operatorname{curl} \mathbf{L}_{\mathbf{J}_s(t)} \cdot \operatorname{curl} \bar{\mathbf{W}} \, dx = \int_{\Omega} \mathbf{J}_s(t) \cdot \bar{\mathbf{W}} \, dx, \forall \bar{\mathbf{W}} \in X^0(\Omega_s). \quad (53)$$

Since  $\mathbf{J}_s(t)$  is assumed to be divergence free in  $\Omega_s$  and to satisfy

$$\mathbf{J}_s(t) \cdot \mathbf{n} = 0 \text{ on } \partial\Omega_s, \quad (54)$$

the identity (53) remains valid for all  $\mathbf{W} \in X(\Omega_s)$ , namely

$$\int_{\Omega_s} \operatorname{curl} \mathbf{L}_{\mathbf{J}_s(t)} \cdot \operatorname{curl} \bar{\mathbf{W}} \, dx = \int_{\Omega} \mathbf{J}_s(t) \cdot \bar{\mathbf{W}} \, dx, \forall \bar{\mathbf{W}} \in X(\Omega_s). \quad (55)$$

Using the coercivity of the left-hand side of (55) in  $X(\Omega_c)$ , the mapping

$$\mathbf{J}_s \rightarrow \mathbf{L}_{\mathbf{J}_s}$$

is linear and continuous from  $X^0(\Omega_s)$  into  $X(\Omega_s)$ , by using finite differences and passing to the limit, we easily deduce that  $\mathbf{B}'_1$  defined by  $\mathbf{B}'_1(t) = \operatorname{curl} \mathbf{L}_{\mathbf{J}_s(t)}$ , for all  $t \in [0, T]$  satisfies

$$\mathbf{B}'_1 \in C^1([0, T]; H_0(\operatorname{div} = 0, \Omega_s)).$$

By setting

$$\langle \mathbf{F}(t), \mathbf{Z} \rangle = \int_{\Omega_s} \mathbf{B}'_1(t) \cdot \operatorname{curl} \bar{\mathbf{A}}'' \, dx, \forall \mathbf{Z} = (\mathbf{A}'', \mathbf{B}'') \in V_m,$$

we see by Lemma 3.3 that it defines an element in  $C^1([0, T]; V'_m)$ . Furthermore owing to (55), we have

$$\langle \mathbf{F}(t), \mathbf{Z} \rangle = \int_{\Omega} \mathbf{J}_s(t) \cdot \bar{\mathbf{A}}'' \, dx, \forall \mathbf{Z} = (\mathbf{A}'', \mathbf{B}'') \in V_m. \quad (56)$$

Now in the case  $\Omega_s \subset \Omega_c$ , we introduce the Hilbert space

$$M = \{\mathbf{A} \in \mathbf{L}^2(\Omega_c) : \operatorname{curl} \mathbf{A} \in \mathbf{L}^2(\Omega_s)\},$$

equipped with the natural inner product

$$(\mathbf{A}, \mathbf{C})_M = \int_{\Omega_c} \mathbf{A} \cdot \bar{\mathbf{C}} \, dx + \int_{\Omega_s} \operatorname{curl} \mathbf{A} \cdot \operatorname{curl} \bar{\mathbf{C}} \, dx, \forall \mathbf{A}, \mathbf{C} \in M.$$

Then for any  $t \in [0, T]$ , we consider the unique solution  $\mathbf{L}_{\mathbf{J}_s(t)} \in M$  of

$$\int_{\Omega_c} \mathbf{L}_{\mathbf{J}_s(t)} \cdot \bar{\mathbf{C}} \, dx + \int_{\Omega_s} \operatorname{curl} \mathbf{L}_{\mathbf{J}_s(t)} \cdot \operatorname{curl} \bar{\mathbf{C}} \, dx = \int_{\Omega} \mathbf{J}_s(t) \cdot \bar{\mathbf{C}} \, dx, \forall \bar{\mathbf{C}} \in M. \quad (57)$$

Since  $\mathbf{J}_s(t)$  is assumed to be divergence free in  $\Omega_s$  and to satisfy (54), by taking test-functions  $\mathbf{C} = \nabla\varphi$ , with  $\varphi \in H^1(\Omega_c)$ , we deduce that  $\mathbf{L}_{\mathbf{J}_s}(t)$  belongs to  $H_0(\text{div} = 0, \Omega_c)$ . Here by setting

$$\langle \mathbf{F}(t), \mathbf{Z} \rangle = \int_{\Omega_c} \mathbf{L}_{\mathbf{J}_s}(t) \cdot \bar{\mathbf{A}}'' dx + \int_{\Omega_s} \text{curl} \mathbf{L}_{\mathbf{J}_s}(t) \cdot \text{curl} \bar{\mathbf{A}}'' dx \forall \mathbf{Z} = (\mathbf{A}'', \mathbf{B}'') \in V_m,$$

we see by Lemma 3.5 that it defines an element in  $C^1([0, T]; V'_m)$  such that (56) holds owing to (57).

In both cases, Theorem 4.1 (see (44)) furnishes a solution  $\mathbf{X} = (\mathbf{A}, \mathbf{B}) \in C((0, T), V)$  of

$$\langle \partial_t(\mathcal{M}\mathbf{X})(\cdot, t), \mathbf{Z} \rangle + \int_{\Omega} \nu \text{curl} \mathbf{A} \cdot \text{curl} \bar{\mathbf{A}}'' dx + \int_{\Omega_s} \nu \mathbf{B} \cdot \mathbf{B}'' dx = \int_{\Omega} \widetilde{\mathbf{J}}_s(t) \cdot \bar{\mathbf{A}}'' dx, \forall \mathbf{Z} = (\mathbf{A}'', \mathbf{B}'') \in V, \forall t \in (0, T), \quad (58)$$

with initial condition

$$\mathcal{M}\mathbf{X}(t=0) = \mathcal{M}\mathbf{X}_0, \quad (59)$$

with  $\mathbf{X}_0 = (\mathbf{A}_0, \mathbf{0})$  that clearly belongs to  $V_m$ . Furthermore  $\mathbf{X}$  belongs to  $L^2(0, T; X^0(\Omega) \times L^2(\Omega_s)^{n-1})$ , satisfies  $\mathcal{M}\mathbf{X} \in C([0, T]; V'_m)$  with the estimate (compare with (45))

$$\|\mathbf{A}\|_{L^2(0, T; X^0(\Omega))} + \|\mathbf{B}\|_{L^2(0, T; L^2(\Omega_s)^{3(n-1)})} + \|\mathcal{M}\mathbf{X}\|_{C([0, T]; V'_m)} \lesssim \|\mathbf{J}_s\|_{L^2(0, T; L^2(\Omega_s)^3)} + |\mathbf{A}_0|_{X^0(\Omega)}. \quad (60)$$

In order to come back to the original problem, we actually need more regularity for  $\mathbf{X}$  in order that

$$\partial_t(\mathcal{M}\mathbf{X}) = \mathcal{M}\partial_t\mathbf{X}.$$

This is the reason of the assumptions  $\mathbf{J}_s \in C^2([0, T], H_0(\text{div} = 0, \Omega_s))$ . Hence by using our previous argument (applied with  $\partial_t\mathbf{J}_s$  and a zero initial datum), we find a unique (strong) solution of  $\mathbf{C} \in C((0, T), V)$  of

$$\partial_t(\mathcal{M}\mathbf{C})(\cdot, t) + \mathcal{A}\mathbf{C}(\cdot, t) = (\partial_t\widetilde{\mathbf{J}}_s(t), \mathbf{0}) \text{ in } V'_m, \quad \forall t > 0, \quad (61)$$

with initial condition

$$\mathcal{M}\mathbf{C}(t=0) = \mathbf{0}.$$

This solution  $\mathbf{C}$  belongs to  $L^2(0, T; X^0(\Omega) \times L^2(\Omega_s)^{3(n-1)})$  and satisfies  $\mathcal{M}\mathbf{C} \in C([0, T]; V'_m)$  with the estimate

$$\|\mathbf{C}\|_{L^2(0, T; X^0(\Omega) \times L^2(\Omega_s)^{3(n-1)})} + \|\mathcal{M}\mathbf{C}\|_{C([0, T]; V'_m)} \lesssim \|\partial_t\mathbf{J}_s\|_{L^2(0, T; L^2(\Omega_s)^3)}. \quad (62)$$

Now we define a primitive of  $\mathbf{C}$ :

$$\mathbf{P}(t) = \int_0^t \mathbf{C}(s) ds + \mathbf{X}_0,$$

that has a meaning due to (62) (we recall that  $\mathbf{X}_0 = (\mathbf{A}_0, \mathbf{0})$ ) and has the regularity

$$\mathbf{P} \in H^1(0, T; X^0(\Omega) \times L^2(\Omega_s)^{3(n-1)}).$$

Integrating the identity (61) between 0 and  $t$  (meaningful due to (62)), we find that

$$\mathcal{M}\mathbf{C} + \mathcal{A}\mathbf{P} - \mathcal{A}\mathbf{X}_0 = (\widetilde{\mathbf{J}}_s(t), \mathbf{0}) - (\widetilde{\mathbf{J}}_s(0), \mathbf{0}) \text{ in } V'_m, \quad \forall t > 0.$$

By the definition of  $\mathbf{A}_0$  and the property  $\mathbf{C} = \partial_t\mathbf{P}$ , we find that  $\mathbf{P}$  is solution of

$$\mathcal{M}\partial_t\mathbf{P} + \mathcal{A}\mathbf{P} = (\widetilde{\mathbf{J}}_s(t), \mathbf{0}) \text{ in } V'_m, \quad \forall t > 0, \quad (63)$$

with initial condition

$$\mathbf{P}(t=0) = \mathbf{X}_0.$$

Since this last property implies that

$$\mathcal{M}\mathbf{P}(t=0) = \mathcal{M}\mathbf{X}_0,$$

this initial condition is the same as (58) and as (63) is the strong formulation of (58), we deduce that  $\mathbf{X} = \mathbf{P}$ , which proves the result due to (60) and (62). ■



We are ready to prove our main result.

*Proof. of Theorem 2.1.*

For  $\mathbf{J}_s \in H^1((0, T); H_0(\operatorname{div} = 0, \Omega_s))$ , we can consider

$$\widehat{\mathbf{J}}_s = \mathbf{J}_s - \mathbf{J}_{s,0},$$

that is still in  $H^1((0, T); H_0(\operatorname{div} = 0, \Omega))$  but is zero at  $t = 0$ . Note that by Corollary A.3 of [6], we have

$$\widehat{\mathbf{J}}_s(t) = \int_0^t \widehat{\mathbf{J}}_s'(u) du, \forall t \in [0, T],$$

hence by using the same arguments as in the scalar case, namely by multiplication by cut-off functions and convolution by a regularization sequence, if we denote by

$$C_l^\infty([0, T]; H_0(\operatorname{div} = 0, \Omega_s)) = \{\mathbf{F} \in C^\infty([0, T]; H_0(\operatorname{div} = 0, \Omega_s)) : \mathbf{F} = 0 \text{ in a neighbourhood of } t = 0\},$$

there exists a sequence  $\widehat{\mathbf{J}}_s^{(n)} \in C_l^\infty([0, T]; H_0(\operatorname{div} = 0, \Omega_s))$  such that

$$\widehat{\mathbf{J}}_s^{(n)} \rightarrow \widehat{\mathbf{J}}_s \text{ in } H^1((0, T); H_0(\operatorname{div} = 0, \Omega)) \text{ as } n \rightarrow \infty.$$

Setting

$$\mathbf{J}_s^{(n)} = \widehat{\mathbf{J}}_s^{(n)} + \mathbf{J}_{s,0}$$

we get a sequence in  $C^\infty([0, T]; H_0(\operatorname{div} = 0, \Omega_s))$  satisfying the assumptions of Theorem 4.2 and such that

$$\mathbf{J}_s^{(n)} \rightarrow \mathbf{J}_s \text{ in } H^1((0, T); H_0(\operatorname{div} = 0, \Omega_s)) \text{ as } n \rightarrow \infty.$$

as well as  $\mathbf{J}_s^{(n)}(t = 0) = \mathbf{J}_s(t = 0)$ .

Hence by Theorem 4.2, for all  $n$  there exists a (strong) solution  $(\mathbf{A}_n, \varphi_n, \mathbf{B}_n) \in H^1(0, T; X^0(\Omega)) \times L^2(0, T; \widetilde{H}^1(\Omega_c)) \times H^1(0, T; L^2(\Omega_s))$  to problem (12)-(18) with the (additional) initial condition (49). This in particular implies that

$$\begin{aligned} & (\sigma_c(\partial_t \mathbf{A}_n + \nabla \varphi_n), \widetilde{\mathbf{A}}' + \nabla \widetilde{\varphi}')_{\Omega_s} + (\sigma_s \mathcal{P}(\operatorname{curl} \partial_t \mathbf{A}_n, \partial_t \mathbf{B}_n)^\top, (\operatorname{curl} \mathbf{A}', \mathbf{B}')^\top)_{\Omega_s} + (\nu \operatorname{curl} \mathbf{A}_n, \operatorname{curl} \mathbf{A}')_{\Omega} \\ & + (\nu \mathbf{B}_n, \mathbf{B}')_{\Omega_s} = (\mathbf{J}_s^{(n)}, \mathbf{A}')_{\Omega_s}, \forall \mathbf{A}' \in X_N(\Omega), \varphi' \in \widetilde{H}^1(\Omega_c), \mathbf{B}' \in L^2(\Omega_s)^{3(n-1)}. \end{aligned} \quad (64)$$

Furthermore by the estimate (50), applied to  $(\mathbf{A}_n, \varphi_n, \mathbf{B}_n) - (\mathbf{A}_m, \varphi_m, \mathbf{B}_m)$  for all  $n, m$ , we have

$$\|\mathbf{A}_n - \mathbf{A}_m\|_{H^1(0, T; X^0(\Omega))} + \|\varphi_n - \varphi_m\|_{L^2(0, T; \widetilde{H}^1(\Omega_s))} + \|\mathbf{B}_n - \mathbf{B}_m\|_{H^1(0, T; L^2(\Omega_s)^{3(n-1)})} \lesssim \|\mathbf{J}_s^{(n)} - \mathbf{J}_s^{(m)}\|_{H^1(0, T; L^2(\Omega_s)^3)}, \quad (65)$$

for all  $n, m$ . Hence there exist  $\mathbf{A} \in H^1(0, T; X^0(\Omega))$ ,  $\varphi \in L^2(0, T; \widetilde{H}^1(\Omega_c))$  and  $\mathbf{B} \in H^1(0, T; L^2(\Omega_s)^{3(n-1)})$  such that

$$\mathbf{A}_n \rightarrow \mathbf{A} \text{ in } H^1(0, T; X^0(\Omega)) \text{ as } n \rightarrow \infty, \quad (66)$$

$$\varphi_n \rightarrow \varphi \text{ in } L^2(0, T; \widetilde{H}^1(\Omega_c)) \text{ as } n \rightarrow \infty, \quad (67)$$

$$\mathbf{B}_n \rightarrow \mathbf{B} \text{ in } H^1(0, T; L^2(\Omega_s)^{3(n-1)}) \text{ as } n \rightarrow \infty. \quad (68)$$

Passing to the limit in (64), we conclude that  $(\mathbf{A}, \varphi, \mathbf{B}) \in H^1(0, T; X^0(\Omega)) \times L^2(0, T; \widetilde{H}^1(\Omega_c)) \times H^1(0, T; L^2(\Omega_s))$  is the unique solution of (19). ■

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