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ON A CONJECTURE OF GROSS, MANSOUR AND TUCKER

SERGEI CHMUTOV, FABIEN VIGNES-TOURNERET

ABSTRACT. Partial duality is a duality of ribbon graphs relative to a subset of their edges generalizing the classical Euler–Poincaré duality. This operation often changes the genus. Recently J. L. Gross, T. Mansour, and T. W. Tucker formulated a conjecture that for any ribbon graph different from plane trees and their partial duals, there is a subset of edges partial duality relative to which does change the genus. A family of counterexamples was found by Qi Yan and Xian’an Jin. In this note we prove that essentially these are the only counterexamples.

1. RIBBON GRAPHS AND PARTIAL DUALITY

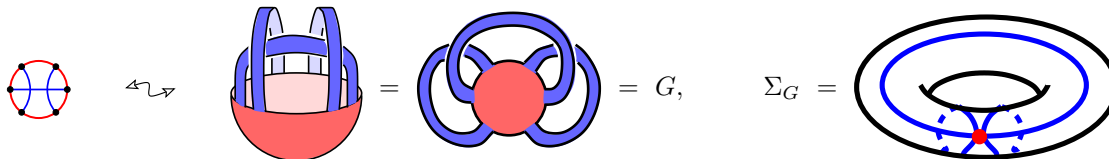
A *ribbon graph* topologically can be regarded as a regular neighborhood of the graph cellularly embedded into a closed surface. Then the neighborhoods of vertices and edges represent *vertex-discs* and *edge-ribbons*. A component of the complement of the ribbon graph is called a *face-disc*. The formal definition is the following:

Definition 1.1. A *ribbon graph* G is a surface with boundary represented as the union of two sets of closed topological discs called *vertex-discs* $V(G)$ and *edge-ribbons* $E(G)$, satisfying the following conditions:

- the vertex-discs and edge-ribbons intersect by disjoint line segments;
- each such line segment lies on the boundary of precisely one vertex-disc and precisely one edge-ribbon;
- every edge-ribbon contains exactly two such line segments.

The last condition allows us to view each edge-ribbon as a topological rectangle (ribbon) attached to vertex-discs by two opposite sides. The face-discs are assumed to be discs which are glued to each of the boundary component of the ribbon graph to get a closed surface Σ_G .

In this paper we are dealing with orientable ribbon graphs only. Also this paper deals a lot with one-vertex ribbon graphs which we will encode by *chord diagrams* following the common practice in knot theory [CDM12, Sec.4.8]. Namely we attach a vertex-disc to the circle of a chord diagram and thicken the chords to narrow edge-ribbons. Here is an example



This ribbon graph has two boundary components. The corresponding closed surface Σ_G has thus two face-discs and its Euler characteristic is equal to $\chi(\Sigma_G) = 1 - 3 + 2 = 0$. This is a torus and the genus of G is equal to 1.

Partial duality of ribbon graphs was introduced in [Ch09] under the name of generalized duality. Then it was further studied and developed in papers [VT09, Mo10, Mo11, EMM12,

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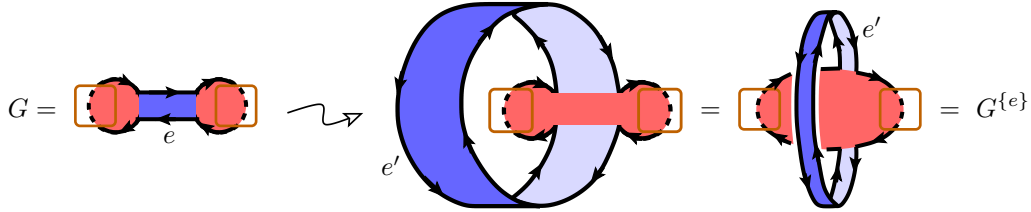
2010 Mathematics Subject Classification. 05C10, 05C65, 57M15, 57Q15.

Key words and phrases. Ribbon graphs, partial duality, partial-dual genus polynomial, Gross–Mansour–Tucker conjecture.

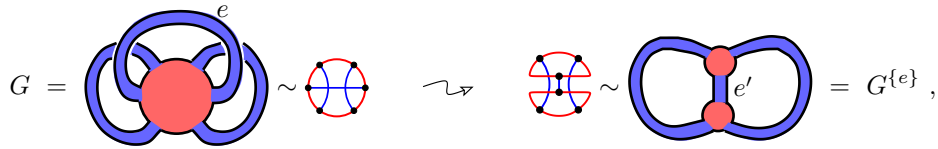
BBC12, Mo13, HM13, CVT14, EZ17, GMT20, YaJi20]. We refer to [EMM13] for an excellent exposition.

Definition 1.2. For a ribbon graph G and a subset of its edge-ribbons $A \subseteq E(G)$, the *partial dual* G^A of G relative to A is a ribbon graph constructed in the following way. The vertex-discs of G^A are bounded by connected components of the boundary of the spanning subgraph of G containing all the vertices of G and only the edges from A . The edge-ribbons of $E(G) \setminus A$ are attached to these new vertices exactly at the same places as in G . The edge-ribbons from A become parts of the new vertex-discs now. For an $e \in A$ we take a copy of e , e' , and attach it to the new vertex-discs in the following way. The rectangle representing e intersects with vertex-discs of G by a pair of opposite sides. But it intersects the boundary of the spanning subgraph, that is the new vertex-discs, along the arcs of the other pair of its opposite sides. We attach e' to these arcs by this second pair. The copies of the first pair of sides in e' becomes the arcs of the boundary of G^A .

The following figure illustrates this definition for the case of an edge e connecting different vertices (the boxes with dashed arcs mean that there might be other edges attached to these vertices).



Partial duality relative to a loop acts on this figure backwards, from right to left. In the example from previous page the partial duality relative to middle loop gives:



a graph cellularly embedded into a sphere.

Partial duality relative to a set of edges can be done step by step one edge at a time. Partial duality relative to the whole set of edges $E(G)$ is the classical Euler–Poincaré duality. Partial duality relative to a spanning tree produces a one-vertex ribbon graph which can be encoded by a chord diagram.

We see that partial duality relative to single edge $e \in E(G)$ changes the structure of vertices attached to the edge-ribbon e . Namely, if there were two separate vertex-discs attached to e in G , then they merge together to single vertex-disc attached to the corresponding edge-ribbon e' in $G^{\{e\}}$, and vice versa. A careful analysis of the orientation of the boundary of the ribbon graphs following the arrows on the above picture shows that a similar rearrangement happened with the face-discs. We formulate it as a lemma.

Lemma 1.3. *If the edge-ribbon $e \in E(G)$ was attached to two distinct vertex- (resp. face-) discs of G , then in the ribbon graph $G^{\{e\}}$ these discs will merge together into a single vertex- (resp. face-) disc of $G^{\{e\}}$ attached to the corresponding edge-ribbon $e' \in E(G^{\{e\}})$. Backwards, if $e \in E(G)$ was attached to a single vertex- (resp. face-) disc of G , then in the ribbon graph $G^{\{e\}}$ it will be split into two vertex- (resp. face-) discs of $G^{\{e\}}$ attached to the corresponding edge-ribbon $e' \in E(G^{\{e\}})$.*

where $g(G^A)$ stands for the orientable genus of G^A , namely half its Euler genus.

The **ribbon-join operation** $G_1 \vee G_2$ on ribbon graphs G_1 and G_2 was introduced by I. Moffatt [Mo13]. To obtain $G_1 \vee G_2$ glue together a vertex-disc of G_1 and a vertex-disc of G_2 along some arcs on their boundaries between consecutive ends of the end of edge ribbons to get a new vertex disc of $G_1 \vee G_2$. For one vertex ribbon graphs this operation is the standard multiplication of corresponding chord diagrams, see [CDM12, Sec. 4.4.2]. Obviously the result is not unique, it depends on choice of vertices and arcs. But, since the partial dualities of factors G_1 and G_2 are independent, namely

$$\forall E_1 \subset E(G_1), E_2 \subset E(G_2), (G_1 \vee G_2)^{E_1 \cup E_2} = G_1^{E_1} \vee G_2^{E_2},$$

the polynomial $\partial\Gamma_G(z)$ is multiplicative with respect to this operation, $\partial\Gamma_{G_1 \vee G_2}(z) = \partial\Gamma_{G_1}(z) \cdot \partial\Gamma_{G_2}(z)$.

J. L. Gross, T. Mansour, and T. W. Tucker formulated the following conjecture.

GMT-conjecture. [GMT20, Conjecture 3.1] *There is no orientable ribbon graph having a non-constant partial-dual genus polynomial with only one non-zero coefficient.*

They noted that if G is a tree, which implies that its genus is 0, all partial duals of G also will be of genus 0. That means for a tree with n edges, $\partial\Gamma_G(z) = 2^n$. This explains the exclusion of constants from the conjecture.

Q. Yan, X. Jin [YaJi20] found a family of counterexamples to the GMT-conjecture. They used the one-vertex ribbon graphs B_n with n edge-ribbons every two of which are interlaced along the boundary circle of the vertex. These are the corresponding first chord diagrams.

$$B_1 = \text{---}, \quad B_2 = \text{---}, \quad B_3 = \text{---}, \quad B_4 = \text{---}, \quad B_5 = \text{---}, \quad \dots$$

An **intersection graph** of a chord diagram [CDM12, Sec.4.8] is a graph whose vertices correspond to the chords and two vertices are connected by an edge if the corresponding chords intersect. The ribbon graphs B_n can be characterized as one-vertex ribbon graphs for which chord diagrams have the complete graph K_n as the intersection graph.

It was proved in [YaJi20, Theorem 22] that for odd values of n , $\partial\Gamma_{B_n}(z) = 2^n z^{\frac{n-1}{2}}$, thus providing the counterexample to the GMT-conjecture.

Example 2.1. To demonstrate that the even values of n do not work, we need to check the attachment of edge-ribbon to the face-discs. For that we double each chord modeling the boundary sides of the corresponding rectangles and look for the boundary components we obtain. For example,

$$B_2 = \text{---} \rightsquigarrow \text{---} \rightsquigarrow \text{---}$$

We can see a single immersed curve representing the boundary of the ribbon graph B_2 . Each boundary component is a circle to which a face-disc is supposed to be glued in to form a closed surface. In this case we have only one face-disc. So each ribbon of B_2 is attached to the same face. Hence it is of Type uu , and the partial dual relative to each would change the genus. Consequently, the partial-dual genus polynomial would have more than one term. Meanwhile for odd values of n , for example $n = 3$, each ribbon would be attached to different boundary components:

$$B_3 = \text{---} \rightsquigarrow \text{---} \rightsquigarrow \text{---}$$

We would have two immersed curves, so we would have corresponding two face-discs, and each edge ribbon would be attached to different face-discs. That means they are of Type up and the partial duality relative to a single edge would not change the genus. To complete the calculation

of [YaJi20] we would need to check that all edge-ribbons will stay withing Types pu and up after performing partial dualities.

By the multiplicativity property, the partial-dual genus polynomial of the ribbon-join of B_n 's with odd n 's also will be a monomial. In fact, even for trees, the total duality gives the one-vertex ribbon graph for which the chords of its chord diagram do not intersect each other. So it is a crossing-less matching of $2n$ points on a circle. As such it is a ribbon-join of n copies of B_1 . Therefore, all the know examples of one-vertex ribbon graph with a monomial as the partial-dual genus polynomial consist of ribbon-joins of various B_n 's with odd values of n . Our main theorem claims that these are the only counterexamples to the GMT-conjecture.

3. MAIN RESULT

We call a connected ribbon graph G *join-prime* if it cannot be represented as the ribbon-join of two graphs G_1, G_2 with at least one edge-ribbon each: $G \neq G_1 \vee G_2$.

Theorem 3.1. *For any join-prime ribbon graph different from partial duals of B_n 's with odd values of n , there are partial duals of different genus.*

One can observe that there are exactly two graphs in the partial dual class of B_n . Namely B_n itself and its partial dual relative to one edge-ribbon. Partial duality relative to two edges of B_n will return to B_n back. So, for any join-prime ribbon graph different from these two the GMT-conjecture holds. Moreover, from the multiplicativity of $\partial\Gamma$ with respect to the join operation, Theorem 3.1 implies that the only orientable ribbon graphs having a non-constant partial-dual genus polynomial with only one non-zero coefficient are partial duals of joins of B_n 's with n odd.

Remark. The non-orientable counterpart of the Gross-Mansour-Tucker conjecture has been treated by Maya Thompson (Royal Holloway University of London). She proved that the only non-orientable ribbon graphs having a non-constant partial-dual genus polynomial with only one non-zero coefficient are partial duals of the one-vertex ribbon graph with one twisted edge. Moreover, a preprint by Q. Yan and X. Jin [YaJi21] posted recently contains in particular proofs of both the orientable and non-orientable cases.

A one-vertex ribbon graph is join-prime if and only if the intersection graph of the corresponding chord diagram is connected.

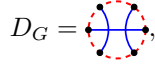
Our proof of Theorem 3.1 is based on the following Lemma.

Lemma 3.2. *Let G be a one-vertex join-prime ribbon graph and $e \in E(G)$. If for all subsets $A \subseteq E(G)$, $g(G) = g(G^A)$ then*

- (1) *e is attached to different face-discs $f_1 \neq f_2$. That is e has to be of Type up .*
- (2) *Any edge-ribbon interlaced with e is attached to the same face-discs f_1 and f_2 .*
- (3) *Any edge-ribbon not interlaced with e is attached to a pair of face-discs different from $\{f_1, f_2\}$.*

Proof of the Lemma. Since $g(G) = g(G^{\{e\}})$, the edge-ribbon e has to be of Type up . The same is true for any edge-ribbon of G . This proves (1). Let e_1 be an edge-ribbon interlaced with e and attached to two different face-discs f_3 and f_4 . Then on the partial dual $G^{\{e\}}$ the corresponding edge e_1 will connect two different vertices. If we want the genus to stay the same, $g(G) = g(G^{\{e, e_1\}})$, the edge e_1 has to be of Type pu in $G^{\{e\}}$. This means it has to be attached to the same face-disc of $G^{\{e\}}$. In other words the faces f_3 and f_4 merge to each other in $G^{\{e\}}$. But the only change in the structure of faces under the partial duality relative to e is that the faces f_1 and f_2 merge to a single face. This implies $\{f_3, f_4\} = \{f_1, f_2\}$, which proves (2). In a similar way, let e_2 be an edge-ribbon not interlaced with e and attached to two different face-discs f_5 and f_6 . Then, after the partial duality relative to e , e_2 is still a loop attached to one of the two vertices of $G^{\{e\}}$. So it has to be of Type up in $G^{\{e\}}$ which means that $f_5 \neq f_6$ in $G^{\{e\}}$. But this partial duality merges f_1 and f_2 . So, $\{f_5, f_6\} \neq \{f_1, f_2\}$, which proves (3). \square

Proof of the Theorem. Since every connected ribbon graph has a one-vertex graph as partial dual, we assume that G is a one-vertex join-prime ribbon graph with n edge-ribbons and such that $g(G) = g(G^A)$ for all subsets $A \subseteq E(G)$. Let D_G be the corresponding chord diagram. We also assume that the intersection graph of D_G is different from the complete graph K_n . Then there is a path of length 2 in the intersection graph. Let e_2, e_1, e_3 be the vertices of the path. They correspond to chords of D_G such that the chord e_1 intersects both e_2 and e_3 , but e_2 and e_3 do not intersect each other. So, the chord diagram D_G looks like this



where the dashed circle means that there might be other chords with end points on these arcs. In terms of the ribbon graph G , the edge-ribbon e_1 is interlaced with both e_2 and e_3 , but the edge-ribbons e_2 and e_3 do not interlace. Every edge-ribbon has to be of Type *up*, that is attached to two different face-discs. Let $\{f_1, f_2\}$ (resp. $\{f_3, f_4\}$ and $\{f_5, f_6\}$) be the pair of face-discs attached to e_1 (resp. e_2 and e_3). Then according to case (2) of the Lemma, $\{f_3, f_4\} = \{f_1, f_2\} = \{f_5, f_6\}$. But this contradicts case (3) of the lemma according to which $\{f_3, f_4\} \neq \{f_5, f_6\}$. This means that any two vertices have to be actually adjacent. This implies that the intersection graph has to be the complete graph K_n which proves the Theorem. \square

T. W. Tucker indicated that the theorem can be reformulated in the following ways.

Corollary 3.3. *The only join-prime ribbon graph G where all edges have Type up is a one-vertex ribbon graph with an odd number of edges where every loop interlaces with every other loop.*

Or in the dual form:

Corollary 3.4. *The only join-prime ribbon graph where all edges have Type pu is the dipole on an odd number of edges where the rotation at one vertex is the opposite of the rotation at the other vertex. (i.e. the Petrie dual of the standard dipole in the sphere).*

4. FUTURE RESEARCH

- The paper [GMT20] contains another very interesting Conjecture 5.1 that the sequence of coefficients of $\partial\Gamma_G(z)$ is *log-concave* as well as analogous conjectures for the partial-dual Euler-genus polynomial in the non-orientable case.
- The paper [CVT14] generalizes partial duality to hypermaps. It would be interesting to find (and prove) the analogous conjectures for hypermaps.
- Ribbon graphs may be considered from the point of view of delta-matroids [CMNR14]. In this way the concepts of partial duality and genus can be interpreted in terms of delta-matroids. It would be interesting to know whether the partial-dual genus polynomial and the related conjectures would make sense for general delta-matroids.

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