

Quasi-regularity verification for 2D polygonal objects based on medial axis analysis

Phuc Ngo – Nicolas Passat – Yukiko Kenmochi

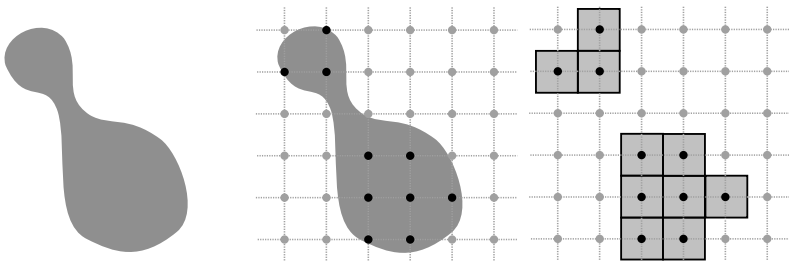
IEEE ICIP – Alaska, USA

19 – 22 September 2021

Motivation

Problem

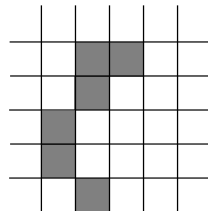
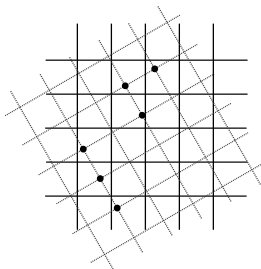
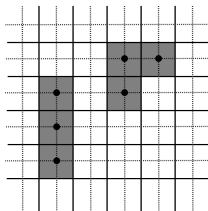
1. Topological preservation by digitization process
2. Rigid motion in discrete space of \mathbb{Z}^2 : topological and geometrical properties of the transformed objects



Motivation

Problem

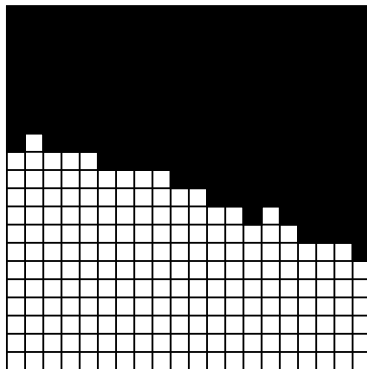
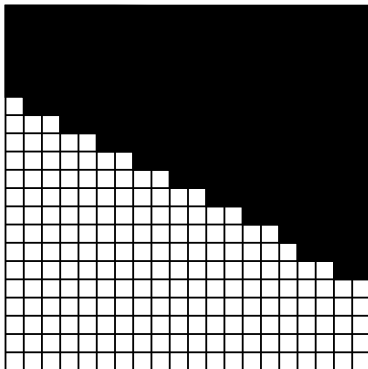
1. Topological preservation by digitization process
2. Rigid motion in discrete space of \mathbb{Z}^2 : topological and geometrical properties of the transformed objects



Motivation

Problem

1. Topological preservation by digitization process
2. Rigid motion in discrete space of \mathbb{Z}^2 : topological and geometrical properties of the transformed objects



Motivation

Problem

1. Topological preservation by digitization process
2. Rigid motion in discrete space of \mathbb{Z}^2 : topological and geometrical properties of the transformed objects

Contributions

1. A class of objects, called *quasi-regular*, that allows the connectivity preservation by digitization
2. Exact method for verifying quasi-regular polygonal objects via the notion of medial axis
3. A framework of rigid motion in \mathbb{Z}^2 using polygonal representation of digital object together with the quasi-regularity for the topological and geometrical preservation

Table of contents

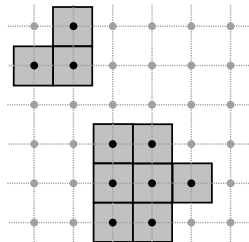
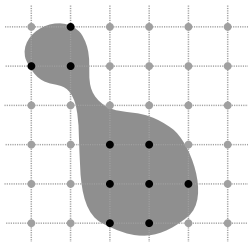
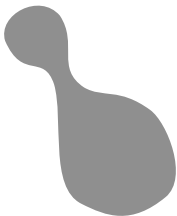
1. Motivation
2. Background notions
3. Quasi-regularity and its verification
4. Application: Rigid motions on \mathbb{Z}^2
5. Conclusion

Gauss digitization

Definition [Klette and Rosenfeld, 2004]

Let $X \subset \mathbb{R}^n$, $n \geq 2$. The Gauss digitization is defined as the intersection of X with \mathbb{Z}^n :

$$X = X \cap \mathbb{Z}^d$$

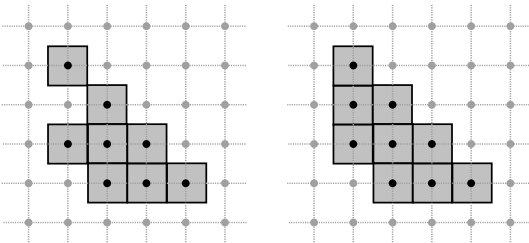


Digital topology of X is often non-coherent with the continuous topology of X .

Well-composed sets

Definition [Latecki et al., 1995]

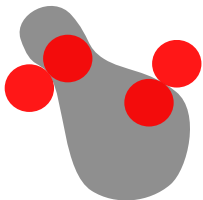
A digital set $X \subset \mathbb{Z}^2$ is *well-composed* if each 8-connected component of X and of its complement \bar{X} is also 4-connected.



r -regularity

Definition [Pavlidis, 1982]

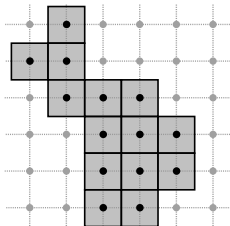
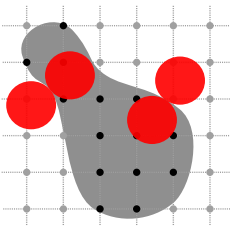
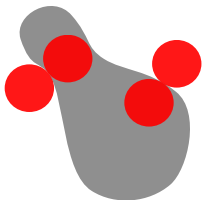
A finite and connected subset $X \subset \mathbb{R}^2$ is r -regular if for each boundary point of X , there exist two tangent open balls of radius r , lying entirely in X and its complement \bar{X} , respectively.



r -regularity

Definition [Pavlidis, 1982]

A finite and connected subset $X \subset \mathbb{R}^2$ is r -regular if for each boundary point of X , there exist two tangent open balls of radius r , lying entirely in X and its complement \bar{X} , respectively.



Proposition [Latecki et al., 1998]

If a finite and connected set $X \subset \mathbb{R}^2$ is r -regular, with $r \geq \frac{\sqrt{2}}{2}$, then $X = X \cap \mathbb{Z}^2$ is a well-composed digital set.

r -regularity

Definition (in mathematical morphology)

Let $X \subset \mathbb{R}^2$ be a bounded, simply connected object (i.e., connected without hole). X is r -regular if it satisfies the following four properties:

- (i) $X \ominus B_r$ is non-empty and connected
- (ii) $\overline{X} \ominus B_r$ is connected
- (iii) $X \subseteq X \oplus B_r \oplus B_r$
- (iv) $\overline{X} \subseteq \overline{X} \oplus B_r \oplus B_r$

where $\overline{X} = \mathbb{R}^2 \setminus X$; \oplus, \ominus are the dilation and erosion operators; and $B_t \subset \mathbb{R}^2$ is the Euclidean ball of centre $0_{\mathbb{R}^2}$ and radius t .

By definition, polygonal objects are not r -regular for any $r > 0$.

r -regularity

Definition (in mathematical morphology)

Let $X \subset \mathbb{R}^2$ be a bounded, simply connected object (i.e., connected without hole). X is r -regular if it satisfies the following four properties:

- (i) $X \ominus B_r$ is non-empty and connected
- (ii) $\overline{X} \ominus B_r$ is connected
- (iii) $X \subseteq X \oplus B_r \oplus B_r$
- (iv) $\overline{X} \subseteq \overline{X} \oplus B_r \oplus B_r$

where $\overline{X} = \mathbb{R}^2 \setminus X$; \oplus, \ominus are the dilation and erosion operators; and $B_t \subset \mathbb{R}^2$ is the Euclidean ball of centre $0_{\mathbb{R}^2}$ and radius t .

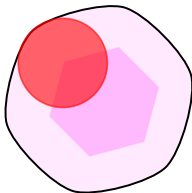
By definition, polygonal objects are not r -regular for any $r > 0$.

Generalization of r -regularity: quasi- r -regularity

Definition [Ngo et al., 2018]

Let $X \subset \mathbb{R}^2$ be a bounded, simply connected object (i.e., connected without hole). X is r -regular if it satisfies the following four properties:

- (i) $X \ominus B_r$ is non-empty and connected
- (ii) $\bar{X} \ominus B_r$ is connected
- (iii) $X \subseteq X \ominus B_r \oplus B_{r\sqrt{2}}$
- (iv) $\bar{X} \subseteq \bar{X} \ominus B_r \oplus B_{r\sqrt{2}}$

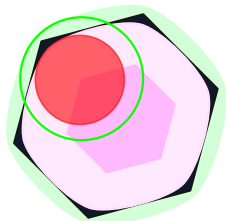
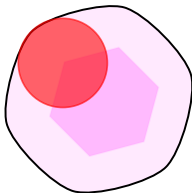


Generalization of r -regularity: quasi- r -regularity

Definition [Ngo et al., 2018]

Let $X \subset \mathbb{R}^2$ be a bounded, simply connected object (i.e., connected without hole). X is r -regular if it satisfies the following four properties:

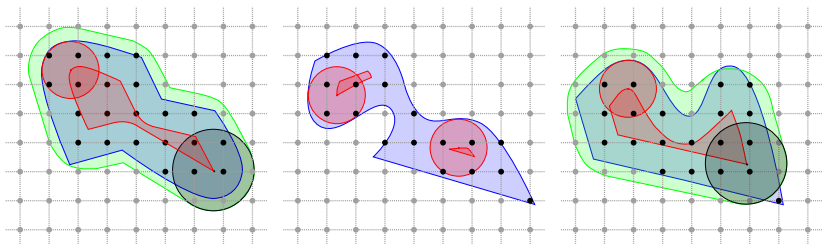
- (i) $X \ominus B_r$ is non-empty and connected
- (ii) $\bar{X} \ominus B_r$ is connected
- (iii) $X \subseteq X \ominus B_r \oplus B_{r\sqrt{2}}$
- (iv) $\bar{X} \subseteq \bar{X} \ominus B_r \oplus B_{r\sqrt{2}}$



Quasi- r -regular object and digitization

Proposition [Ngo et al., 2018]

Let $X \subset \mathbb{R}^2$ be a bounded, simply connected object. If X is quasi-1-regular, then $X = X \cap \mathbb{Z}^2$ and $\bar{X} = \bar{X} \cap \mathbb{Z}^2$ are both 4-connected. In particular, X is then well-composed.

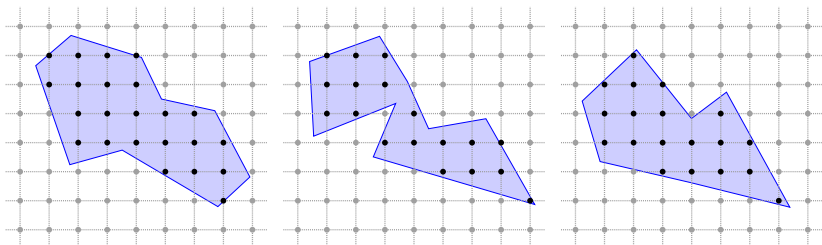


Verify the quasi-1-regularity of polygonal objects? \Rightarrow Medial axis

Quasi- r -regular object and digitization

Proposition [Ngo et al., 2018]

Let $X \subset \mathbb{R}^2$ be a bounded, simply connected object. If X is quasi-1-regular, then $X = X \cap \mathbb{Z}^2$ and $\bar{X} = \bar{X} \cap \mathbb{Z}^2$ are both 4-connected. In particular, X is then well-composed.

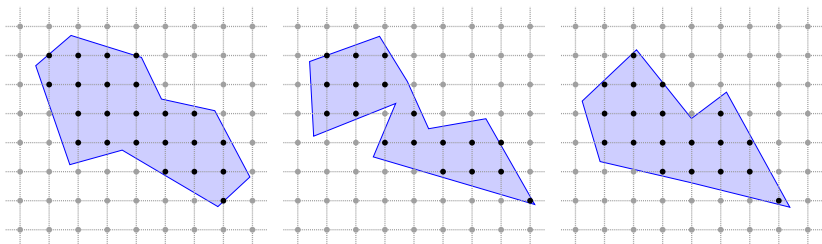


Verify the quasi-1-regularity of polygonal objects? \Rightarrow Medial axis

Quasi- r -regular object and digitization

Proposition [Ngo et al., 2018]

Let $X \subset \mathbb{R}^2$ be a bounded, simply connected object. If X is quasi-1-regular, then $X = X \cap \mathbb{Z}^2$ and $\bar{X} = \bar{X} \cap \mathbb{Z}^2$ are both 4-connected. In particular, X is then well-composed.



Verify the quasi-1-regularity of polygonal objects? \Rightarrow Medial axis

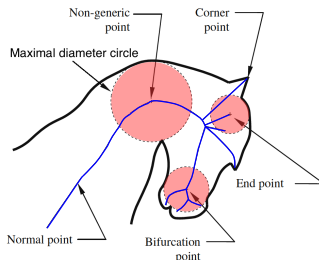
Medial axis

Definition [Blum, 1967]

Let $X \subset \mathbb{R}^2$ be a closed, bounded set such that the boundary ∂X of X is a 1-manifold. The medial axis of X is defined as the locus of the centers of the maximal balls included in X :

$$\mathcal{M}(X) = \{x \in X \mid \nexists y \in X, B(x, r(x)) \subset B(y, r(y))\}$$

where $B(y, r) \subseteq X$ is the ball of center y and radius $r \in \mathbb{R}_+$.



Medial axis

Definition [Blum, 1967]

Let $X \subset \mathbb{R}^2$ be a closed, bounded set such that the boundary ∂X of X is a 1-manifold. The medial axis of X is defined as the locus of the centers of the maximal balls included in X :

$$\mathcal{M}(X) = \{x \in X \mid \nexists y \in X, B(x, r(x)) \subset B(y, r(y))\}$$

where $B(y, r) \subseteq X$ is the ball of center y and radius $r \in \mathbb{R}_+$.

By definition, we have $\mathcal{M}(X) \subseteq X$ and

$$X = \bigcup_{x \in \mathcal{M}(X)} B(x, r(x))$$

Medial axis

Definition [Blum, 1967]

Let $X \subset \mathbb{R}^2$ be a closed, bounded set such that the boundary ∂X of X is a 1-manifold. The medial axis of X is defined as the locus of the centers of the maximal balls included in X :

$$\mathcal{M}(X) = \{x \in X \mid \nexists y \in X, B(x, r(x)) \subset B(y, r(y))\}$$

where $B(y, r) \subseteq X$ is the ball of center y and radius $r \in \mathbb{R}_+$.

We define the λ -level medial axis, noted $\mathcal{M}_\lambda(X)$, by

$$\mathcal{M}_\lambda(X) = \{x \in \mathcal{M}(X) \mid r(x) \geq \lambda\}$$

In particular, $\lambda_1 \leq \lambda_2 \Rightarrow \mathcal{M}_{\lambda_2}(X) \subseteq \mathcal{M}_{\lambda_1}(X)$, and $\mathcal{M}_0(X) = \mathcal{M}(X)$.

We also define

$$\mathcal{M}_{\lambda_1}^{\lambda_2}(X) = \{x \in \mathcal{M}(X) \mid \lambda_1 \leq r(x) \leq \lambda_2\}$$

Properties of medial axis

Proposition [Lieutier, 2004]

X and $\mathcal{M}(X)$ have the same homotopy type, and noted $X \simeq \mathcal{M}(X)$.

Proposition [Serra, 1983]

Let B_λ be the ball of center $0_{\mathbb{R}^2}$ and of radius $\lambda \geq 0$. We have

$$X \ominus B_\lambda = \bigcup_{x \in \mathcal{M}_\lambda(X)} B(x, r(x) - \lambda)$$

$$X \oplus B_\lambda = \bigcup_{x \in \mathcal{M}(X)} B(x, r(x) + \lambda)$$

$$\mathcal{M}(X \ominus B_\lambda) = \mathcal{M}_\lambda(X)$$

We now verify the quasi-1-regularity of polygonal objects via the notion of medial axis.

Properties of medial axis

Proposition [Lieutier, 2004]

X and $\mathcal{M}(X)$ have the same homotopy type, and noted $X \simeq \mathcal{M}(X)$.

Proposition [Serra, 1983]

Let B_λ be the ball of center $0_{\mathbb{R}^2}$ and of radius $\lambda \geq 0$. We have

$$X \ominus B_\lambda = \bigcup_{x \in \mathcal{M}_\lambda(X)} B(x, r(x) - \lambda)$$

$$X \oplus B_\lambda = \bigcup_{x \in \mathcal{M}(X)} B(x, r(x) + \lambda)$$

$$\mathcal{M}(X \ominus B_\lambda) = \mathcal{M}_\lambda(X)$$

We now verify the quasi-1-regularity of polygonal objects via the notion of medial axis.

Properties of medial axis

Proposition [Lieutier, 2004]

X and $\mathcal{M}(X)$ have the same homotopy type, and noted $X \simeq \mathcal{M}(X)$.

Proposition [Serra, 1983]

Let B_λ be the ball of center $0_{\mathbb{R}^2}$ and of radius $\lambda \geq 0$. We have

$$X \ominus B_\lambda = \bigcup_{x \in \mathcal{M}_\lambda(X)} B(x, r(x) - \lambda)$$

$$X \oplus B_\lambda = \bigcup_{x \in \mathcal{M}(X)} B(x, r(x) + \lambda)$$

$$\mathcal{M}(X \ominus B_\lambda) = \mathcal{M}_\lambda(X)$$

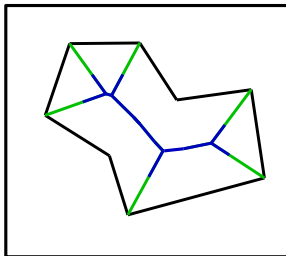
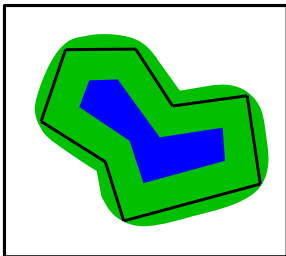
We now verify the quasi-1-regularity of polygonal objects via the notion of medial axis.

Verification of quasi-1-regularity

Property

Let $X \subset \mathbb{R}^2$ be a bounded, simply connected polygon. If $\mathcal{M}(X) \simeq \mathcal{M}_1(X)$ and $\mathcal{M}(\bar{X}) \simeq \mathcal{M}_1(\bar{X})$ then

- (i) $X \ominus B_1$ is non-empty and connected
- (ii) $\bar{X} \ominus B_1$ is connected

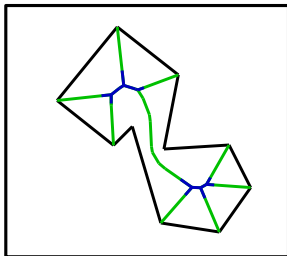
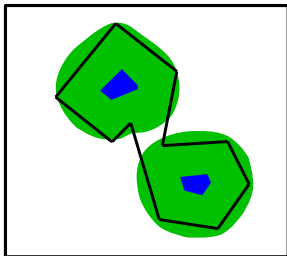


Verification of quasi-1-regularity

Property

Let $X \subset \mathbb{R}^2$ be a bounded, simply connected polygon. If $\mathcal{M}(X) \simeq \mathcal{M}_1(X)$ and $\mathcal{M}(\bar{X}) \simeq \mathcal{M}_1(\bar{X})$ then

- (i) $X \ominus B_1$ is non-empty and connected
- (ii) $\bar{X} \ominus B_1$ is connected

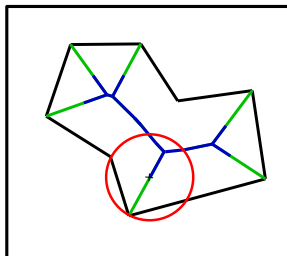
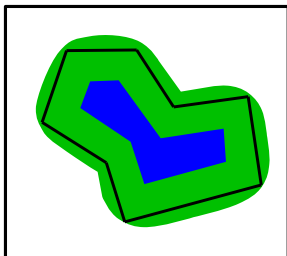


Verification of quasi-1-regularity

Let $Y \in \{X, \bar{X}\}$ and $M \subseteq \mathcal{M}_0^1(Y)$ a connected component of $\mathcal{M}_0^1(Y)$.
 M contains a set of k points, noted z_i ($1 \leq i \leq k$), with $r(z_i) = 0$ (they are convex vertices of the polygon Y), and a point y with $r(y) = 1$.

Let $(\mathcal{P}) : \forall 1 \leq i \leq k, \|y - z_i\|_2 \leq \sqrt{2}$. We have

$$(\mathcal{P}) \Rightarrow \bigcup_{x \in M} B(x, r(x)) \subseteq Y \ominus B_1 \oplus B_{\sqrt{2}}$$

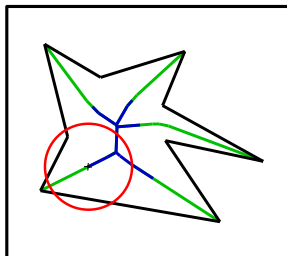
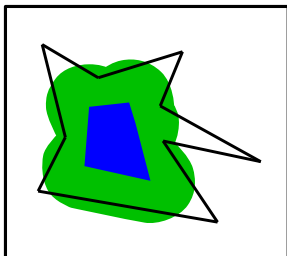


Verification of quasi-1-regularity

Let $Y \in \{X, \overline{X}\}$ and $M \subseteq \mathcal{M}_0^1(Y)$ a connected component of $\mathcal{M}_0^1(Y)$.
 M contains a set of k points, noted z_i ($1 \leq i \leq k$), with $r(z_i) = 0$ (they are convex vertices of the polygon Y), and a point y with $r(y) = 1$.

Let $(\mathcal{P}) : \forall 1 \leq i \leq k, \|y - z_i\|_2 \leq \sqrt{2}$. We have

$$(\mathcal{P}) \Rightarrow \bigcup_{x \in M} B(x, r(x)) \subseteq Y \ominus B_1 \oplus B_{\sqrt{2}}$$



Verification of quasi-1-regularity

Let $Y \in \{X, \overline{X}\}$ and $M \subseteq \mathcal{M}_0^1(Y)$ a connected component of $\mathcal{M}_0^1(Y)$. M contains a set of k points, noted z_i ($1 \leq i \leq k$), with $r(z_i) = 0$ (they are convex vertices of the polygon Y), and a point y with $r(y) = 1$.

Let $(\mathcal{P}) : \forall 1 \leq i \leq k, \|y - z_i\|_2 \leq \sqrt{2}$. We have

$$(\mathcal{P}) \Rightarrow \bigcup_{x \in M} B(x, r(x)) \subseteq Y \ominus B_1 \oplus B_{\sqrt{2}}$$

Proposition

Let $X \subset \mathbb{R}^2$ be a bounded, simply connected polygon. Let us suppose that $\mathcal{M}(X) \frown \mathcal{M}_1(X)$, $\mathcal{M}(\overline{X}) \frown \mathcal{M}_1(\overline{X})$ and that for each connected component of $\mathcal{M}_0^1(X)$ and $\mathcal{M}_0^1(\overline{X})$, property (\mathcal{P}) holds. Then, X is quasi-1-regular.

Quasi-1-regularity verification method

The method consists in verifying the following two conditions:

1. $\mathcal{M}(X) \frown \mathcal{M}_1(X)$ and $\mathcal{M}(\bar{X}) \frown \mathcal{M}_1(\bar{X})$
2. (\mathcal{P}) holds for each connected component of $\mathcal{M}_0^1(X)$ and $\mathcal{M}_0^1(\bar{X})$.

Algorithm 1: Quasi-regularity verification

Input: A simply connected polygonal object $X \subset \mathbb{R}^2$

Output: A Boolean indicating whether X is quasi-regular

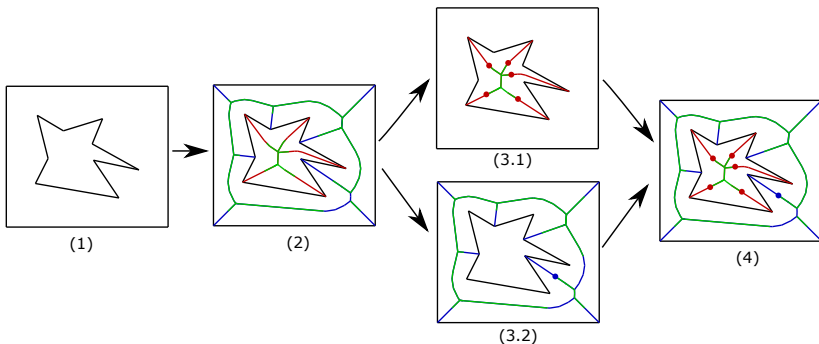
```

1 for  $Y \in \{X, \bar{X}\}$  do
2   if not  $(\mathcal{M}(Y) \frown \mathcal{M}_1(Y))$  then return false
3   foreach connected component  $M \in \mathcal{M}_0^1(Y)$  do
4     Let  $y \in M$  s.t.  $r(y) = 1$ 
5     foreach  $z_i \in M$  s.t.  $r(z_i) = 0$  do
6       if  $\|y - z_i\|_2^2 > 2$  then return false
7 return true
```

Quasi-1-regularity verification method

The method consists in verifying the following two conditions:

1. $\mathcal{M}(X) \simeq \mathcal{M}_1(X)$ and $\mathcal{M}(\bar{X}) \simeq \mathcal{M}_1(\bar{X})$
2. (\mathcal{P}) holds for each connected component of $\mathcal{M}_0^1(X)$ and $\mathcal{M}_0^1(\bar{X})$.

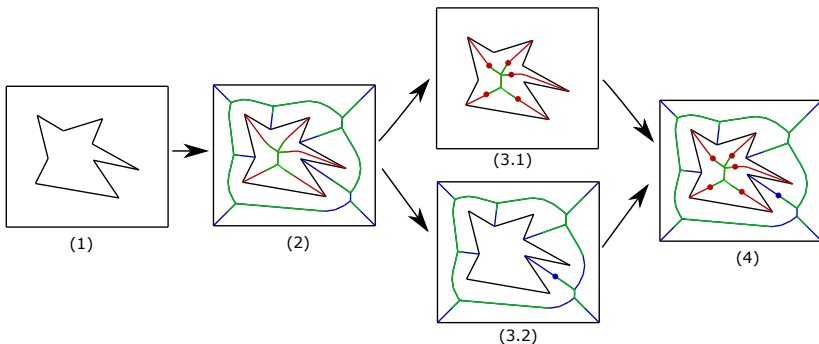


CGAL is used for computing the medial axis of the polygons.

Quasi-1-regularity verification method

The method consists in verifying the following two conditions:

1. $\mathcal{M}(X) \simeq \mathcal{M}_1(X)$ and $\mathcal{M}(\bar{X}) \simeq \mathcal{M}_1(\bar{X})$
2. (\mathcal{P}) holds for each connected component of $\mathcal{M}_0^1(X)$ and $\mathcal{M}_0^1(\bar{X})$.



CGAL is used for computing the medial axis of the polygons.

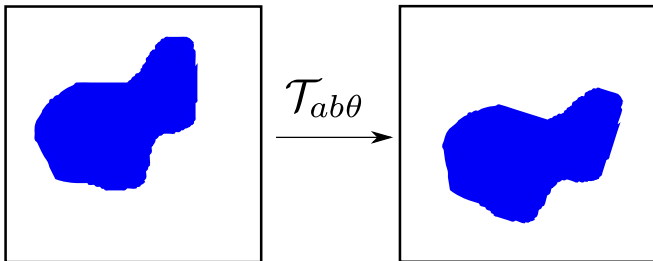
Rigid motion in \mathbb{R}^2

Definition

A rigid motion is a bijection defined for any $\mathbf{x} = (x, y) \in \mathbb{R}^2$, as

$$\mathcal{T}(\mathbf{x}) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}$$

where $a, b \in \mathbb{R}$ and $\theta \in [0, 2\pi[$.



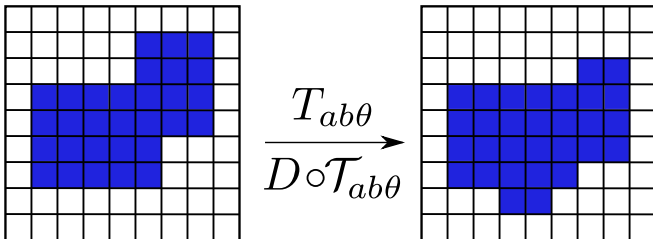
Rigid motion in \mathbb{Z}^2

Definition

A digital rigid motion on \mathbb{Z}^2 is defined for any $\mathbf{p} = (p, q) \in \mathbb{Z}^2$ as

$$T(\mathbf{p}) = D \circ \mathcal{T}(\mathbf{p}) = \begin{pmatrix} [p \cos \theta - q \sin \theta + a] \\ [p \sin \theta + q \cos \theta + b] \end{pmatrix}$$

where $D : \mathbb{R}^2 \rightarrow \mathbb{Z}^2$ is digitization (a rounding function).



Problems induced by rigid motion on \mathbb{Z}^2



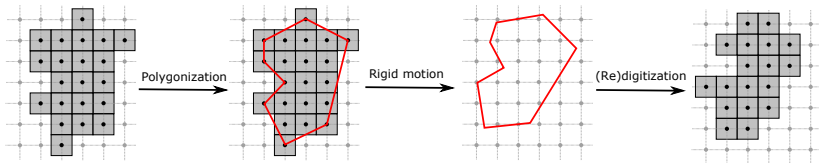
Topological and geometrical alteration by point-wise rigid motion on \mathbb{Z}^2 .

Digital rigid motion preserving topology and geometry

Proposed method

Approach via polygonization:

- polygonal representation of digital object for the overall shape preservation of the object by the transformation
- quasi-1-regularity for topology preservation by the digitization
- exact calculation considering rational rigid motions

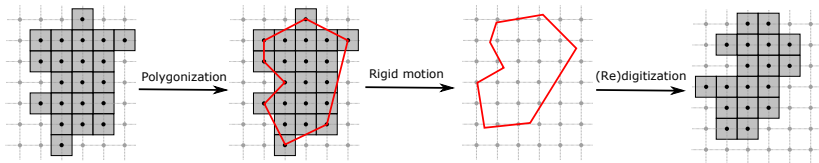


Digital rigid motion preserving topology and geometry

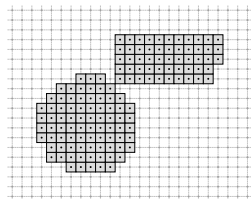
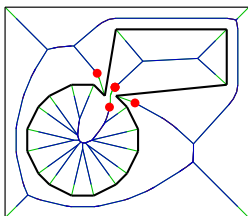
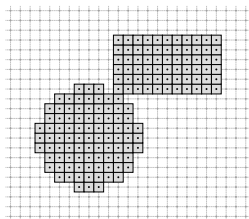
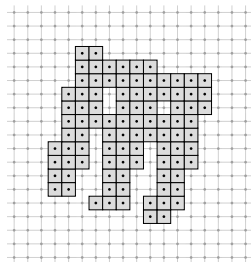
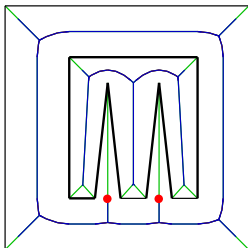
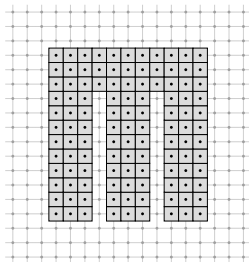
Proposed method

Approach via polygonization:

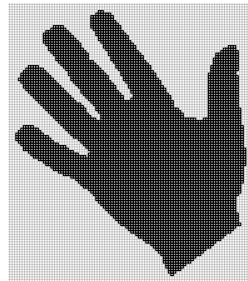
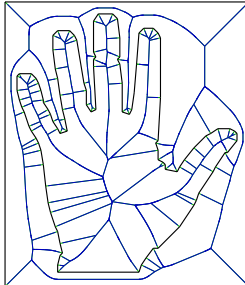
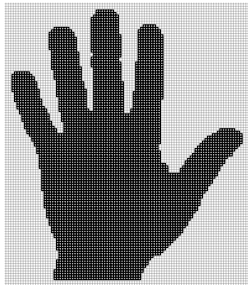
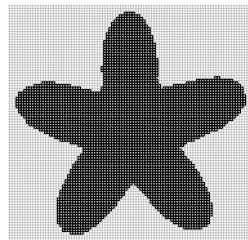
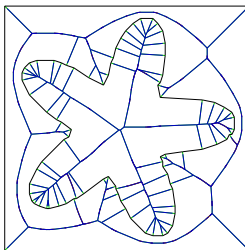
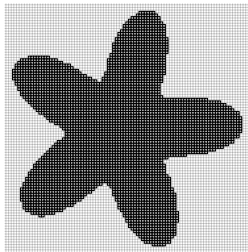
- polygonal representation of digital object for the overall shape preservation of the object by the transformation
- quasi-1-regularity for topology preservation by the digitization
- exact calculation considering rational rigid motions



Experimental results



Experimental results



Conclusion

Present work

1. Exact method for verifying quasi-1-regularity of polygonal objects using medial axis
2. Application in rigid motion in \mathbb{Z}^2 for preservation of both topology and geometry

Perspectives

- Polygonization methods for digital objects
- Repairing non-quasi-regular objects using medial axis
- Multi-resolution approaches for the limit case of quasi-1-regularity
- Extension of the proposed method in 3D

Thank you for your attention !
Any question ?

References i



Blum, H. (1967).

A transformation for extracting new descriptors of shape.

In Wathen-Dunn, W., editor, *Models for the Perception of Speech and Visual Form*, pages 362–380. MIT Press, Cambridge.



Klette, R. and Rosenfeld, A. (2004).

Digital Geometry: Geometric Methods for Digital Picture Analysis.

Elsevier, Amsterdam, Boston.



Latecki, L. J., Conrad, C., and Gross, A. (1998).

Preserving topology by a digitization process.

Journal of Mathematical Imaging and Vision, 8(2):131–159.

References ii



Latecki, L. J., Eckhardt, U., and Rosenfeld, A. (1995).

Well-composed sets.

Comput Vis Image Und, 61(1):70–83.



Lieutier, A. (2004).

Any open bounded subset of \mathbb{R}^n has the same homotopy type as its medial axis.

Computer-Aided Design, 36(11):1029–1046.



Ngo, P., Passat, N., and Kenmochi, Y. (2021).

Quasi-regularity verification for 2D polygonal objects based on medial axis analysis.

In *International Conference on Image Processing (ICIP)*, Anchorage, Alaska, United States.

References iii



Ngo, P., Passat, N., Kenmochi, Y., and Debled-Rennesson, I. (2018).

Convexity invariance of voxel objects under rigid motions.

In *International Conference on Pattern Recognition (ICPR)*, pages 1157–1162, Beijing, China.



Pavlidis, T. (1982).

Algorithms for graphics and image processing.

Berlin: Springer, and Rockville: Computer Science Press.



Serra, J. (1983).

Image Analysis and Mathematical Morphology.

Academic Press, Inc., USA.

Polygonization methods

Polygonal representation

The properties to satisfy for the calculation of the polygonal representation $P(X)$ of X :

- reversibility: $P(X) \cap \mathbb{Z}^2 = X$
- polygon's vertices with rational coordinates (for exact calculus)

For a given $X \subset \mathbb{Z}^2$, different results can be obtained from different polygonization techniques

- Convex objects: convex hull + half-plane representation
- Non convex objects: methods based on object contour (decomposition into convex parts, tree of concavities, ...)

Polygonization methods

Polygonal representation

The properties to satisfy for the calculation of the polygonal representation $P(X)$ of X :

- reversibility: $P(X) \cap \mathbb{Z}^2 = X$
- polygon's vertices with rational coordinates (for exact calculus)

For a given $X \subset \mathbb{Z}^2$, different results can be obtained from different polygonization techniques

- Convex objects: convex hull + half-plane representation
- Non convex objects: methods based on object contour (decomposition into convex parts, tree of concavities, ...)