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Asymptotic of the eigenvalues of Toeplitz matrices with even symbol

Philippe Rambour*

Abstract

In this paper we consider an interval $[\theta_1, \theta_2] \subset]0, \pi[$ and f a periodic and even function in $C^4([0, 2\pi])$ such that $f(\theta) \in [f(\theta_1), f(\theta_2)] \iff \theta \in [\theta_1, \theta_2]$. Then we obtain a higher order asymptotic formula for all the eigenvalues of the Toeplitz matrix $T_N(f)$ as $N \rightarrow +\infty$ which belong to $[f(\theta_1), f(\theta_2)]$.

Mathematical Subject Classification (2020)

Primary 47L80 ; Secondary 47A08, 47A10, 47A15, 47G30.

Keywords Toeplitz matrices, operator eigenvalues.

1 Introduction and statement of the main results

If $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ and $h \in L^1(\mathbb{T})$ we denote by $T_N(h)$ the Toeplitz matrix of order N with symbol h . It is the $(N+1) \times (N+1)$ matrix such that, for $N \geq k, l \geq 0$, $(T_N(h))_{k+1, l+1} = \hat{h}(k-l)$ where $\hat{h}(u)$ is the Fourier coefficient of order u of h ([16, 9]). For a real valued function h the matrix $T_N(h)$ is a Hermitian Toeplitz matrix. We here consider a symmetric Toeplitz matrix, which is equivalent to assuming that the symbol h is an even function and we denote by $\lambda_N^{(1)} \leq \lambda_N^{(2)} \leq \dots \leq \lambda_N^{(N+1)}$ the eigenvalues of $T_N(h)$. This paper addresses the asymptotic behavior of the eigenvalues of $T_N(h)$ as N goes to infinity. This is a topic which has attracted mathematicians and physicists for a long time. Toeplitz matrices and their relatives emerge in particular in statistic [10, 5] and in statistical physics [3, 4]. It is known from a long time that Toeplitz matrices are useful for providing Green's kernels for studying the solutions of certain differential equations ([27, 23]), and also for discretizing differential operators with finite differences [19]. Finally, these matrices are used in more recent fields, such as Ising models [11] and iso-geometric analysis [15]. The questions about the asymptotic behavior of their spectral characteristics, especially their determinants, eigenvalues, and eigenvectors, are always at the heart of the matter. We refer to the papers [11] for an extensive list of references. According to the first Szegő limit theorem (see [16]) the eigenvalues of $T_N(h)$ are asymptotically distributed

*Université de Paris Saclay, Bâtiment 307; F-91405 Orsay Cedex; tel : 01 69 15 57 28 ; fax 01 69 15 60 19
e-mail : philippe.rambour@universite-paris-saclay.fr

as the value of h ; see [16] for L^∞ symbols, [31] for L^1 symbols, and [29, 30] for more general situations. In the Hermitian case extensive works has been done on the search for eigenvalues (or the extreme eigenvalues) of Toeplitz matrices [32, 16, 26, 25, 20, 22, 21] and more recently, for instance, [6, 7, 2, 14, 13, 8]. In [6] the authors give an asymptotic expansion of order 2 for the eigenvalues of a Toeplitz matrix with smooth simple-loop symbol, the results of this article are the closest to ours, but the techniques are different. [7] is a good reminder of the various results obtained on the eigenvalues of Toeplitz matrices with polynomial symbol. In [2] M. Barrera, A. Böttcher, S. M. Grudsky and E. A. Maximenko show that the eigenvalues of the matrix $T_N(4(1 - \cos \theta)^2)$ cannot have an asymptotic expansion to order 4. For banded Toeplitz matrices or block symmetric Toeplitz matrices, the reader is referred to [14, 13]. Here we consider real symbol. For complex symbols an alternative to our results is given in [12]. The results of Theorem 1 are consistent with those of Theorem 2.3 of [6]. But the method of proof is different and the statement concern functions which are outside the framework of [6]. On the other hand Theorem 1 indicates that the problem of the eigenvalues of Toeplitz matrices is a local problem, related to the variation of the function which is the symbol of the matrix.

Here we denote by $\mathcal{A}^+([0, 2\pi])$ (resp. $\mathcal{A}^-([0, 2\pi])$) the set of even differentiable periodic functions of period 2π , such that $f'(\theta) > 0$ (resp. $f'(\theta) < 0$) for all θ in $]0, \pi[$.

More generally $[\theta_1, \theta_2] \subset]0, \pi[$ we say that $f \in \mathcal{A}^+([\theta_1, \theta_2])$ (resp. $f \in \mathcal{A}^-([\theta_1, \theta_2])$) if f is a differentiable, 2π periodic, and even function such that $f'(\theta) > 0$ (resp. $f'(\theta) < 0$) for all $\theta \in [\theta_1, \theta_2]$ and also $f(\theta) \in [f(\theta_1), f(\theta_2)] \iff \theta \in [\theta_1, \theta_2]$ (resp. $f(\theta) \in [f(\theta_2), f(\theta_1)] \iff \theta \in [\theta_1, \theta_2]$).

For $\nu \geq 0$ we denote by W^ν the weighted Wiener algebra of all functions $\psi : \mathbb{T} \mapsto \mathbb{C}$ which admits the representation $\psi(t) = \sum_{j \in \mathbb{Z}} \hat{\psi}(j) t^j$ whose Fourier coefficients satisfy

$$\|\psi\|_\nu = \sum_{j \in \mathbb{Z}} |\hat{\psi}(j)| (|j| + 1)^\nu < \infty.$$

Now for $f \in \mathcal{A}^+([\theta_1, \theta_2])$ we define the functions

$$H(\theta', \theta'') = \frac{f(\theta') - f(\theta'')}{(1 - \cos \theta') - (1 - \cos \theta'')} \quad \text{for } (\theta', \theta'') \in [0, \pi]^2$$

and

$$\rho(\theta) = \frac{1}{4\pi} P.V. \int_0^{2\pi} \frac{\ln((H(t, \theta))}{\tan\left(\frac{t-\theta}{2}\right)} dt - \frac{1}{4\pi} P.V. \int_0^{2\pi} \frac{\ln((H(t, \theta))}{\tan\left(\frac{t+\theta}{2}\right)} dt \quad \text{for } \theta \in [\theta_1, \theta_2].$$

Lastly for an interval $[a, b] \subset [\theta_1, \theta_2]$ and N an integer we denote by $k_{a,N}$ and $k_{b,N}$ the integers such that : $k_{a,N} = \min\{k | \frac{k\pi}{N+2} \in [a, b]\}$, $k_{b,N} = \max\{k | \frac{k\pi}{N+2} \in [a, b]\}$. We have also to define the two functions

$$c_1(t) = f'(t)\rho(t)$$

and

$$c_2(t) = f'(t)\rho'(t)\rho(t) + \frac{1}{2}f''(t)\rho(t)^2;$$

Now we can state our main result and an easy consequence.

Theorem 1 *Let f in $C^4[0, 2\pi]$ be such that $f \in \mathcal{A}^+([\theta_1, \theta_2])$ for an interval $[\theta_1, \theta_2] \subset]0, \pi[$. Then for all interval $[a, b] \subset]\theta_1, \theta_2[$ and for a sufficiently large integer N we have the two following statements.*

1. *For all eigenvalue λ of $T_N(f)$ in $[f(a), f(b)]$ we have a single integer k in $[k_{\theta_1, N}, k_{\theta_2, N}]$ such that $\lambda = \tilde{\lambda}_N^{(k)} + O\left(\frac{1}{(N+2)^3}\right)$, uniformly in λ with*

$$\tilde{\lambda}_N^{(k)} = f\left(\frac{k\pi}{N+2}\right) + \frac{c_1\left(\frac{k\pi}{N+2}\right)}{(N+2)} + \frac{c_2\left(\frac{k\pi}{N+2}\right)}{(N+2)^2}$$

2. *For all $k \in [k_{a, N}, k_{b, N}]$ the matrix $T_N(f)$ has a single eigenvalue λ in $[f(\theta_1), f(\theta_2)]$ such that $\lambda = \tilde{\lambda}_N^{(k)} + O\left(\frac{1}{(N+2)^3}\right)$ uniformly in k .*

Remark 1 *Similar results to Theorem 1 holds for the case where f in $\mathcal{A}^-([\theta_1, \theta_2])$ for an interval $[\theta_1, \theta_2] \subset]0, \pi[$.*

Remark 2 *In Theorem 1 the eigenvalue $\lambda = \tilde{\lambda}_N^{(k)} + O\left(\frac{1}{(N+2)^2}\right)$ is not necessarily $\lambda_N^{(k)}$.*

Remark 3 *If we consider the functions $\psi_\alpha : \theta \mapsto (1 - \cos \theta)^\alpha c(\theta)$ where $\alpha \geq 2$ and c a even positive function such that $c \in C^4[0, 2\pi]$ and $\alpha \sin \theta (1 - \cos \theta)^{\alpha-1} c(\theta) + (1 - \cos \theta)^\alpha c'(\theta) > 0$ for all $\theta \in [0, \pi]$ we can remark that Theorem 1 provides all the eigenvalues of the functions of ψ_α in $[N\epsilon, \pi - \epsilon]$ for all $\epsilon > 0$, that is an extension of the main results of [8].*

Remark 4 *Revisiting the proof of Theorem 1, one can show that under the assumption f in $C^3([0, 2\pi])$ we obtain an analogous version of this Theorem with the formula $\lambda = \hat{\lambda}_N^{(k)} + O\left(\frac{\log N}{(N+2)^2}\right)$, with*

$$\hat{\lambda}_N^{(k)} = f\left(\frac{k\pi}{N+2}\right) + \frac{c_1\left(\frac{k\pi}{N+2}\right)}{(N+2)}$$

where the function c_1 is as in Theorem 1.

Our result can also be compared with that of Trench [28] where it is proved that for this class of symbols the eigenvalues are all distinct.

To conclude we can remark that a tiny modification of the proof of Theorem (1) allows us to obtain the following Theorem which is in fact Theorem 2-3 in [2].

Theorem 2 *Let f in $C^4[0, 2\pi]$ be such that $f \in \mathcal{A}^+([0, 2\pi])$, $f''(0) > 0$ and $f''(\pi) < 0$, then for a sufficiently large N we have*

$$\lambda_N^{(k)} = f\left(\frac{k\pi}{N+2}\right) + \frac{c_1\left(\frac{k\pi}{N+2}\right)}{(N+2)} + \frac{c_2\left(\frac{k\pi}{N+2}\right)}{(N+2)^2} + O\left(\frac{1}{(N+2)^3}\right)$$

uniformly in $k = 1, 2, \dots, N$ and with c_1 and c_2 defined as previously:

Lastly we have to recall the following definition

Definition 1 *We denote by \mathbb{H}^+ is the set of all functions φ in $L^2(\mathbb{T})$ whose Fourier coefficients satisfy $\hat{\varphi}(j) = 0$ for all $j < 0$.*

Lastly in the rest of this paper we denote by χ the function $\theta \mapsto e^{i\theta}$.

2 Proof of Theorem 1

2.1 Preliminaries

In this proof we have to use the following Theorem which provides an inversion formula for a family of Toeplitz matrices.

Theorem 3 *Let P_{N+1} a trigonometric polynomial with degree $N + 1$ and without zeros on the united disc \bar{D} . Let $\omega = r\bar{\chi}_0$, $0 < r < 1$, $|\chi_0| = 1$ and also $f_r = g_1 g_2$, with $g_1 = \chi_0(1 - \omega\chi)(P_{N+1})^{-1}$ and $g_2 = (1 - \omega\bar{\chi})(\overline{P_{N+1}})^{-1}$. Then for all polynomial P in $\mathcal{P}_N = \text{vect}\{1, \chi, \dots, \chi^N\}$ we have*

$$T_N(f_r)^{-1}(P) = \frac{1}{g_1} \pi_+ \left(\frac{P}{g_2} \right) - \frac{1}{g_1} \pi_+ \left(\Phi_N \sum_{s=0}^{+\infty} (H_{\Phi_N}^* H_{\Phi_N})^s \pi_+ \left(\tilde{\Phi}_N \pi_+ \left(\frac{P}{g_2} \right) \right) \right).$$

with

$$\begin{cases} \Phi_N & = \frac{g_1}{g_2} \chi^{N+1}, \\ \tilde{\Phi}_N & = \frac{g_2}{g_1} \chi^{-(N+1)}, \\ H_{\Phi_N}(\Psi) & = \pi_-(\Phi_N \Psi) \\ H_{\Phi_N}^*(\tilde{\Psi}) & = \pi_+(\tilde{\Phi}_N \Psi) \end{cases}$$

for $\Psi \in \mathbb{H}^+$, (resp. $\tilde{\Psi} \in (H^+)^{\perp}$) and where π_+ (resp. π_-) are the orthogonal projection on \mathbb{H}^+ (resp. $(H^+)^{\perp}$).

The reader can see [24] for the statement and the proof of Theorem 3. In the appendix of this article we briefly recall how to use it to calculate $((T_N)^{-1}(f_0))_{(1,1)}$ where the symbol f_0 is defined by $f_0 = \chi_0(1 - \bar{\chi}_0\chi)(1 - \bar{\chi}_0\bar{\chi}) \frac{1}{|P_{N+1}|^2}$. The equation (27) gives the expression of $((T_N)^{-1}(f_0))_{(1,1)}$. We use this expression to obtain the equation (4) which is a fundamental tool of our proof.

Always to obtain (4) we have to use the fundamental property of the predictor polynomials which is the property (1). Before stating this property, we need of course to recall the definition of the predictor polynomial and its main property.

Definition 2 *The predictor polynomial of degree M of a regular function h is the trigonometric polynomial K_M defined by*

$$K_M = \sum_{k=0}^M \frac{(T_M(h))_{k+1,1}^{-1}}{\sqrt{(T_M(h))_{1,1}^{-1}}} \chi^k.$$

Property 1 *For all integers j , such that $-M \leq j \leq M$ we have*

$$\left(\frac{1}{|K_M|^2} \right)(j) = \hat{h}(j).$$

We have also the useful property

Property 2 $K_M(e^{i\theta}) \neq 0$ for all $\theta \in \mathbb{R}$.

Finally, consider the scalar product defined on \mathcal{P}_M by $\langle P|Q \rangle = \int_0^{2\pi} P(\theta)\overline{Q(\theta)}h(\theta)d\theta$ and let's denote $\Phi_0, \Phi_1, \dots, \Phi_M$ the orthogonal polynomials for this scalar product. Then the predictor polynomials of degree $0, 1, \dots, M$ are closely related to these orthogonal polynomials by the relation

$$K_j(z) = z^j \bar{\Phi}_j \left(\frac{1}{z} \right) \quad \forall j, 0 \leq j \leq M \quad \text{and} \quad \forall z \neq 0.$$

The reader can consult [18] for the predictor polynomials.

We can now begin the demonstration of the theorem.

This demonstration is divided into three parts. In the first part we obtain the equation (9) whose solutions are of the form $f^{-1}(\lambda)$ where the reals λ are the eigenvalues of f belonging to $[\theta_1, \theta_2]$. In the second part we obtain an integral expression for the ρ_N functions involved in this equation, which gives us the uniform convergence of the ρ_N in $[\theta_1, \theta_2]$ to a continuous function ρ . The ρ_N are therefore uniformly bounded, allowing us to locate the solutions of (9). In the third part, we study the smoothness of the function ρ , and use Taylor's theorem in (9) to obtain the asymptotic formula stated in Theorem.

2.2 Equation for the eigenvalues

Using the assumptions we can write $f(\theta) = f_1(1 - \cos \theta)$ where f_1 is a differentiable function strictly increasing on $[0, 2]$. For all λ in $[f(\theta_1), f(\theta_2)]$ we put $\theta_\lambda = f^{-1}(\lambda)$ and $\lambda' = f_1^{-1}(\lambda)$, that means $\theta_\lambda = \arccos(1 - \lambda')$, and $\theta_\lambda \in [0, \pi]$.

Remark 5 *In the next of the proof we denote by I_{θ_1, θ_2} the set $[f(\theta_1), f(\theta_2)]$.*

For $\lambda \in I_{\theta_1, \theta_2}$ we have

$$f(\theta) - \lambda = f_1(1 - \cos \theta) - \lambda = ((1 - \cos \theta) - (1 - \cos \theta_\lambda)) H_\lambda(\theta)$$

where $H_\lambda : \theta \mapsto H(\theta, \theta_\lambda)$ is a regular function on $[-\pi, \pi]$. We can write

$$(1 - \cos \theta) - (1 - \cos \theta_\lambda) = (1 - \cos \theta) - \lambda' = \frac{1}{2} (|1 - \chi|^2 - 2\lambda'). \quad (1)$$

If $\chi_\lambda = e^{i\theta_\lambda}$ we have $\chi_\lambda = (1 - \lambda') + i\sqrt{1 - (\lambda' - 1)^2}$ and we can write the equation (1) as

$$(1 - \cos \theta) - (1 - \cos \theta_\lambda) = -\frac{1}{2} \chi_\lambda (1 - \bar{\chi}_\lambda \chi) (1 - \bar{\chi}_\lambda \bar{\chi}). \quad (2)$$

Denote by $P_{N+1, \lambda}$ the predictor polynomial of H_λ . The property (1) allows to write the equation

$$T_N(((1 - \cos \theta) - (1 - \cos \theta_\lambda)) H_\lambda) = T_N \left(-\frac{1}{2} \chi_\lambda (1 - \bar{\chi}_\lambda \chi) (1 - \bar{\chi}_\lambda \bar{\chi}) \frac{1}{|P_{N+1, \lambda}|^2} \right). \quad (3)$$

For a fixed integer N we denote by $T_{1, N, \lambda}$ the quantity $((T_N(f) - \lambda I_N)^{-1})_{1, 1}$.

Since $T_N(f) - \lambda I_N$ is a Toeplitz matrix we have

$$T_{1, N, \lambda} = \frac{\det(T_{N-1}(f) - \lambda I_{N-1})}{\det(T_N(f) - \lambda I_N)}$$

and, since the eigenvalues of $(T_{N-1}(f))$ are not in $\{\lambda_N^{(1)}, \dots, \lambda_N^{(k)} \dots \lambda_N^{(N+1)}\} = \text{Spec}(T_N(f))$ (see [17, 1]), we have

$$\lambda \in \text{Spec}(T_N(f)) \iff \frac{1}{T_{1,N,\lambda}} = 0.$$

Using equation (3) and the inversion formula of Toeplitz matrices see in Theorem (3) we obtain the entry $(T_N^{-1}(f))_{(1,1)}$. Then with the results (see the equation (27) in the appendix) we can write

$$\frac{1}{T_{1,N,\lambda}} = \frac{1 - \bar{\chi}_\lambda^{-2(N+1)} \tau_N(\chi_\lambda)}{\left(1 - \bar{\chi}_\lambda^{-2(N+2)} \tau_N(\chi_\lambda)\right) B_{2,N,\lambda} - B_{1,N,\lambda}}, \quad (4)$$

with

$$\tau_N(\theta_\lambda) = \frac{\bar{P}_{N+1,\lambda}(\chi_\lambda) P_{N+1,\lambda}(\chi_\lambda)}{\bar{P}_{N+1,\lambda}(\bar{\chi}_\lambda) P_{N+1,\lambda}(\bar{\chi}_\lambda)},$$

and

$$B_{1,N,\lambda} = \left| P_{N+1,\lambda}(0) \frac{P_{N+1,\lambda}(\chi_\lambda)}{\bar{P}_{N+1,\lambda}(\bar{\chi}_\lambda)} \right|^2 (1 - \bar{\chi}_\lambda^2)^{-1}, \quad B_{2,N,\lambda} = \chi_\lambda \frac{1}{|P_{N+1,\lambda}(0)|^2}.$$

For $\theta \in [\theta_1, \theta_2]$ the quantities $B_{1,N,\lambda}$ and $B_{2,N,\lambda}$ are defined (see Property 2) if $\left(1 - \bar{\chi}_\lambda^{-2(N+2)} \tau_N(\chi_\lambda)\right) B_{2,N,\lambda} - B_{1,N,\lambda} = 0$ this equality means that $\det(T_{N-1}(f) - \lambda I_{N-1}) = 0$ and λ is an eigenvalue of $T_{N-1}(f)$ so it cannot be an eigenvalue of $T_N(f)$.

Hence we can write

$$\lambda \in (\text{Spec}(T_N(f)) \cap I_{\theta_1, \theta_2}) \iff \chi_\lambda^{2(N+2)} = \tau_N(\theta_\lambda), \lambda \in I_{\theta_1, \theta_2}. \quad (5)$$

Since the function H_λ is even, the constant $\tau_N(\theta_\lambda)$ can be rewritten as

$$\tau_N(\theta_\lambda) = \left(\frac{P_{N+1,\lambda}(\chi_\lambda)}{P_{N+1,\lambda}(\bar{\chi}_\lambda)} \right)^2.$$

As the function $\theta_\lambda \mapsto \frac{P_{N+1,\lambda}(e^{-i\theta})}{P_{N+1,\lambda}(e^{i\theta})}$ is continuous from I_{θ_1, θ_2} to $\{z \mid |z| = 1\}$ we have a function ρ_N defined and continuous on $[\theta_1, \theta_2]$ such that $\tau_N(\theta_\lambda) = e^{2i\rho_N(\theta_\lambda)}$. Then equation (5) can be written

$$\lambda \in (\text{Spec}(T_N(f)) \cap [f(\theta_1), f(\theta_2)]) \iff \theta_\lambda = \frac{\rho_N(\theta_\lambda) + k\pi}{(N+2)}, k \in [0, 2N+3]. \quad (6)$$

More precisely if $M_N = \max_{\theta \in [\theta_1, \theta_2]} |\rho_N(\theta)|$ and if $\theta_1 < a < b < \theta_2$ we can write, according to the construction of χ_λ

$$\lambda \in (\text{Spec}(T_N(f)) \cap [f(a), f(b)]) \Rightarrow \theta_\lambda = \frac{\rho_N(\theta_\lambda) + k\pi}{N+2}, k \in \left[\frac{(N+2)a - M_N}{\pi}, \frac{(N+2)b + M_N}{\pi} \right] \quad (7)$$

and

$$\theta_\lambda = \frac{\rho_N(\theta_\lambda) + k\pi}{N+2} \frac{k\pi}{N+2} \in [a, b] \Rightarrow \lambda \in \left(\text{Spec}(T_N(f)) \cap \left[f\left(a - \frac{M_N}{N+2}\right), f\left(b + \frac{M_N}{N+2}\right) \right] \right) \quad (8)$$

Lastly it is clear that we have now to solve the equation

$$\theta = \frac{\rho_N(\theta) + k\pi}{(N+2)} \quad (9)$$

Now we have to make a more precise study of the function ρ_N . If $s \geq 0$, then every function $f \in \mathbb{A}(\mathbb{T}, s)$ without zeros on \mathbb{T} admits a Wiener-Hopf factorization, that is, there exist function f_+ and f_- such that $f(e^{i\theta}) = f_+(e^{i\theta})e^{i\gamma\theta}f_-(e^{i\theta})$ with some $\gamma \in \mathbb{Z}$ the index of the factorization. The function f_+ (resp. f_-) belongs to set $\mathbb{A}(\mathbb{T}, s)_+$ (resp. $\mathbb{A}(\mathbb{T}, s)_-$) where

$$\mathbb{A}(\mathbb{T}, s)_+ = \left\{ f \in W^s \mid f(e^{i\theta}) = \sum_{j=0}^{+\infty} \hat{f}(j)e^{ij\theta} \right\}$$

and

$$\mathbb{A}(\mathbb{T}, s)_- = \left\{ f \in W^s \mid f(e^{i\theta}) = \sum_{j=0}^{+\infty} \hat{f}(-j)e^{-ij\theta} \right\}.$$

Here we have clearly $|P_{N+1,\lambda}(e^{i\theta})|^2 = P_{N+1,\lambda}(e^{i\theta})P_{N+1,\lambda}(e^{-i\theta})$ and

$$\left(\frac{1}{|P_{N+1,\lambda}(e^{i\theta})|^2} \right)_+ = \frac{1}{P_{N+1,\lambda}(e^{i\theta})} \quad \text{and} \quad \left(\frac{1}{|P_{N+1,\lambda}(e^{i\theta})|^2} \right)_- = \frac{1}{P_{N+1,\lambda}(e^{-i\theta})},$$

with index zero. Now it is well known that in the Wiener-Hopf factorization $\left(\frac{1}{|P_{N+1,\lambda}(e^{i\theta})|^2} \right)_+$ can be written in the form

$$\left(\frac{1}{|P_{N+1,\lambda}(e^{i\theta})|^2} \right)_+ = \exp \left(\frac{1}{2} \log \left(\frac{1}{|P_{N+1,\lambda}(e^{i\theta})|^2} \right) + \frac{1}{2\pi i} P.V. \int_{\mathbb{T}} \frac{\log \left(\frac{1}{|P_{N+1,\lambda}(z)|^2} \right)}{z - e^{i\theta}} dz \right)$$

that can be rewritten as

$$\exp \left(\frac{1}{2} \log \left(\frac{1}{|P_{N+1,\lambda}(e^{i\theta})|^2} \right) + \frac{1}{4\pi i} P.V. \int_0^{2\pi} \frac{\log \left(\frac{1}{|P_{N+1,\lambda}(e^{iu})|^2} \right)}{\tan \frac{u-\theta}{2}} du + \frac{1}{4\pi} \int_0^{2\pi} \log \left(\frac{1}{|P_{N+1,\lambda}(e^{iu})|^2} \right) du \right).$$

That provides $\frac{P_{N+1,\lambda}(e^{i\theta})}{P_{N+1,\lambda}(e^{-i\theta})} = e^{i\rho_{N,\lambda}(\theta)}$ with

$$\rho_{N,\lambda}(\theta) = \frac{1}{4\pi} P.V. \int_0^{2\pi} \frac{\log \left(\frac{1}{|P_{N+1,\lambda}(e^{iu})|^2} \right)}{\tan \frac{u-\theta}{2}} du - \frac{1}{4\pi} P.V. \int_0^{2\pi} \frac{\log \left(\frac{1}{|P_{N+1,\lambda}(e^{iu})|^2} \right)}{\tan \frac{u+\theta}{2}} du$$

and finally $\rho_N(\theta_\lambda) = \rho_{N,\lambda}(\theta_\lambda)$, and $\rho_N(\theta) = \rho_{N,f(\theta)}(\theta)$. The same methods give, for $G_\lambda = (H_\lambda)_+$

$$G_\lambda(e^{i\theta}) = (H_\lambda(\theta))_+ = \exp \left(\frac{1}{2} \log (H_\lambda(v)) + \frac{1}{2\pi i} P.V. \int_{\mathbb{T}} \frac{\log (H_\lambda(v))}{e^{iv} - e^{i\theta}} dv \right)$$

and

$$\rho_\lambda(\theta) = \frac{1}{4\pi} P.V. \int_0^{2\pi} \frac{\log (H_\lambda(u))}{\tan \frac{u-\theta}{2}} du - \frac{1}{4\pi} P.V. \int_0^{2\pi} \frac{\log (H_\lambda(u))}{\tan \frac{u+\theta}{2}} du,$$

$\rho(\theta_\lambda) = \rho_\lambda(\theta_\lambda)$, or $\rho(\theta) = \rho_{f(\theta)}(\theta)$.

2.3 Limit of the sequence $(\rho_N)_{(N \in \mathbb{N})}$

Now we need to relate the two functions ρ_N and ρ and for this we have to obtain the following property

Property 3 *When N goes to the infinity $|\rho_N(\theta) - \rho(\theta)| = O(\frac{\ln^2 N}{N^2})$ uniformly in $\theta \in [\theta_1, \theta_2]$.*

Lemmas 1 to 5 are devoted to the prove of this property.

Lemma 1 *For all $\lambda \in I_{\theta_1, \theta_2}$ the function $\theta \mapsto H_\lambda(\theta)$ is in $C^3([0, 2\pi])$ and for all $j \in \{0, 1, 2, 3\}$ we have a real K_j not depending on λ such that $\|H_\lambda^{(j)}\|_\infty \leq K_j$.*

Remark 6 *We recall that the hypothesis $\lambda \neq f(0), f(\pi)$ corresponds to the fact that the maximum and minimum of the function f cannot be eigenvalues of $T_N(f)$. The values $f(0)$ and $f(\pi)$ are therefore never considered in the proof of Theorem 1.*

Proof of the lemma 1 : If $t = 1 - \cos \theta$ and $t_\lambda = 1 - \cos \theta_\lambda$ we have to prove that the function $H_{1,\lambda} : t \mapsto \frac{f_1(t) - f_1(t_\lambda)}{t - t_\lambda}$ is in $C^3([0, 2\pi])$ and that for all integer j , $0 \leq j \leq 3$ there exists a real $K_{1,j}$ such that, for all λ in I_{θ_1, θ_2} $\|H_{1,\lambda}^{(j)}\|_\infty \leq K_{1,j}$. Clearly $\|H_{1,\lambda}\|_\infty \leq \|f^{(1)}\|_\infty$. Now for $t \neq t_\lambda$

$$\begin{aligned} H_{1,\lambda}^{(1)}(t) &= \frac{f_1^{(1)}(t)(t - t_\lambda) - (f_1(t) - f_1(t_\lambda))}{(t - t_\lambda)^2} \\ &= \frac{\left(f_1^{(1)}(t_\lambda) + (t - t_\lambda)f_1^{(2)}(a_1)\right)(t - t_\lambda) - \left(f_1^{(1)}(t_\lambda)(t - t_\lambda) + \frac{(t - t_\lambda)^2}{2}f_1^{(2)}(a_2)\right)}{(t - t_\lambda)^2} \end{aligned}$$

with a_1 and a_2 between t and t_λ . That provides

- $H_{1,\lambda}^{(1)}(t_\lambda) = \frac{f_1^{(2)}(t_\lambda)}{2}$,
- $\|H_{1,\lambda}^{(1)}\|_\infty \leq \frac{3}{2}\|f_1^{(2)}\|_\infty$.

Now we have, for $t \neq t_\lambda$

$$\begin{aligned} H_{1,\lambda}^{(2)}(t) &= \frac{f_1^{(2)}(t)(t - t_\lambda)^2 - 2\left(f_1^{(1)}(t)(t - t_\lambda) - (f_1(t) - f_1(t_\lambda))\right)}{(t - t_\lambda)^3} \\ &= \frac{\left(f_1^{(2)}(t_\lambda) + f_1^{(3)}(a_3)(t - t_\lambda)\right)(t - t_\lambda)^2 - 2(d_{1,\lambda}(t) - d_{2,\lambda}(t))}{(t - t_\lambda)^3} \end{aligned}$$

with

$$\begin{aligned} d_{1,\lambda}(t) &= f_1^{(1)}(t_\lambda)(t - t_\lambda) + f_1^{(2)}(t_\lambda)(t - t_\lambda)^2 + f_1^{(3)}(a_4)\frac{(t - t_\lambda)^3}{2} \\ d_{2,\lambda}(t) &= f_1^{(1)}(t_\lambda)(t - t_\lambda) + f_1^{(2)}(t_\lambda)\frac{(t - t_\lambda)^2}{2} + f_1^{(3)}(a_5)\frac{(t - t_\lambda)^3}{6}, \end{aligned}$$

and a_3, a_4, a_5 between t and t_λ . That provides

- $H_{1,\lambda}^{(2)}(t_\lambda) = \frac{f_1^{(3)}(t_\lambda)}{3}$,

- $\|H_{1,\lambda}^{(2)}\|_\infty \leq \frac{7}{3}\|f_1^{(3)}\|_\infty$.

Finally we can write, for $t \neq t_\lambda$

$$\begin{aligned} H_{1,\lambda}^{(3)}(t) &= \frac{f_1^{(3)}(t)(t-t_\lambda)^3 - 3\left(f_1^{(2)}(t)(t-t_\lambda)^2 - 2\left(f_1^{(1)}(t)(t-t_\lambda) - (f_1(t) - f_1(t_\lambda))\right)\right)}{(t-t_\lambda)^4} \\ &= \frac{\left(f_1^{(3)}(t_\lambda) + f_1^{(4)}(a_6)(t-t_\lambda)\right)(t-t_\lambda)^3 - 3(d_{3,\lambda}(t) - 2d_{4,\lambda}(t))}{(t-t_\lambda)^4}. \end{aligned}$$

with

$$\begin{aligned} d_{3,\lambda}(t) &= f_1^{(2)}(t_\lambda)(t-t_\lambda)^2 + f_1^{(3)}(t_\lambda)(t-t_\lambda)^3 + f_1^{(4)}(a_7)\frac{(t-t_\lambda)^4}{2} \\ d_{4,\lambda}(t) &= f_1^{(1)}(t_\lambda)(t-t_\lambda) + f_1^{(2)}(t_\lambda)(t-t_\lambda)^2 + f_1^{(3)}(t_\lambda)\frac{(t-t_\lambda)^3}{2} + f_1^{(4)}(a_8)\frac{(t-t_\lambda)^4}{6} \end{aligned}$$

and a_6, a_7, a_8 between t and t_λ . This last equalities give us

- $H_{1,\lambda}^{(3)}(t_\lambda) = \frac{1}{4}f_1^{(4)}(t_\lambda)$,
- $\|H_{1,\lambda}^{(3)}\|_\infty \leq \frac{15}{4}\|f_1\|_\infty$,

which end the proof. □

Remark 7 If h is a function is $L^2([0, 2\pi])$ we denote by $\|h\|_{q,2}$ the quadratic norm $\left(\int_0^{2\pi} |h(t)|^2 dt\right)^{\frac{1}{2}}$.

Lemma 2 We have a real S_0 not depending on k and λ such that

$$\left|\widehat{G}_\lambda(k)\right| \leq \frac{S_0}{k^3}, \quad \left|\widehat{\frac{1}{G_\lambda}}(k)\right| \leq \frac{S_0}{k^3}, \quad \text{for } k > 0, \quad \text{and} \quad \left|\widehat{\frac{G_\lambda}{\overline{G_\lambda}}}(k)\right| \leq \frac{S_0}{k^3}, \quad \text{for } k \neq 0.$$

Proof : We can observe that, for $0 \leq j \leq 3$ $(\pi_+(\log H_\lambda))^{(j)} = \pi_+\left((\log H_\lambda)^{(j)}\right)$. Hence, with the lemma 1, we have, for $0 \leq j \leq 3$,

$$\|(\pi_+(\log H_\lambda))^{(j)}\|_{q,2} \leq \|(\log H_\lambda)^{(j)}\|_q \leq T_j. \quad (10)$$

If m_0 is the minimum of H on $[0, 2\pi] \times [\theta_1, \theta_2]$ it is clear that for all $j \in \{0, 1, 2, 3\}$ T_j is only depending on the constants m_0, K_0, K_1, K_2, K_3 , of Lemma 1. Hence T_j is no depending from λ .

On the other hand since $\log H_\lambda \in C^3([0, 2\pi])$ we have, for all $n \geq 0$

$$|\widehat{\log H_\lambda}(n)| \leq \frac{\|(\log H_\lambda)^{(3)}\|_{q,2}}{n^3} \leq \frac{T_3}{n^3}$$

and

$$|(\widehat{\log H_\lambda})^{(1)}(n)| \leq \frac{\|(\log H_\lambda)^{(3)}\|_{q,2}}{n^2} \leq \frac{T_3}{n^2}.$$

Hence

$$\|\exp(\pi_+(\log H_\lambda))\|_\infty \leq \exp\left(T_3 \sum_{n \geq 0} \frac{1}{n^3}\right) = M_1 \quad (11)$$

and

$$\|(\pi_+(\log H_\lambda))^{(1)}\|_\infty \leq T_3 \sum_{n \geq 0} \frac{1}{n^2} = M_2.$$

Now if we put $\pi_+(\log H_\lambda) = F_\lambda$ we can write

$$(\exp -F_\lambda)^{(3)} = \left(-F_\lambda^{(3)} + 3F_\lambda^{(1)}F_\lambda^{(2)} - (F_\lambda^{(1)})^3\right) \exp -F_\lambda,$$

and

$$(\exp F_\lambda)^{(3)} = \left(-F_\lambda^{(3)} + 3F_\lambda^{(1)}F_\lambda^{(2)} - (F_\lambda^{(1)})^3\right) \exp F_\lambda.$$

According to (10) we have the inequalities

$$\begin{aligned} \|F_\lambda^{(3)} \exp F_\lambda\|_{q,2} &\leq \|F_\lambda^{(3)}\|_{q,2} \|\exp F_\lambda\|_\infty \leq T_2 M_1, \\ \|F_\lambda^{(1)} F_\lambda^{(2)} \exp F_\lambda\|_{q,2} &\leq \|F_\lambda^{(1)}\|_\infty \|\exp F_\lambda\|_\infty \|F_\lambda^{(2)}\|_{q,2} \leq M_1 M_2 T_2, \\ \|(F_\lambda^{(1)})^3 \exp F_\lambda\|_{q,2} &\leq \|(F_\lambda^{(1)})\|_\infty^3 \|\exp F_\lambda\|_\infty \leq M_2^3 M_1. \end{aligned}$$

This means that $\|(\exp F_\lambda)^{(3)}\|_2$, is bounded by a constant S_1 not depending on λ and n . This result implies

$$|\widehat{\exp -F_\lambda}(n)| \leq \frac{\|(\exp -F_\lambda)^{(3)}\|_{q,2}}{n^3} \leq \frac{S_1}{n^3}$$

and

$$|\widehat{\exp F_\lambda}(n)| \leq \frac{\|(\exp F_\lambda)^{(3)}\|_{q,2}}{n^3} \leq \frac{S_1}{n^3}$$

for all $n \geq 0$, that is the first part of the lemma.

On the other hand for $n > 0$ we have

$$\begin{aligned} \left|\frac{\widehat{G_\lambda}}{\widehat{G_\lambda}}(n)\right| &= \left|\sum_{h \geq 0} \widehat{G_\lambda}(h+n) \frac{\widehat{1}}{\widehat{G_\lambda}}(-h)\right| \\ &\leq H_1^2 \frac{\sum_{h > 0} \frac{1}{h^3}}{n^3} + \frac{1}{n^3} \left|\frac{\widehat{1}}{\widehat{G_\lambda}}(0)\right|, \end{aligned}$$

and

$$\begin{aligned} \left|\frac{\widehat{G_\lambda}}{\widehat{G_\lambda}}(-n)\right| &= \left|\sum_{k \geq 0} \widehat{G_\lambda}(k) \frac{\widehat{1}}{\widehat{G_\lambda}}(-(k+n))\right| \\ &\leq H_1^2 \frac{\sum_{k > 0} \frac{1}{k^3}}{n^3} + \frac{1}{n^3} \left|\frac{\widehat{1}}{\widehat{G_\lambda}}(0)\right|, \end{aligned}$$

and with (11) we can write $|\frac{\widehat{1}}{\widehat{G_\lambda}}(0)| \leq \frac{1}{2\pi} \|\exp(\pi_+(\log H_\lambda))\|_\infty \leq \frac{M_1}{2\pi}$ that provides the third inequality of the lemma. \square

Lemma 3 *If $\beta_{k,\lambda} = \widehat{\frac{1}{G_\lambda}}(k)$ we have, for a sufficient large N*

$$((T_N(H_\lambda))_{k,1})^{-1} = \overline{\beta_{0,\lambda}}\beta_{k,\lambda} + R_{k,N,\lambda}$$

with $|R_{k,N,\lambda}| \leq \frac{M}{N^2(N+1-k)^2}$ where M is not depending on λ and k .

Proof : Using the inversion formula given in the appendix of this paper we obtain, for $H_\lambda = G_\lambda \bar{G}_\lambda$, $G_\lambda \in \mathbb{H}^+$.

$$(T_N(H_\lambda))_{l+1,k+1}^{-1} = \left\langle \pi_+ \left(\frac{\chi^l}{\bar{G}_\lambda} \right) \middle| \frac{\chi^k}{\bar{G}_\lambda} \right\rangle - \left\langle \sum_{s=0}^{+\infty} \left(H_{\Phi_{N,\lambda}}^* H_{\Phi_{N,\lambda}} \right)^s \pi_+ \bar{\Phi}_{N,\lambda} \pi_+ \left(\frac{\chi^l}{\bar{G}_\lambda} \right) \middle| \pi_+ \bar{\Phi}_{N,\lambda} \pi_+ \left(\frac{\chi^k}{\bar{G}_\lambda} \right) \right\rangle,$$

with

$$\begin{aligned} \Phi_{N,\lambda} &= \frac{G_\lambda}{\bar{G}_\lambda} \chi^{N+1}, \quad \text{and} \quad \tilde{\Phi}_{N,\lambda} = \frac{\bar{G}_\lambda}{G_\lambda} \chi^{-(N+1)}, \\ H_{\Phi_{N,\lambda}}(\Psi) &= \pi_-(\Phi_{N,\lambda}\Psi) \quad \text{for} \quad \Psi \in \mathbb{H}^+, \\ H_{\tilde{\Phi}_{N,\lambda}}^*(\Psi) &= \pi_+(\tilde{\Phi}_{N,\lambda}\Psi) \quad \text{for} \quad \Psi \in (\mathbb{H}^+)^\perp. \end{aligned}$$

For $l = 0$ this formula becomes

$$(T_N(H_\lambda))_{k+1,1}^{-1} = \left\langle \pi_+ \left(\frac{1}{\bar{G}_\lambda} \right) \middle| \frac{\chi^k}{\bar{G}_\lambda} \right\rangle - \left\langle \sum_{s=0}^{+\infty} \left(H_{\Phi_{N,\lambda}}^* H_{\Phi_{N,\lambda}} \right)^s \pi_+ \bar{\Phi}_{N,\lambda} \pi_+ \left(\frac{1}{\bar{G}_\lambda} \right) \middle| \pi_+ \bar{\Phi}_{N,\lambda} \pi_+ \left(\frac{\chi^k}{\bar{G}_\lambda} \right) \right\rangle.$$

In the next of the proof we use the following notation :

$$\frac{G_\lambda}{\bar{G}_\lambda} = \sum_{u \in \mathbb{Z}} \gamma_{u,\lambda} \chi^u.$$

From Lemma 2 we have a positive constant S_0 such that

$$|\beta_{u,\lambda}| \leq \frac{S_0}{u^3} \quad \forall u \in \mathbb{N}^* \quad \text{and} \quad |\gamma_{u,\lambda}| \leq \frac{S_0}{u^3} \quad \forall u \in \mathbb{Z}^*.$$

With these notations we obtain

$$\begin{aligned} \left\langle \pi_+ \left(\frac{1}{\bar{G}_\lambda} \right) \middle| \frac{\chi^k}{\bar{G}_\lambda} \right\rangle &= \bar{\beta}_{0,\lambda} \beta_{k,\lambda}, \\ \pi_+ \bar{\Phi}_{N,\lambda} \pi_+ \left(\frac{1}{\bar{G}_\lambda} \right) &= \pi_+ (\bar{\Phi}_{N,\lambda} \bar{\beta}_{0,\lambda}) = \bar{\beta}_{0,\lambda} \sum_{v \geq N+1} \bar{\gamma}_{-v,\lambda} \chi^{v-N-1}, \\ \pi_+ \bar{\Phi}_{N,\lambda} \pi_+ \left(\frac{\chi^k}{\bar{G}_\lambda} \right) &= \sum_{w=0}^k \bar{\beta}_{w,\lambda} \left(\sum_{v \geq N+1-k+w} \bar{\gamma}_{-v,\lambda} \chi^{v-N-1+k-w} \right). \end{aligned}$$

Hence we obtain

$$\left\| \pi_+ \bar{\Phi}_{N,\lambda} \pi_+ \left(\frac{1}{\bar{G}_\lambda} \right) \right\|_{q,2} \leq S_1 (N+1)^{-2},$$

and

$$\left\| \pi_+ \bar{\Phi}_{N,\lambda} \pi_+ \left(\frac{\chi^k}{\bar{G}_\lambda} \right) \right\|_{q,2} \leq S_1 ((N+1-k)^{-2}),$$

where S_1 no depending on λ and N On the other hand for $\psi = \sum_{w \geq 0} \alpha_w \chi^w$ a function in \mathbb{H}^+ we have, with the continuity of the projection π_- ,

$$H_{\Phi_{N,\lambda}}(\psi) = \sum_{w \geq 0} \alpha_w \left(\sum_{v > N+1+w} \gamma_{-v,\lambda} \chi^{-v+w+N+1} \right)$$

that provides

$$\begin{aligned} \|H_{\Phi_{N,\lambda}}(\psi)\|_2 &\leq \sum_{w \geq 0} |\alpha_w| \left(\sum_{v > N+1+w} |\gamma_{-v,\lambda}| \right) \\ &\leq \|\psi\|_2 \left(\sum_{w \geq 0} \left(\sum_{v > N+1+w} |\gamma_{-v,\lambda}| \right)^2 \right)^{1/2} \\ &\leq S_0 \|\psi\|_2 (N+1)^{-3/2} \end{aligned}$$

that means $\|H_{\Phi_{N,\lambda}}\| \leq S_0 (N+1)^{-3/2}$. Clearly we have also $\|H_{\Phi_{N,\lambda}}^*\| \leq S_0 (N+1)^{-3/2}$ and we can write

$$\left\| \sum_{s=0}^{+\infty} \left(H_{\Phi_{N,\lambda}}^* H_{\Phi_{N,\lambda}} \right)^s \pi_+ \bar{\Phi}_{N,\lambda} \pi_+ \left(\frac{1}{\bar{G}_\lambda} \right) \right\|_2 \leq \frac{S_0}{(1 - S_0^2 (N+1)^{-3})^2} (N+1)^{-2}.$$

And finally we can write

$$(T_N(H_\lambda))_{1,k+1}^{-1} = \bar{\beta}_{0,\lambda} \beta_{k,\lambda} + O((N+1)^{-2} (N+1-k)^{-2})$$

with $O((N+1)^{-2} (N+1-k)^{-2}) = 2S_0^2 (N+1)^{-2} (N+1-k)^{-2}$ uniformly in λ that is the expected result with $M = 2H^2$. \square

Remark 8 As the coefficient $\beta_{0,\lambda}$ is real the form of $\tau_N(\chi_\lambda)$ allows to assume that $\beta_{0,\lambda} = 1$ is the rest of our demonstration.

Lemma 4 We have $\left\| \ln \left(\frac{1}{|P_{N+1,\lambda}|^2} \right) - \ln(H_\lambda) \right\|_0 = O\left(\frac{1}{N^2}\right)$ uniformly in λ .

Proof : Using Lemma 3, we obtain

$$\|P_{N+1,\lambda} - \frac{1}{G_\lambda}\|_0 \leq M(N+1)^{-2} \sum_{k=0}^N \frac{1}{(N+1-k)^2} + \sum_{k=N+1}^{+\infty} |\beta_{k,\lambda}|.$$

Hence

$$\|P_{N+1,\lambda} - \frac{1}{G_\lambda}\|_0 \leq \frac{M + S_0}{(N+1)^2}, \quad (12)$$

where M and $\beta_{k,\lambda}$ as in Lemma 3 and S_0 is the real not depending on N and from λ which has been introduced in Lemma 2. Always with M and S_0 no depending from λ and the norm $\left\|P_{N+1,\lambda} - \frac{1}{\bar{G}_\lambda}\right\|_0$ is bounded by $O\left(\frac{1}{N^2}\right)$. Now since $\|\Psi\Phi\|_0 \leq \|\Psi\|_0\|\Phi\|_0$ we have

$$\left\|\frac{1}{P_{N+1,\lambda}} - G_\lambda\right\|_0 \leq \left\|P_{N+1,\lambda} - \frac{1}{\bar{G}_\lambda}\right\|_0 \left\|\frac{G_\lambda}{P_{N+1,\lambda}}\right\|_{s_0} \leq \left\|P_{N+1,\lambda} - \frac{1}{\bar{G}_\lambda}\right\|_0 \left\|\frac{1}{P_{N+1,\lambda}}\right\|_0 \|G_\lambda\|_0. \quad (13)$$

Then, according to (13) we have

$$\left\|\frac{1}{P_{N+1,\lambda}}\right\|_0 - \|G_\lambda\|_0 \leq \left\|P_{N+1,\lambda} - \frac{1}{\bar{G}_\lambda}\right\|_0 \left\|\frac{1}{P_{N+1,\lambda}}\right\|_0 \|G_\lambda\|_0 \quad (14)$$

That provides

$$\left\|\frac{1}{P_{N+1,\lambda}}\right\|_0 \left(\left(1 - \left\|P_{N+1,\lambda} - \frac{1}{\bar{G}_\lambda}\right\|_0\right) \|G_\lambda\|_0 \right) \leq \|G_\lambda\|_0 \quad (15)$$

According the lemma 2 we have a real A_1 such that for all λ in $]f(\theta_1), f(\theta_2[$ $\|G_\lambda\|_0 \leq A_1$. Hence with 12 we obtain that for N sufficiently large we have

$$1 - \left\|P_{N+1,\lambda} - \frac{1}{\bar{G}_\lambda}\right\|_0 \|G_\lambda\|_0 \geq \frac{1}{2}. \quad (16)$$

and

$$\left\|\frac{1}{P_{N+1,\lambda}}\right\|_0 \leq 2A_1. \quad (17)$$

Merging (12) and (17) we obtain

$$\left\|\frac{1}{|P_{N+1,\lambda}|^2} - H_\lambda\right\|_0 \leq \left\|\frac{1}{|P_{N+1,\lambda}|^2} - \frac{1}{P_{N+1,\lambda}}\bar{G}_\lambda\right\|_0 + \left\|\frac{1}{P_{N+1,\lambda}}\bar{G}_\lambda - H_\lambda\right\|_0 \leq 3A_1 \left(\frac{M + S_0}{(N+1)^2}\right) \quad (18)$$

hence $\left\|\frac{1}{|P_{N+1,\lambda}|^2} - H_\lambda\right\|_0 = O\left(\frac{1}{N^2}\right)$, uniformly in λ . Now observe that

$$\left\|\ln\left(\frac{1}{|P_{N+1,\lambda}|^2}\right) - \ln(H_\lambda)\right\|_0 = \left\|\ln\left(1 + \frac{\frac{1}{|P_{N+1,\lambda}|^2} - H_\lambda}{H_\lambda}\right)\right\|_0,$$

that is also

$$\left\|\ln\left(1 + \frac{\frac{1}{|P_{N+1,\lambda}|^2} - H_\lambda}{H_\lambda}\right)\right\|_0 \leq \sum_{n \geq 1} \frac{1}{n} \left(\left\|\frac{1}{|P_{N+1,\lambda}|^2} - H_\lambda\right\|_0\right)^n \left(\left\|\frac{1}{H_\lambda}\right\|_0\right)^n.$$

Now we have, according to Lemma (1),

$$\left\|\frac{1}{H_\lambda}\right\|_0 \leq \left(\sum_{n \geq 0} \frac{1}{n^2}\right) \frac{1}{2\pi} \left\|\frac{1}{H_\lambda^{(2)}}\right\|_{q,2} \leq \frac{K}{m_0^3}$$

with m_0 as in Lemma 2 and K no depending on λ and N . That gives us, according to (18)

$$\left\|\ln\left(1 + \frac{\frac{1}{|P_{N+1,\lambda}|^2} - H_\lambda}{H_\lambda}\right)\right\|_0 \leq \sum_{n \geq 1} \frac{1}{n} \left(\left\|3A_1 \left(\frac{M + S_0}{(N+1)^2}\right)\right\|_0\right)^n \left(\frac{K}{m_0^3}\right)^n.$$

Since m_0 and K are not depending on λ we can conclude

$$\left\| \ln \left(\frac{1}{|P_{N+1,\lambda}|^2} \right) - \ln(H_\lambda) \right\|_0 = \left\| \ln \left(1 + \frac{\frac{1}{|P_{N+1,\lambda}|^2} - H_\lambda}{H_\lambda} \right) \right\|_0 = O \left(\frac{1}{N^2} \right)$$

uniformly in λ . □

Since the Cauchy singular operator is bounded on the Wiener classes $A(\mathbb{T}, s)$, $s \geq 0$, we have $\|\rho_N - \rho\|_0 = O(\frac{1}{N^2})$ and $|\rho_N(\lambda) - \rho(\lambda)| = O(\frac{1}{N^2})$ uniformly in λ . That ends the proof of Property 3

2.4 Derivation and solutions of the equation for the eigenvalues

To do this we need the two following lemmas.

Lemma 5 *The function ρ is in $\mathcal{C}^2([\theta_1, \theta_2])$.*

Proof : We prove the result for the function

$$I : \theta \mapsto P.V. \int_0^{2\pi} \frac{\ln(H(t, \theta))}{\tan\left(\frac{t-\theta}{2}\right)} dt,$$

the proof is quite the same for the function

$$\theta \mapsto P.V. \int_0^{2\pi} \frac{\ln(H(t, \theta))}{\tan\left(\frac{t+\theta}{2}\right)} dt.$$

First we write $I(\theta) = I_{1,\theta} + I_{2,\theta}$ with

$$I_{1,\theta} = P.V. \int_0^{2\pi} \frac{\ln(H(\theta, \theta))}{\tan\left(\frac{t-\theta}{2}\right)} dt,$$

$$I_{2,\theta} = \int_0^{2\pi} \frac{\log(H(t, \theta)) - \log(H(\theta, \theta))}{\tan\left(\frac{t-\theta}{2}\right)} dt.$$

A simple calculus provides us $I_{1,\theta} = 0$. On the other hand we can observe that the function $\Psi : (t, \theta) \mapsto \frac{\log(H(t, \theta)) - \log(H(\theta, \theta))}{\tan\left(\frac{t-\theta}{2}\right)}$ can be write $\frac{\log(H(t, \theta)) - \log(H(\theta, \theta))}{\frac{t-\theta}{2}} \frac{\frac{t-\theta}{2}}{\tan\left(\frac{t-\theta}{2}\right)}$. Thanks to the symmetry of the function $H : \theta \mapsto H(\theta, \theta')$ we can say that the function $\theta \mapsto H(t, \theta)$ is in $\mathcal{C}^3([\theta_1, \theta_2])$ for all t in $[0, 2\pi]$. Hence if Ψ_1 is the function defined by $\Psi_1 : \theta \mapsto \frac{\log(H(t, \theta)) - \log(H(\theta, \theta))}{t-\theta}$ the function $\frac{\partial \Psi_1}{\partial \theta}(t, \theta)$ is defined for all $\theta \neq t$ and is equal to

$$\frac{(t - \theta) ((\log H)'_t(t, \theta) - (\log H)'_t(\theta, \theta) - (\log H)'_\theta(\theta, \theta)) + ((\log H)(t, \theta) - (\log H)(\theta, \theta))}{(t - \theta)^2},$$

where we have denoted by $(\log H)'_t$ the quantity $\frac{\partial(\log H)}{\partial t}$ and by $(\log H)'_\theta$ the quantity $\frac{\partial(\log H)}{\partial \theta}$. We see that for $t = \theta$ the function $\frac{\partial \Psi_1}{\partial \theta}$ is equal to $\frac{1}{2} \frac{\partial^2 \log H}{\partial t^2}(\theta, \theta)$. Since the functions $\log H$, $\frac{\partial(\log H)}{\partial t}$, $\frac{\partial(\log H)}{\partial \theta}$, and $\frac{\partial^2 \log H}{\partial \theta^2}$ are continuous on $[0, 2\pi] \times [\theta_1, \theta_2]$ we obtain that the function

$(t, \theta) \mapsto \frac{\partial \Psi}{\partial \theta}$ is defined and continuous on $[0, 2\pi] \times [\theta_1, \theta_2]$, that completes this demonstration for the existence of $\rho^{(1)}$. For $\rho^{(2)}$ the function $\frac{\partial^2 \Psi_1}{\partial \theta^2}(t, \theta)$ is defined for all $t \neq \theta$ and is equal to

$$\frac{(t - \theta)^2 \Psi_1(t, \theta) + 2(t - \theta) \Psi_3(t, \theta) + 2\Psi_4(t, \theta)}{(t - \theta)^3}$$

where,

$$\begin{aligned} \Psi_2(t, \theta) &= (\log H)''_{\theta^2}(t, \theta) - (\log H)''_{t^2}(\theta, \theta) - (\log H)''_{\theta^2}(\theta, \theta) - (\log H)''_{t\theta}(\theta, \theta), \\ \Psi_3(t, \theta) &= (\log H)'_{\theta}(t, \theta) - (\log H)'_t(\theta, \theta) - (\log H)'_{\theta}(\theta, \theta), \\ \Psi_4(t, \theta) &= (\log H)(t, \theta) - (\log H)(\theta, \theta), \end{aligned}$$

and for $t = \theta$ we see that

$$\frac{\partial^2 \Psi_1}{\partial \theta^2} = (\log H)_{\theta^2, t}^{(3)}(\theta, \theta) + (\log H)_{t^3}^{(3)}(\theta, \theta) + (\log H)_{t^2 \theta}^{(3)}(\theta, \theta).$$

Then the same arguments as previously allow us to conclude. \square

To begin stating Theorem 1 we have to remark that with Property 3 we have a real $M > 0$ such that $-M \leq \rho_N(\theta) \leq M$ for all integer N and all $\theta \in [\theta_1, \theta_2]$.

Now if $\lambda \in [f(a), f(b)]$ is an eigenvalue of $T_N(f)$ we know that there is a real $\theta_\lambda \in [a, b]$ such that θ_λ is a solution of (??) that implies $a - \frac{M}{N+2} \leq \frac{k\pi}{N+2} \leq bf + \frac{M}{N+2}$, and we can conclude $k \in [k_{\theta_1, N}, k_{\theta_2, N}]$ for N sufficiently large.

Reciprocally if N is sufficiently large we have for all $k \in [k_{a, N}, k_{b, N}]$ two reals θ'_k and θ''_k in $[\theta_1, \theta_2]$ such that $\theta'_k < \frac{k\pi - M}{N+2}$ and $\frac{k\pi + M}{N+2} < \theta''_k$ that provides a solution to the equation $\theta = \frac{\rho_N(\theta) + k\pi}{N+2}$.

Now we can easily obtain the formula announced in the statement of Theorem 1. For λ an eigenvalue in $]f(\theta_1), f(\theta_2)[$ we have following the equation (9) $\lambda = f\left(\frac{k\pi + \rho_N(\theta_\lambda)}{N+2}\right)$ that is also $\lambda = f\left(\frac{k\pi + \rho(\theta_\lambda)}{N+2} + \frac{R_{N, \lambda}}{(N+2)}\right)$ with $\frac{R_{N, \lambda}}{(N+2)} = O\left(\frac{1}{(N+2)^3}\right)$ uniformly in λ according to Property 3 and where θ_λ is a solution of the equation (9). Putting $d = \frac{\pi k}{N+2}$ we have by Taylor's theorem,

$$\begin{aligned} \lambda &= f(d) + f'(d) \left(\frac{\rho(\theta_\lambda) + R_{N, \lambda}}{N+2}\right) + \frac{1}{2} f''(d) \left(\frac{\rho(\theta_\lambda) + R_{N, \lambda}}{N+2}\right)^2 \\ &\quad + \frac{1}{6} f^{(3)} \left(d + h_1 \frac{\rho(\theta_\lambda) + R_{N, \lambda}}{N+2}\right) \left(\frac{\rho(\theta_\lambda) + R_{N, \lambda}}{N+2}\right)^3, \end{aligned}$$

with $0 < h_1 < 1$. That provides

$$\lambda = f(d) + f'(d) \left(\frac{\rho(\theta_\lambda)}{N+2}\right) + \frac{1}{2} f''(d) \left(\frac{\rho(\theta_\lambda)}{N+2}\right)^2 + \frac{1}{6} f^{(3)} \left(d + h_1 \frac{\rho(\theta_\lambda)}{N+2}\right) \left(\frac{\rho(\theta_\lambda)}{N+2}\right)^3 + O\left(\frac{1}{(N+2)^3}\right), \quad (19)$$

where the quantity $O\left(\frac{1}{(N+2)^3}\right)$ is bounded uniformly in λ . On the other hand, with the equation (9) $\theta_\lambda = \frac{k\pi + \rho(\theta_\lambda)}{N+2} + \frac{R_{N, \lambda}}{(N+2)^2}$ and we can write, always by Taylor's theorem,

$$\rho(\theta_\lambda) = \rho(d) + \rho'(d) \frac{\rho(\theta_\lambda) + R_{N, \lambda}}{N+2} + \frac{1}{2} \rho'' \left(d + h_2 \frac{\rho(\theta_\lambda) + R_{N, \lambda}}{N+2}\right) \left(\frac{\rho(\theta_\lambda) + R_{N, \lambda}}{N+2}\right)^2, \quad (20)$$

with $0 < h_2 < 1$, that implies

$$\rho(\theta_\lambda) = \rho(d) + \rho'(d) \frac{\rho(d)}{N+2} + O\left(\frac{1}{(N+2)^2}\right). \quad (21)$$

with the rest is bounded by $\frac{|S|}{(N+2)^2}$ where S is a constant no depending from λ . Merging the equation (19) and (21) we obtain

$$\lambda = f(d) + \frac{f'(d)\rho(d)}{N+2} + \frac{f'(d)\rho'(d)\rho(d)}{(N+2)^2} + \frac{1}{2} \frac{f''(d)\rho^2(d)}{(N+2)^2} + R_{N,d}, \quad (22)$$

with $R_{N,d} = O\left(\frac{1}{(N+2)^3}\right)$ uniformly in λ . To achieve the proof we have to be sure that the eigenvalues found are distincts as announced. To do this we need the following two lemmas.

Lemma 6 For $k, k+1$ in $]f(\theta_1), f(\theta_2)[$ we have $\tilde{\lambda}_N^{(k+1)} - \tilde{\lambda}_N^{(k)} > 0$ and $|\tilde{\lambda}_N^{(k)} - \tilde{\lambda}_N^{(k+1)}| = O\left(\frac{1}{N}\right)$.

Proof : This lemma follows directly from (22). \square

Lemma 7 For a fixed k the equation (9) has one and only one solution in $[\theta_1, \theta_2]$.

Proof : Assume $\tilde{\lambda}_N$ and $\tilde{\lambda}'_N$ two solutions of (9) for a same integer k . By (22) we have $|\tilde{\lambda}_N - \tilde{\lambda}'_N| = o\left(\frac{1}{(N+2)^3}\right)$. By [1] we know that we have an eigenvalue λ_{N+1} of the matrix $T_{N+1}(f)$ with the bound $\tilde{\lambda}_N < \lambda_{N+1} < \tilde{\lambda}'_N$ that implies $|\tilde{\lambda}_N - \lambda_{N+1}| = o\left(\frac{1}{(N+2)^3}\right)$. By (22) we have $|\tilde{\lambda}_N - \lambda_{N+1}| \geq O\left(\frac{1}{N}\right)$, that is a contradiction with the previous estimation. \square

3 Proof of Theorem 2

First we can observe that we can define the function H on $[0, \pi] \times [0, \pi]$ with $H(0, 0) = .f''(0)$ and $H(\pi, \pi) = f''(\pi, \pi)$. Hence for all $\lambda \in I_{0,2\pi}$ we have

$$f(\theta) - \lambda = f_1(1 - \cos \theta) - \lambda = ((1 - \cos \theta) - (1 - \cos \theta_\lambda)) H_\lambda(\theta)$$

where $H_\lambda : \theta \mapsto H(\theta, \theta_\lambda)$ is a regular function on $[-\pi, \pi]$. for all $\lambda \in [0, \pi]$. With the same notations as in the proof of Theorem 1 we can still write

$$T_{1,N,\lambda} = \frac{\det(T_{N-1}(f) - \lambda I_{N-1})}{\det(T_N(f) - \lambda I_N)}.$$

We have also

$$\lambda \in \text{Spec}(T_N(f)) \iff \frac{1}{T_{1,N,\lambda}} = 0.$$

and, always with the equation (27), we can write

$$\frac{1}{T_{1,N,\lambda}} = \frac{1 - \tilde{\chi}_\lambda^{2(N+1)} \tau_N(\chi_\lambda)}{\left(1 - \tilde{\chi}_\lambda^{2(N+2)} \tau_N(\chi_\lambda)\right) B_{2,N,\lambda} - B_{1,N,\lambda}}, \quad (23)$$

with

$$\tau_N(\theta_\lambda) = \frac{\bar{P}_{N+1,\lambda}(\chi_\lambda)P_{N+1,\lambda}(\chi_\lambda)}{\bar{P}_{N+1,\lambda}(\bar{\chi}_\lambda)P_{N+1,\lambda}(\bar{\chi}_\lambda)},$$

and $B_{1,N,\lambda}, B_{2,N,\lambda}$ as previously. Hence we can write

$$\lambda \in (\text{Spec}(T_N(f)) \cap I_{0,\pi}) \iff \chi_\lambda^{2(N+2)} = \tau_N(\theta_\lambda), \lambda \in I_{0,\pi}. \quad (24)$$

Since the function H_λ is even, the constant $\tau_N(\theta_\lambda)$ can be rewritten as

$$\tau_N(\theta_\lambda) = \left(\frac{P_{N+1,\lambda}(\chi_\lambda)}{P_{N+1,\lambda}(\bar{\chi}_\lambda)} \right)^2.$$

On the other hand the function $\theta \mapsto \frac{P_{N+1,f(\theta)}(\bar{\chi}_\lambda)}{P_{N+1,f(\theta)}(\chi_\lambda)}$ is continuous from $[0, 2\pi]$ to $\{z \mid |z| = 1\}$ hence we have a function ρ_N defined and continuous on $[0, \pi]$ such that $\tau_N(\theta_\lambda) = e^{2i\rho_N(\theta_\lambda)}$. Then equation (24) can be written

$$\lambda \in (\text{Spec}(T_N(f)) \cap [f(\theta_1), f(\theta_2)]) \iff \theta_\lambda = \frac{\rho_N(\theta_\lambda) + k\pi}{(N+2)}, k \in [0, 2N+3]. \quad (25)$$

Hence for k in $\{0, \dots, 2N+3\}$ we have to find the solution in $[0, \pi]$ of the equation

$$(N+2)\theta - \rho_N(\theta) = k\pi \quad (26)$$

But in the particular case where $\theta_1 = 0$ and $\theta_2 = \pi$ it is easy to verify that the function ρ_N is in fact an odd 2π -periodic function.

Now we denote by F_N the function $\theta \mapsto (N+2)\theta - \rho_N(\theta)$. For $1 \leq k \leq N+1$ we have

$$F_N(0) = 0 < \pi k, F_N(\pi) = (N+2)\pi > \pi k.$$

Hence the equation (26) has at least one solution in $[0, \pi]$ for all $k \in \{0, \dots, N+1\}$. In the other hand it is obvious that the solution of the equations $\theta = \frac{\rho_N(\theta) + k\pi}{(N+2)}$ and $\theta = \frac{\rho_N(\theta) + k'\pi}{(N+2)}$ are different for $k \neq k'$. Since f is strictly increasing on $[0, \pi]$ we have found $N+1$ eigenvalues of $T_N(f)$ in $[f(0), f(\pi)]$, and we will not obtain other eigenvalues outside the set $\{0, \dots, N+1\}$. The rest of the proof is the same as the proof of Theorem 1.

4 Appendix

4.1 Inversion formula for Toeplitz matrices.

For the proof of Theorem 1 we have to know $T_N(f)_{1,1}^{-1}$. First we use Theorem 3 to obtain $T_N(f_r)_{1,1}^{-1}$ with $f_r = \chi_\lambda(1 - r\bar{\chi}_\lambda\chi)(1 - r\bar{\chi}_\lambda\bar{\chi}) \frac{1}{|P_{N+1,\lambda}|^2}$, and now $g_1 = \chi_\lambda(1 - r\bar{\chi}_\lambda\chi) \frac{1}{P_{N+1,\lambda}}$, $g_2 = (1 - r\bar{\chi}_\lambda\bar{\chi}) \frac{1}{P_{N+1,\lambda}}$

We have to observe that $T_N(f_r)_{1,1}^{-1}$ that is also $\langle T_N(f_r)^{-1}(1)|1 \rangle$. Write $\langle T_N(f_r)^{-1}(1)|1 \rangle = x_0 - y_0$. Theorem 3 provides

$$x_0 = \langle \pi_+ \left(\frac{1}{g_2} \right) \mid \frac{1}{g_1} \rangle = \chi_\lambda \left| \frac{1}{P_{N+1,\lambda}(0)} \right|^2.$$

To obtain y_0 we need the terms $\pi_+ \left(\tilde{\Phi}_N \pi_+ \left(\frac{1}{g_2} \right) \right)$ and $\pi_+ \left(\bar{\Phi}_N \pi_+ \left(\frac{1}{g_1} \right) \right)$. We have , if $\omega = r\bar{\chi}\lambda$,

$$\pi_+ \left(\tilde{\Phi}_N \pi_+ \left(\frac{1}{g_2} \right) \right) = \overline{P_{N+1,\lambda}(0)} \pi_+ \left(\frac{g_2}{g_1} \chi^{-N-1} \right) = C_1 \frac{1}{1 - \omega\chi}$$

with

$$C_1 = \overline{P_{N+1,\lambda}(0)} \bar{\chi}\lambda \left(\frac{P_{N+1,\lambda}(\frac{1}{\omega})}{\bar{P}_{N+1}(\omega)} \right) \omega^{N+1} (1 - \omega^2).$$

Likewise we can write

$$\pi_+ \left(\bar{\Phi}_N \pi_+ \left(\frac{1}{g_1} \right) \right) = C'_1 \frac{1}{1 - \bar{\omega}\chi},$$

with

$$C'_1 = \overline{P_{N+1,\lambda}(0)} \bar{\chi}\lambda \left(\frac{P_{N+1,\lambda}(\frac{1}{\bar{\omega}})}{\bar{P}_{N+1,\lambda}(\bar{\omega})} \right) \bar{\omega}^{N+1} (1 - \bar{\omega}^2).$$

Hence

$$y_0 = C_1 \bar{C}'_1 \left\langle (I - H_{\Phi_N^*} H_{\Phi_N})^{-1} \frac{1}{1 - \omega\chi} \middle| \frac{1}{1 - \bar{\omega}\chi} \right\rangle.$$

We have now to use the following lemma

Lemma 8 $\frac{1}{1 - \omega\chi}$ is an eigenvector of $H_{\Phi_N^*} H_{\Phi_N}$ for the eigenvalue $\tau_{N,r}(\omega)\omega^{2(N+2)}$ with $\tau_{N,r}(\omega) = \frac{P_{N+1,\lambda}(\frac{1}{\omega})P_{N+1,\lambda}(\frac{1}{\bar{\omega}})}{P_{N+1,\lambda}(\omega)P_{N+1,\lambda}(\bar{\omega})}$, with $|\omega^{2(N+2)}\tau_{N,r}(\omega)| < 1$ for $r \rightarrow 1$ and N sufficiently large.

It is Lemma 1 of [24]. We obtain

$$y_0 = C_1 \bar{C}'_1 \frac{1}{1 - \omega^{2N+2}\tau_{N,r}(\omega)} \frac{1}{1 - \omega^2}.$$

If now we consider the function f_1 defined by the product $f_1 = \tilde{g}_1 \tilde{g}_2$ with $\tilde{g}_1 = \chi_0(1 - \bar{\chi}_0\chi) \frac{1}{P_{N+1}}$ and $\tilde{g}_2 = (1 - \bar{\chi}_0\bar{\chi}) \frac{1}{P_{N+1}}$, then for a fixed N $\lim_{r \rightarrow 1} (T_N f_r)_{1,1}^{-1} = (T_N f)_{1,1}^{-1}$. Indeed

$$(T_N f_r)^{-1} (T_N f) = (T_N f_r)^{-1} (T_N f_r) + (T_N f_r)^{-1} (T_N (f_1 - f_r)).$$

And $\lim_{r \rightarrow 1} (T_N (f - f_r)) = 0$ that implies $\lim_{r \rightarrow 1} (T_N f_r)^{-1} (T_N f) = I_N$. Hence we can conclude that

$$(T_N(f))_{1,1}^{-1} = \frac{\left(1 - \bar{\chi}_\lambda^{2(N+2)} \tau_N(\chi_\lambda)\right) B_{2,N,\lambda} - B_{1,N,\lambda}}{1 - \bar{\chi}_\lambda^{2(N+2)} \tau_N(1\chi_\lambda)}, \quad (27)$$

with $B_{1,N,\lambda} = C_1 \bar{C}'_1 (1 - \bar{\chi}_\lambda^2)^{-1}$, $B_{2,N,\lambda} = \chi_\lambda \left| \frac{1}{P_{N+1,\lambda}(0)} \right|^2$, and $\tau_N(\chi_\lambda) = \frac{\overline{P_{N+1,\lambda}(\bar{\chi}_0)} P_{N+1,\lambda}(\bar{\chi}_\lambda)}{P_{N+1,\lambda}(\chi_0) P_{N+1,\lambda}(\chi_\lambda)}$

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